Meerdimensionale continue wavelettransformaties en veralgemeende fouriertransformaties in cliffordanalyse

Multi-dimensional Continuous Wavelet Transforms and Generalized Fourier Transforms in Clifford Analysis

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Voorwoord

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Chapter 1

Introduction

This work covers three mathematical analysis domains: Continuous Wavelet Transform, Fourier Transform and Clifford Analysis, and consists of three parts in which these domains interact.

The wavelet transform has become quite a standard tool in numerous research and application domains and its popularity has increased rapidly over the last few decades (see for e.g. [45, 51, 76]). The main idea of wavelet theory is to analyse a signal according to scale. Wavelets are functions that oscillate like a wave in a limited portion of time or space and vanish outside of it, i.e. they are wave-like but localized functions. One chooses a particular wavelet, dilates or contracts it (to meet a given scale) and shifts it, while looking into its correlations with the analysed signal. The signal correlations with wavelets dilated to large scales reveal gross (”rude”) features, while at small scales fine signal structures are discovered.
In such a scanning through a signal, the scale and position can vary continuously or in discrete steps. The former gives rise to the continuous wavelet transform (abbreviated CWT), the latter to the discrete wavelet transform (abbreviated DWT). The CWT and DWT enjoy more or less opposite properties and both have their specific field of application. The CWT is a successful tool for the analysis of signals and feature detection in signals, while the DWT provides a powerful technique for e.g. data compression and signal reconstruction. Although the DWT-inspired methods constitute the overwhelming majority among the wavelet community, the CWT and, more generally, redundant representations of signals, offer distinct advantages in certain cases.

Let us now give the pinnacles in the history of wavelets. The first known connection to modern wavelets dates back to the work of Jean-Baptiste Joseph Fourier (1768-1830). In 1807, Fourier became the founding father with his theory of frequency analysis, which proved to be enormously important and influential. His work is based on the fact that functions can be represented as the sum of sines and cosines. The first recorded mention of what we now call a "wavelet" seems to be in 1909, in the PhD-thesis on the orthogonal systems of functions by Alfred Haar (1885-1933). He defined the wavelet family which were named after him, the Haar wavelets. These compactly supported wavelets are the simplest of all wavelet families. After Haar's contribution to wavelets there was again a gap in time, until further progress was made by Paul Levy (1886-1971). This physicist investigated Brownian motion and discovered that the scale-varying Haar wavelets constituted a better basis than the Fourier basis functions. The concept of wavelets in its present theoretical form was first proposed in the early 1980's by the engineer Jean Morlet and the team at the Marseille Theoretical Physics centre working under the physicist Alex Grossmann. Morlet was the first researcher who used the term "wavelet" ("ondelette") to describe his functions; more specifically, he called them "Wavelets of Constant Slope". Morlet and Grossmann also laid the foundations of data compression. The next two important contributors to the field of wavelets were Yves Meyer and Stéphane Mallat who developed the concept of multiresolution analysis yielding the DWT. The two-dimensional CWT was the topic of the PhD-thesis [94] of Romain Murenzi. It was developed in co-operation with Jean-Pierre Antoine, Alex Grossmann and Ingrid Daubechies. The latter - who is born in Houthalen (Belgium) and is currently a professor at Princeton University - used, around 1988, the idea of multiresolution analysis to construct a family of wavelets which satisfy the properties of compact support, orthonormality and regularity.

More or less during the same period of time, also Clifford algebra and Clifford analysis found their origin. In 1882 a paper of William Kingdon Clifford (1845-1879) was published posthumously ([46]). In this paper he introduced the algebras which were named after him, as a generalization of both Grassmann's exterior algebra and Hamilton's algebra of quaternions. The importance of these algebras essentially lies in the fact that they incorporate inside one single structure as well the geometrical as the algebraic properties of Euclidean space; for that reason Clifford called these structures *geometrical algebras.*
These Clifford algebras were rediscovered at several occasions, in particular by physicists. For instance, when Paul A.M. Dirac in 1928 - in his famous article [58] on the electron - introduced the γ-matrices in order to linearize the Klein-

Gordon equation, he actually constructed the generators for the Clifford algebra $\mathbb{R}_{1,3}$.

When constructing the universal Clifford algebra over $\mathbb{R}$, the algebra of complex numbers is obtained. By multiplying the Cauchy-Riemann operator - which lies at the very heart of the theory of complex holomorphic functions - with its complex conjugate, one obtains the Laplace operator in two dimensions. Therefore, holomorphic functions in the complex plane may be considered as functions in the kernel of a first order rotationally invariant differential operator factorizing the Laplacian.

It is in this sense that Clifford analysis must be regarded as a natural generalization to higher dimension of the theory of holomorphic functions in the complex plane. The generalized holomorphic functions, known as *monogenic* functions, are thus to be interpreted as null solutions of the first order rotationally invariant differential operator factorizing the Laplacian in higher dimensions. This factorization of the Laplace operator establishes a special relationship between monogenic functions and harmonic functions of several variables, in that the properties of monogenic functions constitute a refinement of those of harmonic functions.
The first results in Clifford analysis were obtained independently in the 1930's by R. Fueter, G. Moisil and N. Théodorescu in their attempts to linearize the Laplace operator in m-dimensional Euclidean space. A profound study of Clifford analysis, drawing the parallels between the classical complex function theory on the one hand and this monogenic function theory on the other hand, can be found in the book [15] of Brackx, Delanghe and Sommen. Over the years, Clifford analysis has gained more and more interest and has grown out to a proper branch of classical analysis. Standard references are [55, 70, 72, 103].

An intrinsic feature of Clifford analysis is that it encompasses all dimensions at once, as opposed to a multi-dimensional tensorial approach with tensor products of one-dimensional phenomena. This true multi-dimensional nature allows for a very specific construction of higher dimensional wavelets and the development of the corresponding CWT-theory, based on generalizations to higher dimension of classical orthogonal polynomials on the real line. In Part I this wavelet construction procedure is presented within the usual, orthogonal Clifford analysis framework, while in Part III we generalize it to the metric dependent setting of Clifford analysis.

It is interesting to note that the connection between the CWT and Clifford analysis was already considered by M. Mitrea, V. Kisil and J. Cnops for the case of $\mathbb{R}^m$ and Hardy spaces (see [48, 79, 93]), however, by means of a different, group theoretical approach. For the development of wavelets on the unit ball using Clifford algebra, we refer to the work [43] of P. Cerejeiras, M. Ferreira and U. Kähler. Furthermore, Clifford algebra was also successfully used for constructing higher dimensional *discrete* orthogonal wavelets (see [1]).

The last topic dealt with in this thesis, needs little introduction. The Fourier transform, named after Joseph Fourier, is without any doubt one of the most powerful tools in pure and applied mathematics.
The idea of extending the Fourier transform to Clifford analysis was already worked out by several authors. Let us mention Bülow and Sommer, Felsberg, Ebling and Scheuermann, Sommen, Li, McIntosh and Qian; an overview is given at the beginning of Chapter 12. The novel Clifford-Fourier transform introduced in Part II, arises as a theoretical construct quite naturally in the spirit of the above mentioned refinement of harmonic functions by monogenic ones.

Now a more detailed overview of the content of this work is given.

We push off with two purely introductory chapters. In Chapter 2 we introduce the reader into the language of orthogonal Clifford algebra and Clifford analysis. The word "orthogonal" refers to the fact that the fundamental group leaving the Dirac operator invariant is the orthogonal group $\mathrm{SO}(m)$, which is doubly covered by the $\mathrm{Spin}(m)$-group. We also collect some frequently used results concerning the classical multi-dimensional Fourier transform and spherical harmonics.
The next chapter deals with the classical CWT, with emphasis on the one-dimensional case. It is mainly inspired by reference [5].

In **Part I** the domains introduced in the first two chapters are melted together, yielding multi-dimensional Clifford CWTs. In Chapter 4 we describe the general theory, which arose after a detailed study of the Clifford-Hermite CWT of Brackx and Sommen and the development of new specific Clifford wavelets. Indeed, we then realized that a general framework could be worked out in which all formerly constructed Clifford CWTs fit in. Moreover, this Clifford wavelet theory can be qualified as *isotropic*, since the metric in the underlying space is the standard Euclidean one.
This general wavelet construction method roughly proceeds as follows. The first step is the introduction of new polynomials generalizing classical orthogonal polynomials on the real line to the Clifford analysis setting. Here a crucial role is played by a specific Clifford analysis technique, the so-called Cauchy-Kowalewakaia extension of a real-analytic function in $\mathbb{R}^m$ to a monogenic func-

tion in $\mathbb{R}^{m+1}$. For the new Clifford polynomials a recurrence relation and a Rodrigues formula are established. This Rodrigues formula together with the Clifford Stokes theorem then lead to an orthogonality relation of the newly constructed polynomials. From this orthogonality relation we select candidates for mother wavelets and show that these candidates indeed may serve as kernel functions for a multi-dimensional CWT if they satisfy the mother wavelet conditions, mostly called admissibility conditions, in the orthogonal Clifford analysis setting.
In the following three chapters, specific examples in this Clifford wavelet theory are presented. We start with briefly recalling the Clifford-Hermite polynomials and associated Clifford CWT. These Clifford-Hermite polynomials were introduced by Sommen as a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. The reader will notice that they run like a continuous thread through this work. Next, we establish the so-called *bi-axial* generalizations of the Clifford-Hermite polynomials. These newly introduced polynomials give rise to wavelet functions which have an "elliptical form", the shape and orientation of which can be adapted in a flexible way.
Next, in Chapters 6 and 7, we establish multi-dimensional Clifford-wavelets starting from a Clifford generalization of the traditional Gegenbauer and Laguerre weight functions.

We then concentrate on the construction of wavelets based on a Clifford generalization of the remaining class of classical orthogonal polynomials, namely the Jacobi polynomials. However, in this case our general wavelet construction technique results into functions which do not satisfy the general mother wavelet conditions established in Chapter 4. Nevertheless, we are able to use these functions as kernel functions for a multi-dimensional Clifford CWT. To that end we are forced to use the orthogonal decomposition of the space of square integrable functions into the Hardy space $H^2(\mathbb{R}^m)$ and its orthogonal complement. Again, it turns out that these so-called Clifford-Jacobi wavelets can be seen as a specific application of another general theory, which we call the "Half" Clifford CWT theory. It is developed in Chapter 8.

In our quest for Clifford wavelets we came across a new method for constructing Clifford algebra-valued orthogonal polynomials in Euclidean space or in the open unit ball of it. As opposed to most of the preceding chapters, the class of weight functions involved is now enlarged to encompass *Clifford algebra-valued* functions. This construction method, which is the topic of Chapter 10, consists of transforming the orthogonality relation on the Euclidean space or on the open unit ball into an orthogonality relation on the real axis by means of the so-called Clifford-Heaviside functions, a multi-dimensional generalization of the Heaviside step-function on the real axis. Consequently, appropriate orthogonal polynomials on the real axis give rise to Clifford algebra-valued orthogonal polynomials in Euclidean space or in the open unit ball of it. Apparently, this construction method is simple, but nevertheless it should be emphasized that this is entirely due to the power of Clifford analysis and the existence of the idempotent Clifford-Heaviside functions inexisting in complex or harmonic analysis.

The central topic of **Part II** is the development of a new multi-dimensional Fourier transform in the framework of Clifford analysis, the so-called *Clifford-Fourier transform*.
Starting point for its construction was a study of the classical Fractional Fourier transform (abbreviated FrFT), which is a kind of fractional power of the standard Fourier transform. In the one-dimensional case, one obtains an integral representation for this FrFT by means of Mehler's formula for the classical Hermite polynomials. In Chapter 11, we proceed the other way around. First we introduce a multi-dimensional FrFT in the Clifford analysis setting making use of the Clifford-Hermite polynomials. By showing that it coincides with the classical multi-dimensional FrFT in the tensorial approach, we are able to prove Mehler's formula for the Clifford-Hermite polynomials.

When investigating the FrFT, our attention was drawn by an alternative representation of the classical tensorial Fourier transform $\mathcal{F}$. Indeed, this standard Fourier transform can be expressed as an operator exponential involving a scalar-valued differential operator kernel $\mathcal{H}$.
Due to the scalar character of the standard Fourier kernel, the Fourier spectrum inherits its Clifford algebra character from the original signal, without any interaction with the Fourier kernel. Hence, in order to genuinely introduce the Clifford analysis character into the Fourier transform the idea occurred to us to replace the scalar-valued operator $\mathcal{H}$ in the operator exponential by a Clifford algebra-valued one. To that end we aimed at factorizing the operator $\mathcal{H}$, making use of the factorization of the Laplace operator by the Dirac operator. Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second order operators, leads in a natural way to a pair of transforms $(\mathcal{F}_{\mathcal{H}^+}, \mathcal{F}_{\mathcal{H}^-})$, the harmonic average of which is precisely the standard Fourier transform $\mathcal{F}$. So one could say that the Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ offers a refinement of the classical Fourier transform $\mathcal{F}$ in a similar way as monogenic functions constitute a refinement of harmonic ones.
Moreover, the two-dimensional case of this Clifford-Fourier transform is special in that we are able to find a closed form for the kernel of the integral represen-

tation. This closed form enables us to generalize the well-known results for the standard Fourier transform both in the L_1 and in the L_2 context.

The last topic of Part II is Gabor and Hermite filters. They both belong to the class of mathematical models suggested for the receptive field profiles of the human visual system and therefore are also called "filters for early vision". Before establishing them in Clifford analysis, we recall their construction in the classical setting.
The building blocks for the so-called *Clifford-Hermite filters* are the Clifford-Hermite polynomials of Sommen, while the so-called *two-dimensional Clifford-Gabor filters* are based on the two-dimensional Clifford-Fourier transform mentioned above.

Finally, in **Part III** of this thesis, we develop the idea of a metric dependent Clifford analysis, offering the possibility of adjusting the co-ordinate system to preferential, not necessarily orthogonal, directions. In this new area of Clifford analysis we construct so-called anisotropic Clifford-Hermite wavelets which are adaptable to preferential directions in the signals or textures to be analysed.

Chapter 2

The orthogonal Clifford toolbox

Clifford analysis is nowadays a well established mathematical discipline closely related but complementary to harmonic analysis. It has gradually developed to a comprehensive theory which offers a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, it focusses on the null solutions of various special partial differential operators arising naturally within the Clifford algebra language, the most important of them being the so-called Dirac operator $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$. Here $(e_1, \ldots, e_m)$ forms an orthonormal basis for the quadratic space underlying the construction of the Clifford algebra. We will refer to this setting as the *orthogonal* one, since the fundamental group leaving the Dirac operator invariant is the orthogonal group $\mathrm{SO}(m)$ which is doubly covered by the $\mathrm{Spin}(m)$ group. Numerous papers, conference proceedings and books have moulded this theory and shown its ability for applications, let us mention [15, 55, 70, 71, 72, 93, 99, 103]. This chapter is included to make the text self-contained for the reader unfamiliar with Clifford algebra and Clifford analysis.

Before stating the most relevant results of orthogonal Clifford analysis (see Section 2.2), we start with the basic concepts of Clifford algebra (Section 2.1). In a final section, we also collect some results concerning the Fourier transform and spherical harmonics which will be frequently used in the sequel.

2.1 Clifford algebra

Clifford algebra may be seen as a generalization to higher dimension of the norm division algebras of the real numbers, the complex numbers and the quaternions; it is a norm algebra where the multiplication is non-commutative, but still associative. Let $\mathbb{R}^m$ be endowed with a non-degenerate quadratic form of signature (p,q), $p+q=m$, and let $(e_1,\ldots,e_m)$ be an orthonormal basis for $\mathbb{R}^{p,q}$. The non-commutative multiplication in the universal Clifford algebra $\mathbb{R}_{p,q}$, constructed over $\mathbb{R}^{p,q}$ is governed by the rules:

$$\begin{aligned} e_j^2 &= 1 \ , \quad j=1,\ldots,p \\ e_{p+j}^2 &= -1 \ , \quad j=1,\ldots,q \\ e_j e_k + e_k e_j &= 0 \ , \quad j \neq k \ , \quad j,k=1,\ldots,m \ . \end{aligned}$$

A canonical basis for $\mathbb{R}_{p,q}$ is obtained by considering for any set $A=\{j_1,\ldots,j_h\} \subset \{1,\ldots,m\} = M$, ordered by $1 \leq j_1 < j_2 < \ldots < j_h \leq m$, the element $e_A = e_{j_1} e_{j_2} \ldots e_{j_h}$. Moreover for the empty set $\emptyset$ one puts $e_\emptyset = 1$, the latter being the identity element. Any Clifford number λ in $\mathbb{R}_{p,q}$ may thus be written as $\lambda = \sum_{A\subset M} e_A \lambda_A$, $\lambda_A \in \mathbb{R}$, or still as $\lambda = \sum_{k=0}^m [\lambda]_k$, where $[\lambda]_k = \sum_{|A|=k} e_A \lambda_A$ is the so-called k-vector part of λ $(k=0,1,\ldots,m)$.
Denoting by $\mathbb{R}_{p,q}^k$ the *subspace of* all *k-vectors* in $\mathbb{R}_{p,q}$, i.e. the image of $\mathbb{R}_{p,q}$ under the projection operator $[\ .\ ]_k$, one has the *multi-vector structure* decomposition $\mathbb{R}_{p,q} = \mathbb{R}_{p,q}^0 \oplus \mathbb{R}_{p,q}^1 \oplus \ldots \oplus \mathbb{R}_{p,q}^m$, leading to the identification of $\mathbb{R}$ with the subspace of real scalars $\mathbb{R}_{p,q}^0$ and of $\mathbb{R}^m$ with the subspace of real Clifford vectors $\mathbb{R}_{p,q}^1$.
The Clifford number $e_M = e_1 e_2 \ldots e_m$ is mostly called the pseudoscalar; depending on the dimension m, the pseudoscalar commutes or anti-commutes with the k-vectors and squares to ± 1.
In the sequel we will consider the real Clifford algebra $\mathbb{R}_{0,m}$ and the complex Clifford algebra $\mathbb{C}_m$ which may be seen as its complexification: $\mathbb{C}_m = \mathbb{C} \otimes \mathbb{R}_{0,m} = \mathbb{R}_{0,m} \oplus i\ \mathbb{R}_{0,m}$.

The automorphisms on $\mathbb{C}_m$ leaving the multi-vector structure invariant are (i) the *main involution*, defined by

$$\begin{aligned} \widetilde{(\lambda\mu)} &= \widetilde{\lambda}\ \widetilde{\mu} \\ \widetilde{(\lambda_A e_A)} &= \lambda_A\ \widetilde{e_A} \quad (A \subset M) \\ \widetilde{e_j} &= -e_j \quad (j=1,\ldots,m) \ , \end{aligned}$$

(ii) the *reversion*, defined by

$$\begin{aligned}(\lambda\mu)^* &= \mu^*\lambda^* \\ (\lambda_A e_A)^* &= \lambda_A\ e_A^* \quad (A \subset M) \\ e_j^* &= e_j \quad (j = 1, \ldots, m)\end{aligned}$$

and finally, (iii) the *Hermitian conjugation*, defined by

$$\begin{aligned}(\lambda\mu)^\dagger &= \mu^\dagger\ \lambda^\dagger \\ (\lambda_A e_A)^\dagger &= \lambda_A^c\ e_A^\dagger \quad (A \subset M) \\ e_j^\dagger &= -e_j \quad (j = 1, \ldots, m) \quad .\end{aligned}$$

Here λ_A^c denotes the complex conjugate of the complex number λ_A.
In view of the decomposition $\mathbb{C}_m = \mathbb{R}_{0,m} \oplus i\ \mathbb{R}_{0,m}$, any complex Clifford number $\lambda \in \mathbb{C}_m$ may also be written as $\lambda = a + ib$ with $a, b \in \mathbb{R}_{0,m}$. Moreover, the restriction of the Hermitian conjugation to $\mathbb{R}_{0,m}$ coincides with the usual conjugation in $\mathbb{R}_{0,m}$, i.e. the main anti-involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, m$. Hence, one may also write

$$\lambda^\dagger = (a + ib)^\dagger = \overline{a} - i\overline{b} \quad .$$

The Hermitian conjugation leads to a *Hermitian inner product* and its associated *norm* on $\mathbb{C}_m$ given respectively by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0 \quad \text{and} \quad |\lambda|^2 = [\lambda^\dagger \lambda]_0 = \sum_A |\lambda_A|^2 \quad .$$

For $\lambda, \mu \in \mathbb{C}_m$ the following properties hold:

$$|\lambda\mu| \leq 2^m |\lambda|\ |\mu| \quad \text{and} \quad |\lambda + \mu| \leq |\lambda| + |\mu| \quad . \tag{2.1}$$

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \ldots, x_m)$ with the real vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^m e_j x_j \quad ,$$

whereas the Euclidean space $\mathbb{R}^{m+1}$ is identified with $\mathbb{R}_{0,m}^0 \oplus \mathbb{R}_{0,m}^1$ by identifying $(x_0, x_1, \ldots, x_m)$ with the real *paravector* $x_0 + \underline{x}$.

The product of two vectors splits up into a scalar part and a 2-vector, also called *bivector*, part:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y} \quad ,$$

where

$$\underline{x} \cdot \underline{y} = - < \underline{x}, \underline{y} > = - \sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j \ (x_i y_j - x_j y_i) \quad .$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign:

$$\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2 \quad .$$

The *even subalgebra* $\mathbb{R}^{+}_{0,m}$ of the Clifford algebra $\mathbb{R}_{0,m}$ is defined by

$$\mathbb{R}^{+}_{0,m} = \sum_{k \text{ even}} \oplus \, \mathbb{R}^{k}_{0,m} \quad .$$

The *Clifford group* $\Gamma(m)$ of the Clifford algebra $\mathbb{R}_{0,m}$, consists of those invertible elements λ in $\mathbb{R}_{0,m}$ for which the action $\lambda \underline{x} \overline{\lambda}$ on a vector $\underline{x} \in \mathbb{R}^{1}_{0,m}$ is again a vector. Its subgroup $\Gamma^{+}(m)$ is the intersection of $\Gamma(m)$ with the even subalgebra $\mathbb{R}^{+}_{0,m}$. The *Spin-group* $\text{Spin}(m)$ is the subgroup of $\Gamma^{+}(m)$ of those elements $s \in \Gamma^{+}(m)$ for which $ss^{*} = 1$, or equivalently

$$\text{Spin}(m) = \left\{ s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \ ; \ \underline{\omega}_j \in S^{m-1}, \ j = 1, \ldots, 2\ell, \ \ell \in \mathbb{N} \right\} \quad , \tag{2.2}$$

where S^{m-1} denotes the unit sphere in $\mathbb{R}^m$. The Spin-group is a two-fold covering group of the rotation group $\text{SO}(m)$. For $T \in \text{SO}(m)$, there exists $s \in \text{Spin}(m)$ such that $T(\underline{x}) = s \underline{x} \overline{s} = (-s) \ \underline{x} \ (-\overline{s})$, for all $\underline{x} \in \mathbb{R}^m$.

2.2 Clifford analysis

Clifford analysis offers a function theory which is a higher-dimensional analogue of the theory of the holomorphic functions of one complex variable. The functions considered are defined in the Euclidean space $\mathbb{R}^m$ or $\mathbb{R}^{m+1}$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_{0,m}$ or in its complexification $\mathbb{C}_m$.

The central notion in Clifford analysis is the notion of monogenicity, a notion which is the multi-dimensional counterpart to that of holomorphy in the complex plane.
A function $F(x_1,\ldots,x_m)$, respectively $F(x_0,x_1,\ldots,x_m)$, defined and continuously differentiable in an open region of $\mathbb{R}^m$, respectively $\mathbb{R}^{m+1}$, and taking values in $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$, is called *left monogenic* in that region if

$$\partial_{\underline{x}} F = 0 \quad , \quad \text{respectively} \quad (\partial_{x_0} + \partial_{\underline{x}})F = 0 \quad .$$

Here $\partial_{\underline{x}}$ is the *Dirac operator* in $\mathbb{R}^m$:

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j} \quad ,$$

an elliptic, rotation-invariant, vector differential operator of the first order, which may be looked upon as the "square root" of the Laplace operator in $\mathbb{R}^m$:

$$\Delta_m = -\partial_{\underline{x}}^2 \quad , \tag{2.3}$$

whereas $\partial_{x_0} + \partial_{\underline{x}}$ is the *Cauchy-Riemann operator* in $\mathbb{R}^{m+1}$ for which

$$\Delta_{m+1} = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} + \overline{\partial_{\underline{x}}}) = (\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}}) \quad .$$

The factorization (2.3) of the Laplace operator establishes a special relationship between Clifford analysis and harmonic analysis in that monogenic functions refine the properties of harmonic functions. Note, for instance, that each harmonic function $h(\underline{x})$ can be split as $h(\underline{x}) = f(\underline{x}) + \underline{x}\ g(\underline{x})$ with f and g monogenic, and that a real harmonic function is always the real part of a monogenic one, which does not need to be the case for a harmonic function of several complex variables.
The notion of *right monogenicity* is defined in a similar way by letting act the Dirac operator or the Cauchy-Riemann operator from the right. It is easily seen that if a Clifford algebra-valued function F is left monogenic, its Hermitian conjugate $F^\dagger$ is right monogenic.

Introducing spherical co-ordinates in $\mathbb{R}^m$ by:

$$\underline{x} = r\underline{\omega} \quad , \quad r = |\underline{x}| \in [0, +\infty[\quad , \quad \underline{\omega} \in S^{m-1} \quad ,$$

the Dirac operator takes the form:

$$\partial_{\underline{x}} = \underline{\omega} \left(\partial_r + \frac{1}{r}\, \Gamma \right) \quad ,$$

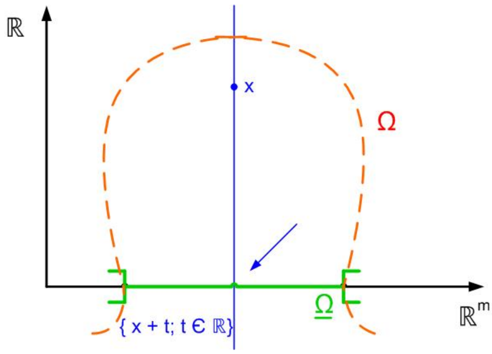


Figure 2.1: An x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$.

where

$$\Gamma = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j \; (x_i \partial_{x_j} - x_j \partial_{x_i})$$

is the so-called *angular Dirac operator* depending only on the angular co-ordinates.
In Chapter 12 we will also use the *angular momentum operators*

$$L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \quad , \quad i, j = 1, 2, \ldots, m \quad .$$

Another fundamental operator is the *Euler operator*

$$E = < \underline{x}, \partial_{\underline{x}} > = \sum_{i=1}^{m} x_i \partial_{x_i} \quad ,$$

which measures the degree of homogeneity of both polynomials and polynomial operators.

A fundamental method for constructing monogenic functions is the so-called Cauchy-Kowalewskaia extension procedure, introduced by Sommen in [104]. It runs as follows.
If $\underline{\Omega} \subset \mathbb{R}^m$ is open, then an open neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$ is said to be x_0*-normal* if for each $x \in \Omega$ the line segment $\{x + t \; ; \; t \in \mathbb{R}\} \cap \Omega$ is connected and contains exactly one point in $\underline{\Omega}$ (see Figure 2.1).

Considering $\mathbb{R}^m$ as the hyperplane $x_0 = 0$ in $\mathbb{R}^{m+1}$, a real-analytic function $f(\underline{x})$ in an open connected domain $\underline{\Omega}$ in $\mathbb{R}^m$ can be uniquely extended to a monogenic function $F(x_0, \underline{x})$ in an open connected and x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$. This so-called *Cauchy-Kowalewskaia (CK-) extension* of $f(\underline{x})$ is given by

$$F(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} (-1)^\ell \, \frac{x_0^\ell}{\ell!} \, \partial_{\underline{x}}^\ell f(\underline{x}) \quad . \tag{2.4}$$

The CK-extension procedure leads to the *CK-product* which, despite the non-commutativity of the Clifford algebra, preserves the monogenicity of the factors: the CK-product of two monogenic functions in $\mathbb{R}^{m+1}$ is the CK-extension to $\mathbb{R}^{m+1}$ of the product of the real-analytic restrictions to $\mathbb{R}^m$. For example, if $z_j, \; j = 1, 2, \ldots, m$, denote the monogenic variables in $\mathbb{R}^{m+1}$:

$$z_j = x_j - x_0 \, e_j \quad ,$$

then the CK-product of z_j and z_k $(k \neq j)$ is the CK-extension of $x_j x_k$, given by

$$z_j \odot z_k = \frac{1}{2} \, (z_j z_k + z_k z_j) \quad .$$

The CK-products of the monogenic variables are the building blocks of the Taylor series expansion of a monogenic function.

In this work, the monogenic homogeneous polynomials, or spherical monogenics for short, will play an important role.
A left, respectively right, monogenic homogeneous polynomial P_k of degree k $(k \geq 0)$ in $\mathbb{R}^m$ is called a left, respectively right, *solid inner spherical monogenic* of order k. A left, respectively right, monogenic homogeneous function Q_k of degree $-(k+m-1)$ in $\mathbb{R}^m \setminus \{0\}$ is called a left, respectively right, *solid outer spherical monogenic* of order k.
The set of all left, respectively right, solid inner spherical monogenics of order k will be denoted by $M_\ell^+(k)$, respectively $M_r^+(k)$, while the set of all left, respectively right, solid outer spherical monogenics of order k will be denoted by $M_\ell^-(k)$, respectively $M_r^-(k)$.
The dimension of $M_\ell^+(k)$ is given by

$$\dim \left(M_\ell^+(k) \right) = \frac{(m+k-2)!}{(m-2)! \; k!} \quad .$$

It should also be noted that if P_k is a left solid inner spherical monogenic of order k in $\mathbb{R}^m$, then

$$Q_k(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^m} \, P_k\left(\frac{\underline{x}}{|\underline{x}|^2}\right) = \frac{\underline{x} \, P_k(\underline{x})}{|\underline{x}|^{2k+m}} = \frac{\underline{\omega} \, P_k(\underline{\omega})}{r^{k+m-1}}$$

is a left solid outer spherical monogenic of order k in $\mathbb{R}^m \setminus \{0\}$.
For $P_k \in M_\ell^+(k)$ and $s \in \mathbb{N}$ the following fundamental formula holds:

$$\partial_{\underline{x}}[\underline{x}^s P_k] = \begin{cases} -s \; \underline{x}^{s-1} P_k & \text{for } s \text{ even} \\ -(s+2k+m-1) \; \underline{x}^{s-1} P_k & \text{for } s \text{ odd.} \end{cases} \tag{2.5}$$

The set of harmonic homogeneous polynomials S_k of degree k in $\mathbb{R}^m$:

$$\Delta_m S_k(\underline{x}) = 0 \quad \text{and} \quad S_k(t\underline{x}) = t^k S_k(\underline{x}) \quad ,$$

usually called *solid spherical harmonics*, is denoted by $\mathcal{H}(k)$.
Obviously we have that

$$M_\ell^+(k) \subset \mathcal{H}(k) \quad \text{and} \quad M_r^+(k) \subset \mathcal{H}(k) \quad .$$

Let $H_{(r)}$ be a *unitary right Clifford-module*, i.e. $H_{(r)}, +$ is an abelian group and a law $(f, \lambda) \to f\lambda$ from $H \times \mathbb{R}_{0,m}$ into $H_{(r)}$ is defined such that for all $\lambda, \mu \in \mathbb{R}_{0,m}$ and $f, g \in H_{(r)}$:
(i) $f(\lambda + \mu) = f\lambda + f\mu$
(ii) $f(\lambda\mu) = (f\lambda)\mu$
(iii) $(f+g)\lambda = f\lambda + g\lambda$
(iv) $fe_\emptyset = f$.
Note that $H_{(r)}$ becomes a real vector space if $\mathbb{R}$ is identified with $\mathbb{R}e_\emptyset \subset \mathbb{R}_{0,m}$.
Then a function $(\, . \, , \, . \,) : H_{(r)} \times H_{(r)} \to \mathbb{R}_{0,m}$ is said to be an *inner product* on $H_{(r)}$ if for all $f, g, h \in H_{(r)}$ and $\lambda \in \mathbb{R}_{0,m}$:
(i) $(f, g\lambda + h) = (f,g)\lambda + (f,h)$
(ii) $(f,g) = \overline{(g,f)}$
(iii) $[(f,f)]_0 \geq 0$ and $[(f,f)]_0 = 0$ if and only if $f = 0$.
From this $\mathbb{R}_{0,m}$-valued inner product $(\, . \, , \, . \,)$, one can deduce the real inner product

$$(f,g)_R = [(f,g)]_0$$

on $H_{(r)}$, the latter being considered as a real vector space.
Putting for each $f \in H_{(r)}$

$$\|f\|^2 = [(f,f)]_0 \quad ,$$

$\| \, . \, \|$ is a *norm* on $H_{(r)}$ turning it into a *normed right Clifford-module.*
Now, let $H_{(r)}$ be a unitary right Clifford-module provided with an inner product $(\, . \, , \, . \,)$. Then it is called a *right Hilbert Clifford-module* if $H_{(r)}$ considered as a real vector space provided with the real inner product $(\, . \, , \, . \,)_R$ is a Hilbert space (see [54]).

Let h be a positive function on $\mathbb{R}^m$. Then we consider the Clifford algebra-valued inner product of the functions f and g defined in $\mathbb{R}^m$ and taking values in the Clifford algebra $\mathbb{C}_m$:

$$< f, g > = \int_{\mathbb{R}^m} h(\underline{x}) \; f^\dagger(\underline{x}) \; g(\underline{x}) \; dV(\underline{x}) \quad ,$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^m$, and moreover the associated norm

$$\|f\|^2 = [< f, f >]_0 \quad .$$

The unitary right Clifford-module of Clifford algebra-valued measurable functions on $\mathbb{R}^m$ for which $\|f\|^2 < \infty$ is a right Hilbert Clifford-module which we denote by $L_2\left(\mathbb{R}^m, h(\underline{x}) \; dV(\underline{x})\right)$.

In particular, if we take $h(\underline{x}) \equiv 1$, we obtain the right Hilbert-module of *square integrable functions*:

$$L_2\left(\mathbb{R}^m, dV(\underline{x})\right) \;=\; \Big\{ f : \text{Lebesgue measurable in } \mathbb{R}^m \text{ for which } \|f\|_2 = \left(\int_{\mathbb{R}^m} |f(\underline{x})|^2 \; dV(\underline{x}) \right)^{1/2} < \infty \Big\} \quad .$$

The group Spin(m) has two often used representations on the space $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$; they are

- the *L-representation*

$$L(s)f(\underline{x}) = sf(\overline{s}\underline{x}s) \quad , \qquad s \in \text{Spin}(m)$$

- the *H-representation*

$$H(s)f(\underline{x}) = sf(\overline{s}\underline{x}s)\overline{s} \quad , \qquad s \in \text{Spin}(m) \quad .$$

Naturally, $L_1\left(\mathbb{R}^m, dV(\underline{x})\right)$ denotes the right Clifford-module of *integrable functions*:

$$L_1\left(\mathbb{R}^m, dV(\underline{x})\right) = \Big\{ f : \text{Lebesgue measurable in } \mathbb{R}^m \text{ for which} \\ \|f\|_1 = \int_{\mathbb{R}^m} |f(\underline{x})| \; dV(\underline{x}) < \infty \Big\} \;,$$

while $C_0\big(\mathbb{R}^m\big)$, $L_\infty\big(\mathbb{R}^m\big)$ and $\mathcal{S}\big(\mathbb{R}^m\big)$ denote the right Clifford-modules of $\mathbb{C}_m$-valued, respectively continuous, bounded and rapidly decreasing functions.

In Clifford analysis extensive use is made of the standard tensorial multi-dimensional *Fourier transform* given by:

$$\mathcal{F}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \; f(\underline{x}) \; dV(\underline{x}) \quad . \tag{2.6}$$

The Gaussian function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$ is an eigenfunction of this Fourier transform:

$$\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad . \tag{2.7}$$

Next, let us introduce the *fundamental solution E of the Cauchy-Riemann operator* in $\mathbb{R}^{m+1}$:

$$E(x_0, \underline{x}) = \frac{1}{A_{m+1}} \frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{m+1}} \quad .$$

It satisfies

$$(\partial_{x_0} + \partial_{\underline{x}}) E(x_0, \underline{x}) = \delta(x_0, \underline{x})$$

in the distributional sense. Here

$$A_{m+1} = \frac{2 \; \pi^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}\right)}$$

stands for the area of the unit sphere S^m in $\mathbb{R}^{m+1}$.
We may now define for a square integrable function $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$, its *Cauchy integral* (see for e.g. [70]) in the half spaces

$$\mathbb{R}^{m+1}_{\pm} = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1} \; : \; x_0 \gtrless 0\}$$

by

$$\mathcal{C}[f](x_0,\underline{x}) = E(x_0,\ .\ ) * f(\ .\ )(\underline{x}) = \int_{\mathbb{R}^m} E(x_0,\underline{x}-\underline{y})\ f(\underline{y})\ dV(\underline{y}) \quad .$$

This Cauchy integral is a linear isomorphism between $L_2(\mathbb{R}^m, dV(\underline{x}))$ and the so-called *Hardy spaces* $H^2(\mathbb{R}^{m+1}_\pm)$, defined by

$$\begin{aligned} H^2(\mathbb{R}^{m+1}_\pm) \quad = \quad & \Big\{ F(x_0,\underline{x}) \text{ left monogenic in } \mathbb{R}^{m+1}_\pm \text{ such that} \\ & \sup_{x_0 \gtrless 0} \int_{\mathbb{R}^m} |F(x_0,\underline{x})|^2\ dV(\underline{x}) < +\infty \Big\} \quad . \end{aligned}$$

Furthermore, the $L_2(\mathbb{R}^m, dV(\underline{x}))$ non-tangential boundary values for $x_0 \to 0+$ and $x_0 \to 0-$ of the Cauchy integral take the following form:

$$\mathcal{C}^+[f](\underline{x}) := \lim_{\substack{x_0 \to 0 \\ >}} \mathcal{C}[f](x_0,\underline{x}) = \frac{1}{2} f(\underline{x}) + \frac{1}{2} H[f](\underline{x})$$

and

$$\mathcal{C}^-[f](\underline{x}) := \lim_{\substack{x_0 \to 0 \\ <}} \mathcal{C}[f](x_0,\underline{x}) = -\frac{1}{2} f(\underline{x}) + \frac{1}{2} H[f](\underline{x}) \quad .$$

Here $H[f]$ denotes the *Hilbert transform* of the function f given by

$$\begin{aligned} H[f](\underline{x}) \quad &= \quad \frac{2}{A_{m+1}}\ \mathrm{Pv} \frac{\overline{\underline{x}}}{r^{m+1}} * f \\ &= \quad \frac{2}{A_{m+1}}\ \mathrm{Pv} \int_{\mathbb{R}^m} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{|\underline{x}-\underline{y}|^{m+1}}\ f(\underline{y})\ dV(\underline{y}) \\ &= \quad -\frac{2}{A_{m+1}} \lim_{\substack{\epsilon \to 0 \\ >}} \int_{|\underline{x}-\underline{y}|>\epsilon} \frac{\underline{x}-\underline{y}}{|\underline{x}-\underline{y}|^{m+1}}\ f(\underline{y})\ dV(\underline{y}) \quad . \end{aligned}$$

The Hilbert transform is a bounded unitary operator on $L_2(\mathbb{R}^m, dV(\underline{x}))$ which squares to the identity operator $\mathbb{I}$: $H^2 = \mathbb{I}$.
Next, we introduce another *Hardy space*, namely $H^2(\mathbb{R}^m)$ which is defined as the closure in $L_2(\mathbb{R}^m, dV(\underline{x}))$ of the space of the non-tangential boundary values for $x_0 \to 0+$ of all functions in $H^2(\mathbb{R}^{m+1}_+)$.
As $H^2(\mathbb{R}^m)$ is a closed subspace of $L_2(\mathbb{R}^m, dV(\underline{x}))$, we obtain the following orthogonal decomposition:

$$L_2(\mathbb{R}^m, dV(\underline{x})) = H^2(\mathbb{R}^m) \oplus H^2(\mathbb{R}^m)^\perp \quad .$$

Hence, there exist two projection operators, the so-called *Szegö projections*, given by

$$\mathbb{P}^+ : L_2(\mathbb{R}^m, dV(\underline{x})) \rightarrow H^2(\mathbb{R}^m) \quad \text{and} \quad \mathbb{P}^- : L_2(\mathbb{R}^m, dV(\underline{x})) \rightarrow H^2(\mathbb{R}^m)^\perp \quad .$$

It is clear that these Szegö projections coincide with the so-called Hardy projections $\mathcal{C}^+$ and (- $\mathcal{C}^-$) :

$$\mathbb{P}^+[f] = \mathcal{C}^+[f] = \frac{1}{2}(f + H[f]) \quad \text{and} \quad \mathbb{P}^-[f] = -\mathcal{C}^-[f] = \frac{1}{2}(f - H[f]) \quad .$$

Moreover, the Fourier transform of the Hilbert transform is given by

$$\mathcal{F}\big[H[f]\big](\underline{\xi}) = i\underline{\eta}\ \mathcal{F}[f](\underline{\xi}) \quad ,$$

where we have used spherical co-ordinates in frequency space given by

$$\underline{\xi} = \rho\ \underline{\eta} \quad , \quad \rho = |\underline{\xi}| \in [0, +\infty[\quad , \quad \underline{\eta} \in S^{m-1} \quad .$$

The orthogonal decomposition of an $L_2(\mathbb{R}^m, dV(\underline{x}))$ - function f :

$$f = \mathbb{P}^+[f] + \mathbb{P}^-[f]$$

thus reads in frequency space

$$\mathcal{F}[f] = \frac{1}{2}(1 + i\underline{\eta})\ \mathcal{F}[f] + \frac{1}{2}(1 - i\underline{\eta})\ \mathcal{F}[f] \quad .$$

Here the so-called *Clifford-Heaviside functions*

$$P^+ = \frac{1}{2}\left(1 + i\underline{\eta}\right) \quad , \quad P^- = \frac{1}{2}\left(1 - i\underline{\eta}\right) \quad , \quad \underline{\eta} \in S^{m-1} \quad ,$$

appear; they were introduced independently by Sommen in [105] and McIntosh in [81] and [92]. They generalize to higher dimension the Heaviside step-function on the real axis and are a typical feature of Clifford analysis. Moreover, these Clifford-Heaviside functions are self-adjoint mutually orthogonal primitive idempotents:

$$P^+ + P^- = 1 \ ; \ (P^\pm)^\dagger = P^\pm \ ; \ P^+P^- = P^-P^+ = 0 \ ; \ (P^\pm)^2 = P^\pm \quad . \tag{2.8}$$

Furthermore, they satisfy

$$i\underline{\eta}\ P^\pm = \pm P^\pm$$

and hence

$$i\underline{\xi}\ P^\pm = \pm|\underline{\xi}|\ P^\pm \quad \text{or} \quad \underline{\xi}P^\pm = \mp i|\underline{\xi}|P^\pm \quad . \tag{2.9}$$

2.3 Some useful results

We start this section with a few fundamental results concerning the Fourier transform.

Theorem 2.1 *The Fourier transform $\mathcal{F}$ is an isometry on the space of square integrable functions, in other words, for all $f, g \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ the Parseval formula holds:*

$$< f, g > = < \mathcal{F}[f], \mathcal{F}[g] > \quad .$$

In particular, for each $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ one has:

$$\|f\|_2 = \|\mathcal{F}[f]\|_2 \quad .$$

Proposition 2.1 *The Fourier transform $\mathcal{F}$ satisfies:*
(i) the multiplication rule:

$$\mathcal{F}[\underline{x} f(\underline{x})](\underline{\xi}) = i\partial_{\underline{\xi}} \ \mathcal{F}[f(\underline{x})](\underline{\xi}) \quad ;$$

(ii) the differentiation rule:

$$\mathcal{F}[\partial_{\underline{x}} f(\underline{x})](\underline{\xi}) = i\underline{\xi} \ \mathcal{F}[f(\underline{x})](\underline{\xi}) \quad .$$

A basic integral formula which plays an important role in the sequel of this work is the *Clifford-Stokes formula.*
Let $\Omega \subset \mathbb{R}^m$ be open, let C be a compact orientable m-dimensional manifold with boundary ∂C and define the oriented $\mathbb{C}_m$-valued surface element $d\sigma_{\underline{x}}$ on ∂C by the Clifford differential form

$$d\sigma_{\underline{x}} = \sum_{j=1}^{m} (-1)^j \ e_j \ dx^{M\setminus\{j\}} \quad ,$$

where

$$dx^{M\setminus\{j\}} = dx_1 \wedge \ldots \wedge [dx_j] \wedge \ldots \wedge dx_m \quad , \quad j = 1, 2, \ldots, m \quad .$$

We also put

$$dx^M = dx_1 \wedge \ldots \wedge dx_m \quad .$$

If $n(\underline{x})$ stands for the outward pointing unit normal at $\underline{x} \in \partial C$, then

$$d\sigma_{\underline{x}} = n(\underline{x}) \ d\Sigma(\underline{x}) \quad ,$$

$d\Sigma(\underline{x})$ being the elementary Lebesgue surface measure.

Theorem 2.2 (Clifford-Stokes theorem) *Let $f, g \in C_1(\Omega)$. Then for each $C \subset \Omega$, one has*

$$\int_{\partial C} f(\underline{x})\ d\sigma_{\underline{x}}\ g(\underline{x}) = \int_C [(f\partial_{\underline{x}})\ g + f\ (\partial_{\underline{x}} g)]\ dx^M \quad .$$

In particular, for $f \equiv 1$:

$$\int_{\partial C} d\sigma_{\underline{x}}\ g(\underline{x}) = \int_C \partial_{\underline{x}}[g]\ dx^M \quad .$$

Putting $g \equiv 1$, the above Clifford-Stokes theorem immediately yields *Cauchy's theorem.*

Corollary 2.1 (Cauchy's theorem) *Suppose that $f \in C_1(\Omega)$ is right monogenic in Ω. Then for each $C \subset \Omega$, one has*

$$\int_{\partial C} f(\underline{x})\ d\sigma_{\underline{x}} = 0 \quad .$$

An important particular example of Cauchy's theorem occurs in the following case: take $f = 1$ and $C = \overline{B}(1) = \{\underline{x} \in \mathbb{R}^m; |\underline{x}| \leq 1\}$, the closed unit ball in $\mathbb{R}^m$. Then $\partial C = S^{m-1}$ and at each point $\underline{\omega} \in S^{m-1}$, $n(\underline{\omega}) = \underline{\omega}$, whence $d\sigma_{\underline{\omega}} = \underline{\omega}\ dS(\underline{\omega})$ with $dS(\underline{\omega})$ the Lebesgue measure on S^{m-1}.
Consequently, we have

$$\int_{S^{m-1}} \underline{\omega}\ dS(\underline{\omega}) = 0 \quad ,$$

confirming the fact that $\underline{\omega}$ is a spherical harmonic.
The above result will be of crucial importance in our method described in Chapter 10, for constructing Clifford algebra-valued orthogonal polynomials.

Next, we formulate the *Funk-Hecke theorem* (see [74]).

Theorem 2.3 (Funk-Hecke theorem) *Let $S_k \in \mathcal{H}(k)$ be a solid spherical harmonic of degree k, then*

$$\int_{S^{m-1}} f(<\underline{\omega}, \underline{\eta}>)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \quad A_{m-1}\ S_k(\underline{\eta}) \int_{-1}^{1} f(t)\ (1-t^2)^{(m-3)/2}\ P_k(t)\ dt \quad ,$$

where $P_k(t)$ *denotes the Legendre polynomial of degree* k *in* $\mathbb{R}^m$ *and*

$$A_{m-1} = \frac{2\ \pi^{(m-1)/2}}{\Gamma\left(\frac{m-1}{2}\right)}$$

stands for the area of the unit sphere S^{m-2} *in* $\mathbb{R}^{m-1}$.

This leads to the following result (see for e.g. [110]).

Proposition 2.2 *Let* $S_k \in \mathcal{H}(k)$ *be a solid spherical harmonic of degree* k*, then for* $r,\ \rho > 0$ *and* $\underline{\omega},\ \underline{\eta} \in S^{m-1}$*, one has*

$$\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \quad (-i)^k\ (2\pi)^{m/2}\ S_k(\underline{\eta})\ (\rho r)^{1-m/2}\ J_{k+m/2-1}(\rho r) \quad .$$

Here $J_{k+m/2-1}$ *denotes the Bessel function of the first kind of order* $k + \dfrac{m}{2} - 1$.

Proof. Application of the Funk-Hecke theorem leads to

$$\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \quad A_{m-1}\ S_k(\underline{\eta}) \int_{-1}^{+1} \exp\left(-ir\rho t\right)\ (1-t^2)^{(m-3)/2}\ P_k(t)\ dt \quad .$$

As

$$P_k(t) = \frac{k!\ (m-3)!}{(k+m-3)!}\ C_k^{(m-2)/2}(t)$$

and the Gegenbauer polynomials C_k^λ satisfy

$$C_k^\lambda(-x) = (-1)^k\ C_k^\lambda(x) \quad ,$$

we obtain

$$\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \quad (-1)^k\ A_{m-1}\ S_k(\underline{\eta}) \int_{-1}^{+1} \exp\left(ir\rho t\right)\ (1-t^2)^{(m-3)/2}\ P_k(t)\ dt \quad . \tag{2.10}$$

As (see [74])

$$\int_{-1}^{+1} \exp(ir\rho t)\ (1-t^2)^{(m-3)/2}\ P_k(t)\ dt$$
$$= \ i^k\ 2^{m/2-1}\ \sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)\ (\rho r)^{1-m/2}\ J_{k+m/2-1}(\rho r)$$

with $J_{k+m/2-1}$ the Bessel function of the first kind of order $k+\frac{m}{2}-1$, equation (2.10) becomes

$$\int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\eta} >)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \ (-i)^k\ 2^{m/2-1}\ \sqrt{\pi}\ \Gamma\left(\frac{m-1}{2}\right)\ A_{m-1}\ S_k(\underline{\eta})\ (\rho r)^{1-m/2}\ J_{k+m/2-1}(\rho r) \quad .$$

Inserting the expression for the area A_{m-1} of the unit sphere S^{m-2} in $\mathbb{R}^{m-1}$, we finally obtain

$$\int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\eta} >)\ S_k(\underline{\omega})\ dS(\underline{\omega})$$
$$= \ (-i)^k\ (2\pi)^{m/2}\ S_k(\underline{\eta})\ (\rho r)^{1-m/2}\ J_{k+m/2-1}(\rho r) \quad . \quad \square$$

By means of this result one can prove the following Proposition (see for e.g. [19]).

Proposition 2.3 *Let $S_k \in \mathcal{H}(k)$ be a solid spherical harmonic of degree k, then*

$$S_k(\partial_{\underline{x}}) \exp\left(-\frac{|\underline{x}|^2}{2}\right) = (-1)^k\ S_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \quad .$$

Proof. By means of Proposition 2.1, we have

$$\mathcal{F}\left[S_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) = S_k(i\partial_{\underline{\xi}})\ \mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi})$$
$$= i^k\ S_k(\partial_{\underline{\xi}}) \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)$$

or equivalently

$$S_k(\partial_{\underline{\xi}}) \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) = (-i)^k\ \mathcal{F}\left[S_k(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \quad .$$

Introducing spherical co-ordinates:

$$\underline{x} = r\,\underline{\omega} \quad , \quad \underline{\xi} = \rho\,\underline{\eta} \quad ; \quad r = |\underline{x}| \quad , \quad \rho = |\underline{\xi}| \quad , \quad \underline{\omega},\ \underline{\eta} \in S^{m-1} \quad ,$$

we obtain

$$\begin{aligned}
&\mathcal{F}\left[S_k(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&\quad = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) S_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) dV(\underline{x}) \\
&\quad = \frac{1}{(2\pi)^{m/2}} \int_0^{+\infty} r^{k+m-1} \exp\left(-\frac{r^2}{2}\right) dr \\
&\qquad\quad \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) S_k(\underline{\omega})\ dS(\underline{\omega}) \quad .
\end{aligned}$$

Next, Proposition 2.2 leads to

$$\begin{aligned}
&\mathcal{F}\left[S_k(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&\quad = (-i)^k\ \rho^{1-m/2}\ S_k(\underline{\eta}) \int_0^{+\infty} r^{k+m/2}\ J_{k+m/2-1}(\rho r)\ \exp\left(-\frac{r^2}{2}\right) dr
\end{aligned}$$

with $J_{k+m/2-1}$ the Bessel function of the first kind of order $k + \dfrac{m}{2} - 1$. From the theory of Bessel functions (see [65]) we know that

$$\int_0^{+\infty} r^{k+m/2}\ J_{k+m/2-1}(\rho r)\ \exp\left(-\frac{r^2}{2}\right) dr = \rho^{k+m/2-1}\ \exp\left(-\frac{\rho^2}{2}\right) \quad .$$

Hence, we finally obtain

$$\begin{aligned}
\mathcal{F}\left[S_k(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) &= (-i)^k\ \rho^k\ S_k(\underline{\eta})\ \exp\left(-\frac{\rho^2}{2}\right) \\
&= (-i)^k\ S_k(\underline{\xi}) \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad ,
\end{aligned}$$

which leads to the desired result. □

Chapter 3

The classical Continuous Wavelet Transform

This chapter is purely introductory and aims at making our text self-contained for the reader who is unfamiliar with wavelet theory. It contains a rather detailed account of the one-dimensional Continuous Wavelet Transform, offering a touchstone for our theory developed in the following chapters.

3.1 The one-dimensional continuous wavelet transform

Wavelet analysis is a particular time- or space-scale representation of signals, which has found numerous applications in mathematics, physics and engineering. The range of pure and applied fields touched upon by wavelets is extremely wide: Fourier analysis, approximation theory, functional analysis, operator theory, group representation theory, theory of fractals, numerical analysis, signal analysis, image compression, computer science, electrical engineering, physics etc. In this work, emphasis is laid on the theoretical point of view.
For a standard introduction to wavelet analysis we refer the reader to for e.g. [45, 51, 76] and for a detailed overview of the continuous wavelet transform in particular, we recommend [5].

Let us start with a survey of the one-dimensional (1-D) continuous wavelet transform.

3.1.1 The idea of wavelet analysis

Most signals in practice are non-stationary and cover a wide range of frequencies; in general low frequency segments of a signal last a relatively long time, whereas high frequencies occur for a short moment. Classical Fourier analysis is inadequate for analysing such kind of signals: the information about the time localization of a given frequency is lost, it is unstable with respect to perturbation due to its global character and for reproducing even an almost flat signal it requires an infinite series.

For such non-stationary, inhomogeneous signals, analysts turn to *time-frequency* (TF) or *time-scale representations*. A general linear TF transform of a signal f takes the form

$$f(x) \longmapsto F(a,b) = \int_{-\infty}^{+\infty} \left(\psi_{a,b}(x)\right)^c \, f(x) \, dx \quad .$$

The parameter x is, after TF transformation, thus replaced by two parameters, namely the parameter a referring to the frequency or the scale and the parameter b indicating the time or the position in the signal. The function $\psi_{a,b}$ is called the *analysing function*. This concept of a TF representation is in fact quite old and familiar; the most obvious example is simply a musical score.
Among the TF transforms, two are particularly successful: the *windowed* or *short-time Fourier transform* (STFT) and the *wavelet transform* (WT).
The analysing function of the STFT takes the form:

$$\psi_{a,b}(x) = \exp\left(i\frac{x}{a}\right) \, \psi(x-b) \quad .$$

Here ψ is a window function (such as, for example, the Gaussian $\exp\left(-x^2\right)$) and the factor $\exp\left(i\frac{x}{a}\right)$ causes the oscillations in the window. This transform was originally introduced by Gabor using a Gaussian window function (see [116]) and for this reason it is sometimes called the *Gabor transform*. Note that it may also be interpreted as the Fourier transform of a windowed signal f around the point b.
In the case of the WT, the analysing function takes the form:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \, \psi\left(\frac{x-b}{a}\right) \quad ,$$

where ψ (which must be oscillating, see Section 3.1.2) is the so-called *mother wavelet*.

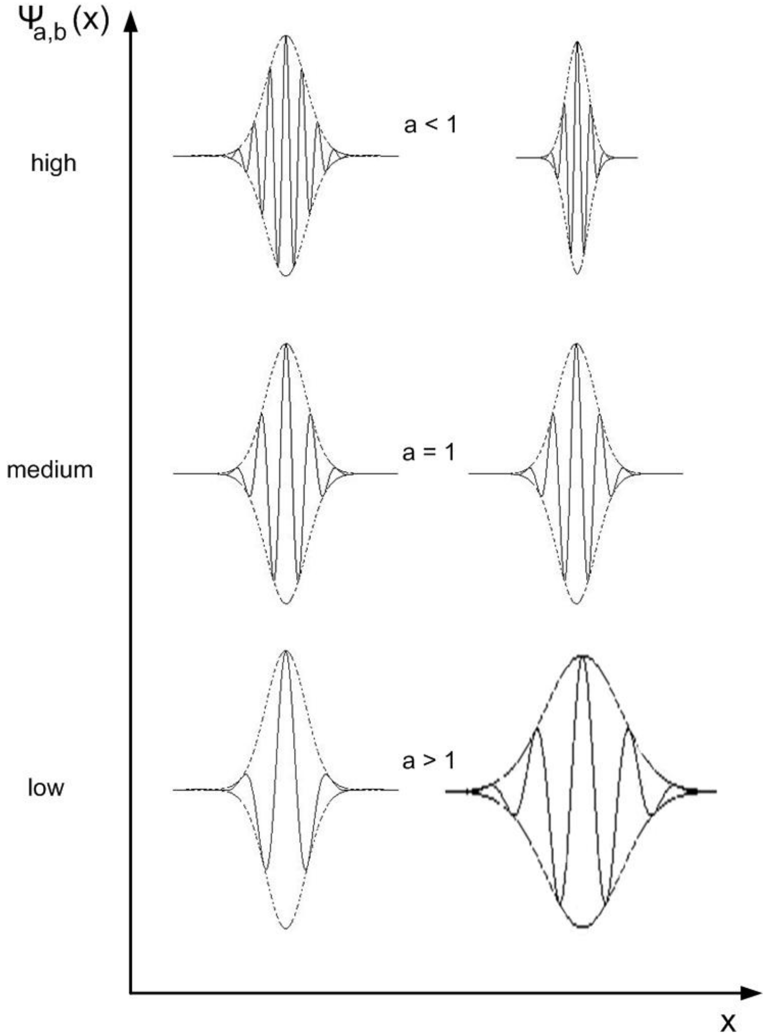


Figure 3.1: The function $\psi_{a,b}(x)$ for different values of the scale parameter a, in the case of the short-time Fourier transform (left) and the wavelet transform (right). The quantity $\frac{1}{a}$, which corresponds to a frequency, increases from bottom to top.

The essential difference between these two TF transforms lies in the way the frequency parameter a is introduced in the analysing funtion. For the STFT, the window has a fixed width, but the lower a, the larger the number of oscillations in the window (see Figure 3.1 (left)). For the WT on the other hand, the function ψ is dilated ($a > 1$) or contracted ($a < 1$) by the parameter a. The shape of the function is unchanged, it is simply spread or squeezed (see Figure 3.1 (right)). It is exactly this difference that makes the WT much more flexible than the STFT.

Similar to Fourier analysis, one should distinguish between two different versions of the WT, viz. the continuous WT (CWT) and the discrete WT (DWT). They enjoy more or less opposite properties and both have their specific field of application. The CWT plays an analogous role as the Fourier transform and is a successful tool for the analysis of signals and feature detection in signals. The DWT (see e.g. [100]) is the analogue of the discrete Fourier transform and provides a powerful technique for e.g. data compression and signal reconstruction.

3.1.2 Definition

The CWT of a 1-D L_2-function f, representing a signal of finite energy, is defined as the inner product of the signal with the transformed wavelets $\psi_{a,b}$:

$$F(a,b) = < \psi_{a,b} \, , f > = \int_{-\infty}^{+\infty} \left(\psi_{a,b}(x)\right)^c \, f(x) \; dx \quad , \tag{3.1}$$

where the wavelets $\psi_{a,b}$ are derived from the mother wavelet ψ by change of scale a (i.e. by dilation) and by change of position b (i.e. by translation):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \, \psi\left(\frac{x-b}{a}\right) \quad , \quad a > 0, \; b \in \mathbb{R} \quad . \tag{3.2}$$

Recall that the Fourier transform of the signal f is defined by

$$\mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \exp\left(-i\xi x\right) \, f(x) \; dx \quad .$$

In particular, we have

$$\mathcal{F}[\psi_{a,b}](\xi) = \sqrt{a} \; \exp\left(-i\xi b\right) \, \mathcal{F}[\psi](a\xi) \quad .$$

As already mentioned in Section 2.3, one of the fundamental properties of the Fourier transform is that it is an isometry on $L_2(\mathbb{R}, dx)$, in other words, for all

$f, g \in L_2(\mathbb{R}, dx)$ the Parseval formula holds:

$$< f, g > = < \mathcal{F}[f], \mathcal{F}[g] > \quad .$$

Hence in the frequency domain the CWT reads:

$$\begin{aligned} F(a,b) &= \int_{-\infty}^{+\infty} \left(\mathcal{F}[\psi_{a,b}](\xi)\right)^c \mathcal{F}[f](\xi)\ d\xi \\ &= \sqrt{a} \int_{-\infty}^{+\infty} \exp\left(i\xi b\right) \left(\mathcal{F}[\psi](a\xi)\right)^c \mathcal{F}[f](\xi)\ d\xi \quad . \end{aligned} \tag{3.3}$$

In wavelet theory some conditions on the mother wavelet ψ have to be imposed.

1. First, ψ is required to be an L_2-function which is well localized both in the time and in the frequency domain.
2. Moreover, ψ has to satisfy the so-called *admissibility condition*:

$$C_\psi \equiv 2\pi \int_{-\infty}^{+\infty} \frac{|\mathcal{F}[\psi](\xi)|^2}{|\xi|}\ d\xi < \infty \quad . \tag{3.4}$$

   If ψ is also an L_1-function, this admissibility condition implies that ψ should have "zero momentum", i.e.

$$\mathcal{F}[\psi](0) = 0 \quad \Longleftrightarrow \quad \int_{-\infty}^{+\infty} \psi(x)\ dx = 0$$

   which can only be fulfilled if ψ is an oscillating function, explaining the terminology "wavelet". A wavelet is a function that oscillates like a wave in a limited portion of time or space and vanishes outside of it, i.e. it is a wave-like but localized function.

3. For applications additional requirements are imposed, among which a given number of vanishing moments, viz.

$$\int_{-\infty}^{+\infty} x^n\ \psi(x)\ dx = 0 \quad , \quad n = 0, 1, \ldots, N \quad .$$

   Note that in the frequency domain the above condition is equivalent to zero derivatives up to order N of the Fourier transform of the mother wavelet at zero frequency:

$$\frac{d^n}{d\xi^n} \mathcal{F}[\psi](0) = 0 \quad , \quad n = 0, 1, \ldots, N \quad .$$

This property makes the CWT adequate at detecting singularities in the signal, since it then filters out polynomial behaviour up to degree N, which constitutes the smoothest part of the signal.

4. Finally, ψ is often required to be progressive, which means that $\mathcal{F}[\psi]$ is real-valued and vanishes identically for $\xi \leq 0$. In the signal processing community a signal with this property is called *analytic*.

We emphasize that conditions 1 and 2 are necessary requirements, while conditions 3 and 4 are desirable, but not compulsory.

3.1.3 Localization properties and interpretation as local filter

As mentioned above, it is important that ψ and $\mathcal{F}[\psi]$ are as well localized as possible. Assume that ψ has a support of width T, centred around 0, while $\mathcal{F}[\psi]$ has a support of width Ω, centred around ξ_0. Then it follows from (3.2) that the wavelet $\psi_{a,b}$ has a support of width aT, centred around b. Hence, the CWT gives local information of a signal $f(x)$ with a time window

$$\left[b - \frac{aT}{2}, b + \frac{aT}{2}\right] \quad .$$

Similarly, expression (3.3) implies that, with the exception of a multiple of $\sqrt{a}$ and a linear phase-shift of $\exp(ib\xi)$, the CWT also gives local information of $\mathcal{F}[f](\xi)$ with a frequency window

$$\left[\frac{\xi_0}{a} - \frac{\Omega}{2a}, \frac{\xi_0}{a} + \frac{\Omega}{2a}\right] \quad .$$

Hence the CWT localizes the signal with "time-frequency window"

$$\left[b - \frac{aT}{2}, b + \frac{aT}{2}\right] \times \left[\frac{\xi_0}{a} - \frac{\Omega}{2a}, \frac{\xi_0}{a} + \frac{\Omega}{2a}\right] \quad .$$

Note that the product of the widths of the supports of $\psi_{a,b}$ and $\mathcal{F}[\psi_{a,b}]$ is a constant. Therefore (see Figure 3.2),

- if $a >> 1$, then $\psi_{a,b}$ is a wide window, whereas $\mathcal{F}[\psi_{a,b}]$ is peaked around a small frequency $\frac{\xi_0}{a}$. Consequently this transform will be sensitive to low frequencies.

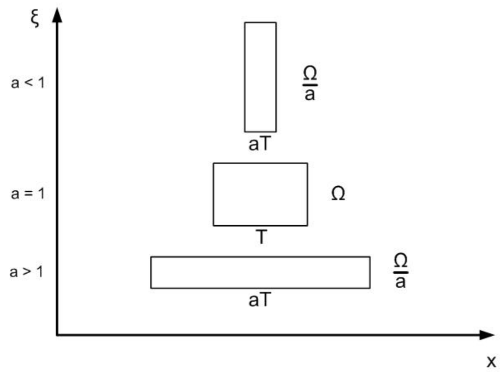


Figure 3.2: Support properties of $\psi_{a,b}$ and $\mathcal{F}[\psi_{a,b}]$. The time axis is horizontal; the frequency axis vertical.

- if $a << 1$, then $\psi_{a,b}$ is a narrow window, whereas $\mathcal{F}[\psi_{a,b}]$ is wide and centred around a high frequency $\frac{\xi_0}{a}$. Hence this wavelet has a good localization capability in the time domain and is mostly sensitive to high frequencies.

Consequently the CWT has the so-called *zoom-in and zoom-out capability.* It has a flexible time-frequency window that automatically widens when studying low frequency environments and narrows when observing high frequency phenomena. The signal property mentioned in Section 3.1.1, namely that the frequency of a signal is directly proportional to its duration, is thus reproduced. In comparison, the time-frequency window of any STFT is rigid and hence not very effective for detecting signals with high frequencies and investigating signals with low frequencies. Furthermore, the Gabor transform (i.e. the STFT with a Gaussian window function) is the STFT with the smallest time-frequency window and hence the most accurate one. However, in some applications a larger window must be chosen in order to achieve other desirable properties, such as computational and implementational efficiency.

The definition (3.1) of the CWT can be rephrased as a convolution:

$$F(a,b) = (\widetilde{\psi}_a * f)(b) = \int_{-\infty}^{+\infty} \widetilde{\psi}_a(b-x)\ f(x)\ dx$$

with $\widetilde{\psi}_a$ the scaled, flipped and conjugated wavelet:

$$\widetilde{\psi}_a(x) = \frac{1}{\sqrt{a}} \left(\psi\left(-\frac{x}{a}\right)\right)^c \quad .$$

In other words, the CWT acts as a filter with a function of zero momentum. Indeed, a linear shift invariant filter $\mathcal{I}$ is completely characterized by its impulse response $i = \mathcal{I}[\delta]$, in the sense that for a given signal g :

$$\mathcal{I}[g] = i * g \quad .$$

One can easily verify that the transformed wavelet $\widetilde{\psi}_a$ is nothing else but the impulse response of the wavelet filter.
Combining this with the support properties, it becomes clear that the CWT performs a local filtering, both in time and in scale. The CWT selects that part of the signal, if any, that lives around the time b and the scale a.

Furthermore, if $\mathcal{F}[\psi]$ has a bandwidth of width Ω, then $\mathcal{F}[\psi_{a,b}]$ has a numerical support of width $\frac{\Omega}{a}$. Recalling that $\frac{1}{a}$ behaves like a frequency, we can conclude that the CWT works at constant *relative* bandwidth, i.e. $\frac{\Delta\xi}{\xi}$ = constant. Hence at high frequencies, $\Delta\xi$ is wide, which, by the uncertainty principle ($\Delta\xi \ \Delta x \geq$ constant), implies that Δx is small. So, at high frequencies, we have a good localization in the time domain. This property makes the CWT an ideal tool for detecting singularities which are contained in the high frequency part of the signal.

3.1.4 Basic properties

Assume that ψ is an admissible wavelet.
When considering two L_2-functions f and g with respective CWT-images F and G, the following inner product in the space of wavelet transforms may be introduced:

$$[F, G] = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} \big(F(a,b)\big)^c \ G(a,b) \ \frac{da}{a^2} \ db \quad .$$

The CWT has the following main properties:

1. energy conservation, reconstruction formula

Taking into account the admissibility condition (3.4) for the mother wavelet ψ, the corresponding Parseval formula is readily obtained:

$$[F, G] = < f, g > \quad .$$

In other words, as a consequence of the admissibility condition, the CWT is an isometry from the space of signals into the space of transforms. The latter is a closed subspace $\mathcal{H}_\psi$ of $L_2(\mathbb{R}_+ \times \mathbb{R}, C_\psi^{-1}\ a^{-2}\ da\ db)$.
As it is an isometry, the CWT is invertible on its range $\mathcal{H}_\psi$ and the inverse transformation is the adjoint. This means that for $f \in L_2(\mathbb{R}, dx)$:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} \psi_{a,b}(x)\ F(a,b)\ \frac{da}{a^2}\ db \tag{3.5}$$

holds weakly in $L_2(\mathbb{R}, dx)$. Hence the signal $f(x)$ may be reconstructed exactly from its wavelet transform $F(a,b)$ or, in other words, the CWT provides a decomposition of the signal $f(x)$ in terms of the analysing wavelets $\psi_{a,b}(x)$ with coefficients $F(a,b)$.

2. reproducing property

The projection from $L_2(\mathbb{R}_+ \times \mathbb{R}, C_\psi^{-1}\ a^{-2}\ da\ db)$ onto the range $\mathcal{H}_\psi$ of the CWT is an integral operator whose reproducing kernel is given by:

$$K(a, b\ ;\ a', b') = < \psi_{a',b'}, \psi_{a,b} > \quad .$$

Note that this kernel is nothing else but the wavelet transform of the wavelet itself, that is, the autocorrelation function of the wavelet. It describes the intrinsic redundancy between the values of the wavelets at (a, b) and at (a', b').
Hence a function $F(a,b) \in L_2(\mathbb{R}_+ \times \mathbb{R}, C_\psi^{-1}\ a^{-2}\ da\ db)$ is the CWT of a signal $f \in L_2(\mathbb{R}, dx)$ iff it satisfies the reproducing property:

$$\begin{aligned} F(a,b) &= [K(a, b\ ;\ a', b'), F(a', b')] \\ &= \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} \big(K(a, b\ ;\ a', b')\big)^c\ F(a', b')\ \frac{da'}{a'^2}\ db' \quad . \end{aligned}$$

This means that the range $\mathcal{H}_\psi$ of the CWT, i.e. the space of transforms, is a reproducing kernel Hilbert space.

3. covariance under translations and dilations

The CWT is covariant under translation and dilation, which means that the correspondence $f(x) \longmapsto F(a,b)$ implies:

1. $f(x-b_0) \longmapsto F(a, b-b_0)$
2. $\frac{1}{\sqrt{a_0}}\, f\left(\frac{x}{a_0}\right) \longmapsto F\left(\frac{a}{a_0}, \frac{b}{a_0}\right)$.

3.1.5 Examples of 1-D CWT wavelets

The two most well-known and widely used mother wavelets are the Mexican hat and the Morlet wavelet.

1. The Mexican hat

Up to a minus sign, this wavelet is simply the second order derivative of a Gaussian:

$$\psi_H(x) = -\frac{d^2}{dx^2} \exp\left(-\frac{x^2}{2}\right) = (1-x^2) \exp\left(-\frac{x^2}{2}\right) \quad .$$

The graph of ψ_H (see Figure 3.3) looks a bit like a transverse section of a Mexican hat, whence the name. It is an admissible, even, real-valued wavelet with two vanishing moments ($n = 0, 1$).

2. The Morlet wavelet

This wavelet is just a modulated Gaussian:

$$\psi_M(x) = \exp\left(i\xi_0 x\right)\ \exp\left(-\frac{x^2}{2}\right) + \text{correction term} \quad .$$

The first term alone does not satisfy the admissibility condition, hence a correction term is needed. However, in practice one will arrange that this term is numerically negligible and thus can be omitted.

These two wavelets have very different properties and, naturally, will be used in quite different situations. Typically, the Mexican hat is sensitive to singularities in the signal. On the other hand, as it is complex-valued, the Morlet wavelet will catch the phase of the signal and hence will be sensitive to frequencies.

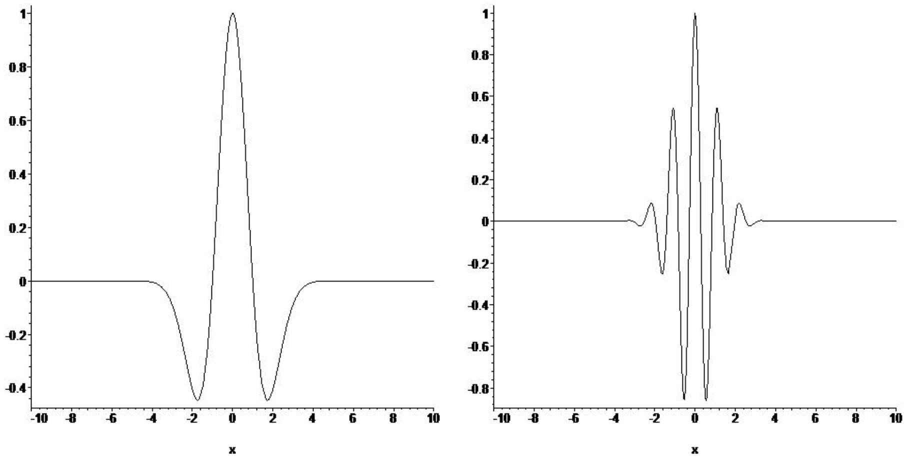


Figure 3.3: The 1-D Mexican hat (left); the real part of the 1-D Morlet wavelet, for $\xi_0 = 5.6$ (right).

3.2 The two-dimensional continuous wavelet transform

The CWT may be extended to two dimensions while still enjoying the same properties as in the 1-D case. This 2-D CWT is characterized by a rotation parameter, in addition to the usual translations and dilations. This enables it to detect oriented features in the signal provided the basic wavelet contains itself an intrinsic orientation. We refer the reader to [3, 5, 4, 6, 7] for a detailed analysis.

3.2.1 Definition

Starting point of deriving the CWT in two dimensions is to decide which elementary operations one wants to apply to a finite energy signal, represented by a square integrable function $f \in L_2(\mathbb{R}^2, dV(\underline{x}))$. Choosing translations in the image plane, global magnification (zooming in and out) and rotations (direction of sight), leads to the following definition.
Given a square integrable function $f \in L_2(\mathbb{R}^2, dV(\underline{x}))$, its CWT with respect to

the (fixed) mother wavelet ψ is defined by:

$$F(a, \underline{b}, \theta) = < \psi_{a,\underline{b},\theta} \ , f > = \int_{\mathbb{R}^2} \left(\psi_{a,\underline{b},\theta}(\underline{x})\right)^c f(\underline{x}) \ dV(\underline{x}) \quad , \tag{3.6}$$

where, as mentioned above, ψ is not only translated by $\underline{b} \in \mathbb{R}^2$ and dilated by $a > 0$, but also rotated by an angle $\theta \in [0, 2\pi[$:

$$\psi_{a,\underline{b},\theta}(\underline{x}) = \frac{1}{a} \ \psi \left(\frac{r_{-\theta}(\underline{x} - \underline{b})}{a} \right) \quad .$$

Here r_θ denotes the usual 2×2 rotation matrix

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad .$$

In case the mother wavelet is rotation invariant, we omit the parameter θ, i.e. we consider

$$\psi_{a,\underline{b}}(\underline{x}) = \frac{1}{a} \ \psi \left(\frac{\underline{x} - \underline{b}}{a} \right) \quad .$$

Invoking the Parseval formula for the Fourier transform $\mathcal{F}$ and using the expression

$$\mathcal{F}[\psi_{a,\underline{b},\theta}](\underline{\xi}) = a \exp\left(-i < \underline{b}, \underline{\xi} >\right) \mathcal{F}[\psi]\big(a r_{-\theta}(\underline{\xi})\big)$$

for the Fourier transform of the wavelets, we can rewrite the CWT in frequency space as follows:

$$F(a, \underline{b}, \theta) = a \int_{\mathbb{R}^2} \exp\left(i < \underline{b}, \underline{\xi} >\right) \left(\mathcal{F}[\psi]\big(a r_{-\theta}(\underline{\xi})\big) \right)^c \mathcal{F}[f](\underline{\xi}) \ dV(\underline{\xi}) \quad .$$

Similar to the 1-D case, the mother wavelet must satisfy the following conditions:

1. The wavelet $\psi \in L_2(\mathbb{R}^2, dV(\underline{x}))$ and its Fourier transform $\mathcal{F}[\psi]$ are both supposed to be well localized.

2. The admissibility condition now reads:

$$C_\psi := (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\mathcal{F}[\psi](\underline{\xi})|^2}{|\underline{\xi}|^2} \ dV(\underline{\xi}) < \infty \quad . \tag{3.7}$$

   If ψ is regular enough ($\psi \in L_1(\mathbb{R}^2, dV(\underline{x})) \cap L_2(\mathbb{R}^2, dV(\underline{x}))$ suffices), this admissibility condition again implies that ψ has zero momentum, expressing the fact that a wavelet must be an oscillating function.

3. Moreover, the wavelet ψ is often required to have a certain number of vanishing moments, which improves its capacity of detecting singularities. Indeed, if ψ has vanishing moments up to order N :

$$\int_{\mathbb{R}^2} x_1^\alpha \; x_2^\beta \; \psi(\underline{x}) \; dV(\underline{x}) = 0 \quad , \quad 0 \leq \alpha + \beta \leq N \quad ,$$

then the associated CWT will filter out polynomial behaviour of the signal up to degree N. For example, if the first moments ($N = 1$) vanish, the transform will erase any linear feature in the signal. In general, if the wavelet has vanishing moments, the smoother parts of the signal will have very small wavelet coefficients, whereas sharp, non-stationary behaviour will give rise to local maxima of the modulus of $F(a, \underline{b}, \theta)$.

Note that the map $\psi \longmapsto \psi_{a,\underline{b},\theta}$ preserves the admissibility condition (3.7). Thus the given wavelet ψ generates, by translation, rotation and dilation the whole family $\{\psi_{a,\underline{b},\theta} \; ; \; a > 0, \; \underline{b} \in \mathbb{R}^2, \; \theta \in [0, 2\pi[\; \}$. Moreover, the linear span of this family is dense in $L_2(\mathbb{R}^2, dV(\underline{x}))$. Hence any square integrable function is uniquely determined by its projections on the transformed wavelets $\psi_{a,\underline{b},\theta}$, which justifies the definition of the 2-D CWT.

3.2.2 Basic properties

The properties of the 2-D CWT, which are completely similar to those familiar in the 1-D case, may be summarized as follows.

1. energy conservation, reconstruction formula

By means of the admissibility condition one can prove, by a straightforward computation, that the 2-D CWT is an isometry from the space of signals $L_2(\mathbb{R}^2, dV(\underline{x}))$ into the space of transforms, the latter being a closed subspace $\mathcal{H}_\psi$ of $L_2(\mathbb{R}_+ \times \mathbb{R}^2 \times [0, 2\pi[, C_\psi^{-1} \; a^{-3} \; da \; dV(\underline{b}) \; d\theta)$.
As a consequence, the signal $f(\underline{x})$ may be reconstructed exactly from its transform $F(a, \underline{b}, \theta)$:

$$f(\underline{x}) = \frac{1}{C_\psi} \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{+\infty} \psi_{a,\underline{b},\theta}(\underline{x}) \; F(a, \underline{b}, \theta) \; \frac{da}{a^3} \; dV(\underline{b}) \; d\theta \quad . \tag{3.8}$$

The above reconstruction formula holds in the weak sense.
It can be generalized by taking a different wavelet χ for the reconstruction. Indeed, provided that the wavelets ψ and χ satisfy a cross-admissibility condition

(see [75]):

$$0 < |C_{\psi\chi}| < \infty$$

where

$$C_{\psi\chi} = (2\pi)^2 \int_{\mathbb{R}^2} \left(\mathcal{F}[\psi](\underline{\xi})\right)^c \mathcal{F}[\chi](\underline{\xi}) \frac{dV(\underline{\xi})}{|\underline{\xi}|^2} \quad ,$$

one obtains the following more general reconstruction formula:

$$f(\underline{x}) = \frac{1}{C_{\psi\chi}} \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{+\infty} \chi_{a,\underline{b},\theta}(\underline{x})\ F(a,\underline{b},\theta)\ \frac{da}{a^3}\ dV(\underline{b})\ d\theta \quad .$$

In particular, if one takes for the reconstruction wavelet χ a delta distribution, one can reconstruct the signal by resumming only over scales and angles:

$$f(\underline{x}) = \frac{1}{C_{\psi\delta}} \int_0^{\infty} \frac{da}{a^2} \int_0^{2\pi} F(a,\underline{x},\theta)\ d\theta \quad .$$

2. reproducing property

The projection from $L_2(\mathbb{R}_+ \times \mathbb{R}^2 \times [0,2\pi[\ , C_\psi^{-1}\ a^{-3}\ da\ dV(\underline{b})\ d\theta)$ onto the range $\mathcal{H}_\psi$ of the CWT is an integral operator, whose reproducing kernel K is the autocorrelation function of ψ :

$$K(a,\underline{b},\theta\ ;\ a',\underline{b}',\theta') = <\psi_{a',\underline{b}',\theta'}\ ,\ \psi_{a,\underline{b},\theta}> \quad .$$

Therefore, wavelet transforms satisfy the reproducing property:

$$F(a,\underline{b},\theta) = \frac{1}{C_\psi} \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{+\infty} \left(K(a,\underline{b},\theta\ ;\ a',\underline{b}',\theta')\right)^c F(a',\underline{b}',\theta')\ \frac{da'}{a'^3}\ dV(\underline{b}')\ d\theta' \quad .$$

3. covariance under translations, dilations and rotations

The correspondence $f(\underline{x}) \longmapsto F(a,\underline{b},\theta)$ implies:

1. $f(\underline{x} - \underline{b}_0) \longmapsto F(a, \underline{b} - \underline{b}_0, \theta)$
2. $a_0^{-1} f(a_0^{-1}\underline{x}) \longmapsto F(a_0^{-1}a, a_0^{-1}\underline{b}, \theta)$
3. $f(r_{-\theta_0}(\underline{x})) \longmapsto F(a, r_{-\theta_0}(\underline{b}), \theta - \theta_0)$.

These covariance relations have a crucial importance for applications.

Finally, we remark that the 2-D CWT is uniquely determined by the three conditions of linearity, covariance and energy conservation, plus some continuity (see [94]).

3.2.3 Applications

The 2-D CWT has been used by a number of authors, in a wide variety of problems. One can distinguish between two different fields of applications, namely image processing and physics. To the first type belong for e.g. contour detection and character recognition, automatic target detection and recognition, medical imaging, detection of symmetries in patterns and image denoising. The second class of applications concerns various fields of classical physics, such as astronomy and astrophysics, geophysics, fluid dynamics, fractals and texture analysis. Recently several new wavelet-like transforms have emerged, for example there exists a zoological garden of more or less exotic -lets like curvelets, contourlets, ridgelets, chirplets etc.

Part I

The Continuous Clifford-Wavelet Transform

Chapter 4

The isotropic Continuous Clifford-Wavelet Transform: general theory

After having studied and generalized the Clifford-Hermite CWT introduced by Brackx and Sommen in [32, 33], and after having established a series of new CWTs in the orthogonal Clifford analysis setting, we realized that in fact a general theory could be worked out where all those specific Clifford CWTs fit in. It is this general theory which is presented in this chapter. Moreover in Section 4.4 we present a methodology for constructing Clifford wavelet functions. Both the general theory and the wavelet construction procedure, fully enjoy the possibilities offered by and the characteristic features of Clifford analysis. Such a highly important intrinsic feature of Clifford analysis is that it encompasses all dimensions at once, as opposed to a tensorial approach with tensor products of one-dimensional phenomena.
The Clifford wavelet theory outlined in this chapter might be characterized as isotropic, since the metric in the underlying space is the standard Euclidean one for which

$$e_j^2 = -1 \quad , \quad j = 1, \ldots, m$$

and

$$e_j e_k = -e_k e_j \quad 1 \leq j \neq k \leq m \quad ,$$

leading to the standard scalar product of a vector $\underline{x} = \sum_{j=1}^{m} x_j e_j$ with itself:

$$< \underline{x}, \underline{x} > = \sum_{j=1}^{m} x_j^2 \quad .$$

4.1 Definition

In the orthogonal Clifford analysis framework, a general Clifford algebra-valued mother wavelet $\psi \in L_1\big(\mathbb{R}^m, dV(\underline{x})\big) \cap L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ generates a family of wavelet kernels, by taking into account dilation or scaling (global zooming in and out), translation and rotation. For the rotation, we consider elements of the Spin(m)-group, see (2.2), and in particular the associated H-representation on the space $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$.
Hence, the family of wavelets originating from a mother wavelet ψ is defined as follows

$$\psi^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{m/2}} \; s \; \psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \; \overline{s}$$

where a is a positive real number, $\underline{b}$ a real Clifford vector and s an element belonging to the group Spin(m).

The corresponding Continuous Clifford-Wavelet Transform (CCWT) applies to functions $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ by

$$\begin{aligned} T_\psi[f](a,\underline{b},s) = F_\psi(a,\underline{b},s) &= \; < \psi^{a,\underline{b},s}, f > \\ &= \int_{\mathbb{R}^m} \big(\psi^{a,\underline{b},s}(\underline{x})\big)^\dagger \; f(\underline{x}) \; dV(\underline{x}) \\ &= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \; \left(\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \; \overline{s} \; f(\underline{x}) \; dV(\underline{x}) \; . \end{aligned}$$

Taking into account the Parseval formula for the Fourier transform (see Theorem 2.1), this definition can be rewritten in the frequency domain as

$$\begin{aligned} F_\psi(a,\underline{b},s) &= \; < \mathcal{F}[\psi^{a,\underline{b},s}] \; , \mathcal{F}[f] > \\ &= \int_{\mathbb{R}^m} \big(\mathcal{F}[\psi^{a,\underline{b},s}](\underline{\xi})\big)^\dagger \; \mathcal{F}[f](\underline{\xi}) \; dV(\underline{\xi}) \quad . \end{aligned} \tag{4.1}$$

The Fourier transform of the continuous family of wavelets:

$$\begin{aligned}
\mathcal{F}[\psi^{a,\underline{b},s}](\underline{\xi}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \psi^{a,\underline{b},s}(\underline{x})\, dV(\underline{x}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \frac{1}{a^{m/2}}\, s\, \psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \overline{s}\, dV(\underline{x}) \quad ,
\end{aligned}$$

can be calculated by means of the successive substitutions $\underline{t} = \underline{x} - \underline{b}$, $\underline{z} = \overline{s}\underline{t}s$ and $\underline{u} = \dfrac{\underline{z}}{a}$:

$$\begin{aligned}
\mathcal{F}[\psi^{a,\underline{b},s}](\underline{\xi}) \\
&= \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) \int_{\mathbb{R}^m} \exp\left(-i < \underline{t}, \underline{\xi} >\right) \frac{1}{a^{m/2}}\, s\, \psi\left(\frac{\overline{s}\underline{t}s}{a}\right) \overline{s}\, dV(\underline{t}) \\
&= \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) \int_{\mathbb{R}^m} \exp\left(-i < \underline{z}, \overline{s}\underline{\xi}s >\right) \frac{1}{a^{m/2}}\, s\, \psi\left(\frac{\underline{z}}{a}\right) \overline{s}\, dV(\underline{z}) \\
&= \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) \int_{\mathbb{R}^m} \exp\left(-i < \underline{u}, a\overline{s}\underline{\xi}s >\right) a^{m/2} s\, \psi(\underline{u})\, \overline{s}\, dV(\underline{u}) \\
&= \frac{1}{(2\pi)^{m/2}}\, a^{m/2}\, \exp\left(-i < \underline{b}, \underline{\xi} >\right) s \\
&\qquad \left(\int_{\mathbb{R}^m} \exp\left(-i < \underline{u}, a\overline{s}\underline{\xi}s >\right) \psi(\underline{u})\, dV(\underline{u})\right) \overline{s} \\
&= a^{m/2} \exp\left(-i < \underline{b}, \underline{\xi} >\right) s\, \mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\, \overline{s} \quad .
\end{aligned}$$

Consequently (4.1) becomes:

$$\begin{aligned}
F_\psi(a, \underline{b}, s) &= a^{m/2} \int_{\mathbb{R}^m} \exp\left(i < \underline{b}, \underline{\xi} >\right) s \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \overline{s}\, \mathcal{F}[f](\underline{\xi})\, dV(\underline{\xi}) \\
&= a^{m/2}\, s\, (2\pi)^{m/2}\, \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \overline{s}\, \mathcal{F}[f](\underline{\xi})\right](-\underline{b}) \quad . \qquad (4.2)
\end{aligned}$$

Similar to the classical theory, some restrictions must be made on the mother wavelet ψ .

Definition 4.1
A Clifford algebra-valued function $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ is a mother wavelet in $\mathbb{R}^m$ if it satisfies in frequency space the following two conditions:

1. *$\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger$ is scalar-valued;*

2.
$$C_\psi = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m}\ dV(\underline{\xi}) < \infty \quad . \tag{4.3}$$

The above relation is called the *admissibility condition* for the Clifford wavelets and the constant C_ψ involved is called the *admissibility constant.* Note that this admissibility condition implies that the wavelet ψ has zero momentum:

$$\int_{\mathbb{R}^m} \psi(\underline{x})\ dV(\underline{x}) = 0 \quad .$$

Remark 4.1 Note that any wavelet $\psi^{a,\underline{b},s}$ obtained from a mother wavelet ψ by dilation, translation and spinor-rotation is again a mother wavelet.
Indeed, we then have that

$$\mathcal{F}[\psi^{a,\underline{b},s}](\underline{\xi})\ \left(\mathcal{F}[\psi^{a,\underline{b},s}](\underline{\xi})\right)^\dagger = a^m\ \mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger$$

is scalar-valued, and moreover, by the substitution $\underline{t} = a\overline{s}\underline{\xi}s$

$$\begin{aligned} C_{\psi^{a,\underline{b},s}} &= (2\pi)^m a^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger}{|\underline{\xi}|^m}\ dV(\underline{\xi}) \\ &= a^m (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](\underline{t})\ \left(\mathcal{F}[\psi](\underline{t})\right)^\dagger}{|\underline{t}|^m}\ dV(\underline{t}) < \infty \quad . \end{aligned}$$

We end this section with the following result, which justifies the definition of the CCWT.

Proposition 4.1 *The linear span of the continuous family of wavelets* $\{\psi^{a,\underline{b},s}\ :\ a > 0\ ,\ \underline{b} \in \mathbb{R}^m\ ,\ s \in \text{Spin}(m)\}$ *is a dense subspace of* $L_2(\mathbb{R}^m, dV(\underline{x}))$.

Proof. Suppose that $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ is orthogonal to every wavelet in the family, i.e.

$$< \psi^{a,\underline{b},s}, f > = 0 \qquad \text{for all } (a, \underline{b}, s) \in \mathbb{R}_+ \times \mathbb{R}^m \times \text{Spin}(m) \quad ,$$

in other words: f belongs to the orthogonal complement of the span of the family of wavelets. In order to prove the statement, we must show that $f = 0$.

The assumption implies that for all $(a, \underline{b}, s) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m)$

$$
\begin{aligned}
&< \mathcal{F}[\psi^{a,\underline{b},s}], \mathcal{F}[f] > \\
&\qquad = a^{m/2} \int_{\mathbb{R}^m} \exp\left(i < \underline{b}, \underline{\xi} >\right) \, s \, \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \, \overline{s} \, \mathcal{F}[f](\underline{\xi}) \, dV(\underline{\xi}) = 0 \quad .
\end{aligned}
$$

Hence

$$
s \, \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \, \overline{s} \, \mathcal{F}[f](\underline{\xi}) = 0 \quad \text{a.e. for all } (a, s) \in \mathbb{R}_+ \times \mathrm{Spin}(m) \quad .
$$

Furthermore, the joint action of dilations and spinor-rotations on $\mathbb{R}^m$ is transitive, i.e. for a fixed $\underline{\xi} \in \mathbb{R}^m$ one has

$$
\left\{ a\overline{s}\underline{\xi}s \; ; \; a > 0 \; , \; s \in \mathrm{Spin}(m) \right\} = \mathbb{R}^m \quad .
$$

Hence, when a and s vary over their range, the supports of $\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger$ will cover the whole of $\mathbb{R}^m$. Therefore, $\mathcal{F}[f] = 0$ a.e. and thus $f = 0$ a.e. □

4.2 Properties of the CCWT

Assume that ψ is a Clifford algebra-valued mother wavelet in $\mathbb{R}^m$. In the sequel $\mathcal{H}_\psi$ denotes the space of wavelet transforms associated with ψ; in other words: if f varies over the space of signals $L_2\left(\mathbb{R}^m, dV(\underline{x})\right)$, then its wavelet transform $T_\psi[f]$ varies over the space $\mathcal{H}_\psi$. Below we will show that $\mathcal{H}_\psi$ is a closed subspace of the weighted L_2-space $L_2\left(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1} a^{-(m+1)} \, da \, dV(\underline{b}) \, ds\right)$. Here ds stands for the Haar measure on $\mathrm{Spin}(m)$.
Hence, the following inner product in the space of wavelet transforms may be introduced:

$$
[F_\psi, G_\psi] = \frac{1}{C_\psi} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi(a, \underline{b}, s)\right)^\dagger \, G_\psi(a, \underline{b}, s) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \, ds \quad ,
$$

where F_ψ and G_ψ denote the CWT-images of two L_2-functions f and g.

We now prove the general properties of the CCWT.
Let us start with the following main result, which follows by a straightforward computation from the admissibility condition (4.3).

Theorem 4.1 *The CCWT is an isometry from the space of signals $L_2\left(\mathbb{R}^m, dV(\underline{x})\right)$ into the space of transforms $\mathcal{H}_\psi$; in other words the Parseval formula*

$$
< f, g > = [F_\psi, G_\psi]
$$

is fulfilled.

Proof. By means of (4.2), we obtain

$$[F_\psi, G_\psi] = \frac{(2\pi)^m}{C_\psi} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[f](\underline{\xi}) \right](-\underline{b}) \right)^\dagger \overline{s}s\; \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[g](\underline{\xi}) \right](-\underline{b}) \; \frac{da}{a} \; dV(\underline{b}) \; ds \quad .$$

Next, taking into account that $\overline{s}s = 1$ for all $s \in \mathrm{Spin}(m)$, we get

$$[F_\psi, G_\psi] = \frac{(2\pi)^m}{C_\psi} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[f](\underline{\xi}) \right](-\underline{b}) \right)^\dagger \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[g](\underline{\xi}) \right](-\underline{b}) \; \frac{da}{a} \; dV(\underline{b}) \; ds \quad .$$

Moreover, in view of the Parseval formula for the Fourier transform, we have consecutively

$$\begin{aligned}
&[F_\psi, G_\psi] \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \left\langle \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[f](\underline{\xi}) \right] , \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[g](\underline{\xi}) \right] \right\rangle \frac{da}{a} \; ds \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \left\langle \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[f](\underline{\xi}) \; , \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s}\; \mathcal{F}[g](\underline{\xi}) \right\rangle \frac{da}{a} \; ds \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}[f](\underline{\xi})\right)^\dagger \; s\; \mathcal{F}[\psi](a\overline{s}\underline{\xi}s) \; \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \\
&\quad \overline{s}\; \mathcal{F}[g](\underline{\xi}) \; \frac{da}{a} \; dV(\underline{\xi}) \; ds \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathbb{R}^m} \left(\mathcal{F}[f](\underline{\xi})\right)^\dagger \left(\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\mathcal{F}[\psi](a\overline{s}\underline{\xi}s) \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \overline{s} \; \frac{da}{a} ds \right) \\
&\quad \mathcal{F}[g](\underline{\xi}) \; dV(\underline{\xi}) \quad .
\end{aligned} \tag{4.4}$$

As ψ is a mother wavelet, we know that $\mathcal{F}[\psi] \; \left(\mathcal{F}[\psi]\right)^\dagger$ is scalar-valued. Hence the integral between brackets in (4.4) becomes

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\right)^\dagger\ \bar{s}\ \frac{da}{a}\ ds$$
$$= \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\right)^\dagger\ \frac{da}{a}\ ds \quad .$$

By means of the substitution

$$\underline{\xi} = \frac{t}{a}\ \underline{\eta}\quad ;\quad \underline{\eta} \in S^{m-1} \quad ,$$

we obtain

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\right)^\dagger\ \bar{s}\ \frac{da}{a}\ ds$$
$$= \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \mathcal{F}[\psi](t\bar{s}\underline{\eta}s)\ \left(\mathcal{F}[\psi](t\bar{s}\underline{\eta}s)\right)^\dagger\ \frac{dt}{t}\ ds \quad .$$

The substitution $\bar{s}\underline{\eta}s = \underline{\nu}$ turns the integration over Spin(m) into an integration over the unit sphere S^{m-1} of $\mathbb{R}^m$, since $|\underline{\nu}| = 1$. The above integral may thus be further simplified to

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\bar{s}\underline{\xi}s)\right)^\dagger\ \bar{s}\ \frac{da}{a}\ ds$$
$$= \int_0^{+\infty} \int_{S^{m-1}} \mathcal{F}[\psi](t\underline{\nu})\ \left(\mathcal{F}[\psi](t\underline{\nu})\right)^\dagger\ dS(\underline{\nu})\ \frac{dt}{t}$$
$$= \int_{\mathbb{R}^m} \mathcal{F}[\psi](\underline{u})\ \left(\mathcal{F}[\psi](\underline{u})\right)^\dagger\ \frac{dV(\underline{u})}{|\underline{u}|^m} = \frac{C_\psi}{(2\pi)^m} \quad .$$

Consequently, we get the desired Parseval formula:

$$[F_\psi, G_\psi] = \int_{\mathbb{R}^m} \left(\mathcal{F}[f](\underline{\xi})\right)^\dagger\ \mathcal{F}[g](\underline{\xi})\ dV(\underline{\xi})\ = <\mathcal{F}[f], \mathcal{F}[g]> = <f, g> \quad .$$

□

Isometries between Hilbert spaces are mappings which are very well-behaved. If $\mathcal{T}$ is an isometry between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and $\mathcal{T}^*$ denotes its adjoint:

$$<\mathcal{T}[h_1]\ , h_2>_{\mathcal{H}_2} = <h_1\ , \mathcal{T}^*[h_2]>_{\mathcal{H}_1} \quad , \quad h_1 \in \mathcal{H}_1 \text{ and } h_2 \in \mathcal{H}_2 \quad ,$$

then

$$\mathcal{T}^*\mathcal{T} = I_{\mathcal{H}_1} \quad \text{and} \quad \mathcal{T}\mathcal{T}^* = P_{\text{range}(\mathcal{T})}$$

with $I_{\mathcal{H}_1}$ the identity operator on $\mathcal{H}_1$ and $P_{\text{range}(\mathcal{T})}$ the orthogonal projection operator from $\mathcal{H}_2$ onto the range of $\mathcal{T}$, which is a closed subspace of $\mathcal{H}_2$.

This classical result leads to the following important corollaries.

Corollary 4.1 *The signal $f(\underline{x})$ may be reconstructed exactly from its CCWT $F_\psi(a,\underline{b},s)$ by the inversion formula:*

$$\begin{aligned} f(\underline{x}) &= \frac{1}{C_\psi}\int_{\text{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty} \psi^{a,\underline{b},s}(\underline{x})\; F_\psi(a,\underline{b},s)\; \frac{da}{a^{m+1}}\; dV(\underline{b})\; ds \qquad (4.5)\\ &= [\left(\psi^{a,\underline{b},s}(\underline{x})\right)^\dagger, F_\psi(a,\underline{b},s)] \ , \end{aligned}$$

which holds weakly in $L_2(\mathbb{R}^m, dV(\underline{x}))$. In other words, the CCWT decomposes the signal $f(\underline{x})$ in terms of the analysing wavelets $\psi^{a,\underline{b},s}(\underline{x})$ with coefficients $F_\psi(a,\underline{b},s)$.

Proof. As the CCWT T_ψ is an isometry between the L_2-spaces $L_2(\mathbb{R}^m, dV(\underline{x}))$ and $L_2\left(\mathbb{R}_+ \times \mathbb{R}^m \times \text{Spin}(m), C_\psi^{-1}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds\right)$, it is invertible on its range and the inverse is simply the adjoint T_ψ^* .
For two square integrable functions f and g with CCWT-images respectively F_ψ and G_ψ , we obtain:

$$\begin{aligned} &< f, g > \\ &\quad = [F_\psi, G_\psi] \\ &\quad = \frac{1}{C_\psi}\int_{\text{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty} < f, \psi^{a,\underline{b},s} >\; G_\psi(a,\underline{b},s)\; \frac{da}{a^{m+1}}\; dV(\underline{b})\; ds \ . \end{aligned}$$

Consequently, we indeed have

$$g(\underline{x}) = \frac{1}{C_\psi}\int_{\text{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty} \psi^{a,\underline{b},s}(\underline{x})\; G_\psi(a,\underline{b},s)\; \frac{da}{a^{m+1}}\; dV(\underline{b})\; ds$$

to hold weakly in $L_2(\mathbb{R}^m, dV(\underline{x}))$. □

Remark 4.2 The wavelet used for the analysis of a signal and the one used for its reconstruction need not necessarily coincide.

Indeed, if the wavelets ψ and χ satisfy the assumption that $\mathcal{F}[\chi] \ \big(\mathcal{F}[\psi]\big)^\dagger$ is scalar-valued, as well as the cross-admissibility condition $0 < |C_{\psi\chi}| < \infty$ with

$$C_{\psi\chi} = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\chi](\underline{\xi}) \ \big(\mathcal{F}[\psi](\underline{\xi})\big)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) \quad ,$$

then the more general reconstruction formula holds:

$$f(\underline{x}) = \frac{1}{C_{\psi\chi}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \chi^{a,\underline{b},s}(\underline{x}) \ F_\psi(a,\underline{b},s) \ \frac{da}{a^{m+1}} \ dV(\underline{b}) \ ds \quad .$$

This formula follows from the observation that

$$\begin{aligned} &< f, g > \\ &\quad = \frac{1}{C_{\psi\chi}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \big(F_\chi(a,\underline{b},s)\big)^\dagger \ G_\psi(a,\underline{b},s) \ \frac{da}{a^{m+1}} \ dV(\underline{b}) \ ds \quad , \end{aligned}$$

the proof of which runs along the same lines as the one of Theorem 4.1.
In particular, if we take for the reconstruction wavelet χ a delta distribution, we obtain the simplified reconstruction formula:

$$f(\underline{x}) = \frac{1}{C_{\psi\delta}} \int_{\mathrm{Spin}(m)} \int_0^{+\infty} F_\psi(a,\underline{x},s) \ \frac{da}{a^{m/2+1}} \ ds \quad .$$

Next, the image of $L_2(\mathbb{R}^m, dV(\underline{x}))$ under the CCWT may be characterized.

Corollary 4.2 *The image $\mathcal{H}_\psi$ of $L_2(\mathbb{R}^m, dV(\underline{x}))$ under the CCWT T_ψ is a closed subspace of $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1} \ a^{-(m+1)} da \ dV(\underline{b}) \ ds\big)$.*

Proof. This follows from the fact that the CCWT T_ψ is an isometry between the L_2-spaces $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ and $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1} \ a^{-(m+1)} \ da \ dV(\underline{b}) \ ds\big)$.
□

Corollary 4.3
(1) The orthogonal projection from $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1} \ a^{-(m+1)} \ da \ dV(\underline{b}) \ ds\big)$ onto $\mathcal{H}_\psi$ is an integral operator whose kernel K_ψ is the autocorrelation function of ψ, also called reproducing kernel, given by:

$$K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s}) = T_\psi[\psi^{a,\underline{b},s}](\tilde{a},\tilde{\underline{b}},\tilde{s}) = < \psi^{\tilde{a},\tilde{\underline{b}},\tilde{s}}, \psi^{a,\underline{b},s} > \quad .$$

The above result can be rephrased as follows.
(2) A function $F\big(a,\underline{b},s\big) \in L_2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1}\ a^{-(m+1)}\ da\ dV(\underline{b})ds\big)$ *is the CCWT of a signal* $f(\underline{x}) \in L_2(\mathbb{R}^m, dV(\underline{x}))$ *iff it satisfies the reproducing property:*

$$
\begin{aligned}
&F(a,\underline{b},s)\\
&= [K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s}), F(\tilde{a},\tilde{\underline{b}},\tilde{s})]\\
&= \frac{1}{C_\psi}\int_{\mathrm{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty}\big(K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s})\big)^\dagger\ F(\tilde{a},\tilde{\underline{b}},\tilde{s})\ \frac{d\tilde{a}}{\tilde{a}^{m+1}}\ dV(\tilde{\underline{b}})\ d\tilde{s} \quad .
\end{aligned}
$$

Proof. Let P_ψ denote the projection operator from $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m),$ $C_\psi^{-1}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds\big)$ onto the image $\mathcal{H}_\psi$ of the CCWT T_ψ . As T_ψ is an isometry, this projection operator is simply the composition of T_ψ with its adjoint, i.e. $P_\psi = T_\psi T_\psi^*$. Hence a function $F \in L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_\psi^{-1}$ $a^{-(m+1)}\ da\ dV(\underline{b})\ ds\big)$ belongs to $\mathcal{H}_\psi$ iff

$$
\begin{aligned}
F(a,\underline{b},s) &= P_\psi[F](a,\underline{b},s) = T_\psi\big[T_\psi^*[F]\big](a,\underline{b},s)\\
&= <\psi^{a,\underline{b},s},\ T_\psi^*[F]> \quad . \qquad (4.6)
\end{aligned}
$$

Furthermore the expression for the adjoint, which on $\mathcal{H}_\psi$ equals the inverse T_ψ^{-1}, takes the form

$$
T_\psi^*[F](\underline{x}) = \frac{1}{C_\psi}\int_{\mathrm{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty}\psi^{\tilde{a},\tilde{\underline{b}},\tilde{s}}(\underline{x})\ F(\tilde{a},\tilde{\underline{b}},\tilde{s})\ \frac{d\tilde{a}}{\tilde{a}^{m+1}}\ dV(\tilde{\underline{b}})\ d\tilde{s} \quad .
$$

Hence (4.6) becomes

$$
\begin{aligned}
&F(a,\underline{b},s)\\
&= \frac{1}{C_\psi}\int_{\mathrm{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty} <\psi^{a,\underline{b},s},\psi^{\tilde{a},\tilde{\underline{b}},\tilde{s}}>\ F(\tilde{a},\tilde{\underline{b}},\tilde{s})\ \frac{d\tilde{a}}{\tilde{a}^{m+1}}\ dV(\tilde{\underline{b}})\ d\tilde{s}\\
&= \frac{1}{C_\psi}\int_{\mathrm{Spin}(m)}\int_{\mathbb{R}^m}\int_0^{+\infty}\big(K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s})\big)^\dagger\ F(\tilde{a},\tilde{\underline{b}},\tilde{s})\ \frac{d\tilde{a}}{\tilde{a}^{m+1}}\ dV(\tilde{\underline{b}})\ d\tilde{s} \quad ,
\end{aligned}
$$

which proves the statement. □

Translations, dilations and spinor-rotations are represented by the following operator actions on the space $L_2(\mathbb{R}^m, dV(\underline{x}))$ of square integrable functions:

1. translation: $T(\underline{b})f(\underline{x}) = f(\underline{x}-\underline{b}) \ , \ \underline{b} \in \mathbb{R}^m$

2. dilation: $D(a)f(\underline{x}) = \frac{1}{a^{m/2}} f\left(\frac{\underline{x}}{a}\right) \quad , \quad a \in \mathbb{R}_+$

3. spinor-rotation: $L(s)f(\underline{x}) = s f(\overline{s}\underline{x}s) \quad , \quad s \in \mathrm{Spin}(m)$
   and
   $H(s)f(\underline{x}) = s f(\overline{s}\underline{x}s)\overline{s} \quad , \quad s \in \mathrm{Spin}(m) \quad .$

Note that the operators $T(\underline{b})$, $D(a)$ and $L(s)$ are unitary, i.e. they preserve the L_2-norm.

The CCWT has the property of being covariant under translations, dilations and spinor-rotations.

Proposition 4.2 *The CCWT satisfies*

1. $T_\psi[T(\tilde{\underline{b}})f](a, \underline{b}, s) = T_\psi[f](a, \underline{b} - \tilde{\underline{b}}, s)$

2. $T_\psi[D(\tilde{a})f](a, \underline{b}, s) = T_\psi[f](\tilde{a}^{-1}a, \tilde{a}^{-1}\underline{b}, s)$

3. $T_\psi[L(\tilde{s})f](a, \underline{b}, s) = \tilde{s}\ T_\psi[f](a, \overline{\tilde{s}}\underline{b}\tilde{s}, \overline{\tilde{s}}s)$
   and
   $T_\psi[H(\tilde{s})f](a, \underline{b}, s) = \tilde{s}\ T_\psi[f](a, \overline{\tilde{s}}\underline{b}\tilde{s}, \overline{\tilde{s}}s)\ \overline{\tilde{s}} \quad .$

Proof. By means of the substitutions $\underline{y} = \underline{x} - \tilde{\underline{b}}$, $\underline{y} = \frac{\underline{x}}{\tilde{a}}$ and $\underline{y} = \overline{\tilde{s}}\underline{x}\tilde{s}$ respectively, one obtains by a straightforward calculation:

1.

$$
\begin{aligned}
&T_\psi[T(\tilde{\underline{b}})f](a, \underline{b}, s) \\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\underline{x} - \underline{b})s}{a}\right)\right)^\dagger \overline{s}\ f(\underline{x} - \tilde{\underline{b}})\ dV(\underline{x}) \\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}\big(\underline{y} - (\underline{b} - \tilde{\underline{b}})\big)s}{a}\right)\right)^\dagger \overline{s}\ f(\underline{y})\ dV(\underline{y}) \\
&\quad = T_\psi[f](a, \underline{b} - \tilde{\underline{b}}, s)
\end{aligned}
$$

2.

$$
\begin{aligned}
&T_\psi[D(\tilde{a})f](a,\underline{b},s)\\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ \frac{1}{\tilde{a}^{m/2}}\ f\left(\frac{\underline{x}}{\tilde{a}}\right)\ dV(\underline{x})\\
&\quad = \frac{\tilde{a}^{m/2}}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\tilde{a}\underline{y}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ f(\underline{y})\ dV(\underline{y})\\
&\quad = \frac{1}{(\tilde{a}^{-1}a)^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\underline{y}-\tilde{a}^{-1}\underline{b})s}{\tilde{a}^{-1}a}\right)\right)^\dagger \overline{s}\ f(\underline{y})\ dV(\underline{y})\\
&\quad = T_\psi[f](\tilde{a}^{-1}a,\tilde{a}^{-1}\underline{b},s)
\end{aligned}
$$

3.

$$
\begin{aligned}
&T_\psi[L(\tilde{s})f](a,\underline{b},s)\\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\overline{\tilde{s}}\underline{x}\tilde{s})\ dV(\underline{x})\\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\tilde{s}\underline{y}\overline{\tilde{s}}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\underline{y})\ dV(\underline{y})\\
&\quad = \tilde{s}\ \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \overline{\tilde{s}}\ s \left(\psi\left(\frac{\overline{s}\tilde{s}(\underline{y}-\overline{\tilde{s}}\underline{b}\tilde{s})\overline{\tilde{s}}s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\underline{y})\ dV(\underline{y})\\
&\quad = \tilde{s}\ T_\psi[f](a,\overline{\tilde{s}}\underline{b}\tilde{s},\overline{\tilde{s}}s)\quad .
\end{aligned}
$$

4.

$$
\begin{aligned}
&T_\psi[H(\tilde{s})f](a,\underline{b},s)\\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\overline{\tilde{s}}\underline{x}\tilde{s})\ \overline{\tilde{s}}\ dV(\underline{x})\\
&\quad = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(\psi\left(\frac{\overline{s}(\tilde{s}\underline{y}\overline{\tilde{s}}-\underline{b})s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\underline{y})\ \overline{\tilde{s}}\ dV(\underline{y})\\
&\quad = \tilde{s}\ \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \overline{\tilde{s}}\ s \left(\psi\left(\frac{\overline{s}\tilde{s}(\underline{y}-\overline{\tilde{s}}\underline{b}\tilde{s})\overline{\tilde{s}}s}{a}\right)\right)^\dagger \overline{s}\ \tilde{s}\ f(\underline{y})\ dV(\underline{y})\ \overline{\tilde{s}}\\
&\quad = \tilde{s}\ T_\psi[f](a,\overline{\tilde{s}}\underline{b}\tilde{s},\overline{\tilde{s}}s)\ \overline{\tilde{s}}\quad . \qquad \square
\end{aligned}
$$

Remark 4.3 In agreement with the general representation theory of groups, we define the representation of Spin(m) on the CCWT as follows:

$$L(\tilde{s})T_\psi[f](a,\underline{b},s) := \tilde{s}\ T_\psi[f](a,\overline{\tilde{s}}\underline{b}\tilde{s},\overline{\tilde{s}}s) \quad .$$

Consequently, we have proved the following commutation relation:

$$T_\psi[L(\tilde{s})f](a,\underline{b},s) = L(\tilde{s})T_\psi[f](a,\underline{b},s) \quad .$$

In order to obtain the above Spin(m)-invariance of the wavelet transform, the use of the H-representation (instead of the L-representation) on the kernel of the wavelet transform is necessary.

4.3 Special case: Spin(m)-invariance of the mother wavelet

Special attention should be paid to the special case where the mother wavelet ψ is invariant under the rotation group Spin(m), which means

$$s\ \psi(\overline{s}\underline{x}s)\ \overline{s} = \psi(\underline{x}) \quad , \quad s \in \mathrm{Spin}(m) \quad .$$

In this case we do not have to take this group into consideration while defining the continuous family of wavelets:

$$\psi^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{m/2}}\ \psi\left(\frac{\underline{x}-\underline{b}}{a}\right) \quad ; \quad a \in \mathbb{R}_+ \ , \ \ \underline{b} \in \mathbb{R}^m \quad .$$

The CCWT of $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ then takes the form

$$\begin{aligned} T_\psi[f](a,\underline{b}) = F_\psi(a,\underline{b}) &= \ < \psi^{a,\underline{b}}, f > \\ &= \int_{\mathbb{R}^m} \left(\psi^{a,\underline{b}}(\underline{x})\right)^\dagger\ f(\underline{x})\ dV(\underline{x}) \quad . \end{aligned}$$

This definition is rewritten in terms of the Fourier transform as:

$$F_\psi(a,\underline{b}) = a^{m/2}\ (2\pi)^{m/2}\ \mathcal{F}\left[\left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger\ \mathcal{F}[f](\underline{\xi})\right](-\underline{b}) \quad .$$

In order to obtain an isometry, the mother wavelet must now satisfy slightly different restrictions.

Definition 4.2 *A* Spin(m)*-invariant Clifford algebra-valued function* $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ *is a mother wavelet in* $\mathbb{R}^m$ *provided the following two conditions are fulfilled:*

1. $\mathcal{F}[\psi](\underline{\xi}) \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger$ *is radial symmetric (i.e. only depending on* $|\underline{\xi}|$*)*

2. $$C_\psi = \frac{(2\pi)^m}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](\underline{\xi}) \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) < +\infty \quad , \tag{4.7}$$

   with A_m *the area if the unit sphere* S^{m-1} *in* $\mathbb{R}^m$ *.*

Obviously, the inner product in the space of transforms now takes the form

$$[F_\psi, G_\psi] = \frac{1}{C_\psi} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi(a, \underline{b})\right)^\dagger \, G_\psi(a, \underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \quad .$$

Next, we briefly summarize the CCWT properties in case of a Spin(m)-invariant mother wavelet. We only include the proof of the main result, the proofs for the other properties being similar to the ones of the previous section.

Theorem 4.2 *The CCWT associated with a* Spin(m)*-invariant mother wavelet is an isometry from the space of signals* $L_2\left(\mathbb{R}^m, dV(\underline{x})\right)$ *into the weighted* L_2*-space* $L_2\left(\mathbb{R}_+ \times \mathbb{R}^m, C_\psi^{-1} \, a^{-(m+1)} \, da \, dV(\underline{b})\right)$*, i.e. the Parseval formula*

$$< f, g > = [F_\psi, G_\psi]$$

holds.

Proof. The inner product in the space of transforms can be written as

$$\begin{aligned}
&[F_\psi, G_\psi] \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}\left[\left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \mathcal{F}[f](\underline{\xi})\right](-\underline{b})\right)^\dagger \\
&\qquad \mathcal{F}\left[\left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \mathcal{F}[g](\underline{\xi})\right](-\underline{b}) \, \frac{da}{a} \, dV(\underline{b}) \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}[f](\underline{\xi})\right)^\dagger \, \mathcal{F}[\psi](a\underline{\xi}) \, \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \mathcal{F}[g](\underline{\xi}) \, \frac{da}{a} \, dV(\underline{\xi}) \\
&= \frac{(2\pi)^m}{C_\psi} \int_{\mathbb{R}^m} \left(\mathcal{F}[f](\underline{\xi})\right)^\dagger \left(\int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \, \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \frac{da}{a}\right) \mathcal{F}[g](\underline{\xi}) \, dV(\underline{\xi}) \quad .
\end{aligned}$$

By means of the substitution

$$\underline{\xi} = \frac{t}{a}\, \underline{\eta}, \quad \underline{\eta} \in S^{m-1},$$

we obtain

$$\int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \, \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \frac{da}{a} = \int_0^{+\infty} \mathcal{F}[\psi](t\underline{\eta}) \, \left(\mathcal{F}[\psi](t\underline{\eta})\right)^\dagger \, \frac{dt}{t} \quad .$$

As $\mathcal{F}[\psi] \, \left(\mathcal{F}[\psi]\right)^\dagger$ is radial symmetric, this integral can be further simplified to

$$\begin{aligned} \int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \, \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \, \frac{da}{a} &= \frac{1}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](\underline{u}) \, \left(\mathcal{F}[\psi](\underline{u})\right)^\dagger}{|\underline{u}|^m} \, dV(\underline{u}) \\ &= \frac{C_\psi}{(2\pi)^m} \quad , \end{aligned}$$

which proves the statement. □

Corollary 4.4 *In case of a* Spin(m)*-invariant mother wavelet, the reconstruction formula, which holds in the weak sense, takes the form*

$$\begin{aligned} f(\underline{x}) &= \frac{1}{C_\psi} \int_{\mathbb{R}^m} \int_0^{+\infty} \psi^{a,\underline{b}}(\underline{x}) \; F_\psi(a,\underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \\ &= [\left(\psi^{a,\underline{b}}(\underline{x})\right)^\dagger, F_\psi(a,\underline{b})] \end{aligned}$$

with $F_\psi(a,\underline{b}) = T_\psi[f](a,\underline{b})$, $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$.

Remark 4.4 Again a more general reconstruction formula holds:

$$f(\underline{x}) = \frac{1}{C_{\psi\chi}} \int_{\mathbb{R}^m} \int_0^{+\infty} \chi^{a,\underline{b}}(\underline{x}) \; F_\psi(a,\underline{b}) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \quad .$$

Here the analysis wavelet ψ and the reconstruction wavelet χ must satisfy the restriction:

$$\mathcal{F}[\chi](\underline{\xi}) \, \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger \; : \; \text{radial symmetric}$$

and the cross-admissibility condition $0 < |C_{\psi\chi}| < \infty$ with

$$C_{\psi\chi} = \frac{(2\pi)^m}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}[\chi](\underline{\xi}) \, \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) < +\infty \quad .$$

Taking as reconstruction wavelet a delta distribution, we obtain the simplified formula

$$f(\underline{x}) = \frac{1}{C_{\psi\delta}} \int_0^{+\infty} F_\psi(a, \underline{x}) \, \frac{da}{a^{m/2+1}} \quad ,$$

in which the signal is reconstructed by integrating only over scales.

Corollary 4.5 *The image of* $L_2(\mathbb{R}^m, dV(\underline{x}))$ *under the CCWT* T_ψ *corresponding to a* Spin(m)*-invariant wavelet, is a closed subspace of* $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m, C_\psi^{-1}\, a^{-(m+1)}\, da\, dV(\underline{b})\big)$ *and thus a Hilbert module with reproducing kernel or autocorrelation function given by:*

$$K_\psi(a, \underline{b}\ ; \tilde{a}, \underline{\tilde{b}}) = T_\psi[\psi^{a,\underline{b}}](\tilde{a}, \underline{\tilde{b}}) = <\psi^{\tilde{a},\underline{\tilde{b}}}, \psi^{a,\underline{b}}> \quad .$$

Hence, a function $F(a, \underline{b}) \in L_2\big(\mathbb{R}_+ \times \mathbb{R}^m, C_\psi^{-1}\, a^{-(m+1)}\, da\, dV(\underline{b})\big)$ *is the CCWT of a signal* $f(\underline{x}) \in L_2(\mathbb{R}^m, dV(\underline{x}))$ *iff it satisfies the reproducing property:*

$$\begin{aligned} F(a, \underline{b}) &= [K_\psi(a, \underline{b}\ ; \tilde{a}, \underline{\tilde{b}}), F(\tilde{a}, \underline{\tilde{b}})] \\ &= \frac{1}{C_\psi} \int_{\mathbb{R}^m} \int_0^{+\infty} \big(K_\psi(a, \underline{b}\ ; \tilde{a}, \underline{\tilde{b}})\big)^\dagger\, F(\tilde{a}, \underline{\tilde{b}})\, \frac{d\tilde{a}}{\tilde{a}^{m+1}}\, dV(\underline{\tilde{b}}) \quad . \end{aligned}$$

Finally, we mention the covariance property of the CCWT.

Proposition 4.3 *The CCWT associated with a* Spin(m)*-invariant mother wavelet, is covariant under translations and dilations:*

1. $T_\psi[T(\underline{\tilde{b}})f](a, \underline{b}) = T_\psi[f](a, \underline{b} - \underline{\tilde{b}})$
2. $T_\psi[D(\tilde{a})f](a, \underline{b}) = T_\psi[f](\tilde{a}^{-1}a, \tilde{a}^{-1}\underline{b})$.

4.4 Constructing Clifford-wavelets: methodology

In the previous sections, we defined the notion of mother wavelet in the framework of Clifford analysis. These Clifford-wavelets give rise to the Continuous Clifford-Wavelet Transform, which shows characteristics analogous to those of the classical CWT. A natural question is of course how to construct such Clifford-wavelets. The answer to that question is given in this section and is at the core of the construction of specific Clifford wavelets in the following chapters.

The first step in the construction of multi-dimensional Clifford-wavelets, is the introduction of new polynomials, generalizing classical orthogonal polynomials on the real line to the Clifford analysis setting. Their construction rests upon a specific Clifford analysis technique described in Chapter 2, viz. the Cauchy-Kowalewskaia extension.
One starts from a real-analytic function $F(\underline{x})$ in an open connected domain $\underline{\Omega}$ in $\mathbb{R}^m$, as an analogue of the classical weight function. The new Clifford algebra-valued polynomials are then generated by the CK-extension $F^*(x_0, \underline{x})$ of this weight function $F(\underline{x})$. As explained in Chapter 2, this CK-extension exists in an open connected and x_0-normal neighbourhood Ω of $\underline{\Omega}$ in $\mathbb{R}^{m+1}$.
By definition, the function F^* is left monogenic in Ω and thus satisfies

$$(\partial_{x_0} + \partial_{\underline{x}})F^*(x_0, \underline{x}) = 0 \quad \text{in } \Omega \quad , \tag{4.8}$$

while its restriction to $\underline{\Omega}$ is precisely the weight function:

$$F^*(0, \underline{x}) = F(\underline{x}) \quad .$$

By means of the monogenicity relation (4.8) a recurrence relation for the newly introduced polynomials is obtained, from which the polynomials can be computed recursively.
Furthermore, from the formula (2.4) for the CK-extension we obtain the so-called Rodrigues formula, which together with the Clifford-Stokes theorem lead to an orthogonality relation on $\mathbb{R}^m$ of the newly generalized Clifford-polynomials.

Next, from this orthogonality relation we obtain $L_1 \cap L_2$-functions ψ which have zero momentum and can thus be used as mother wavelets in $\mathbb{R}^m$, if at least they satisfy the conditions outlined in the previous sections (see Definition 4.1 and 4.2 in case of a Spin(m)-invariant function).

We have applied the above technique for constructing multi-dimensional Clifford-wavelets on the basis of Clifford generalizations of the Hermite polynomials, the Gegenbauer polynomials and the Laguerre polynomials. They are described in detail in the next three chapters.

Chapter 5

Clifford-Hermite polynomials and associated CCWT

In this chapter we present the CCWT associated with the so-called bi-axial Clifford-Hermite polynomials. In this way we generalize the Clifford-Hermite CWT of Brackx and Sommen, which is described in some introductory sections to make the text self-contained.

5.1 Introduction

On the real line the Hermite polynomials associated with the weight function $\exp\left(-\frac{x^2}{2}\right)$ may be defined by the Rodrigues formula

$$He_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left[\exp\left(-\frac{x^2}{2}\right)\right] \quad , \quad n = 0, 1, 2, \ldots \tag{5.1}$$

They constitute an orthogonal basis for the weighted Hilbert space $L_2\left( ]-\infty, +\infty[\, , \exp\left(-\frac{x^2}{2}\right) \, dx \right)$, and they satisfy the orthogonality relation

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2}\right) He_n(x) \, He_m(x) \, dx = n! \, \sqrt{2\pi} \, \delta_{n,m}$$

and moreover the recurrence relation

$$He_{n+1}(x) = x \; He_n(x) - \frac{d}{dx}\left[He_n(x)\right] \; .$$

Furthermore, $He_n(x)$ is an even or an odd function according to the parity of n, i.e. $He_n(-x) = (-1)^n He_n(x)$.

In [108] Sommen introduced the radial Clifford-Hermite polynomials, which are a specific generalization to Clifford analysis of the Hermite polynomials on the real line (see Section 5.2.1). These so-called radial Clifford-Hermite polynomials are the building blocks for new specific wavelet kernel functions (see Section 5.2.2). The resulting Spin(m)-invariant Clifford-Hermite wavelets are alternatively real- and vector-valued, show vanishing moments of any order and offer a refinement of the Marr wavelets.
Using the generalized Clifford-Hermite polynomials (see Section 5.3.1), also introduced in [108], the generalized Clifford-Hermite CWT is obtained in Section 5.3.2. This higher dimensional CWT offers the possibility of a pointwise and a directional analysis of signals.
The bi-axial generalizations of the above mentioned radial, respectively generalized, Clifford-Hermite polynomials, are constructed in Sections 5.4 and 5.5 respectively. These newly introduced polynomials give rise to wavelet functions which have an "elliptic form", the shape and orientation of which can be adapted in a flexible way.

5.2 The radial Clifford-Hermite polynomials and associated CCWTs

5.2.1 The radial Clifford-Hermite polynomials

The *radial Clifford-Hermite polynomials* $H_\ell(\underline{x})$ in $\mathbb{R}^m$ were introduced by Sommen in [108]; they are generated by the CK-extension F^* of the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$:

$$F^*(x_0, \underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \; H_\ell(\underline{x}) \quad .$$

The word "radial" refers to the fact that the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$ is invariant under the rotation group SO(m).

These radial Clifford-Hermite polynomials are alternatively scalar- or vector-valued according to the parity of ℓ and satisfy the recurrence relation:

$$H_{\ell+1}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})H_\ell(\underline{x}) \quad . \tag{5.2}$$

The above recurrence relation allows for a straightforward computation:

$$\begin{aligned}
H_0(\underline{x}) &= 1 \\
H_1(\underline{x}) &= \underline{x} \\
H_2(\underline{x}) &= \underline{x}^2 + m = -|\underline{x}|^2 + m \\
H_3(\underline{x}) &= \underline{x}^3 + (m+2)\ \underline{x} = \underline{x}\ (-|\underline{x}|^2 + m + 2) \\
H_4(\underline{x}) &= \underline{x}^4 + 2(m+2)\ \underline{x}^2 + m(m+2) \\
&\text{etc.}
\end{aligned}$$

Furthermore, the radial Clifford-Hermite polynomials satisfy the Rodrigues formula:

$$H_\ell(\underline{x}) = (-1)^\ell \exp\left(\frac{|\underline{x}|^2}{2}\right)\ \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right] \tag{5.3}$$

and are mutually orthogonal with respect to the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$:

$$\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ \overline{H_t(\underline{x})}\ H_\ell(\underline{x})\ dV(\underline{x}) = \delta_{t,\ell}\ \gamma_\ell \tag{5.4}$$

with

$$\gamma_{2p} = \frac{2^{2p+m/2}\ p!\ \pi^{m/2}\ \Gamma\left(\frac{m}{2}+p\right)}{\Gamma\left(\frac{m}{2}\right)}$$

and

$$\gamma_{2p+1} = \frac{2^{2p+m/2+1}\ p!\ \pi^{m/2}\ \Gamma\left(\frac{m}{2}+p+1\right)}{\Gamma\left(\frac{m}{2}\right)} \quad .$$

5.2.2 The Clifford-Hermite CWT

For $\ell > 0$ the orthogonality relation (5.4) leads to the so-called *Clifford-Hermite wavelets*:

$$\psi_\ell(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_\ell(\underline{x}) = (-1)^\ell\ \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right] \quad .$$

Note that for $\ell = 2N$ $(N > 0)$, we obtain the classical Marr wavelets

$$\psi_{2N}(\underline{x}) = (-\Delta_{\underline{x}})^N \left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right] \quad .$$

If we put

$$\psi_{2N}(\underline{x}) = \psi_N^{\mathrm{Marr}}(\underline{x}) \quad ,$$

we get for $\ell = 2N+1$:

$$\psi_{2N+1}(\underline{x}) = -\partial_{\underline{x}}\left[\psi_N^{\mathrm{Marr}}(\underline{x})\right] \quad .$$

Hence the Clifford-Hermite wavelets, which were introduced by Brackx and Sommen in [32], are indeed a refinement of the traditional Marr wavelets.
It is easily seen that they are alternatively scalar- or vector-valued and invariant under the rotation group $\mathrm{Spin}(m)$.
Their Fourier transform is given by

$$\mathcal{F}\left[\psi_\ell\right](\underline{\xi}) = (-i)^\ell \; \underline{\xi}^\ell \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad .$$

Consequently

$$\mathcal{F}\left[\psi_\ell\right](\underline{\xi}) \; \left(\mathcal{F}\left[\psi_\ell\right](\underline{\xi})\right)^\dagger = |\underline{\xi}|^{2\ell} \exp\left(-|\underline{\xi}|^2\right)$$

is radial symmetric.
Furthermore, their admissibility constants are

$$\begin{aligned} C_\ell &= \frac{(2\pi)^m}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}\left[\psi_\ell\right](\underline{\xi}) \; \left(\mathcal{F}\left[\psi_\ell\right](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \; dV(\underline{\xi}) \\ &= (2\pi)^m \; \frac{(\ell-1)!}{2} < +\infty \quad . \end{aligned}$$

Moreover, the Clifford-Hermite wavelets have vanishing moments up to order $(\ell - 1)$:

$$\int_{\mathbb{R}^m} \underline{x}^j \; \psi_\ell(\underline{x}) \; dV(\underline{x}) = 0 \quad ; \quad j = 0, 1, \ldots, \ell-1 \quad .$$

Hence the corresponding *Clifford-Hermite CWT*

$$\begin{aligned} T_\ell : \; L_2(\mathbb{R}^m, dV(\underline{x})) &\longmapsto L_2\left(\mathbb{R}_+ \times \mathbb{R}^m, C_\ell^{-1} \; a^{-(m+1)} \; da \; dV(\underline{b})\right) \\ f(\underline{x}) &\longmapsto F_\ell(a, \underline{b}) = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}-\underline{b}|^2}{2a^2}\right) \\ &\qquad\qquad \overline{H_\ell}\left(\frac{\underline{x}-\underline{b}}{a}\right) \; f(\underline{x}) \; dV(\underline{x}) \end{aligned}$$

filters out polynomial behaviour of the signal up to degree $(\ell - 1)$. It is worth noticing that this degree may be even or odd, whereas the polynomial factor in the Marr wavelets always has even degree. This is of particular importance for the technique, developed in the 1-D case by Arnéodo et al. (see [8]), where a signal is analysed with several wavelets ψ_N^{Marr} for different N .

5.3 The generalized Clifford-Hermite polynomials and associated CCWT

5.3.1 The generalized Clifford-Hermite polynomials

In order to obtain a basis for the weighted Hilbert module $L_2\left(\mathbb{R}^m,\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)dV(\underline{x})\right)$, Sommen introduced in [108] the *generalized Clifford-Hermite polynomials*. These polynomials depend on a parameter $k \in \mathbb{N}$, such that for $k = 0$ the radial Clifford-Hermite polynomials of Section 5.2.1 are reobtained.
The generalized Clifford-Hermite polynomials are defined by the CK-extension G^* of the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)P_k(\underline{x})$, with $P_k(\underline{x})$ any left solid inner spherical monogenic of order k :

$$G^*(x_0, \underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right)\sum_{\ell=0}^{\infty}\frac{x_0^\ell}{\ell!}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) \quad .$$

From the monogenicity of G^*, we obtain

$$H_{\ell+1,k}(\underline{x})\ P_k(\underline{x}) = (\underline{x} - \partial_{\underline{x}})\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] \quad , \tag{5.5}$$

which in its turn leads to the following recurrence relations:

$$H_{2\ell+1,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})H_{2\ell,k}(\underline{x}) \tag{5.6}$$

and

$$H_{2\ell+2,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})H_{2\ell+1,k}(\underline{x}) - 2k\ \frac{\underline{x}}{|\underline{x}|^2}\ H_{2\ell+1,k}(\underline{x}) \quad . \tag{5.7}$$

A straightforward calculation yields

$$\begin{aligned}
H_{0,k}(\underline{x}) &= 1 \\
H_{1,k}(\underline{x}) &= \underline{x} \\
H_{2,k}(\underline{x}) &= \underline{x}^2 + 2k + m = -|\underline{x}|^2 + 2k + m \\
H_{3,k}(\underline{x}) &= \underline{x}^3 + (2k + m + 2)\ \underline{x} = \underline{x}\ (-|\underline{x}|^2 + 2k + m + 2) \\
H_{4,k}(\underline{x}) &= \underline{x}^4 + 2(2k + m + 2)\ \underline{x}^2 + (2k + m)(2k + m + 2) \\
&= |\underline{x}|^4 - 2(2k + m + 2)|\underline{x}|^2 + (2k + m)(2k + m + 2) \\
H_{5,k}(\underline{x}) &= \underline{x}^5 + 2(2k + m + 4)\ \underline{x}^3 + (2k + m + 4)(2k + m + 2)\ \underline{x} \\
&= \underline{x}\ \left(|\underline{x}|^4 - 2(2k + m + 4)|\underline{x}|^2 + (2k + m + 4)(2k + m + 2)\right)
\end{aligned} \tag{5.8}$$

etc.

The functions $H_{\ell,k}(\underline{x})$ do not depend upon the particular choice of the left solid inner spherical monogenic P_k, only upon the order k. Note that they are polynomials of degree ℓ in the variable $\underline{x}$ with real coefficients depending on k . Moreover, they satisfy the Rodrigues formula

$$H_{\ell,k}(\underline{x})\ P_k(\underline{x}) = \exp\left(\frac{|\underline{x}|^2}{2}\right)\ (-\partial_{\underline{x}})^{\ell} \left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) P_k(\underline{x})\right] \tag{5.9}$$

and the orthogonality relation

$$\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ \left(H_{\ell,k_1}(\underline{x})\ P_{k_1}(\underline{x})\right)^{\dagger}\ H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ dV(\underline{x}) = \gamma_{\ell,k_1}\ \delta_{\ell,t}\ \delta_{k_1,k_2} \tag{5.10}$$

with

$$\gamma_{2p,k} = \frac{2^{2p+m/2+k}\ p!\ \pi^{m/2}\ \Gamma\left(\frac{m}{2} + k + p\right)}{\Gamma\left(\frac{m}{2}\right)}$$

and

$$\gamma_{2p+1,k} = \frac{2^{2p+m/2+k+1}\ p!\ \pi^{m/2}\ \Gamma\left(\frac{m}{2} + k + p + 1\right)}{\Gamma\left(\frac{m}{2}\right)} \quad .$$

Furthermore, the set

$$\left\{\frac{1}{(\gamma_{t,k})^{1/2}}\ H_{t,k}(\underline{x})\ P_k^{(j)}(\underline{x})\ ;\ t, k \in \mathbb{N},\ j \leq \dim(M_\ell^+(k))\right\}$$

constitutes an orthonormal basis for $L_2\left(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ dV(\underline{x})\right)$. Here

$$\left\{P_k^{(j)}(\underline{x})\ ;\ j = 1, 2, \ldots, \dim\left(M_\ell^+(k)\right)\right\}$$

denotes an orthonormal basis of $M_\ell^+(k)$.

5.3.2 The generalized Clifford-Hermite CWT

For $(t, k_2) \neq (0,0)$ the orthogonality relation (5.10) implies that the $L_1 \cap L_2$-functions

$$\psi_{t,k_2}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x}) = (-1)^t\ \partial_{\underline{x}}^t \left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) P_{k_2}(\underline{x})\right]$$

have zero momentum. These so-called *generalized Clifford-Hermite wavelets* were introduced by Brackx and Sommen in [33].
Their Fourier transform is given by

$$\begin{aligned}
\mathcal{F}\left[\psi_{t,k_2}\right](\underline{\xi}) &= (-i)^t\ \underline{\xi}^t\ P_{k_2}(i\partial_{\underline{\xi}})\ \mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right](\underline{\xi}) \\
&= (-i)^t\ i^{k_2}\ \underline{\xi}^t\ P_{k_2}(\partial_{\underline{\xi}})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \\
&= (-i)^{t+k_2}\ \underline{\xi}^t P_{k_2}(\underline{\xi})\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) \quad ,
\end{aligned}$$

where we have used Proposition 2.1 and Proposition 2.3 . Note that this is again a product of the Gaussian function with a polynomial of degree $(t + k_2)$.
The corresponding *generalized Clifford-Hermite CWT* (GCHCWT) applies to functions $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ by

$$\begin{aligned}
T_{t,k_2}[f](a, \underline{b}, s) &= F_{t,k_2}(a, \underline{b}, s) \\
&= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s\ \exp\left(-\frac{|\underline{x}-\underline{b}|^2}{2a^2}\right) \left(P_{k_2}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \\
&\qquad \overline{H_{t,k_2}}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \overline{s}\ f(\underline{x})\ dV(\underline{x}) \quad .
\end{aligned}$$

On condition that $\mathcal{F}[\psi_{t,k_2}]\ \left(\mathcal{F}[\psi_{t,k_2}]\right)^\dagger$ is scalar-valued, and the admissibility condition

$$C_{t,k_2} = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}\left[\psi_{t,k_2}\right](\underline{\xi})\ \left(\mathcal{F}\left[\psi_{t,k_2}\right](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m}\ dV(\underline{\xi}) < +\infty$$

is fulfilled, this GCHCWT maps $L_2(\mathbb{R}^m, dV(\underline{x}))$ isometrically into $L_2\left(\mathbb{R}_+ \times \mathbb{R}^m \times \text{Spin}(m),\, C_{t,k_2}^{-1}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds\right)$ (see Definition 4.1).
The condition that

$$\mathcal{F}[\psi_{t,k_2}](\underline{\xi})\ \left(\mathcal{F}[\psi_{t,k_2}](\underline{\xi})\right)^\dagger = \exp\left(-|\underline{\xi}|^2\right)\ \underline{\xi}^t\ P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger\ (-\underline{\xi})^t$$

should be scalar-valued is equivalent with the condition that $P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger$ should be scalar-valued. Furthermore, if $P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger$ is scalar-valued, we have that, using spherical co-ordinates:

$$\begin{aligned} C_{t,k_2} &= (2\pi)^m \int_{\mathbb{R}^m} \exp\left(-|\underline{\xi}|^2\right)\ |\underline{\xi}|^{2t-m}\ P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger\ dV(\underline{\xi}) \\ &= (2\pi)^m \int_0^{+\infty} \exp\left(-\rho^2\right)\ \rho^{2t+2k_2-1}\ d\rho \int_{S^{m-1}} |P_{k_2}(\underline{\eta})|^2\ dS(\underline{\eta}) \\ &= (2\pi)^m\ \frac{(t+k_2-1)!}{2} \int_{S^{m-1}} |P_{k_2}(\underline{\eta})|^2\ dS(\underline{\eta}) < +\infty\quad . \end{aligned}$$

5.4 The bi-axial Clifford-Hermite polynomials and associated CCWT

In this section and the next one we treat of the bi-axial generalizations of the radial and generalized Clifford-Hermite polynomials and associated CCWTs (see Sections 5.2 and 5.3) as they were constructed in [20].
These bi-axial generalizations are obtained by considering functions of two vector variables

$$\underline{x} = \sum_{j=1}^{p} x_j e_j \quad \text{and} \quad \underline{y} = \sum_{j=1}^{q} y_j e_{p+j}\quad .$$

The sum $\underline{x} + \underline{y}$ represents a vector variable in $\mathbb{R}^{p+q} = \mathbb{R}^m$ equipped with the orthonormal basis $(e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q})$, where $p + q = m$. We start with the bi-axial generalizations of the radial Clifford-Hermite polynomials of Section 5.2.1. For the sake of clarity, we now add the dimension of the considered Euclidean space as a second subindex to the polynomials.

5.4.1 The bi-axial Clifford-Hermite polynomials

The CK-extension H^* of the weight function

$$\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \quad \text{with } a>0 \text{ and } b>0 \quad ,$$

generates the *bi-axial Clifford-Hermite polynomials* in $\mathbb{R}^{p+q}$:

$$H^*(s,\underline{x},\underline{y}) = \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\ H_{\ell,p+q,a,b}(\underline{x},\underline{y}) \quad .$$

By definition H^* satisfies in $\mathbb{R}^{p+q+1}$:

$$(\partial_s + \partial_{\underline{x}} + \partial_{\underline{y}})\ H^*(s,\underline{x},\underline{y}) = 0 \tag{5.11}$$

and

$$H^*(0,\underline{x},\underline{y}) = \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \quad . \tag{5.12}$$

From (5.12) we obtain

$$H_{0,p+q,a,b}(\underline{x},\underline{y}) = 1 \quad ,$$

while the monogenicity relation (5.11) can be rewritten as follows

$$\sum_{\ell=1}^{\infty}\frac{s^{\ell-1}}{(\ell-1)!}\ H_{\ell,p+q,a,b}(\underline{x},\underline{y}) - (a\ \underline{x} + b\ \underline{y})\ \sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\ H_{\ell,p+q,a,b}(\underline{x},\underline{y}) + \sum_{\ell=0}^{\infty}\frac{s^\ell}{\ell!}\ (\partial_{\underline{x}} + \partial_{\underline{y}})\big[H_{\ell,p+q,a,b}(\underline{x},\underline{y})\big] = 0 \quad .$$

The above relation can be further simplified obtaining the recurrence relation:

$$H_{\ell+1,p+q,a,b}(\underline{x},\underline{y}) = (a\underline{x} + b\underline{y})\ H_{\ell,p+q,a,b}(\underline{x},\underline{y}) - (\partial_{\underline{x}} + \partial_{\underline{y}})\ \big[H_{\ell,p+q,a,b}(\underline{x},\underline{y})\big] \quad .$$

This recurrence relation allows for a recursive computation:

$$\begin{aligned}
H_{1,p+q,a,b}(\underline{x},\underline{y}) &= a\underline{x} + b\underline{y} \\
H_{2,p+q,a,b}(\underline{x},\underline{y}) &= a^2\underline{x}^2 + ab\ (\underline{x}\underline{y} + \underline{y}\underline{x}) + b^2\underline{y}^2 - a\ \partial_{\underline{x}}[\underline{x}] - b\ \partial_{\underline{y}}[\underline{y}] \\
&= a^2\underline{x}^2 + b^2\underline{y}^2 + ap + bq \\
H_{3,p+q,a,b}(\underline{x},\underline{y}) &= \underline{x}\left(a^3\underline{x}^2 + ab^2\underline{y}^2 + a(ap+bq)\right) \\
&+ \underline{y}\left(a^2b\underline{x}^2 + b^3\underline{y}^2 + b(ap+bq)\right) - a^2\ \partial_{\underline{x}}[\underline{x}^2] - b^2\ \partial_{\underline{y}}[\underline{y}^2] \\
&= \underline{x}\left(a^3\underline{x}^2 + ab^2\underline{y}^2 + a(ap+bq) + 2a^2\right) \\
&+ \underline{y}\left(b^3\underline{y}^2 + a^2b\underline{x}^2 + b(ap+bq) + 2b^2\right)
\end{aligned}$$

etc.

Note that we have used here the fact that $\underline{x}\underline{y} + \underline{y}\underline{x} = 0$.
We observe that $H_{\ell,p+q,a,b}(\underline{x},\underline{y})$ is a polynomial of degree ℓ ; $H_{2\ell,p+q,a,b}(\underline{x},\underline{y})$ is scalar-valued, while $H_{2\ell+1,p+q,a,b}(\underline{x},\underline{y})$ is vector-valued.
The explicit formula (2.4) for the CK-extension leads to the so-called Rodrigues formula for the bi-axial Clifford-Hermite polynomials:

$$H_{\ell,p+q,a,b}(\underline{x},\underline{y}) = (-1)^\ell \exp\left(\frac{1}{2}a|\underline{x}|^2\right)\exp\left(\frac{1}{2}b|\underline{y}|^2\right) (\partial_{\underline{x}} + \partial_{\underline{y}})^\ell \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right] .$$

From this Rodrigues formula, we will obtain a decomposition of $H_{\ell,p+q,a,b}(\underline{x},\underline{y})$ in terms of polynomials $H_{d,p,a}(\underline{x})$ and $H_{e,q,b}(\underline{y})$.
The polynomials $H_{d,p,a}(\underline{x})$ are generated by the CK-extension F^* to $\mathbb{R}^{p+1}$ of the weight function $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$:

$$F^*(x_0,\underline{x}) = \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\sum_{d=0}^{\infty}\frac{x_0^d}{d!}\ H_{d,p,a}(\underline{x}) .$$

In an analogous manner as above, we obtain that the polynomials $H_{d,p,a}(\underline{x})$ satisfy the recurrence relation

$$H_{d+1,p,a}(\underline{x}) = (a\underline{x} - \partial_{\underline{x}})\ H_{d,p,a}(\underline{x})$$

and the Rodrigues formula

$$H_{d,p,a}(\underline{x}) = (-1)^d \exp\left(\frac{1}{2}a|\underline{x}|^2\right) \partial_{\underline{x}}^d \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \quad . \tag{5.13}$$

Examples are

$$\begin{aligned} H_{0,p,a}(\underline{x}) &= 1 \\ H_{1,p,a}(\underline{x}) &= a\underline{x} \\ H_{2,p,a}(\underline{x}) &= a^2\underline{x}^2 + ap \\ H_{3,p,a}(\underline{x}) &= a^3\underline{x}^3 + a^2(p+2)\underline{x} \\ H_{4,p,a}(\underline{x}) &= a^4\underline{x}^4 + 2a^3(p+2)\underline{x}^2 + a^2p(p+2) \\ &\text{etc.} \end{aligned}$$

Note that $H_{d,p,a}(\underline{x})$ is a polynomial of degree d in the variable $\underline{x}$, $H_{2d,p,a}(\underline{x})$ contains only even powers of $\underline{x}$, while $H_{2d+1,p,a}(\underline{x})$ contains only odd ones.
By means of the Rodrigues formula we obtain that the polynomials $H_{d,p,a}(\underline{x})$ are mutually orthogonal in $\mathbb{R}^p$ with respect to the weight function
$\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$.

Lemma 5.1 *The polynomials $H_{d,p,a}(\underline{x})$ are mutually orthogonal in $\mathbb{R}^p$ with respect to the weight function* $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$*, i.e.*

$$\int_{\mathbb{R}^p} \overline{H_{d,p,a}}(\underline{x})\ H_{w,p,a}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) dV(\underline{x}) = 0$$

for $d \neq w$.

Proof. This proof runs along the same lines as the proof of the mutual orthogonality with respect to the weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$ of the radial Clifford-Hermite polynomials (see [108]).
Suppose that $d < w$; the case $d > w$ is completely similar.
As $H_{d,p,a}(\underline{x})$ is a polynomial of degree d in $\underline{x}$, it is sufficient to show that for each $w \in \mathbb{N}$ and $d < w$:

$$\int_{\mathbb{R}^p} \underline{x}^d\ H_{w,p,a}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) dV(\underline{x}) = 0 \quad .$$

We prove this by induction on d. For $d = 0$ we have for each $w > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^p} H_{w,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \, dV(\underline{x}) \\
&\qquad = \ (-1)^w \int_{\mathbb{R}^p} \partial_{\underline{x}}^w \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \, dV(\underline{x}) \\
&\qquad = \ (-1)^w \int_{\partial\mathbb{R}^p} d\underline{\sigma} \ \partial_{\underline{x}}^{w-1} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] = 0 \ ,
\end{aligned}$$

where we have used the Rodrigues formula (5.13) and the Clifford-Stokes theorem.
Assume that the orthogonality holds for $(d-1)$ and $w > (d-1)$ and let $w > d$. Again by means of the Rodrigues formula and the Clifford-Stokes theorem, we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^p} \underline{x}^d \ H_{w,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \, dV(\underline{x}) \\
&\quad = \ (-1)^w \int_{\mathbb{R}^p} \underline{x}^d \ \partial_{\underline{x}}^w \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \, dV(\underline{x}) \\
&\quad = \ (-1)^w \left\{ \int_{\partial\mathbb{R}^p} \underline{x}^d \ d\underline{\sigma} \ \partial_{\underline{x}}^{w-1} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \right. \\
&\qquad \left. - \int_{\mathbb{R}^p} (\underline{x}^d \partial_{\underline{x}}) \ \partial_{\underline{x}}^{w-1} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \, dV(\underline{x}) \right\} \\
&\quad = \ (-1)^{w+1} \int_{\mathbb{R}^p} (\underline{x}^d \partial_{\underline{x}}) \ \partial_{\underline{x}}^{w-1} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right] \, dV(\underline{x}) \\
&\quad = \ \int_{\mathbb{R}^p} (\underline{x}^d \partial_{\underline{x}}) \ H_{w-1,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \, dV(\underline{x}) \ .
\end{aligned}$$

From (2.5) we obtain in particular

$$\partial_{\underline{x}}[\underline{x}^d] = \begin{cases} -d \ \underline{x}^{d-1} & \text{for } d \text{ even} \\ -(d+p-1) \ \underline{x}^{d-1} & \text{for } d \text{ odd.} \end{cases}$$

Hence, by conjugation we find

$$[\underline{x}^d]\partial_{\underline{x}} = \begin{cases} -d \ \underline{x}^{d-1} & \text{for } d \text{ even} \\ -(d+p-1) \ \underline{x}^{d-1} & \text{for } d \text{ odd.} \end{cases}$$

Summarizing: $[\underline{x}^d]\partial_{\underline{x}} \approx \underline{x}^{d-1}$, so that in view of the induction hypothesis

$$\int_{\mathbb{R}^p} \underline{x}^d \; H_{w,p,a}(\underline{x}) \; \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \, dV(\underline{x})$$
$$\approx \int_{\mathbb{R}^p} \underline{x}^{d-1} \; H_{w-1,p,a}(\underline{x}) \; \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \, dV(\underline{x}) = 0 \quad . \qquad \square$$

Naturally, the same formulae and results hold for the polynomials $H_{e,q,b}(\underline{y})$ which are generated by the CK-extension G^* to $\mathbb{R}^{q+1}$ of the weight function $\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)$:

$$G^*(y_0, \underline{y}) = \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \sum_{e=0}^{\infty} \frac{y_0^e}{e!} \; H_{e,q,b}(\underline{y}) \quad .$$

We are now able to express the bi-axial Clifford-Hermite polynomials $H_{\ell,p+q,a,b}(\underline{x},\underline{y})$ in terms of the polynomials $H_{d,p,a}(\underline{x})$ and $H_{e,q,b}(\underline{y})$.

Lemma 5.2 *The decomposition of the bi-axial Clifford-Hermite polynomials in terms of the polynomials $H_{d,p,a}(\underline{x})$ and $H_{e,q,b}(\underline{y})$ reads:*

$$H_{2k,p+q,a,b}(\underline{x},\underline{y}) = \sum_{j=0}^{k} \binom{k}{j} \; H_{2j,p,a}(\underline{x}) \; H_{2(k-j),q,b}(\underline{y}) \tag{5.14}$$

and

$$H_{2k+1,p+q,a,b}(\underline{x},\underline{y}) = \sum_{j=0}^{k} \binom{k}{j} \Big(H_{2j+1,p,a}(\underline{x}) \; H_{2(k-j),q,b}(\underline{y}) + \; H_{2j,p,a}(\underline{x}) \; H_{2(k-j)+1,q,b}(\underline{y}) \Big) \quad . \tag{5.15}$$

Proof. By means of the Rodrigues formulae and the fact that

$$(\partial_{\underline{x}} + \partial_{\underline{y}})^2 = \partial_{\underline{x}}^2 + \partial_{\underline{y}}^2 \quad ,$$

we obtain for ℓ even, i.e. $\ell = 2k$:

$$H_{2k,p+q,a,b}(\underline{x},\underline{y})$$
$$= \exp\left(\frac{1}{2}a|\underline{x}|^2\right) \exp\left(\frac{1}{2}b|\underline{y}|^2\right) \, (\partial_{\underline{x}}^2 + \partial_{\underline{y}}^2)^k \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right]$$

$$
\begin{aligned}
&= \exp\left(\frac{1}{2}a|\underline{x}|^2\right) \exp\left(\frac{1}{2}b|\underline{y}|^2\right) \sum_{j=0}^{k} \binom{k}{j} \\
&\quad \partial_{\underline{x}}^{2j}\ \partial_{\underline{y}}^{2(k-j)} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right] \\
&= \sum_{j=0}^{k} \binom{k}{j}\ H_{2j,p,a}(\underline{x})\ H_{2(k-j),q,b}(\underline{y}) \quad .
\end{aligned}
$$

Similarly for ℓ odd, i.e. $\ell = 2k+1$, we get:

$$
\begin{aligned}
&H_{2k+1,p+q,a,b}(\underline{x},\underline{y}) \\
&= -\exp\left(\frac{1}{2}a|\underline{x}|^2\right) \exp\left(\frac{1}{2}b|\underline{y}|^2\right) (\partial_{\underline{x}}^2 + \partial_{\underline{y}}^2)^k\ (\partial_{\underline{x}} + \partial_{\underline{y}}) \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right. \\
&\quad \left.\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right] \\
&= -\exp\left(\frac{1}{2}a|\underline{x}|^2\right) \exp\left(\frac{1}{2}b|\underline{y}|^2\right) \sum_{j=0}^{k} \binom{k}{j}\ \partial_{\underline{x}}^{2j}\ \partial_{\underline{y}}^{2(k-j)} \left[\partial_{\underline{x}}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right]\right. \\
&\quad \left.\exp\left(-\frac{1}{2}b|\underline{y}|^2\right) + \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \partial_{\underline{y}}\left[\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right]\right] \\
&= \sum_{j=0}^{k} \binom{k}{j} \left(H_{2j+1,p,a}(\underline{x})\ H_{2(k-j),q,b}(\underline{y}) + H_{2j,p,a}(\underline{x})\ H_{2(k-j)+1,q,b}(\underline{y})\right) . \ \square
\end{aligned}
$$

Lemma 5.2, combined with the mutually weighted orthogonality in $\mathbb{R}^p$, respectively $\mathbb{R}^q$, of the polynomials $H_{d,p,a}(\underline{x})$ and $H_{e,q,b}(\underline{y})$ respectively, lead to the mutual orthogonality in $\mathbb{R}^{p+q}$ of the polynomials $H_{\ell,p+q,a,b}(\underline{x},\underline{y})$ with respect to the weight function $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)$.

Theorem 5.1 *The bi-axial Clifford-Hermite polynomials* $H_{\ell,p+q,a,b}(\underline{x},\underline{y})$ *are mutually orthogonal in* $\mathbb{R}^{p+q}$ *with respect to the weight function* $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$

$\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)$, *i.e.*

$$\int_{\mathbb{R}^{p+q}} \overline{H_{\ell,p+q,a,b}}(\underline{x},\underline{y})\ H_{t,p+q,a,b}(\underline{x},\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) = 0 \quad ,$$

whenever $\ell \neq t$.

Proof. Suppose that ℓ is odd, i.e. $\ell = 2n+1$, and t is even, i.e. $t = 2s$. For the three other possible combinations according to the parity of ℓ and t, the proof is similar.
By means of Lemma 5.2 we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{p+q}} \overline{H_{2n+1,p+q,a,b}}(\underline{x},\underline{y})\ H_{2s,p+q,a,b}(\underline{x},\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \\
&\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) \\
&= \sum_{j=0}^{n}\sum_{i=0}^{s} \binom{n}{j}\binom{s}{i} \Bigg\{ \int_{\mathbb{R}^{p+q}} \overline{H_{2(n-j),q,b}}(\underline{y})\ \overline{H_{2j+1,p,a}}(\underline{x})\ H_{2i,p,a}(\underline{x}) \\
&\quad H_{2(s-i),q,b}(\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) \\
&\quad + \int_{\mathbb{R}^{p+q}} \overline{H_{2(n-j)+1,q,b}}(\underline{y})\ \overline{H_{2j,p,a}}(\underline{x})\ H_{2i,p,a}(\underline{x})\ H_{2(s-i),q,b}(\underline{y}) \\
&\quad \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) \Bigg\} \\
&= \sum_{j=0}^{n}\sum_{i=0}^{s} \binom{n}{j}\binom{s}{i} \Bigg\{ \left(\int_{\mathbb{R}^{p}} \overline{H_{2j+1,p,a}}(\underline{x})\ H_{2i,p,a}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x}) \right) \\
&\quad \left(\int_{\mathbb{R}^{q}} \overline{H_{2(n-j),q,b}}(\underline{y})\ H_{2(s-i),q,b}(\underline{y})\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{y}) \right) \\
&+ \left(\int_{\mathbb{R}^{p}} \overline{H_{2j,p,a}}(\underline{x})\ H_{2i,p,a}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x}) \right) \\
&\quad \left(\int_{\mathbb{R}^{q}} \overline{H_{2(n-j)+1,q,b}}(\underline{y})\ H_{2(s-i),q,b}(\underline{y})\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{y}) \right) \Bigg\} \quad ,
\end{aligned}$$

which indeed is zero by the mutually weighted orthogonality of the polynomials $H_{d,p,a}(\underline{x})$ and $H_{e,q,b}(\underline{y})$ and the assumption $2n+1 \neq 2s$. □

5.4.2 The bi-axial Clifford-Hermite wavelets

Theorem 5.1 implies that for $t > 0$:

$$\int_{\mathbb{R}^{p+q}} H_{t,p+q,a,b}(\underline{x},\underline{y}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \ dV(\underline{x}) \ dV(\underline{y}) = 0 \quad .$$

In terms of wavelet theory this means that the $L_1 \cap L_2$-functions

$$\begin{aligned} \psi_{t,a,b}(\underline{x},\underline{y}) &= H_{t,p+q,a,b}(\underline{x},\underline{y}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \\ &= (-1)^t \ (\partial_{\underline{x}} + \partial_{\underline{y}})^t \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right] \end{aligned}$$

are candidate mother wavelets in $\mathbb{R}^{p+q}$. We call them the *bi-axial Clifford-Hermite wavelets*.
These bi-axial Clifford-Hermite wavelets are alternatively scalar- or vector-valued and are generated by the CK-extension H^* of the weight function $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$ $\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)$:

$$H^*(s,\underline{x},\underline{y}) = \sum_{t=0}^{\infty} \frac{s^t}{t!} \ \psi_{t,a,b}(\underline{x},\underline{y}) \quad .$$

Their Fourier transform is given by

$$\begin{aligned} &\mathcal{F}\left[\psi_{t,a,b}(\underline{x},\underline{y})\right](\underline{u},\underline{v}) \\ &\quad= (-1)^t \ (i(\underline{u}+\underline{v}))^t \ \mathcal{F}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right](\underline{u}) \ \mathcal{F}\left[\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\right](\underline{v}) \\ &\quad= (-i)^t \left(\frac{1}{a}\right)^{p/2} \left(\frac{1}{b}\right)^{q/2} (\underline{u}+\underline{v})^t \ \exp\left(-\frac{|\underline{u}|^2}{2a}\right) \exp\left(-\frac{|\underline{v}|^2}{2b}\right) \end{aligned} \tag{5.16}$$

with

$$\underline{u} = \sum_{j=1}^{p} u_j e_j \quad \text{and} \quad \underline{v} = \sum_{j=1}^{q} v_j e_{p+j} \quad .$$

Moreover, the bi-axial Clifford-Hermite wavelet $\psi_{t,a,b}(\underline{x},\underline{y})$ has vanishing moments up to order $(t-1)$, which means that the corresponding CCWT will filter out polynomial behaviour of the signal up to degree $(t-1)$, making it more adequate at detecting singularities.

Proposition 5.1 *The bi-axial Clifford-Hermite wavelet $\psi_{t,a,b}(\underline{x},\underline{y})$ has vanishing moments up to order $(t-1)$, i.e.*

$$\int_{\mathbb{R}^{p+q}} (\underline{x}+\underline{y})^j \ \psi_{t,a,b}(\underline{x},\underline{y}) \ dV(\underline{x}) \ dV(\underline{y}) = 0$$

for $j = 0, 1, \ldots, t-1$.

Proof. Suppose $j < t$ is even, i.e. $j = 2n$. The case j odd is proven in a similar way.
As

$$(\underline{x}+\underline{y})^{2n} = (\underline{x}^2+\underline{y}^2)^n = \sum_{i=0}^{n} \binom{n}{i} \ \underline{x}^{2i} \ \underline{y}^{2(n-i)} \quad ,$$

we obtain

$$\begin{aligned}\int_{\mathbb{R}^{p+q}} & (\underline{x}+\underline{y})^{2n} \ \psi_{t,a,b}(\underline{x},\underline{y}) \ dV(\underline{x}) \ dV(\underline{y}) \\ &= \sum_{i=0}^{n} \binom{n}{i} \int_{\mathbb{R}^{p+q}} \underline{x}^{2i} \ \underline{y}^{2(n-i)} \ H_{t,p+q,a,b}(\underline{x},\underline{y}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \\ &\qquad \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \ dV(\underline{x}) \ dV(\underline{y}) \quad .\end{aligned}$$

By means of the decomposition formula (5.14) we get for t even, i.e. $t = 2s$:

$$\begin{aligned}\int_{\mathbb{R}^{p+q}} & (\underline{x}+\underline{y})^{2n} \ \psi_{2s,a,b}(\underline{x},\underline{y}) \ dV(\underline{x}) \ dV(\underline{y}) \\ &= \sum_{i=0}^{n} \sum_{\ell=0}^{s} \binom{n}{i} \binom{s}{\ell} \left(\int_{\mathbb{R}^{p}} \underline{x}^{2i} \ H_{2\ell,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ dV(\underline{x}) \right) \\ &\qquad \left(\int_{\mathbb{R}^{q}} \underline{y}^{2(n-i)} \ H_{2(s-\ell),q,b,}(\underline{y}) \ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \ dV(\underline{y}) \right) \quad . \qquad (5.17)\end{aligned}$$

Similarly for t odd, i.e. $t = 2s + 1$, decomposition formula (5.15) leads to

$$\int_{\mathbb{R}^{p+q}} (\underline{x} + \underline{y})^{2n} \ \psi_{2s+1,a,b}(\underline{x}, \underline{y}) \ dV(\underline{x}) \ dV(\underline{y})$$
$$= \sum_{i=0}^{n} \sum_{\ell=0}^{s} \binom{n}{i} \binom{s}{\ell} \Bigg\{ \left(\int_{\mathbb{R}^p} \underline{x}^{2i} \ H_{2\ell+1,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2} a |\underline{x}|^2 \right) \ dV(\underline{x}) \right)$$
$$\left(\int_{\mathbb{R}^q} \underline{y}^{2(n-i)} \ H_{2(s-\ell),q,b}(\underline{y}) \ \exp\left(-\frac{1}{2} b |\underline{y}|^2 \right) \ dV(\underline{y}) \right)$$
$$+ \left(\int_{\mathbb{R}^p} \underline{x}^{2i} \ H_{2\ell,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2} a |\underline{x}|^2 \right) \ dV(\underline{x}) \right)$$
$$\left(\int_{\mathbb{R}^q} \underline{y}^{2(n-i)} \ H_{2(s-\ell)+1,q,b,}(\underline{y}) \ \exp\left(-\frac{1}{2} b |\underline{y}|^2 \right) \ dV(\underline{y}) \right) \Bigg\} \quad . \tag{5.18}$$

Expression (5.17), respectively (5.18), is zero by the assumption that $2n < 2s$, respectively $2n < 2s + 1$, and by the properties (see proof Lemma 5.1)

$$\int_{\mathbb{R}^p} \underline{x}^g \ H_{d,p,a}(\underline{x}) \ \exp\left(-\frac{1}{2} a |\underline{x}|^2 \right) \ dV(\underline{x}) = 0 \quad \text{for } g < d$$

and

$$\int_{\mathbb{R}^q} \underline{y}^h \ H_{e,q,b,}(\underline{y}) \ \exp\left(-\frac{1}{2} b |\underline{y}|^2 \right) \ dV(\underline{y}) = 0 \quad \text{for } h < e \quad . \ \square$$

The bi-axial Clifford-Hermite wavelets $\psi_{t,a,b}(\underline{x}, \underline{y})$ have basically an elliptical form, the eccentricity being governed by the two parameters a and b. They are to be considered as a refinement of the circular Clifford-Hermite wavelets of Section 5.2.2 which correspond to the parameter values $a = b = 1$. By introducing the extra two parameters a and b, more details can be recognized in the image. The bi-axial Clifford-Hermite wavelets also form an intermediate step between the spherical symmetrical case ($a = b$) and the linear case (for e.g. $a << 1$ and $b >> 1$).

5.4.3 The bi-axial Clifford-Hermite CWT

Still for $t > 0$, the corresponding *bi-axial Clifford-Hermite CWT* (BCHCWT) applies to functions $f \in L_2\left(\mathbb{R}^{p+q}, dV(\underline{x}) \ dV(\underline{y})\right)$ by

$$T_{t,a,b}[f](\alpha, \underline{z}', s) = F_{t,a,b}(\alpha, \underline{z}', s) = < \Psi_{t,a,b}^{\alpha, \underline{z}', s}, f > \quad .$$

Here the continuous family of wavelets $\Psi_{t,a,b}^{\alpha,\underline{z}',s}$ is given by

$$\Psi_{t,a,b}^{\alpha,\underline{z}',s}(\underline{z}) = \frac{1}{\alpha^{(p+q)/2}} \; s \; \Psi_{t,a,b}\left(\frac{\overline{s}(\underline{z}-\underline{z}')s}{\alpha}\right)\overline{s}$$

with

$$\alpha \in \mathbb{R}_+ \quad , \; \underline{z}' \in \mathbb{R}^{p+q} \quad , \; s \in \mathrm{Spin}(p+q)$$

and by definition

$$\Psi_{t,a,b}(\underline{z}) := \psi_{t,a,b}(\underline{x},\underline{y}) \quad , \quad \underline{z} = \underline{x} + \underline{y} \in \mathbb{R}^{p+q} \quad .$$

Now using the expression (5.16) for the Fourier transform of the bi-axial Clifford-Hermite wavelets, we obtain that

$$\begin{aligned} &\mathcal{F}\big[\Psi_{t,a,b}(\underline{z})\big](\underline{\xi}) \; \big(\mathcal{F}\big[\Psi_{t,a,b}(\underline{z})\big](\underline{\xi})\big)^{\dagger} \\ &\quad = \mathcal{F}\big[\psi_{t,a,b}(\underline{x},\underline{y})\big](\underline{u},\underline{v}) \; \big(\mathcal{F}\big[\psi_{t,a,b}(\underline{x},\underline{y})\big](\underline{u},\underline{v})\big)^{\dagger} \\ &\quad = \left(\frac{1}{a}\right)^{p}\left(\frac{1}{b}\right)^{q} |\underline{u}+\underline{v}|^{2t} \; \exp\left(-\frac{|\underline{u}|^2}{a}\right) \; \exp\left(-\frac{|\underline{v}|^2}{b}\right) \end{aligned}$$

is scalar-valued. Moreover the bi-axial Clifford-Hermite wavelets satisfy the admissibility condition

$$\begin{aligned} &C_{t,a,b} \\ &= (2\pi)^{p+q} \int_{\mathbb{R}^{p+q}} \frac{\mathcal{F}\big[\Psi_{t,a,b}(\underline{z})\big](\underline{\xi}) \; \big(\mathcal{F}\big[\Psi_{t,a,b}(\underline{z})\big](\underline{\xi})\big)^{\dagger}}{|\underline{\xi}|^{p+q}} \, dV(\underline{\xi}) \\ &= (2\pi)^{p+q} \int_{\mathbb{R}^{p+q}} \frac{\mathcal{F}\big[\psi_{t,a,b}(\underline{x},\underline{y})\big](\underline{u},\underline{v}) \; \big(\mathcal{F}\big[\psi_{t,a,b}(\underline{x},\underline{y})\big](\underline{u},\underline{v})\big)^{\dagger}}{|\underline{u}+\underline{v}|^{p+q}} \, dV(\underline{u}) \; dV(\underline{v}) \\ &= (2\pi)^{p+q} \left(\frac{1}{a}\right)^{p}\left(\frac{1}{b}\right)^{q} \int_{\mathbb{R}^{p+q}} |\underline{u}+\underline{v}|^{2t-(p+q)} \; \exp\left(-\frac{|\underline{u}|^2}{a}\right) \\ &\qquad \exp\left(-\frac{|\underline{v}|^2}{b}\right) dV(\underline{u}) \; dV(\underline{v}) < \infty \quad , \end{aligned}$$

since $t \geq 1$.
Consequently the BCHCWT is an isometry between the two L_2-spaces $L_2(\mathbb{R}^{p+q}, dV(\underline{x}) \; dV(\underline{y}))$ and $L_2\bigg(\mathbb{R}_+ \times \mathbb{R}^{p+q} \times \mathrm{Spin}(p+q), C_{t,a,b}^{-1} \; \alpha^{-(p+q+1)}$ $d\alpha \; dV(\underline{z}') \; ds\bigg)$.

5.5 The generalized bi-axial Clifford-Hermite polynomials and associated CCWT

In the following, $P_k(\underline{x})$ and $P_\ell(\underline{y})$ represent solid inner spherical monogenics of order k in the variable $\underline{x}$ and of order ℓ in the variable $\underline{y}$. This means that $P_k(\underline{x})$ and $P_\ell(\underline{y})$ are homogeneous polynomials taking values in the Clifford algebra spanned by respectively $\{e_1, \ldots, e_p\}$ and $\{e_{p+1}, \ldots, e_{p+q}\}$; they satisfy the relations

$$\begin{aligned} \partial_{\underline{x}} P_k(\underline{x}) &= 0 \qquad & P_k(t\underline{x}) &= t^k P_k(\underline{x}) \\ \partial_{\underline{y}} P_\ell(\underline{y}) &= 0 \qquad & P_\ell(t\underline{y}) &= t^\ell P_\ell(\underline{y}) \quad . \end{aligned}$$

We also assume $P_k(\underline{x})$ and $P_\ell(\underline{y})$ to be even Clifford algebra-valued, which implies that $P_k(\underline{x})$ and $P_\ell(\underline{y})$ commute.

5.5.1 The generalized bi-axial Clifford-Hermite polynomials

The CK-extension K^* of the function

$$\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\, P_k(\underline{x})\ P_\ell(\underline{y}) \quad , \qquad a > 0,\ b > 0$$

can be written as

$$\begin{aligned} &K^*(s, \underline{x}, \underline{y}) \\ &= \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\sum_{t=0}^{\infty}\frac{s^t}{t!}\ H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y})\ P_k(\underline{x})\ P_\ell(\underline{y}) \quad . \end{aligned}$$

This CK-extension defines the *generalized bi-axial Clifford-Hermite polynomials* $H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y})$.

Similarly as in Section 5.4.1, we will derive a decomposition of $H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y})$ in terms of the polynomials $H_{d,p,a,k}(\underline{x})$ and $H_{e,q,b,\ell}(\underline{y})$ which are generated by the CK-extension of the functions $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)P_k(\underline{x})$ and $\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)P_\ell(\underline{y})$ respectively.

In what follows, we will discuss the polynomials $H_{d,p,a,k}(\underline{x})$. Of course, the same

formulae and results hold for the polynomials $H_{e,q,b,\ell}(\underline{y})$.
From the monogenicity of the CK-extension F^* of $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)P_k(\underline{x})$:

$$F^*(x_0, \underline{x}) = \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\sum_{d=0}^{\infty}\frac{x_0^d}{d!}\ H_{d,p,a,k}(\underline{x})\ P_k(\underline{x}) \quad ,$$

it follows immediately that

$$H_{d+1,p,a,k}(\underline{x})\ P_k(\underline{x}) = a\underline{x}\ H_{d,p,a,k}(\underline{x})\ P_k(\underline{x}) - \partial_{\underline{x}}\big[H_{d,p,a,k}(\underline{x})\ P_k(\underline{x})\big] \quad . \tag{5.19}$$

In a similar way as was done in [108] for the generalized Clifford-Hermite polynomials, we will now derive from the above relation two recurrence relations for the polynomials $H_{d,p,a,k}(\underline{x})$.

Lemma 5.3 *The polynomials $H_{d,p,a,k}(\underline{x})$ satisfy the recurrence relations*

$$H_{2d+1,p,a,k}(\underline{x}) = a\underline{x}\ H_{2d,p,a,k}(\underline{x}) - \partial_{\underline{x}}[H_{2d,p,a,k}(\underline{x})] \tag{5.20}$$

and

$$H_{2d+2,p,a,k}(\underline{x}) = (a\underline{x} - \partial_{\underline{x}})[H_{2d+1,p,a,k}(\underline{x})] - 2k\ \frac{\underline{x}}{|\underline{x}|^2}\ H_{2d+1,p,a,k}(\underline{x}) \quad . \tag{5.21}$$

Proof. From (5.19) we immediately obtain

$$H_{2d+1,p,a,k}(\underline{x}) = a\underline{x}\ H_{2d,p,a,k}(\underline{x}) - \partial_{\underline{x}}[H_{2d,p,a,k}(\underline{x})] \quad ,$$

whereas

$$H_{2d+2,p,a,k}(\underline{x})\ P_k(\underline{x}) = a\underline{x}\ H_{2d+1,p,a,k}(\underline{x})\ P_k(\underline{x}) - \partial_{\underline{x}}\big[H_{2d+1,p,a,k}(\underline{x})\ P_k(\underline{x})\big] \quad ,$$

since we assume that $H_{2d,p,a,k}(\underline{x})$ contains only even powers of $\underline{x}$, while $H_{2d+1,p,a,k}(\underline{x})$ contains only odd ones (for $k = 0$, they reduce to the polynomials $H_{d,p,a}(\underline{x})$ of Section 5.4.1).
Moreover, we have

$$\begin{aligned}
\partial_{\underline{x}}\big[H_{2d+1,p,a,k}(\underline{x})\ P_k(\underline{x})\big] &= -\partial_{\underline{x}}\left[r^{2k+p-1}\ \underline{\omega}^2\ H_{2d+1,p,a,k}(\underline{x})\ \frac{P_k(\underline{x})}{r^{2k+p-1}}\right] \\
&= -\partial_{\underline{x}}\left[r^{2k+p-1}\ \underline{\omega}\ H_{2d+1,p,a,k}(\underline{x})\ \underline{\omega}\ \frac{P_k(\underline{x})}{r^{2k+p-1}}\right] \\
&= -\partial_{\underline{x}}\big[r^{2k+p-1}\ \underline{\omega}\ H_{2d+1,p,a,k}(\underline{x})\big]\ \frac{\underline{\omega}P_k(\underline{x})}{r^{2k+p-1}} \quad ,
\end{aligned}$$

where we have used the fact that

$$\frac{\underline{\omega} P_k(\underline{x})}{r^{2k+p-1}} = \frac{\underline{x}\ P_k(\underline{x})}{|\underline{x}|^{2k+p}}$$

is a left solid outer spherical monogenic of order k in $\mathbb{R}^p \setminus \{0\}$.
Next, as the angular Dirac operator Γ only depends on the angular co-ordinates, we also obtain:

$$\begin{aligned}
&\partial_{\underline{x}}\big[r^{2k+p-1}\ \underline{\omega}\ H_{2d+1,p,a,k}(\underline{x})\big] \\
&= \underline{\omega}\left(\partial_r + \frac{1}{r}\ \Gamma\right)\left[r^{2k+p-1}\ \underline{\omega}\ H_{2d+1,p,a,k}(\underline{x})\right] \\
&= \underline{\omega}\ \partial_r\big[r^{2k+p-1}\ \underline{\omega}\ H_{2d+1,p,a,k}(\underline{x})\big] \\
&= -(2k+p-1)\ r^{2k+p-2}\ H_{2d+1,p,a,k}(\underline{x}) + \underline{\omega}\ r^{2k+p-1}\ \underline{\omega}\ \partial_r\ [H_{2d+1,p,a,k}(\underline{x})] \quad .
\end{aligned}$$

Furthermore, as $\Gamma[\underline{x}] = (p-1)\ \underline{x}$, we find

$$\begin{aligned}
\underline{\omega}\ \partial_r\big[H_{2d+1,p,a,k}(\underline{x})\big] &= \left(\partial_{\underline{x}} - \underline{\omega}\ \frac{1}{r}\ \Gamma\right)\big[H_{2d+1,p,a,k}(\underline{x})\big] \\
&= \left(\partial_{\underline{x}} - \underline{\omega}\ \frac{1}{r}\ (p-1)\right) H_{2d+1,p,a,k}(\underline{x}) \quad .
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&H_{2d+2,p,a,k}(\underline{x})\ P_k(\underline{x}) \\
&\quad = a\underline{x}\ H_{2d+1,p,a,k}(\underline{x})\ P_k(\underline{x}) + \Big\{-(2k+p-1)\ r^{2k+p-2}\ H_{2d+1,p,a,k}(\underline{x}) \\
&\qquad + \underline{\omega}\ r^{2k+p-1}\left(\partial_{\underline{x}} - \underline{\omega}\ \frac{1}{r}\ (p-1)\right)H_{2d+1,p,a,k}(\underline{x})\Big\}\ \frac{\underline{\omega} P_k(\underline{x})}{r^{2k+p-1}} \\
&\quad = a\underline{x}\ H_{2d+1,p,a,k}(\underline{x})\ P_k(\underline{x}) - 2k\ r^{2k+p-2}\ H_{2d+1,p,a,k}(\underline{x})\ \frac{\underline{\omega} P_k(\underline{x})}{r^{2k+p-1}} \\
&\qquad + \underline{\omega}\ \partial_{\underline{x}}\big[H_{2d+1,p,a,k}(\underline{x})\big]\ \underline{\omega} P_k(\underline{x}) \quad ,
\end{aligned}$$

which finally leads to

$$H_{2d+2,p,a,k}(\underline{x}) = (a\underline{x} - \partial_{\underline{x}})\big[H_{2d+1,p,a,k}(\underline{x})\big] - 2k\ \frac{\underline{x}}{|\underline{x}|^2}\ H_{2d+1,p,a,k}(\underline{x}) \quad . \ \square$$

As it follows from

$$F^*(0,\underline{x}) = \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)P_k(\underline{x})$$

that $H_{0,p,a,k}(\underline{x}) = 1$, the recurrence relations (5.20) and (5.21) allow for a recursive computation:

$$\begin{aligned} H_{1,p,a,k}(\underline{x}) &= a\underline{x} \\ H_{2,p,a,k}(\underline{x}) &= (a\underline{x} - \partial_{\underline{x}})[a\underline{x}] - 2k \ \frac{\underline{x}}{|\underline{x}|^2} \ a\underline{x} \\ &= a^2\underline{x}^2 + a(p+2k) \\ H_{3,p,a,k}(\underline{x}) &= a\underline{x} \left(a^2\underline{x}^2 + a(p+2k) \right) - \partial_{\underline{x}}\left[a^2\underline{x}^2 + a(p+2k)\right] \\ &= a^3\underline{x}^3 + a^2(p+2k+2)\underline{x} \\ H_{4,p,a,k}(\underline{x}) &= (a\underline{x} - \partial_{\underline{x}})\left[a^3\underline{x}^3 + a^2(p+2k+2)\underline{x}\right] \\ & \quad -2k \ \frac{\underline{x}}{|\underline{x}|^2} \left(a^3\underline{x}^3 + a^2(p+2k+2)\underline{x} \right) \\ &= a^4\underline{x}^4 + 2a^3(p+2k+2)\underline{x}^2 + a^2(p+2k)(p+2k+2) \end{aligned}$$

etc.

One also has the Rodrigues formula:

$$H_{d,p,a,k}(\underline{x}) \ P_k(\underline{x}) = (-1)^d \exp\left(\frac{1}{2}a|\underline{x}|^2\right) \partial_{\underline{x}}^d \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) P_k(\underline{x})\right] \ , \tag{5.22}$$

which together with the Clifford-Stokes theorem leads to the following orthogonality relation.

Lemma 5.4 *Whenever $d \neq w$ or $k \neq k'$, one has the orthogonality relations*

$$\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger \ \overline{H_{d,p,a,k}}(\underline{x}) \ H_{w,p,a,k'}(\underline{x}) \ P_{k'}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) dV(\underline{x}) \quad = \quad 0 \ .$$

Proof. This proof is similar to the one of the orthogonality relation (5.10) of the generalized Clifford-Hermite polynomials (see [108]).
If $d = w$ and $k \neq k'$ we have

$$\begin{aligned} &\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger \ \overline{H_{d,p,a,k}}(\underline{x}) \ H_{d,p,a,k'}(\underline{x}) \ P_{k'}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) dV(\underline{x}) \\ &\quad = \int_0^{+\infty} r^{k+k'+p-1} \left(\overline{H_{d,p,a,k}}(\underline{x}) \ H_{d,p,a,k'}(\underline{x}) \right)(r) \ \exp\left(-\frac{1}{2}a \ r^2\right) dr \\ &\qquad\qquad \int_{S^{p-1}} \left(P_k(\underline{\omega})\right)^\dagger \ P_{k'}(\underline{\omega}) \ dS(\underline{\omega}) = 0 \ \ , \end{aligned}$$

since $\overline{H_{d,p,a,k}}(\underline{x})\ H_{d,p,a,k'}(\underline{x})$ is scalar-valued and

$$\int_{S^{p-1}} \left(P_k(\underline{\omega})\right)^\dagger\ P_{k'}(\underline{\omega})\ dS(\underline{\omega}) = 0$$

for $k \neq k'$.
When $d \neq w$, the proof is similar to the one of Lemma 5.1 (see Section 5.4.1).
Suppose that $d < w$; the case $d > w$ is similar.
As $H_{d,p,a,k}(\underline{x})$ is a polynomial of degree d in $\underline{x}$, it is sufficient to show that for each $w \in \mathbb{N}$ and $d < w$:

$$\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ \underline{x}^d\ H_{w,p,a,k'}(\underline{x})\ P_{k'}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x}) = 0 \quad . \tag{5.23}$$

We prove this by induction on d. For $d = 0$, we have by means of the Rodrigues formula (5.22) and the Clifford-Stokes theorem that for $w > 0$

$$\begin{aligned}
&\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ H_{w,p,a,k'}(\underline{x})\ P_{k'}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x}) \\
&\quad = (-1)^w \int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ \partial_{\underline{x}}^w \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\ dV(\underline{x}) \\
&\quad = (-1)^w \left\{\int_{\partial\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ d\underline{\sigma}\ \partial_{\underline{x}}^{w-1}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\right. \\
&\qquad \left. - \int_{\mathbb{R}^p} \left(\left(P_k(\underline{x})\right)^\dagger \partial_{\underline{x}}\right)\ \partial_{\underline{x}}^{w-1}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\ dV(\underline{x})\right\} = 0 \quad ,
\end{aligned}$$

since $\left(P_k(\underline{x})\right)^\dagger \partial_{\underline{x}} = 0$.
Assume that the orthogonality (5.23) holds for $(d-1)$ and $w > (d-1)$ and let $w > d$.
Then, again by means of the Rodrigues formula and the Clifford-Stokes theorem, we find

$$\begin{aligned}
&\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ \underline{x}^d\ H_{w,p,a,k'}(\underline{x})\ P_{k'}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x}) \\
&\quad = (-1)^w \int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ \underline{x}^d\ \partial_{\underline{x}}^w \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\ dV(\underline{x}) \\
&\quad = (-1)^w \left\{\int_{\partial\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger\ \underline{x}^d\ d\underline{\sigma}\ \partial_{\underline{x}}^{w-1}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\right. \\
&\qquad \left. - \int_{\mathbb{R}^p} \left[\left(P_k(\underline{x})\right)^\dagger\ \underline{x}^d\right] \partial_{\underline{x}}\ \ \partial_{\underline{x}}^{w-1}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_{k'}(\underline{x})\right]\ dV(\underline{x})\right\}
\end{aligned}$$

$$= (-1)^{w+1} \int_{\mathbb{R}^p} \left[\left(P_k(\underline{x})\right)^\dagger \ \underline{x}^d \right] \partial_{\underline{x}} \ \partial_{\underline{x}}^{w-1} \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ P_{k'}(\underline{x}) \right] \ dV(\underline{x})$$

$$= \int_{\mathbb{R}^p} \left[\left(P_k(\underline{x})\right)^\dagger \ \underline{x}^d \right] \partial_{\underline{x}} \ H_{w-1,p,a,k'}(\underline{x}) \ P_{k'}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) dV(\underline{x}) \quad (5.24)$$

By Hermitian conjugation we obtain from (2.5)

$$\left[\left(P_k(\underline{x})\right)^\dagger \ \underline{x}^d \right] \partial_{\underline{x}} = \begin{cases} -d \ \left(P_k(\underline{x})\right)^\dagger \ \underline{x}^{d-1} & \text{for } d \text{ even} \\ -(d+2k+p-1) \ \left(P_k(\underline{x})\right)^\dagger \ \underline{x}^{d-1} & \text{for } d \text{ odd.} \end{cases}$$

This implies that (5.24) equals, up to a scalar,

$$\int_{\mathbb{R}^p} \left(P_k(\underline{x})\right)^\dagger \ \underline{x}^{d-1} \ H_{w-1,p,a,k'}(\underline{x}) \ P_{k'}(\underline{x}) \ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ dV(\underline{x}) \quad ,$$

which is zero in view of the induction hypothesis. □

We now return to the generalized bi-axial Clifford-Hermite polynomials. By means of the Rodrigues formula

$$H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y}) \ P_k(\underline{x}) \ P_\ell(\underline{y}) = (-1)^t \ \exp\left(\frac{1}{2}a|\underline{x}|^2\right) \ \exp\left(\frac{1}{2}b|\underline{y}|^2\right)$$
$$(\partial_{\underline{x}} + \partial_{\underline{y}})^t \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right) \ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \ P_k(\underline{x}) \ P_\ell(\underline{y}) \right] \quad , \quad (5.25)$$

we obtain their decomposition in terms of the polynomials $H_{d,p,a,k}(\underline{x})$ and $H_{e,q,b,\ell}(\underline{y})$.

Lemma 5.5 *The generalized bi-axial Clifford-Hermite polynomials may be written in terms of the polynomials $H_{d,p,a,k}(\underline{x})$ and $H_{e,q,b,\ell}(\underline{y})$ as follows:*

$$H_{2t,p+q,a,b,k,\ell}(\underline{x},\underline{y}) = \sum_{j=0}^{t} \binom{t}{j} \ H_{2j,p,a,k}(\underline{x}) \ H_{2(t-j),q,b,\ell}(\underline{y}) \quad (5.26)$$

and

$$H_{2t+1,p+q,a,b,k,\ell}(\underline{x},\underline{y}) = \sum_{j=0}^{t} \binom{t}{j} \ \Big(H_{2j+1,p,a,k}(\underline{x}) \ H_{2(t-j),q,b,\ell}(\underline{y})$$
$$+ \ H_{2j,p,a,k}(\underline{x}) \ H_{2(t-j)+1,q,b,\ell}(\underline{y}) \Big) \quad . \quad (5.27)$$

Proof. Applying the Rodrigues formulae (5.25), (5.22) and the analogous formula for the polynomials $H_{e,q,b,\ell}(\underline{y})$ yields:

$$
\begin{aligned}
& H_{2t,p+q,a,b,k,\ell}(\underline{x},\underline{y})\ P_k(\underline{x})\ P_\ell(\underline{y}) \\
&= \exp\left(\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(\frac{1}{2}b|\underline{y}|^2\right)\ \sum_{j=0}^{t}\binom{t}{j}\ \partial_{\underline{x}}^{2j}\ \partial_{\underline{y}}^{2(t-j)}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right. \\
&\qquad \left. P_k(\underline{x})\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ P_\ell(\underline{y})\right] \\
&= \sum_{j=0}^{t}\binom{t}{j}\ H_{2j,p,a,k}(\underline{x})\ P_k(\underline{x})\ H_{2(t-j),q,b,\ell}(\underline{y})\ P_\ell(\underline{y})
\end{aligned}
$$

and

$$
\begin{aligned}
& H_{2t+1,p+q,a,b,k,\ell}(\underline{x},\underline{y})\ P_k(\underline{x})\ P_\ell(\underline{y}) \\
&= -\exp\left(\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(\frac{1}{2}b|\underline{y}|^2\right)\ \sum_{j=0}^{t}\binom{t}{j}\ \partial_{\underline{x}}^{2j}\ \partial_{\underline{y}}^{2(t-j)}\ (\partial_{\underline{x}}+\partial_{\underline{y}}) \\
&\qquad \left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ P_k(\underline{x})\ P_\ell(\underline{y})\right] \\
&= -\exp\left(\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(\frac{1}{2}b|\underline{y}|^2\right)\ \sum_{j=0}^{t}\binom{t}{j}\ \partial_{\underline{x}}^{2j}\ \partial_{\underline{y}}^{2(t-j)} \\
&\qquad \left[\partial_{\underline{x}}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_k(\underline{x})\right]\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)P_\ell(\underline{y})\right. \\
&\qquad \left. +\ \partial_{\underline{y}}\left[\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ P_\ell(\underline{y})\right]\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_k(\underline{x})\right] \\
&= \sum_{j=0}^{t}\binom{t}{j}\Big(H_{2j+1,p,a,k}(\underline{x})\ P_k(\underline{x})\ H_{2(t-j),q,b,\ell}(\underline{y})\ P_\ell(\underline{y}) \\
&\qquad +\ H_{2(t-j)+1,q,b,\ell}(\underline{y})\ P_\ell(\underline{y})\ H_{2j,p,a,k}(\underline{x})\ P_k(\underline{x})\Big)\ ,
\end{aligned}
$$

which proves the statement. □

These decomposition formulae allow for the computation of some explicit examples

$$\begin{aligned}
H_{0,p+q,a,b,k,\ell}(\underline{x},\underline{y}) &= 1 \\
H_{1,p+q,a,b,k,\ell}(\underline{x},\underline{y}) &= a\underline{x}+b\underline{y} \\
H_{2,p+q,a,b,k,\ell}(\underline{x},\underline{y}) &= a^2\underline{x}^2+b^2\underline{y}^2+a(p+2k)+b(q+2\ell) \\
H_{3,p+q,a,b,k,\ell}(\underline{x},\underline{y}) &= a^3\underline{x}^3+b^3\underline{y}^3+ab^2\underline{x}\underline{y}^2+a^2b\underline{y}\underline{x}^2 \\
&\quad + a\Big(a(p+2k+2)+b(q+2\ell)\Big)\underline{x} \\
&\quad + b\Big(b(q+2\ell+2)+a(p+2k)\Big)\underline{y}
\end{aligned}$$

etc.

As expected, for $k=\ell=0$ these polynomials reduce to the bi-axial Clifford-Hermite polynomials of Section 5.4.1.

Now we are able to prove that $H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y})\ P_k(\underline{x})\ P_\ell(\underline{y})$ are mutually orthogonal with respect to the weight function $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)$.

Theorem 5.2 *Whenever $s \neq t$ or $k \neq k'$ or $\ell \neq \ell'$, one has the orthogonality relations*

$$\int_{\mathbb{R}^{p+q}} \left(P_\ell(\underline{y})\right)^\dagger \left(P_k(\underline{x})\right)^\dagger\ \overline{H_{s,p+q,a,b,k,\ell}(\underline{x},\underline{y})}\ H_{t,p+q,a,b,k',\ell'}(\underline{x},\underline{y})\ P_{k'}(\underline{x})\ P_{\ell'}(\underline{y})$$
$$\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) = 0 \quad .$$

Proof. The proof is similar to the proof of Theorem 5.1 and uses the decomposition formulae (5.26) and (5.27), as well as the orthogonality relations for $H_{d,p,a,k}(\underline{x})\ P_k(\underline{x})$ and $H_{e,q,b,\ell}(\underline{y})\ P_\ell(\underline{y})$ (see Lemma 5.4).
Suppose that s is even, i.e. $s=2n$ and t is odd, i.e. $t=2m+1$ (the three other possible combinations are proven in a similar way).
Using the decomposition formulae (5.26) and (5.27), we obtain

$$\int_{\mathbb{R}^{p+q}} \left(P_\ell(\underline{y})\right)^\dagger\ \left(P_k(\underline{x})\right)^\dagger\ \overline{H_{2n,p+q,a,b,k,\ell}(\underline{x},\underline{y})}\ H_{2m+1,p+q,a,b,k',\ell'}(\underline{x},\underline{y})\ P_{k'}(\underline{x})$$
$$P_{\ell'}(\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y})$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\left\{\int_{\mathbb{R}^{p+q}} \left(P_\ell(\underline{y})\right)^\dagger\ \left(P_k(\underline{x})\right)^\dagger\ \overline{H_{2(n-i),q,b,\ell}(\underline{y})}\ \overline{H_{2i,p,a,k}(\underline{x})}\right.$$

$$H_{2j+1,p,a,k}(\underline{x})\ H_{2(m-j),q,b,\ell}(\underline{y})\ P_{k'}(\underline{x})\ P_{\ell'}(\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$$
$$\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y}) + \int_{\mathbb{R}^{p+q}} \left(P_\ell(\underline{y})\right)^\dagger\ \left(P_k(\underline{x})\right)^\dagger\ \overline{H_{2(n-i),q,b,\ell}}(\underline{y})$$
$$\overline{H_{2i,p,a,k}}(\underline{x})\ H_{2j,p,a,k}(\underline{x})\ H_{2(m-j)+1,q,b,\ell}(\underline{y})\ P_{k'}(\underline{x})\ P_{\ell'}(\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)$$
$$\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{x})\ dV(\underline{y})\Bigg\}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\left\{\left(\int_{\mathbb{R}^p}\left(P_k(\underline{x})\right)^\dagger\ \overline{H_{2i,p,a,k}}(\underline{x})\ H_{2j+1,p,a,k}(\underline{x})\ P_{k'}(\underline{x})\right.\right.$$
$$\left.\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x})\right)\ \left(\int_{\mathbb{R}^q}\left(P_\ell(\underline{y})\right)^\dagger\ \overline{H_{2(n-i),q,b,\ell}}(\underline{y})\ H_{2(m-j),q,b,\ell}(\underline{y})\right.$$
$$\left.P_{\ell'}(\underline{y})\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{y})\right)\ +\ \left(\int_{\mathbb{R}^p}\left(P_k(\underline{x})\right)^\dagger\ \overline{H_{2i,p,a,k}}(\underline{x})\ H_{2j,p,a,k}(\underline{x})\right.$$
$$\left.P_{k'}(\underline{x})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ dV(\underline{x})\right)\ \left(\int_{\mathbb{R}^q}\left(P_\ell(\underline{y})\right)^\dagger\ \overline{H_{2(n-i),q,b,\ell}}(\underline{y})\right.$$
$$\left.\left.H_{2(m-j)+1,q,b,\ell}(\underline{y})\ P_{\ell'}(\underline{y})\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ dV(\underline{y})\right)\right\}\ .$$

This last expression is zero in view of the mutually weighted orthogonality of $H_{d,p,a,k}(\underline{x})\ P_k(\underline{x})$ and $H_{e,q,b,\ell}(\underline{y})\ P_\ell(\underline{y})$ and the assumption $2n \neq 2m+1$ or $k \neq k'$ or $\ell \neq \ell'$. □

5.5.2 The generalized bi-axial Clifford-Hermite wavelets

Theorem 5.2 implies that for $(t,k,\ell) \neq (0,0,0)$ the $L_1 \cap L_2$-functions

$$\begin{aligned}
&\psi_{t,a,b,k,\ell}(\underline{x},\underline{y}) \\
&\quad = H_{t,p+q,a,b,k,\ell}(\underline{x},\underline{y})\ P_k(\underline{x})\ P_\ell(\underline{y})\ \exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right) \\
&\quad = (-1)^t\ (\partial_{\underline{x}} + \partial_{\underline{y}})^t\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ P_k(\underline{x})\ P_\ell(\underline{y})\right]
\end{aligned}$$

are good candidates for mother wavelets in $\mathbb{R}^{p+q}$. We call them the *generalized bi-axial Clifford-Hermite wavelets.*

These generalized bi-axial Clifford-Hermite wavelets are generated by the CK-extension K^* of the weight function $\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ \exp\left(-\frac{1}{2}b|\underline{y}|^2\right)\ P_k(\underline{x})$ $P_\ell(\underline{y})$:

$$K^*(s,\underline{x},\underline{y}) = \sum_{t=0}^{\infty} \frac{s^t}{t!}\ \psi_{t,a,b,k,\ell}(\underline{x},\underline{y}) \quad .$$

It is quite remarkable that both bi-axial and generalized bi-axial Clifford-Hermite wavelets are generated by the CK-extension of a weight function which is not spherical symmetrical.
Their Fourier transform is given by

$$\begin{aligned}\mathcal{F}\left[\psi_{t,a,b,k,\ell}(\underline{x},\underline{y})\right](\underline{u},\underline{v}) = (-1)^t \left(i(\underline{u}+\underline{v})\right)^t \mathcal{F}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)P_k(\underline{x})\right](\underline{u})\\ \mathcal{F}\left[\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)P_\ell(\underline{y})\right](\underline{v})\end{aligned} \tag{5.28}$$

with

$$\underline{u} = \sum_{j=1}^{p} u_j e_j \quad \text{and} \quad \underline{v} = \sum_{j=1}^{q} v_j e_{p+j} \quad .$$

By means of Proposition 2.1 and 2.3, we obtain

$$\begin{aligned}&\mathcal{F}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\ P_k(\underline{x})\right](\underline{u})\\ &= P_k(i\partial_{\underline{u}})\ \mathcal{F}\left[\exp\left(-\frac{1}{2}a|\underline{x}|^2\right)\right](\underline{u}) = i^k\ \left(\frac{1}{a}\right)^{p/2} P_k(\partial_{\underline{u}}) \exp\left(-\frac{|\underline{u}|^2}{2a}\right)\\ &= (-i)^k\ \left(\frac{1}{a}\right)^{p/2}\ \frac{1}{a^k}\ P_k(\underline{u})\ \exp\left(-\frac{|\underline{u}|^2}{2a}\right) \quad .\end{aligned} \tag{5.29}$$

Substituting (5.29) and the similar expression for

$$\mathcal{F}\left[\exp\left(-\frac{1}{2}b|\underline{y}|^2\right)P_\ell(\underline{y})\right](\underline{v})$$

in (5.28) leads to

$$\mathcal{F}\big[\psi_{t,a,b,k,\ell}(\underline{x},\underline{y})\big](\underline{u},\underline{v}) = (-i)^{t+k+\ell} \left(\frac{1}{a}\right)^{p/2} \left(\frac{1}{b}\right)^{q/2} \frac{1}{a^k}\,\frac{1}{b^\ell}$$
$$(\underline{u}+\underline{v})^t\ P_k(\underline{u})\ \exp\left(-\frac{|\underline{u}|^2}{2a}\right)\ P_\ell(\underline{v})\ \exp\left(-\frac{|\underline{v}|^2}{2b}\right) \quad . \qquad (5.30)$$

5.5.3 The generalized bi-axial Clifford-Hermite CWT

Still for $(t,k,\ell) \neq (0,0,0)$, we introduce the continuous family of wavelets

$$\Psi^{\alpha,\underline{z}',s}_{t,a,b,k,\ell}(\underline{z}) = \frac{1}{\alpha^{(p+q)/2}}\ s\ \Psi_{t,a,b,k,\ell}\left(\frac{\overline{s}(\underline{z}-\underline{z}')s}{\alpha}\right)\ \overline{s} \quad ,$$

with $\alpha \in \mathbb{R}_+$ the dilation parameter, $\underline{z}' \in \mathbb{R}^{p+q}$ the translation parameter, $s \in \mathrm{Spin}(p+q)$ the spinor-rotation parameter and

$$\Psi_{t,a,b,k,\ell}(\underline{z}) := \psi_{t,a,b,k,\ell}(\underline{x},\underline{y}) \quad , \quad \underline{z} = \underline{x}+\underline{y} \in \mathbb{R}^{p+q} \quad .$$

The *generalized bi-axial Clifford-Hermite CWT* (GBCHCWT) applies to functions $f \in L_2(\mathbb{R}^{p+q}, dV(\underline{x})\ dV(\underline{y}))$ by

$$\begin{aligned} &T_{t,a,b,k,\ell}[f](\alpha,\underline{z}',s) \\ &\quad = F_{t,a,b,k,\ell}(\alpha,\underline{z}',s) = <\Psi^{\alpha,\underline{z}',s}_{t,a,b,k,\ell}, f> \\ &\quad = \frac{1}{\alpha^{(p+q)/2}} \int_{\mathbb{R}^{p+q}} s\ \left(\Psi_{t,a,b,k,\ell}\left(\frac{\overline{s}(\underline{z}-\underline{z}')s}{\alpha}\right)\right)^\dagger\ \overline{s}\ f(\underline{z})\ dV(\underline{z}) \quad . \end{aligned}$$

Expression (5.30) yields

$$\begin{aligned} &\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi})\ \left(\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi})\right)^\dagger \\ &= \mathcal{F}\big[\psi_{t,a,b,k,\ell}(\underline{x},\underline{y})\big](\underline{u},\underline{v})\ \left(\mathcal{F}\big[\psi_{t,a,b,k,\ell}(\underline{x},\underline{y})\big](\underline{u},\underline{v})\right)^\dagger \\ &= \left(\frac{1}{a}\right)^{p+2k} \left(\frac{1}{b}\right)^{q+2\ell}\ (\underline{u}+\underline{v})^t\ P_k(\underline{u})\ P_\ell(\underline{v})\ \big(P_\ell(\underline{v})\big)^\dagger\ \big(P_k(\underline{u})\big)^\dagger\ (-(\underline{u}+\underline{v}))^t \\ &\quad \exp\left(-\frac{|\underline{u}|^2}{a}\right)\ \exp\left(-\frac{|\underline{v}|^2}{b}\right) \quad . \end{aligned}$$

The above formula implies that

$$\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi}) \left(\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi})\right)^\dagger$$

is scalar-valued if and only if

$$P_k(\underline{x}) \, \big(P_k(\underline{x})\big)^\dagger \quad \text{and} \quad P_\ell(\underline{y}) \, \big(P_\ell(\underline{y})\big)^\dagger$$

are scalar-valued.
In that case, we have that

$$\begin{aligned} & C_{t,a,b,k,\ell} \\ = \;& (2\pi)^{p+q} \int_{\mathbb{R}^{p+q}} \frac{\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi}) \left(\mathcal{F}\big[\Psi_{t,a,b,k,\ell}(\underline{z})\big](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^{p+q}} \, dV(\underline{\xi}) \\ = \;& (2\pi)^{p+q} \left(\frac{1}{a}\right)^{p+2k} \left(\frac{1}{b}\right)^{q+2\ell} \int_{\mathbb{R}^{p+q}} |\underline{u}+\underline{v}|^{2t-(p+q)} \, P_k(\underline{u}) \, \big(P_k(\underline{u})\big)^\dagger \, P_\ell(\underline{v}) \\ & \big(P_\ell(\underline{v})\big)^\dagger \, \exp\left(-\frac{|\underline{u}|^2}{a}\right) \, \exp\left(-\frac{|\underline{v}|^2}{b}\right) \, dV(\underline{u}) \, dV(\underline{v}) < +\infty \quad , \end{aligned}$$

since $(t,k,\ell) \neq (0,0,0)$.
Consequently, if $P_k(\underline{x}) \, \big(P_k(\underline{x})\big)^\dagger$ and $P_\ell(\underline{y}) \, \big(P_\ell(\underline{y})\big)^\dagger$ are scalar-valued, the generalized bi-axial Clifford-Hermite wavelets are mother wavelets in $\mathbb{R}^{p+q}$.

We end this chapter with the following remark.

Remark 5.1 In the same order of ideas also *poly-axial Clifford-Hermite wavelets* may be considered. For that purpose similar constructions as above are to be made for a vector variable $\underline{t}$ which is the sum of three or more vector variables; for example $\underline{t} = \underline{x} + \underline{y} + \underline{z} \in \mathbb{R}^{p+q+r}$ with

$$\underline{x} = \sum_{j=1}^{p} x_j e_j \quad , \quad \underline{y} = \sum_{j=1}^{q} y_j e_{p+j} \quad , \quad \underline{z} = \sum_{j=1}^{r} z_j e_{p+q+j} \quad .$$

Chapter 6

Clifford-Gegenbauer polynomials and associated CCWT

In this chapter we apply the general Clifford wavelet theory exposed in Chapter 4 starting from a Clifford generalization of the traditional Gegenbauer weight function.

6.1 Introduction

On the real line the Gegenbauer or ultraspherical polynomials $C_n^\lambda(x)$ $\left(\lambda > -\dfrac{1}{2}\ ,\ n = 0, 1, 2, \ldots\right)$ may be defined by the Rodrigues formula

$$C_n^\lambda(x) = \frac{(-1)^n\ \Gamma(\lambda + \frac{1}{2})\ \Gamma(n + 2\lambda)}{2^n\ n!\ \Gamma(2\lambda)\ \Gamma(n + \lambda + \frac{1}{2})}\ (1 - x^2)^{1/2-\lambda}\ \frac{d^n}{dx^n}\left[(1 - x^2)^{n+\lambda-1/2}\right] \quad .$$

They constitute an orthogonal basis for the Hilbert space $L_2\Big(\,]-1, 1, [\ ,$ $(1 - x^2)^{\lambda-1/2}\ dx\Big)$ and satisfy the orthogonality relation

$$\int_{-1}^{1} C_n^\lambda(x)\ C_m^\lambda(x)\ (1 - x^2)^{\lambda-1/2}\ dx = \frac{\pi\ 2^{1-2\lambda}\ \Gamma(n + 2\lambda)}{n!\ (\lambda + n)\ (\Gamma(\lambda))^2}\ \delta_{n,m} \quad , \qquad \lambda \neq 0 \quad .$$

Furthermore, they satisfy the recurrence relation

$$(n+1)\ C_{n+1}^{\lambda}(x) = (n+2\lambda)\ x\ C_n^{\lambda}(x) - (1-x^2)\ \frac{d}{dx}\left[C_n^{\lambda}(x)\right]$$

and

$$C_n^{\lambda}(-x) = (-1)^n\ C_n^{\lambda}(x) \quad .$$

These Gegenbauer polynomials are a special case of the Jacobi polynomials.

In this chapter new specific wavelet kernel functions are constructed on the basis of Clifford generalizations of the Gegenbauer polynomials on the real line. These so-called Clifford-Gegenbauer wavelets (see Section 6.2) and generalized Clifford-Gegenbauer wavelets (see Section 6.3) were introduced in [22].

6.2 The radial Clifford-Gegenbauer polynomials and associated CCWT

6.2.1 The radial Clifford-Gegenbauer polynomials

As a generalization of the classical Gegenbauer weight function to Clifford analysis, we take the real-analytic function $F(\underline{x}) = (1+|\underline{x}|^2)^{\alpha},\ \alpha \in \mathbb{R}$. The so-called *radial Clifford-Gegenbauer polynomials*, denoted by $G_{\ell,\alpha}(\underline{x})$, are generated by the CK-extension F^* of this weight function F :

$$F^*(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} \frac{x_0^{\ell}}{\ell!}\ G_{\ell,\alpha}(\underline{x})\ (1+|\underline{x}|^2)^{\alpha-\ell} \quad .$$

The word "radial" again refers to the fact that the weight function $(1+|\underline{x}|^2)^{\alpha}$ is invariant under the rotation group $\mathrm{SO}(m)$.

As explained in Chapter 2, the function F^* is left monogenic in $\mathbb{R}^{m+1}$ and thus satisfies:

$$(\partial_{x_0} + \partial_{\underline{x}})F^*(x_0, \underline{x}) = 0 \quad , \tag{6.1}$$

while its restriction to $\mathbb{R}^m$, identified with the hyperplane $x_0 = 0$, is precisely the weight function:

$$F^*(0, \underline{x}) = (1+|\underline{x}|^2)^{\alpha} \quad . \tag{6.2}$$

From the monogenicity relation (6.1) we derive the recurrence relation

$$G_{\ell+1,\alpha}(\underline{x}) = -2(\alpha-\ell)\ \underline{x}\ G_{\ell,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{\ell,\alpha}(\underline{x})] \quad ,$$

since

$$\partial_{\underline{x}}(1+|\underline{x}|^2)^{\alpha-\ell} = 2(\alpha-\ell)\ (1+|\underline{x}|^2)^{\alpha-\ell-1}\ \underline{x} \quad .$$

As it follows from (6.2) that $G_{0,\alpha}(\underline{x}) = 1$, we thus obtain

$$\begin{aligned}
G_{1,\alpha}(\underline{x}) &= -2\alpha\ \underline{x}\ G_{0,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}\ [G_{0,\alpha}(\underline{x})] \\
&= -2\alpha\ \underline{x} \\
G_{2,\alpha}(\underline{x}) &= -2(\alpha-1)\ \underline{x}\ G_{1,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{1,\alpha}(\underline{x})] \\
&= 2\alpha\ \Big(2(\alpha-1)+m\Big)\ \underline{x}^2 - 2\alpha\ m \\
G_{3,\alpha}(\underline{x}) &= -2(\alpha-2)\ \underline{x}\ G_{2,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{2,\alpha}(\underline{x})] \\
&= -4\alpha(\alpha-1)\ \Big(2(\alpha-1)+m\Big)\ \underline{x}^3 + 4\alpha(\alpha-1)(m+2)\ \underline{x} \\
G_{4,\alpha}(\underline{x}) &= -2(\alpha-3)\ \underline{x}\ G_{3,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{3,\alpha}(\underline{x})] \\
&= 4\alpha(\alpha-1)\ \Big(2(\alpha-1)+m\Big)\Big(2(\alpha-2)+m\Big)\ \underline{x}^4 - 8\alpha(\alpha-1) \\
&\quad (m+2)\Big(2(\alpha-2)+m\Big)\ \underline{x}^2 + 4\alpha(\alpha-1)(m+2)m
\end{aligned}$$

and so on.
Note that $G_{\ell,\alpha}(\underline{x})$ is a polynomial of degree ℓ in the variable $\underline{x}$, that $G_{2\ell,\alpha}(\underline{x})$ only contains even powers, while $G_{2\ell+1,\alpha}(\underline{x})$ only contains odd ones.
From the explicit formula (2.4) for the CK-extension, we obtain the Rodrigues formula for the radial Clifford-Gegenbauer polynomials:

$$G_{\ell,\alpha}(\underline{x}) = (-1)^\ell\ (1+|\underline{x}|^2)^{\ell-\alpha}\ \partial_{\underline{x}}^\ell (1+|\underline{x}|^2)^\alpha \quad . \tag{6.3}$$

By means of this formula we obtain the following orthogonality relation.

Theorem 6.1 *When $\ell < t$ and $2t < -2\alpha - m + 1$, one has the orthogonality relation*

$$\int_{\mathbb{R}^m} \overline{G_{\ell,\alpha+\ell}}(\underline{x})\ G_{t,\alpha+t}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) = 0 \quad .$$

Proof. As $G_{\ell,\alpha+\ell}(\underline{x})$ is a polynomial of degree ℓ in $\underline{x}$, it is sufficient to show that for each $\ell < t$ and $2t < -2\alpha - m + 1$:

$$\int_{\mathbb{R}^m} \underline{x}^\ell\ G_{t,\alpha+t}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) = 0 \quad .$$

Using the Rodrigues formula (6.3) and the Clifford-Stokes theorem yields

$$\begin{aligned}\int_{\mathbb{R}^m} \underline{x}^\ell \ G_{t,\alpha+t}(\underline{x}) \ (1+|\underline{x}|^2)^\alpha \ dV(\underline{x}) &= (-1)^t \int_{\mathbb{R}^m} \underline{x}^\ell \ \partial_{\underline{x}}^t (1+|\underline{x}|^2)^{\alpha+t} \ dV(\underline{x}) \\ &= (-1)^t \left(\int_{\partial\mathbb{R}^m} \underline{x}^\ell \ d\underline{\sigma} \ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t} \right. \\ & \left. - \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t} dV(\underline{x}) \right).\end{aligned}$$

The first term vanishes if the degree of homogeneity of the integrand $\underline{x}^\ell \ d\underline{\sigma}$ $\partial_{\underline{x}}^{t-1}(1+|\underline{x}|^2)^{\alpha+t}$ is strictly smaller then zero. In other words, this first term is zero if

$$\ell + (m-1) - (t-1) + 2(\alpha+t) < 0 \quad . \tag{6.4}$$

Condition (6.4) is fulfilled, since by assumption ℓ is at most $t-1$ and $2t < -2\alpha - m + 1$.
Hence, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^\ell \ G_{t,\alpha+t}(\underline{x}) \ (1+|\underline{x}|^2)^\alpha \ dV(\underline{x}) \\ &\qquad = (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t} \ dV(\underline{x}) \quad .\end{aligned}$$

Applying again the Clifford-Stokes theorem yields

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^\ell \ G_{t,\alpha+t}(\underline{x}) \ (1+|\underline{x}|^2)^\alpha \ dV(\underline{x}) \\ &\qquad = (-1)^{t+1} \left(\int_{\partial\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ d\underline{\sigma} \ \partial_{\underline{x}}^{t-2} (1+|\underline{x}|^2)^{\alpha+t} \right. \\ &\qquad\qquad \left. - \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}^2) \ \partial_{\underline{x}}^{t-2} (1+|\underline{x}|^2)^{\alpha+t} \ dV(\underline{x}) \right) \quad .\end{aligned}$$

Again the integral over $\partial\mathbb{R}^m$ is zero by means of the assumptions.
Successively repeating this argument leads to

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^\ell \ G_{t,\alpha+t}(\underline{x}) \ (1+|\underline{x}|^2)^\alpha \ dV(\underline{x}) \\ &\qquad = (-1)^{t+\ell+1} \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}^{\ell+1}) \ \partial_{\underline{x}}^{t-(\ell+1)} (1+|\underline{x}|^2)^{\alpha+t} \ dV(\underline{x}) = 0 \quad . \ \square\end{aligned}$$

6.2.2 The Clifford-Gegenbauer wavelets

Note that for $0 < t < (-2\alpha - m + 1)/2$ Theorem 6.1 implies that

$$\int_{\mathbb{R}^m} G_{t,\alpha+t}(\underline{x})\ (1 + |\underline{x}|^2)^{\alpha}\ dV(\underline{x}) = 0 \quad .$$

Consequently, the alternatively scalar- or vector-valued $L_1 \cap L_2$-functions

$$\psi_{t,\alpha}(\underline{x}) = G_{t,\alpha+t}(\underline{x})\ (1 + |\underline{x}|^2)^{\alpha} = (-1)^t\ \partial_{\underline{x}}^t (1 + |\underline{x}|^2)^{\alpha+t}$$

have zero momentum. In what follows we will prove that they satisfy the conditions for being a mother wavelet (see Chapter 4). We call them the *Clifford-Gegenbauer wavelets*. Note that the condition $0 < t < (-2\alpha - m + 1)/2$ forces us to make the restriction $\alpha < 0$.
As the Dirac operator is invariant under the rotation group Spin(m), i.e.

$$s\ \partial_{\overline{s}\underline{x}s}\ \overline{s} = \partial_{\underline{x}} \quad \text{for } s \in \text{Spin}(m) \quad ,$$

we find

$$\begin{aligned} s\ \partial_{\overline{s}\underline{x}s}^t\ \overline{s} &= s\ \partial_{\overline{s}\underline{x}s}\ \partial_{\overline{s}\underline{x}s}\ \ldots\ \partial_{\overline{s}\underline{x}s}\ \overline{s} = s\ \partial_{\overline{s}\underline{x}s}\ \overline{s}s\ \partial_{\overline{s}\underline{x}s}\ \overline{s}s\ \ldots\ \overline{s}s\ \partial_{\overline{s}\underline{x}s}\ \overline{s} \\ &= \partial_{\underline{x}}\ \partial_{\underline{x}}\ \ldots\ \partial_{\underline{x}} = \partial_{\underline{x}}^t \quad . \end{aligned}$$

Consequently, we have

$$\begin{aligned} s\ \psi_{t,\alpha}(\overline{s}\underline{x}s)\ \overline{s} &= (-1)^t\ s\ \partial_{\overline{s}\underline{x}s}^t (1 + |\overline{s}\underline{x}s|^2)^{\alpha+t}\ \overline{s} \\ &= (-1)^t\ \partial_{\underline{x}}^t (1 + |\underline{x}|^2)^{\alpha+t} = \psi_{t,\alpha}(\underline{x}) \quad . \end{aligned}$$

In other words, these Clifford-Gegenbauer wavelets are Spin(m)-invariant.
Furthermore, the wavelets $\psi_{t,\alpha}(\underline{x})$ have vanishing moments if the condition $2\alpha < -m - t$ is fulfilled.

Proposition 6.1 *If $2\alpha < -m - t$, the Clifford-Gegenbauer wavelet $\psi_{t,\alpha}(\underline{x})$ has vanishing moments:*

$$\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,\alpha}(\underline{x})\ dV(\underline{x}) = 0$$

for $0 \leq j < -m - t - 2\alpha$ and $j < t$.

Proof. The proof of this property is similar to the proof of Theorem 6.1 and relies again on the Clifford-Stokes theorem:

$$\begin{aligned}\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,\alpha}(\underline{x})\ dV(\underline{x}) &= (-1)^t \int_{\mathbb{R}^m} \underline{x}^j\ \partial_{\underline{x}}^t (1+|\underline{x}|^2)^{\alpha+t}\ dV(\underline{x}) \\ &= (-1)^t \left(\int_{\partial\mathbb{R}^m} \underline{x}^j\ d\underline{\sigma}\ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t} \right. \\ & \quad \left. - \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}})\ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t}\ dV(\underline{x}) \right) .\end{aligned}$$

The integral over $\partial\mathbb{R}^m$ is zero if

$$j + (m-1) - (t-1) + 2(\alpha + t) < 0$$

or equivalently

$$j < -m - t - 2\alpha \quad . \tag{6.5}$$

As $0 \leq j$, this condition implies $2\alpha < -m - t$.
If (6.5) is fulfilled, we obtain

$$\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,\alpha}(\underline{x})\ dV(\underline{x}) = (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}})\ \partial_{\underline{x}}^{t-1} (1+|\underline{x}|^2)^{\alpha+t}\ dV(\underline{x}) \quad .$$

Repeating this argument yields

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,\alpha}(\underline{x})\ dV(\underline{x}) \\ &\quad = (-1)^{t+j+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}^{j+1})\ \partial_{\underline{x}}^{t-(j+1)} (1+|\underline{x}|^2)^{\alpha+t}\ dV(\underline{x}) \ = \ 0 \quad . \qquad \square\end{aligned}$$

Next, we compute the Fourier transform of the Clifford-Gegenbauer wavelets.

Lemma 6.1 *The Fourier transform of the Clifford-Gegenbauer wavelets takes the form:*

$$\mathcal{F}[\psi_{t,\alpha}](\underline{\xi}) = \frac{(-i)^t\ 2^{\alpha+t+1}}{\Gamma(-\alpha-t)}\ |\underline{\xi}|^{-m/2-\alpha-t}\ K_{m/2+\alpha+t}(|\underline{\xi}|)\ \underline{\xi}^t$$

with $K_\nu(t)$ the modified Bessel function of the second kind.

Proof. First, Proposition 2.1 yields

$$\mathcal{F}\big[\psi_{t,\alpha}\big](\underline{\xi}) = (-1)^t \ (i\underline{\xi})^t \ \mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\big](\underline{\xi})$$

with by definition

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\big](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \ (1+|\underline{x}|^2)^{\alpha+t} \ dV(\underline{x}) \quad . \qquad (6.6)$$

Introducing spherical co-ordinates:

$$\underline{x} = r \ \underline{\omega} \quad , \quad \underline{\xi} = \rho \ \underline{\eta} \quad ; \quad r = |\underline{x}| \quad , \quad \rho = |\underline{\xi}| \quad , \quad \underline{\omega} \ , \ \underline{\eta} \in S^{m-1} \quad ,$$

expression (6.6) becomes

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\big](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_0^{+\infty} (1+r^2)^{\alpha+t} \ r^{m-1} \ dr \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) \ dS(\underline{\omega}) \quad .$$

Applying Proposition 2.2 on the spherical harmonic $S_0(\underline{\omega}) = 1$ gives:

$$\int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) \ dS(\underline{\omega}) = (2\pi)^{m/2} \ (\rho r)^{1-m/2} \ J_{m/2-1}(\rho r) \quad ,$$

with $J_{m/2-1}$ the Bessel function of the first kind of order $\frac{m}{2} - 1$.
Hence we find

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\big](\underline{\xi}) = \rho^{(1-m)/2} \int_0^{+\infty} (1+r^2)^{\alpha+t} \ r^{(m-1)/2} \ J_{m/2-1}(r\rho) \ \rho^{1/2} \ r^{1/2} \ dr \quad .$$

Now, we use the fact that

$$\int_0^{+\infty} r^{\nu+1/2} \ (1+r^2)^{-\mu-1} \ J_\nu(\rho r) \ \rho^{1/2} \ r^{1/2} \ dr = \frac{\rho^{\mu+1/2}}{2^\mu \ \Gamma(\mu+1)} \ K_{\nu-\mu}(\rho) \quad , \quad \rho > 0 \qquad (6.7)$$

if $-1 < \mathrm{Re}(\nu) < 2\ \mathrm{Re}(\mu) + \frac{3}{2}$ (see [59]). Here $K_\nu(t)$ stands for the modified Bessel function of the second kind, also called Macdonald function.
Putting $\nu = \frac{m}{2} - 1 \in \mathbb{R}$ and $\mu = -\alpha - t - 1 \in \mathbb{R}$, (6.7) becomes:

$$\int_0^{+\infty} r^{(m-1)/2}\ (1+r^2)^{\alpha+t}\ J_{m/2-1}(\rho r)\ \rho^{1/2}\ r^{1/2}\ dr$$
$$= \frac{\rho^{-\alpha-t-1/2}}{2^{-\alpha-t-1}\ \Gamma(-\alpha-t)}\ K_{m/2+\alpha+t}(\rho) \quad ,$$

provided the following condition is fulfilled:

$$-1 < \frac{m}{2} - 1 < 2\ (-\alpha - t - 1) + \frac{3}{2} = -2\alpha - 2t - \frac{1}{2} \quad . \tag{6.8}$$

The condition $-1 < \frac{m}{2} - 1$ is fulfilled, since $m > 1$.
Furthermore, the assumption $0 < t < (-2\alpha - m + 1)/2$ made in the beginning of this section leads to

$$2t + 2\alpha < -m + 1 \quad .$$

As $1 - m < \frac{1-m}{2}$, we have

$$2t + 2\alpha < \frac{1-m}{2} \quad \text{or} \quad \frac{m-1}{2} < -2t - 2\alpha \quad .$$

Consequently, condition (6.8) is fulfilled and we find:

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\big](\underline{\xi}) = \frac{2^{\alpha+t+1}}{\Gamma(-\alpha-t)}\ \rho^{-m/2-\alpha-t}\ K_{m/2+\alpha+t}(\rho) \quad . \tag{6.9}$$

Hence, we indeed obtain the following expression for the Fourier transform of the Clifford-Gegenbauer wavelets:

$$\mathcal{F}\big[\psi_{t,\alpha}\big](\underline{\xi}) = \frac{(-i)^t\ 2^{\alpha+t+1}}{\Gamma(-\alpha-t)}\ |\underline{\xi}|^{-m/2-\alpha-t}\ K_{m/2+\alpha+t}(|\underline{\xi}|)\ \underline{\xi}^t \quad . \quad \square$$

6.2.3 The Clifford-Gegenbauer CWT

In view of Lemma 6.1, we have

$$\mathcal{F}\big[\psi_{t,\alpha}\big](\underline{\xi})\ \big(\mathcal{F}\big[\psi_{t,\alpha}\big](\underline{\xi})\big)^\dagger = \frac{2^{2\alpha+2t+2}}{\Big(\Gamma(-\alpha-t)\Big)^2}\ |\underline{\xi}|^{-m-2\alpha}\ |K_{m/2+\alpha+t}(|\underline{\xi}|)|^2$$

is radial symmetric.
Furthermore, the admissibility constant for the Clifford-Gegenbauer wavelets takes the form (see Definition 4.2)

$$\begin{aligned} C_{t,\alpha} &= \frac{(2\pi)^m}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi_{t,\alpha}](\underline{\xi}) \, \left(\mathcal{F}[\psi_{t,\alpha}](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) \\ &= \frac{(2\pi)^m \, 2^{2\alpha+2t+2}}{A_m \left(\Gamma(-\alpha-t)\right)^2} \int_{\mathbb{R}^m} \frac{|\underline{\xi}|^{-m-2\alpha} \, |K_{m/2+\alpha+t}(|\underline{\xi}|)|^2}{|\underline{\xi}|^m} \, dV(\underline{\xi}) \quad . \end{aligned}$$

Introducing spherical co-ordinates

$$\underline{\xi} = \rho \, \underline{\eta} \quad , \quad \rho = |\underline{\xi}| \quad , \quad \underline{\eta} \in S^{m-1} \quad ,$$

this can be further simplified to

$$C_{t,\alpha} = \frac{(2\pi)^m \, 2^{2\alpha+2t+2}}{\left(\Gamma(-\alpha-t)\right)^2} \int_0^{+\infty} \rho^{-m-2\alpha-1} \, |K_{m/2+\alpha+t}(\rho)|^2 \, d\rho \quad .$$

The above admissibility constant is finite, since the modified Bessel functions of the second kind K_ν with Im(ν)=0, have the following limiting behaviour:

$$K_\nu(x) \approx \frac{\pi}{2\sin(\pi|\nu|)} \left(\frac{1}{-|\nu|!}\right) \left(\frac{2}{x}\right)^{|\nu|} \quad \text{for } x \to 0$$

and

$$K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} \exp(-x) \quad \text{for } x \to \infty \quad ,$$

and since moreover we consider $0 < t < (-2\alpha - m + 1)/2$, which in fact is a crucial restriction.
Consequently, the *Clifford-Gegenbauer CWT* (CGCWT) which applies to functions $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ by

$$T_{t,\alpha}[f](a,\underline{b}) = \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \overline{G_{t,\alpha+t}}\left(\frac{\underline{x}-\underline{b}}{a}\right) \left(1 + \frac{|\underline{x}-\underline{b}|^2}{a^2}\right)^\alpha f(\underline{x}) \, dV(\underline{x})$$

is an isometry between the spaces $L_2(\mathbb{R}^m, dV(\underline{x}))$ and $L_2(\mathbb{R}_+ \times \mathbb{R}^m, C_{t,\alpha}^{-1}$ $a^{-(m+1)} \, da \, dV(\underline{b}))$.

6.3 The generalized Clifford-Gegenbauer polynomials and associated CCWT

6.3.1 The generalized Clifford-Gegenbauer polynomials

Analogously as for the Hermite polynomials, the *generalized Clifford-Gegenbauer polynomials* depend on a parameter $k \in \mathbb{N}$, such that for $k = 0$ they reduce to the radial Clifford-Gegenbauer polynomials of section 6.2.1.
They are defined by the CK-extension G^* of the weight function $(1 + |\underline{x}|^2)^\alpha$ $P_k(\underline{x})$, $\alpha \in \mathbb{R}$, with $P_k(\underline{x})$ an arbitrary but fixed left solid inner spherical monogenic of order k :

$$G^*(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \, G_{\ell,k,\alpha}(\underline{x}) \, (1 + |\underline{x}|^2)^{\alpha-\ell} \, P_k(\underline{x}) \quad .$$

The monogenicity of $G^*(x_0, \underline{x})$ leads to

$$\begin{aligned} & G_{\ell+1,k,\alpha}(\underline{x}) \, P_k(\underline{x}) \\ & \quad = -2(\alpha - \ell) \, \underline{x} \, G_{\ell,k,\alpha}(\underline{x}) \, P_k(\underline{x}) - \, (1 + |\underline{x}|^2) \, \partial_{\underline{x}} \big[G_{\ell,k,\alpha}(\underline{x}) \, P_k(\underline{x}) \big] \quad . \end{aligned} \tag{6.10}$$

From the above equation we now derive recurrence relations for the generalized Clifford-Gegenbauer polynomials in a similar way as was done in Section 5.5.1 for the polynomials $H_{d,p,a,k}(\underline{x})$.

Lemma 6.2 *The generalized Clifford-Gegenbauer polynomials satisfy the recurrence relations*

$$G_{2\ell+1,k,\alpha}(\underline{x}) = -2(\alpha - 2\ell) \, \underline{x} \, G_{2\ell,k,\alpha}(\underline{x}) - (1 + |\underline{x}|^2) \, \partial_{\underline{x}}[G_{2\ell,k,\alpha}(\underline{x})] \tag{6.11}$$

and

$$\begin{aligned} G_{2\ell+2,k,\alpha}(\underline{x}) = & -2(\alpha - 2\ell - 1) \, \underline{x} \, G_{2\ell+1,k,\alpha}(\underline{x}) - \, (1 + |\underline{x}|^2) \\ & \left(2k \, \frac{\underline{x}}{|\underline{x}|^2} \, G_{2\ell+1,k,\alpha}(\underline{x}) + \partial_{\underline{x}}[G_{2\ell+1,k,\alpha}(\underline{x})] \right) \quad . \end{aligned} \tag{6.12}$$

Proof. As for $k = 0$, the generalized Clifford-Gegenbauer polynomials reduce to the radial Clifford-Gegenbauer polynomials, we assume that $G_{2\ell,k,\alpha}(\underline{x})$ contains only even powers of $\underline{x}$, while $G_{2\ell+1,k,\alpha}(\underline{x})$ contains only odd ones.
Hence, from (6.10) we obtain

$$G_{2\ell+1,k,\alpha}(\underline{x}) = -2(\alpha - 2\ell) \, \underline{x} \, G_{2\ell,k,\alpha}(\underline{x}) - (1 + |\underline{x}|^2) \, \partial_{\underline{x}}[G_{2\ell,k,\alpha}(\underline{x})] \quad ,$$

whereas

$$\begin{aligned} & G_{2\ell+2,k,\alpha}(\underline{x})\ P_k(\underline{x}) \\ &= -2(\alpha-2\ell-1)\ \underline{x}\ G_{2\ell+1,k,\alpha}(\underline{x})\ P_k(\underline{x}) - \ (1+|\underline{x}|^2)\ \partial_{\underline{x}}\big[G_{2\ell+1,k,\alpha}(\underline{x})\ P_k(\underline{x})\big] \ . \end{aligned}$$

Next, we have that

$$\begin{aligned} \partial_{\underline{x}}\big[G_{2\ell+1,k,\alpha}(\underline{x})\ P_k(\underline{x})\big] &= -\partial_{\underline{x}}\left[r^{2k+m-1}\ \underline{\omega}\ G_{2\ell+1,k,\alpha}(\underline{x})\ \underline{\omega}\ \frac{P_k(\underline{x})}{r^{2k+m-1}}\right] \\ &= -\partial_{\underline{x}}\big[r^{2k+m-1}\ \underline{\omega}\ G_{2\ell+1,k,\alpha}(\underline{x})\big]\ \frac{\underline{\omega} P_k(\underline{x})}{r^{2k+m-1}} \end{aligned}$$

and

$$\begin{aligned} \partial_{\underline{x}}\big[r^{2k+m-1}\ \underline{\omega}\ G_{2\ell+1,k,\alpha}(\underline{x})\big] &= \underline{\omega}\ \partial_r\big[r^{2k+m-1}\ \underline{\omega}\ G_{2\ell+1,k,\alpha}(\underline{x})\big] \\ &= -(2k+m-1)\ r^{2k+m-2}\ G_{2\ell+1,k,\alpha}(\underline{x}) \\ &\quad + \underline{\omega}\ r^{2k+m-1}\ \underline{\omega}\ \partial_r[G_{2\ell+1,k,\alpha}(\underline{x})] \ . \end{aligned}$$

Furthermore, as

$$\underline{\omega}\ \partial_r[G_{2\ell+1,k,\alpha}(\underline{x})] = \left(\partial_{\underline{x}} - \frac{(m-1)\ \underline{\omega}}{r}\right) G_{2\ell+1,k,\alpha}(\underline{x}) \ ,$$

we finally obtain

$$\begin{aligned} & G_{2\ell+2,k,\alpha}(\underline{x}) \\ &= -2(\alpha-2\ell-1)\ \underline{x}\ G_{2\ell+1,k,\alpha}(\underline{x}) + (1+|\underline{x}|^2)\ \Big\{-(2k+m-1)\ r^{2k+m-2} \\ &\quad G_{2\ell+1,k,\alpha}(\underline{x}) + \underline{\omega}\ r^{2k+m-1}\left(\partial_{\underline{x}} - \frac{(m-1)\ \underline{\omega}}{r}\right) G_{2\ell+1,k,\alpha}(\underline{x})\Big\}\ \frac{\underline{\omega}}{r^{2k+m-1}} \\ &= -2(\alpha-2\ell-1)\ \underline{x}\ G_{2\ell+1,k,\alpha}(\underline{x}) - \ (1+|\underline{x}|^2)\ \Big(2k\ \frac{\underline{x}}{|\underline{x}|^2}\ G_{2\ell+1,k,\alpha}(\underline{x}) \\ &\quad + \partial_{\underline{x}}[G_{2\ell+1,k,\alpha}(\underline{x})]\Big) \ . \qquad \square \end{aligned}$$

As $G_{0,k,\alpha}(\underline{x}) = 1$, the recurrence formulae (6.11) and (6.12) allow us to compute recursively

$$\begin{aligned}
G_{1,k,\alpha}(\underline{x}) &= -2\alpha\ \underline{x}\ G_{0,k,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{0,k,\alpha}(\underline{x})] \\
&= -2\alpha\ \underline{x} \\
G_{2,k,\alpha}(\underline{x}) &= -2(\alpha-1)\ \underline{x}\ G_{1,k,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \Bigg(2k\ \frac{\underline{x}}{|\underline{x}|^2}\ G_{1,k,\alpha}(\underline{x}) \\
&\quad + \partial_{\underline{x}}[G_{1,k,\alpha}(\underline{x})]\Bigg) \\
&= 4\alpha(\alpha-1)\ \underline{x}^2 - (1+|\underline{x}|^2)\Bigg(-4k\alpha\ \frac{\underline{x}^2}{|\underline{x}|^2} + 2\alpha m\Bigg) \\
&= 2\alpha\Bigg(2(\alpha-1)+2k+m\Bigg)\ \underline{x}^2 - 2\alpha(2k+m) \\
G_{3,k,\alpha}(\underline{x}) &= -2(\alpha-2)\ \underline{x}\ G_{2,k,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \partial_{\underline{x}}[G_{2,k,\alpha}(\underline{x})] \\
&= -4\alpha(\alpha-2)\ \Bigg(2(\alpha-1)+2k+m\Bigg)\ \underline{x}^3 + 4\alpha(\alpha-2)(2k+m)\ \underline{x} \\
&\quad - (1+|\underline{x}|^2)\ 2\alpha\ \Bigg(2(\alpha-1)+2k+m\Bigg)(-2\underline{x}) \\
&= -4\alpha(\alpha-1)\Bigg(2(\alpha-1)+2k+m\Bigg)\ \underline{x}^3 \\
&\quad + 4\alpha(\alpha-1)(2k+m+2)\ \underline{x} \\
G_{4,k,\alpha}(\underline{x}) &= -2(\alpha-3)\ \underline{x}\ G_{3,k,\alpha}(\underline{x}) - (1+|\underline{x}|^2)\ \Bigg(2k\ \frac{\underline{x}}{|\underline{x}|^2}\ G_{3,k,\alpha}(\underline{x}) \\
&\quad + \partial_{\underline{x}}[G_{3,k,\alpha}(\underline{x})]\Bigg) \\
&= 4\alpha(\alpha-1)\Bigg(2(\alpha-1)+2k+m\Bigg)\Bigg(2(\alpha-2)+2k+m\Bigg)\ \underline{x}^4 \\
&\quad - 8\alpha(\alpha-1)(2k+m+2)\Bigg(2(\alpha-2)+2k+m\Bigg)\ \underline{x}^2 \\
&\quad + 4\alpha(\alpha-1)(2k+m+2)(2k+m)
\end{aligned}$$

etc.

So it is clear that the generalized Clifford-Gegenbauer polynomials do not depend upon the particular choice of the left solid inner spherical monogenic P_k, only upon the order k.

One also has the Rodrigues formula:

$$G_{\ell,k,\alpha}(\underline{x})\ P_k(\underline{x}) = (-1)^\ell\ (1+|\underline{x}|^2)^{\ell-\alpha}\ \partial_{\underline{x}}^\ell\big[(1+|\underline{x}|^2)^\alpha\ P_k(\underline{x})\big] \quad , \tag{6.13}$$

which leads to the following orthogonality relation.

Theorem 6.2 *When $\ell < t$ and $2t < -2\alpha - m + 1 - k_1 - k_2$, one has the orthogonality relation*

$$\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \overline{G_{\ell,k_1,\alpha+\ell}(\underline{x})}\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) = 0 \quad .$$

Proof. The proof is similar to that of Theorem 6.1.
As $G_{\ell,k_1,\alpha+\ell}(\underline{x})$ is a polynomial of degree ℓ in $\underline{x}$, it is sufficient to prove that for each $\ell < t$ and $2t < -2\alpha - m + 1 - k_1 - k_2$:

$$\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) = 0 \quad .$$

Using the Rodrigues formula (6.13) and the Clifford-Stokes theorem respectively, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) \\
&\quad = (-1)^t \int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ \partial_{\underline{x}}^t\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) \\
&\quad = (-1)^t \left\{ \int_{\partial\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ d\underline{\sigma}\ \partial_{\underline{x}}^{t-1}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big] \right. \\
&\qquad \left. - \int_{\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}\ \partial_{\underline{x}}^{t-1}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) \right\} \quad .
\end{aligned}$$

The degree of homogeneity of the integrand of the integral over $\partial\mathbb{R}^m$ is

$$k_1 + \ell + (m-1) - (t-1) + 2(\alpha+t) + k_2 \quad .$$

This degree of homogeneity is strictly smaller then zero, since ℓ is at most $t-1$ and $2t < -2\alpha - m + 1 - k_1 - k_2$.
Consequently the integral over $\partial\mathbb{R}^m$ vanishes and we have:

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) \\
&\quad = (-1)^{t+1} \int_{\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}\ \partial_{\underline{x}}^{t-1}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) \quad .
\end{aligned}$$

Applying again the Clifford-Stokes theorem yields:

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) \\
&\quad = (-1)^{t+1}\Big\{\int_{\partial\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger \underline{x}^\ell\big]\partial_{\underline{x}}\ d\underline{\sigma}\ \partial_{\underline{x}}^{t-2}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big] \\
&\qquad - \int_{\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}^2\ \partial_{\underline{x}}^{t-2}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x})\Big\}\ . \qquad (6.14)
\end{aligned}$$

In view of

$$\big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}} = \begin{cases} -\ell\ \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^{\ell-1} & \text{for } \ell \text{ even} \\ -(\ell+2k_1+m-1)\ \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^{\ell-1} & \text{for } \ell \text{ odd,} \end{cases}$$

the first term of (6.14) equals, up to a scalar,

$$\int_{\partial\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^{\ell-1}\ d\underline{\sigma}\ \partial_{\underline{x}}^{t-2}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ .$$

Analogous as before, this integral is zero by the assumptions that $\ell < t$ and $2t < -2\alpha - m + 1 - k_1 - k_2$.
Consequently equation (6.14) reduces to

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) \\
&\quad = (-1)^{t+2}\int_{\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}^2\ \partial_{\underline{x}}^{t-2}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x})\ .
\end{aligned}$$

By successively repeating of this argument, we finally obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\ G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) \\
&= (-1)^{t+\ell+1}\int_{\mathbb{R}^m} \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}^{\ell+1}\ \partial_{\underline{x}}^{t-(\ell+1)}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) = 0\ ,
\end{aligned}$$

since

$$\big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}^\ell\big]\partial_{\underline{x}}^{\ell+1} \approx \big[\big(P_{k_1}(\underline{x})\big)^\dagger\ \underline{x}\big]\partial_{\underline{x}}^2 \approx \big[\big(P_{k_1}(\underline{x})\big)^\dagger\big]\partial_{\underline{x}} = 0\ . \qquad \square$$

6.3.2 The generalized Clifford-Gegenbauer wavelets

Theorem 6.2 implies that for $0 < t < (-2\alpha - m + 1 - k_2)/2$:

$$\int_{\mathbb{R}^m} G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha\ dV(\underline{x}) = 0 \quad .$$

This means that the $L_1 \cap L_2$-functions

$$\begin{aligned} \psi_{t,k_2,\alpha}(\underline{x}) &= G_{t,k_2,\alpha+t}(\underline{x})\ P_{k_2}(\underline{x})\ (1+|\underline{x}|^2)^\alpha \\ &= (-1)^t\ \partial_{\underline{x}}^t\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big] \end{aligned}$$

are candidate mother wavelets in $\mathbb{R}^m$. We call them the *generalized Clifford-Gegenbauer wavelets.* The condition $0 < t < (-2\alpha - m + 1 - k_2)/2$ implies again that we are forced to make the restriction $\alpha < 0$.

Similarly as in Section 6.2.2 we can prove that the mother wavelet $\psi_{t,k_2,\alpha}(\underline{x})$ has a number of vanishing moments.

Proposition 6.2 *If $2\alpha < -m - t - k_2$, the generalized Clifford-Gegenbauer wavelet $\psi_{t,k_2,\alpha}(\underline{x})$ has vanishing moments*

$$\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,k_2,\alpha}(\underline{x})\ dV(\underline{x}) = 0$$

for $0 \leq j < -m - t - 2\alpha - k_2$ and $j < t$.

Proof. We have consecutively:

$$\begin{aligned} &\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,k_2,\alpha}(\underline{x})\ dV(\underline{x}) \\ &\quad = (-1)^t \int_{\mathbb{R}^m} \underline{x}^j\ \partial_{\underline{x}}^t\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) \\ &\quad = (-1)^t \Big\{ \int_{\partial\mathbb{R}^m} \underline{x}^j\ d\underline{\sigma}\ \partial_{\underline{x}}^{t-1}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big] \\ &\qquad - \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}})\ \partial_{\underline{x}}^{t-1}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big]\ dV(\underline{x}) \Big\} \quad . \end{aligned}$$

The first term is zero if

$$j < -m - t - 2\alpha - k_2 \quad . \tag{6.15}$$

Assuming that (6.15) holds and $j < t$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} \underline{x}^j \; \psi_{t,k_2,\alpha}(\underline{x}) \; dV(\underline{x}) \\
&= (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}) \; \partial_{\underline{x}}^{t-1} \big[(1+|\underline{x}|^2)^{\alpha+t} \; P_{k_2}(\underline{x})\big] \; dV(\underline{x}) \\
&\;\;\vdots \\
&= (-1)^{t+j+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}^{j+1}) \; \partial_{\underline{x}}^{t-(j+1)} \big[(1+|\underline{x}|^2)^{\alpha+t} \; P_{k_2}(\underline{x})\big] \; dV(\underline{x}) \; = \; 0 \;\; ,
\end{aligned}$$

since $\underline{x}^j \partial_{\underline{x}}^{j+1} = 0$.
Note that the assumption (6.15) implies that $2\alpha < -m - t - k_2$. □

Next we calculate the Fourier transform of the newly introduced wavelets.

Lemma 6.3 *The Fourier transform of the generalized Clifford-Gegenbauer wavelets takes the form:*

$$\mathcal{F}[\psi_{t,k_2,\alpha}](\underline{\xi}) = \frac{(-i)^{t+k_2} \; 2^{\alpha+t+1}}{\Gamma(-\alpha-t)} \; \underline{\xi}^t \; P_{k_2}(\underline{\xi}) \; \frac{K_{m/2+\alpha+t+k_2}(|\underline{\xi}|)}{|\underline{\xi}|^{m/2+\alpha+t+k_2}}$$

with $K_\nu(t)$ the modified Bessel function of the second kind.

Proof. In view of Proposition 2.1 we have:

$$\mathcal{F}[\psi_{t,k_2,\alpha}](\underline{\xi}) = (-1)^t \; (i\underline{\xi})^t \; \mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t} \; P_{k_2}(\underline{x})\big](\underline{\xi}) \;\; .$$

Introducing spherical co-ordinates:

$$\underline{x} = r\,\underline{\omega} \;\; , \quad \underline{\xi} = \rho\,\underline{\eta} \;\; , \quad r = |\underline{x}| \;\; , \quad \rho = |\underline{\xi}| \;\; , \quad \underline{\omega} \; , \; \underline{\eta} \in S^{m-1}$$

yields:

$$\begin{aligned}
&\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t} \; P_{k_2}(\underline{x})\big](\underline{\xi}) \\
&\quad = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \; (1+|\underline{x}|^2)^{\alpha+t} \; P_{k_2}(\underline{x}) \; dV(\underline{x}) \\
&\quad = \frac{1}{(2\pi)^{m/2}} \int_0^{+\infty} (1+r^2)^{\alpha+t} \; r^{k_2+m-1} \; dr \\
&\qquad\quad \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) \; P_{k_2}(\underline{\omega}) \; dS(\underline{\omega}) \;\; .
\end{aligned}$$

Furthermore, using Proposition 2.2 this becomes:

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big](\underline{\xi}) \\ = (-i)^{k_2}\ \rho^{1-m/2}\ P_{k_2}(\underline{\eta}) \int_0^{+\infty} (1+r^2)^{\alpha+t}\ J_{k_2+m/2-1}(\rho r)\ r^{k_2+m/2}\ dr \quad .$$

Next, taking $\nu = k_2 + \frac{m}{2} - 1 \in \mathbb{R}$ and $\mu = -\alpha - t - 1 \in \mathbb{R}$, (6.7) becomes

$$\int_0^{+\infty} r^{k_2+m/2-1/2}\ (1+r^2)^{\alpha+t}\ J_{k_2+m/2-1}(\rho r)\ \rho^{1/2}\ r^{1/2}\ dr \\ = \frac{\rho^{-\alpha-t-1/2}}{2^{-\alpha-t-1}\ \Gamma\big(-\alpha-t\big)}\ K_{\alpha+t+k_2+m/2}(\rho) \quad ,$$

on condition that

$$-1 < k_2 + \frac{m}{2} - 1 < 2(-\alpha - t - 1) + \frac{3}{2} = -2\alpha - 2t - \frac{1}{2} \quad .$$

Condition $-1 < k_2 + \frac{m}{2} - 1$ is fulfilled, since $m > 1$ and $k_2 \in \mathbb{N}$.
To prove the second condition

$$k_2 + \frac{m}{2} - 1 < -2\alpha - 2t - \frac{1}{2} \quad ,$$

which can be rewritten as

$$2\alpha + 2t < -k_2 - \frac{m}{2} + \frac{1}{2} \quad ,$$

we use the assumption $0 < t < (-2\alpha - m + 1 - k_2)/2$. This assumption implies:

$$2t + 2\alpha < -m + 1 - k_2 \quad .$$

As $\frac{m}{2} - \frac{1}{2} > 0$, we indeed have

$$2t + 2\alpha < -m + 1 - k_2 < -m + 1 - k_2 + \left(\frac{m}{2} - \frac{1}{2}\right) = -k_2 - \frac{m}{2} + \frac{1}{2} \quad .$$

Consequently, we finally obtain

$$\mathcal{F}\big[(1+|\underline{x}|^2)^{\alpha+t}\ P_{k_2}(\underline{x})\big](\underline{\xi}) \\ = \frac{(-i)^{k_2}\ 2^{\alpha+t+1}}{\Gamma(-\alpha-t)}\ \rho^{-\alpha-t-m/2}\ P_{k_2}(\underline{\eta})\ K_{\alpha+t+k_2+m/2}(\rho) \quad , \qquad (6.16)$$

which leads to the desired result:

$$\mathcal{F}[\psi_{t,k_2,\alpha}](\underline{\xi}) = \frac{(-i)^{t+k_2}\ 2^{\alpha+t+1}}{\Gamma(-\alpha-t)}\ \underline{\xi}^t\ P_{k_2}(\underline{\xi})\ \frac{K_{m/2+\alpha+t+k_2}(|\underline{\xi}|)}{|\underline{\xi}|^{m/2+\alpha+t+k_2}} \quad . \quad \square$$

6.3.3 The generalized Clifford-Gegenbauer CWT

Still for $0 < t < (-2\alpha - m + 1 - k_2)/2$, we introduce the continuous family of wavelets

$$\psi_{t,k_2,\alpha}^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{m/2}}\ s\ \psi_{t,k_2,\alpha}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\ \overline{s} \quad ,$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$ and $s \in \mathrm{Spin}(m)$.
The *generalized Clifford-Gegenbauer CWT* (GCGCWT) applies to functions $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ by

$$\begin{aligned} T_{t,k_2,\alpha}[f](a,\underline{b},s) &= F_{t,k_2,\alpha}(a,\underline{b},s) \ = < \psi_{t,k_2,\alpha}^{a,\underline{b},s}, f > \\ &= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} s \left(P_{k_2}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right)^\dagger \overline{G_{t,k_2,\alpha+t}}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \\ &\qquad \left(1 + \frac{|\underline{x}-\underline{b}|^2}{a^2}\right)^\alpha\ \overline{s}\ f(\underline{x})\ dV(\underline{x}) \quad . \end{aligned}$$

From Chapter 4, we know that this GCGCWT is an isometry between the two spaces $L_2(\mathbb{R}^m, dV(\underline{x}))$ and $L_2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), C_{t,k_2,\alpha}^{-1}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds)$ if

$$\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\ \left(\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\right)^\dagger$$

is scalar-valued and the admissibility condition

$$C_{t,k_2,\alpha} = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\ \left(\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m}\ dV(\underline{\xi}) < +\infty$$

is fulfilled.
Lemma 6.3 leads to:

$$\begin{aligned} &\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\ \left(\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\right)^\dagger \\ &\qquad = \frac{2^{2\alpha+2t+2}}{\left(\Gamma(-\alpha-t)\right)^2}\ \frac{|K_{m/2+\alpha+t+k_2}(|\underline{\xi}|)|^2}{|\underline{\xi}|^{m+2\alpha+2t+2k_2}}\ \underline{\xi}^t\ P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger\ (-\underline{\xi})^t \quad . \end{aligned}$$

The above formula implies that the condition that

$$\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\ \left(\mathcal{F}\left[\psi_{t,k_2,\alpha}\right](\underline{\xi})\right)^{\dagger}$$

is scalar-valued is equivalent with the condition that

$$P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^{\dagger}$$

is scalar-valued.
In that case, we have that

$$\begin{aligned}
&C_{t,k_2,\alpha} \\
&= (2\pi)^m \frac{2^{2\alpha+2t+2}}{\left(\Gamma(-\alpha-t)\right)^2} \int_{\mathbb{R}^m} \frac{|K_{m/2+\alpha+t+k_2}(|\underline{\xi}|)|^2}{|\underline{\xi}|^{2m+2\alpha+2k_2}}\ P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^{\dagger}\ dV(\underline{\xi}) \\
&= (2\pi)^m \frac{2^{2\alpha+2t+2}}{\left(\Gamma(-\alpha-t)\right)^2} \int_0^{+\infty} |K_{m/2+\alpha+t+k_2}(\rho)|^2\ \rho^{-m-2\alpha-1}\ d\rho \\
&\qquad \int_{S^{m-1}} |P_{k_2}(\underline{\eta})|^2\ dS(\underline{\eta}) \quad ,
\end{aligned}$$

where we have introduced spherical co-ordinates

$$\underline{\xi} = \rho\ \underline{\eta}, \quad \rho = |\underline{\xi}| \quad \text{and} \quad \underline{\eta} \in S^{m-1} \quad .$$

The integral over the unit sphere S^{m-1} in $\mathbb{R}^m$ is finite, since we integrate a continuous function over a compactum.
Furthermore, the integral over ρ is also finite in view of the assumption that $0 < t < (-2\alpha-m+1-k_2)/2$, and the earlier mentioned limiting behaviour of the modified Bessel functions of the second kind (see Section 6.2.3). Summarizing, the admissibility constant $C_{t,k_2,\alpha}$ is finite.

Chapter 7

Clifford-Laguerre polynomials and associated CCWT

In this chapter we expose new Clifford algebra-valued wavelet kernel functions which are based on Clifford generalizations of the classical Laguerre polynomials on the real line. They were constructed in [23] and fit in the general Clifford wavelet theory of Chapter 4.

7.1 Introduction

On the real line the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ ($\alpha > -1$, $n = 0, 1, 2, \ldots$) may be defined by the Rodrigues formula

$$L_n^{(\alpha)}(x) = x^{-\alpha} \, \frac{\exp(x)}{n!} \, \frac{d^n}{dx^n} \left[\exp(-x) \, x^{n+\alpha} \right] \quad .$$

They constitute an orthogonal basis for $L_2\Big([0, \infty[\, , \, x^\alpha \exp(-x) \, dx \Big)$ and satisfy the orthogonality relation

$$\int_0^\infty L_m^{(\alpha)}(x) \, L_n^{(\alpha)}(x) \, x^\alpha \, \exp(-x) \, dx = \Gamma(1+\alpha) \binom{n+\alpha}{n} \, \delta_{m,n} \quad . \qquad (7.1)$$

Furthermore, their recurrence relation reads

$$n\ L_n^{(\alpha)}(x) = (2n+\alpha-1-x)\ L_{n-1}^{(\alpha)}(x) - (n+\alpha-1)\ L_{n-2}^{(\alpha)}(x) \quad . \tag{7.2}$$

In this chapter we first construct the so-called Clifford-Laguerre polynomials, a specific generalization to Clifford analysis of the above Laguerre polynomials on the real line (see Section 7.2). After establishing an orthogonality relation for these Clifford-Laguerre polynomials, we select some of them to be candidates for mother wavelets (see Section 7.3). In a final section we prove that these candidates indeed satisfy the necessary conditions for being a mother wavelet.

7.2 The Clifford-Laguerre polynomials

Starting point for the construction of a generalization to Clifford analysis of the classical Laguerre polynomials on the real line are the functions ($\alpha \in \mathbb{R}$)

$$F(\underline{x}) = \exp(-|\underline{x}|)\ |\underline{x}|^\alpha\ P^+ \quad \text{and} \quad G(\underline{x}) = \exp(-|\underline{x}|)\ |\underline{x}|^\alpha\ P^-$$

with P^+ and P^- the Clifford-Heaviside functions (see Section 2.2). The functions F and G are real-analytic in the open connected domain $\mathbb{R}^m \setminus \{\underline{0}\}$ in $\mathbb{R}^m$. Consequently their CK-extensions, denoted by F^* and G^*, exist in an open connected and x_0-normal neighbourhood Ω of $\mathbb{R}^m \setminus \{\underline{0}\}$ in $\mathbb{R}^{m+1}$. The CK-extension F^* generates a first type of so-called *Clifford-Laguerre polynomials* which are indicated by a plus sign as first superindex, while the CK-extension G^* defines a second class of Clifford-Laguerre polynomials which have a minus sign as first superindex:

$$F^*(x_0,\underline{x}) = \exp(-|\underline{x}|)\ \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!}\ |\underline{x}|^{\alpha-2\ell}\ \left(L_{\ell,\alpha}^{+,+}(\underline{x})\ P^+ + L_{\ell,\alpha}^{+,-}(\underline{x})\ P^-\right)$$

and

$$G^*(x_0,\underline{x}) = \exp(-|\underline{x}|)\ \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!}\ |\underline{x}|^{\alpha-2\ell}\ \left(L_{\ell,\alpha}^{-,+}(\underline{x})\ P^+ + L_{\ell,\alpha}^{-,-}(\underline{x})\ P^-\right) \quad .$$

Note that the second superindex is a plus sign if the polynomials belong to the Clifford-Heaviside function P^+, while a minus sign indicates that the polynomials belong to P^-.

By definition F^* satisfies in Ω :

$$F^*(0,\underline{x}) = \exp(-|\underline{x}|)\ |\underline{x}|^\alpha\ P^+ \tag{7.3}$$

and

$$(\partial_{x_0} + \partial_{\underline{x}}) F^*(x_0, \underline{x}) = 0 \quad . \tag{7.4}$$

Using

$$\partial_{\underline{x}} \left[\exp\left(-|\underline{x}|\right) \right] = -\frac{\underline{x}}{|\underline{x}|} \ \exp\left(-|\underline{x}|\right)$$

and

$$\partial_{\underline{x}} \left[\ |\underline{x}|^{\alpha - 2\ell} \ \right] = (\alpha - 2\ell) \ |\underline{x}|^{\alpha - 2\ell - 2} \ \underline{x} \quad ,$$

the monogenicity relation (7.4) yields the following recurrence relation for the first type of Clifford-Laguerre polynomials:

$$\begin{aligned} L^{+,+}_{\ell+1,\alpha}(\underline{x}) \ P^+ + L^{+,-}_{\ell+1,\alpha}(\underline{x}) \ P^- \quad = \quad & |\underline{x}| \ \underline{x} \ \left(L^{+,+}_{\ell,\alpha}(\underline{x}) \ P^+ + L^{+,-}_{\ell,\alpha}(\underline{x}) \ P^- \right) \\ & -(\alpha - 2\ell)\underline{x} \ \left(L^{+,+}_{\ell,\alpha}(\underline{x}) \ P^+ + L^{+,-}_{\ell,\alpha}(\underline{x}) \ P^- \right) \\ & + \underline{x}^2 \ \partial_{\underline{x}} \left[L^{+,+}_{\ell,\alpha}(\underline{x}) \ P^+ + L^{+,-}_{\ell,\alpha}(\underline{x}) \ P^- \right] \quad . \end{aligned}$$

As it follows from (7.3) that

$$L^{+,+}_{0,\alpha}(\underline{x}) = 1 \quad \text{and} \quad L^{+,-}_{0,\alpha}(\underline{x}) = 0 \quad ,$$

we obtain:

$$\begin{aligned} & L^{+,+}_{1,\alpha}(\underline{x}) \ P^+ + L^{+,-}_{1,\alpha}(\underline{x}) \ P^- \\ & \quad = \quad |\underline{x}| \ \underline{x} \ P^+ - \alpha \ \underline{x} \ P^+ + \underline{x}^2 \ \partial_{\underline{x}}[P^+] \\ & \quad = \quad i\underline{x}^2 \ P^+ - \alpha \ \underline{x} \ P^+ - i \ \frac{(1-m)}{2} \ (\ |\underline{x}| \ P^+ + |\underline{x}| \ P^- \ ) \\ & \quad = \quad i\underline{x}^2 \ P^+ - \alpha \ \underline{x} \ P^+ + \frac{(1-m)}{2} \ \underline{x} \ P^+ - \frac{(1-m)}{2} \ \underline{x} \ P^- \quad . \end{aligned}$$

Here we have used (2.8), (2.9) and

$$\partial_{\underline{x}}[P^+] = \frac{i}{2} \ \partial_{\underline{x}} \left[\frac{\underline{x}}{|\underline{x}|} \right] = i \ \frac{(1-m)}{2} \ \frac{1}{|\underline{x}|} \quad .$$

Hence we have

$$L^{+,+}_{1,\alpha}(\underline{x}) = i\underline{x}^2 + \left(\frac{1-m}{2} - \alpha \right) \ \underline{x}$$

and

$$L^{+,-}_{1,\alpha}(\underline{x}) = \left(\frac{m-1}{2}\right) \underline{x} \quad .$$

Similarly, we find consecutively:

$$\begin{aligned}
L^{+,+}_{2,\alpha}(\underline{x}) &= -\underline{x}^4 + i(-2\alpha - m + 1)\ \underline{x}^3 + \left(\alpha(\alpha-2) + \frac{1}{2} - \frac{m}{2} + m\alpha\right)\ \underline{x}^2 \\
L^{+,-}_{2,\alpha}(\underline{x}) &= \left(\frac{m-1}{2}\right)\ \underline{x}^2 \\
L^{+,+}_{3,\alpha}(\underline{x}) &= -i\underline{x}^6 + \left(3\alpha + 3\frac{m}{2} - \frac{3}{2}\right)\ \underline{x}^5 + i\bigg(\alpha(\alpha-2) + 3m\alpha - \frac{5}{2}m + 2 \\
&\quad + 2\alpha^2 - 4\alpha + \frac{m^2}{2}\bigg)\ \underline{x}^4 + \bigg(-\alpha(\alpha-2)(\alpha-4) - \frac{3}{2}m\alpha^2 + 4m\alpha \\
&\quad + \frac{5}{2}\alpha - 2m + \frac{3}{2} - \frac{3}{2}\alpha^2 - \frac{m^2\alpha}{2} + \frac{m^2}{2}\bigg)\ \underline{x}^3 \\
L^{+,-}_{3,\alpha}(\underline{x}) &= \left(\frac{1-m}{2}\right)\ \underline{x}^5 + i\left(-\frac{3}{2}m + 1 - \alpha + m\alpha + \frac{m^2}{2}\right)\ \underline{x}^4 \\
&\quad + \left(-2m\alpha + 2m + \frac{3}{2}\alpha - \frac{\alpha^2}{2} - \frac{3}{2} + \frac{m\alpha^2}{2} + \frac{m^2\alpha}{2} - \frac{m^2}{2}\right)\ \underline{x}^3
\end{aligned}$$

etc.

Note that $L^{+,+}_{\ell,\alpha}(\underline{x})$ is a polynomial of degree 2ℓ in $\underline{x}$, while $L^{+,-}_{\ell,\alpha}(\underline{x})$ is a polynomial of alternative degree $2\ell - 1$ and $2\ell - 2$ in $\underline{x}$.
Furthermore, the Clifford-Laguerre polynomials $L^{+,+}_{\ell,\alpha}(\underline{x})$ and $L^{+,-}_{\ell,\alpha}(\underline{x})$ satisfy the Rodrigues formula:

$$\begin{aligned}
&L^{+,+}_{\ell,\alpha}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha}(\underline{x})\ P^- \\
&\qquad = (-1)^\ell \exp(|\underline{x}|)\ |\underline{x}|^{2\ell-\alpha}\ \partial^\ell_{\underline{x}}\left[\exp(-|\underline{x}|)\ |\underline{x}|^\alpha\ P^+\right] \quad . \qquad (7.5)
\end{aligned}$$

Naturally, similar formulae hold for the Clifford-Laguerre polynomials generated by the CK-extension G^*. It turns out that $L^{-,+}_{\ell,\alpha}(\underline{x})$, respectively $L^{-,-}_{\ell,\alpha}(\underline{x})$ is the complex conjugate of $L^{+,-}_{\ell,\alpha}(\underline{x})$ and $L^{+,+}_{\ell,\alpha}(\underline{x})$ respectively.

By means of the Rodrigues formula (7.5) we obtain the following orthogonality relation.

Theorem 7.1 *Whenever $\alpha > -m$ and $2k < \ell$ one has the orthogonality relation*

$$\int_{\mathbb{R}^m} \left(L^{+,+}_{k,\alpha+2k}(\underline{x})\right)^\dagger \left(L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ +\ L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^-\right)$$
$$|\underline{x}|^\alpha\ \exp(-|\underline{x}|)\ dV(\underline{x}) = 0 \quad .$$

Proof. As $L^{+,+}_{k,\alpha+2k}(\underline{x})$ is a polynomial of degree $2k$ in $\underline{x}$, it is sufficient to show that for each $t < \ell$:

$$\int_{\mathbb{R}^m} \underline{x}^t \left(L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^-\right) |\underline{x}|^\alpha\ \exp(-|\underline{x}|)\ dV(\underline{x}) = 0 \quad . \tag{7.6}$$

Introducing spherical co-ordinates in $\mathbb{R}^m$, it is easily seen that the function $L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^-$ takes the "axial" form

$$A_{2\ell}(r) + B_{2\ell}(r)\ \underline{\omega} \quad , \tag{7.7}$$

where $A_{2\ell}$ and $B_{2\ell}$ are polynomials with complex coefficients of degree 2ℓ in the variable r.
Hence, if $t = 2s$ is even, the above integral becomes

$$(-1)^s \int_0^{+\infty} \int_{S^{m-1}} r^{2s} \left(A_{2\ell}(r) + B_{2\ell}(r)\ \underline{\omega}\right) r^\alpha\ \exp(-r)\ r^{m-1}\ dS(\underline{\omega})\ dr \quad . \tag{7.8}$$

As $\underline{\omega}$ is a specific spherical harmonic, we have:

$$\int_{S^{m-1}} \underline{\omega}\ dS(\underline{\omega}) = 0 \quad .$$

Hence (7.8) can be further simplified to

$$A_m\ (-1)^s \int_0^{+\infty} A_{2\ell}(r)\ r^{2s+\alpha+m-1}\ \exp(-r)\ dr \quad . \tag{7.9}$$

As

$$\int_0^{+\infty} \exp(-r)\ r^{z-1}\ dr \tag{7.10}$$

converges if $\mathrm{Re}(z) > 0$, the integral (7.9) converges on condition that $2s+\alpha+m > 0$ or $\alpha > -m - 2s$.

On the other hand, for t odd, i.e. $t = 2s+1$, we obtain in an analogous manner:

$$\int_{\mathbb{R}^m} \underline{x}^{2s+1} \left(L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^- \right) |\underline{x}|^\alpha \ \exp\left(-|\underline{x}|\right) dV(\underline{x})$$
$$= (-1)^{s+1}\ A_m \int_0^{+\infty} B_{2\ell}(r)\ r^{2s+\alpha+m} \ \exp\left(-r\right)\ dr \ ,$$

which is convergent whenever $2s + \alpha + m + 1 > 0$ or $\alpha > -m - (2s+1)$.
Consequently, we conclude that the integral (7.6) converges for each $0 \leq t < \ell$ under the assumption $\alpha > -m$.
Moreover, using the Rodrigues relation (7.5) and the Clifford-Stokes theorem respectively, we obtain consecutively

$$\begin{aligned}
&\int_{\mathbb{R}^m} \underline{x}^t \left(L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^- \right) |\underline{x}|^\alpha \ \exp\left(-|\underline{x}|\right) dV(\underline{x}) \\
&= (-1)^\ell \int_{\mathbb{R}^m} \underline{x}^t\ \partial^\ell_{\underline{x}} \left[\exp\left(-|\underline{x}|\right)\ |\underline{x}|^{\alpha+2\ell}\ P^+\right] dV(\underline{x}) \\
&= (-1)^\ell \left\{ \int_{\partial\mathbb{R}^m} \underline{x}^t\ d\underline{\sigma}\ \partial^{\ell-1}_{\underline{x}} \left[\exp\left(-|\underline{x}|\right)\ |\underline{x}|^{\alpha+2\ell}\ P^+\right] \right. \\
&\quad \left. - \int_{\mathbb{R}^m} (\underline{x}^t \partial_{\underline{x}})\ \partial^{\ell-1}_{\underline{x}} \left[\exp\left(-|\underline{x}|\right)\ |\underline{x}|^{\alpha+2\ell}\ P^+\right] dV(\underline{x}) \right\} \\
&= (-1)^{\ell+1} \int_{\mathbb{R}^m} (\underline{x}^t \partial_{\underline{x}})\ \partial^{\ell-1}_{\underline{x}} \left[\exp\left(-|\underline{x}|\right)\ |\underline{x}|^{\alpha+2\ell}\ P^+\right] dV(\underline{x}) \ .
\end{aligned}$$

As $\underline{x}^t \partial_{\underline{x}} \approx \underline{x}^{t-1}$ and $t < \ell$, repeating this argument leads to the desired result.
□

7.3 The Clifford-Laguerre wavelets

For $\alpha > -m$ and $0 < \ell$, Theorem 7.1 implies that the functions

$$\begin{aligned}
\psi_{\ell,\alpha}(\underline{x}) &= \left(L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^- \right) |\underline{x}|^\alpha \ \exp\left(-|\underline{x}|\right) \\
&= (-1)^\ell\ \partial^\ell_{\underline{x}} \left[\exp\left(-|\underline{x}|\right)\ |\underline{x}|^{\alpha+2\ell}\ P^+\right]
\end{aligned}$$

have zero momentum.
By means of properties (2.1) of the Clifford norm and the fact that

$L^{+,+}_{\ell,\alpha+2\ell}(\underline{x})\ P^+ + L^{+,-}_{\ell,\alpha+2\ell}(\underline{x})\ P^-$ takes the "axial" form (7.7), we find consecutively

$$\begin{aligned}
&\int_{\mathbb{R}^m} |\psi_{\ell,\alpha}(\underline{x})|\ dV(\underline{x}) \\
&\quad = \int_0^{+\infty} \int_{S^{m-1}} \left| \big(A_{2\ell}(r) + B_{2\ell}(r)\ \underline{\omega}\big)\ r^\alpha \exp(-r)\ \right| r^{m-1}\ dS(\underline{\omega})\ dr \\
&\quad \leq A_m \int_0^{+\infty} \left| A_{2\ell}(r)\ r^\alpha\ \exp(-r) \right| r^{m-1}\ dr \\
&\quad + A_m \int_0^{+\infty} \left| B_{2\ell}(r)\ r^\alpha\ \exp(-r) \right| r^{m-1}\ dr \quad .
\end{aligned}$$

In view of the convergence of the integral (7.10) whenever $\mathrm{Re}(z) > 0$ and the assumption $\alpha > -m$, it is easily seen that

$$\int_{\mathbb{R}^m} |\psi_{\ell,\alpha}(\underline{x})|\ dV(\underline{x}) < \infty \quad .$$

Hence, $\psi_{\ell,\alpha} \in L_1\big(\mathbb{R}^m, dV(\underline{x})\big)$.

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^m} |\psi_{\ell,\alpha}(\underline{x})|^2\ dV(\underline{x}) \\
&\quad \leq A_m \int_0^{+\infty} \left| A_{2\ell}(r)\ r^\alpha\ \exp(-r) \right|^2 r^{m-1}\ dr \\
&\quad + A_m \int_0^{+\infty} \left| B_{2\ell}(r)\ r^\alpha\ \exp(-r) \right|^2 r^{m-1}\ dr \\
&\quad + 2\ A_m \int_0^{+\infty} \left| A_{2\ell}(r)\ r^\alpha\ \exp(-r) \right| \left| B_{2\ell}(r)\ r^\alpha\ \exp(-r) \right| r^{m-1}\ dr \quad .
\end{aligned}$$

As

$$\int_0^{+\infty} \exp(-2r)\ r^{z-1}\ dr$$

converges whenever $\mathrm{Re}(z) > 0$, one can verify that

$$\int_{\mathbb{R}^m} |\psi_{\ell,\alpha}(\underline{x})|^2\ dV(\underline{x}) < \infty$$

on condition that $\alpha > -m/2$. Hence, in order that $\psi_{\ell,\alpha}$ is also an L_2 -function, we have to make the additional restriction $\alpha > -m/2$.

Summarizing, for $\alpha > -m/2$ and $\ell > 0$, the functions $\psi_{\ell,\alpha}$ are good candidates for mother wavelets in $\mathbb{R}^m$, if at least they satisfy the conditions outlined in Chapter 4. We call them the *Clifford-Laguerre wavelets*. They were introduced in [23].

Theorem 7.1 implies that the Clifford-Laguerre wavelets have a number of vanishing moments.

Proposition 7.1 *The Clifford-Laguerre wavelet $\psi_{\ell,\alpha}$ has vanishing moments up to order $(\ell - 1)$, i.e.*

$$\int_{\mathbb{R}^m} \underline{x}^j \; \psi_{\ell,\alpha}(\underline{x}) \; dV(\underline{x}) = 0$$

for $j = 0, 1, \ldots, \ell - 1$.

Next, we pass on to the calculation of the Fourier transform of the new wavelets.

Lemma 7.1 *The Fourier transform of the Clifford-Laguerre wavelets takes the form:*

$$\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi}) = (-i)^\ell \; \frac{\Gamma(\alpha + 2\ell + m)}{2} \; \frac{|\underline{\xi}|^{1-m/2}}{(1 + |\underline{\xi}|^2)^{(\alpha+2\ell+m/2+1)/2}} \; \underline{\xi}^\ell$$
$$\left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1 + |\underline{\xi}|^2)^{-1/2}\big) + (\alpha + 2\ell + m) \; \frac{\underline{\xi}}{|\underline{\xi}|} \right.$$
$$\left. P^{-m/2}_{\alpha+2\ell+m/2}\big((1 + |\underline{\xi}|^2)^{-1/2}\big) \right)$$

with P^μ_ν the associated Legendre function of the first kind.

Proof. This proof is based on Proposition 2.2 on the one hand and the following result (see [59])

$$\int_0^\infty r^{\mu-3/2} \; \exp(-r) \; J_\nu(\rho r) \; (\rho r)^{1/2} \; dr$$
$$= \rho^{1/2} \; (1 + \rho^2)^{-\mu/2} \; \Gamma(\mu + \nu) \; P^{-\nu}_{\mu-1}\big((1 + \rho^2)^{-1/2}\big) \quad , \quad \mathrm{Re}(\mu + \nu) > 0 \qquad (7.11)$$

on the other hand. Here

$$P^\mu_\nu(x) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{1 + x}{1 - x} \right)^{\mu/2} F\left(-\nu, \nu + 1; 1 - \mu; \frac{1 - x}{2} \right) \quad ; \quad -1 < x < 1$$

is the associated Legendre function of the first kind with

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\ \Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\ \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \quad ; \quad |z| < 1$$

the hypergeometric function. As before, J_ν denotes the Bessel function of the first kind of order ν.
First, we have

$$\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi}) = (-i)^\ell\ \underline{\xi}^\ell\ \mathcal{F}\big[\exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ P^+\big](\underline{\xi})$$

with by definition

$$\begin{aligned}&\mathcal{F}\big[\exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ P^+\big](\underline{\xi})\\ &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp(-i < \underline{x}, \underline{\xi} >)\ \exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ \frac{1}{2}\left(1 + i\frac{\underline{x}}{|\underline{x}|}\right) dV(\underline{x}) \quad .\end{aligned}$$

Introducing spherical co-ordinates, this expression becomes

$$\begin{aligned}&\mathcal{F}\big[\exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ P^+\big](\underline{\xi})\\ &= \frac{1}{(2\pi)^{m/2}}\ \frac{1}{2} \int_0^{+\infty} \exp(-r)\ r^{\alpha+2\ell+m-1}\ dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\eta} >)\ dS(\underline{\omega})\\ &+ \frac{1}{(2\pi)^{m/2}}\ \frac{i}{2} \int_0^{+\infty} \exp(-r)\ r^{\alpha+2\ell+m-1}\ dr \int_{S^{m-1}} \exp(-ir\rho < \underline{\omega}, \underline{\eta} >)\ \underline{\omega}\ dS(\underline{\omega}) \quad .\end{aligned}$$

Applying Proposition 2.2 twice, once on the spherical harmonic $S_0(\underline{\omega}) = 1$ and once on $S_1(\underline{\omega}) = \underline{\omega}$, yields

$$\begin{aligned}&\mathcal{F}\big[\exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ P^+\big](\underline{\xi})\\ &\quad = \frac{1}{2}\ \rho^{(1-m)/2} \int_0^{+\infty} r^{\alpha+2\ell+m/2-1/2}\ \exp(-r)\ J_{m/2-1}(\rho r)\ (\rho r)^{1/2}\ dr\\ &\quad + \frac{1}{2}\ \rho^{(1-m)/2}\ \underline{\eta} \int_0^{+\infty} r^{\alpha+2\ell+m/2-1/2}\ \exp(-r)\ J_{m/2}(\rho r)\ (\rho r)^{1/2}\ dr \quad .\end{aligned}$$

Next, using (7.11) once for $\mu = \alpha + 2\ell + m/2 + 1$ and $\nu = m/2 - 1$ and once for

$\mu = \alpha + 2\ell + m/2 + 1$ and $\nu = m/2$, implies

$$\begin{aligned}
&\mathcal{F}\big[\exp(-|\underline{x}|)\ |\underline{x}|^{\alpha+2\ell}\ P^{+}\big](\underline{\xi}) \\
&= \frac{\Gamma(\alpha+2\ell+m)}{2}\ \rho^{1-m/2}\ \frac{1}{(1+\rho^2)^{(\alpha+2\ell+m/2+1)/2}} P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+\rho^2)^{-1/2}\big) \\
&+ \frac{\Gamma(\alpha+2\ell+m+1)}{2}\ \rho^{1-m/2}\ \frac{1}{(1+\rho^2)^{(\alpha+2\ell+m/2+1)/2}} P^{-m/2}_{\alpha+2\ell+m/2}\big((1+\rho^2)^{-1/2}\big)\ \underline{\eta}\,.
\end{aligned}$$

Note that in both cases $\mathrm{Re}(\mu+\nu)$ is indeed strictly positive in view of the assumptions $\alpha > -m$ and $\ell > 0$.
Hence, we finally have:

$$\begin{aligned}
\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi}) = (-i)^{\ell}\ \frac{\Gamma(\alpha+2\ell+m)}{2}\ \frac{|\underline{\xi}|^{1-m/2}}{(1+|\underline{\xi}|^2)^{(\alpha+2\ell+m/2+1)/2}}\ \underline{\xi}^{\ell} \\
\left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big) + (\alpha+2\ell+m)\ \frac{\underline{\xi}}{|\underline{\xi}|}\right. \\
\left. P^{-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)\ . \quad \square
\end{aligned}$$

7.4 The Clifford-Laguerre CWT

Take $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$, then its *Clifford-Laguerre CWT* (CLCWT) is defined by

$$\begin{aligned}
T_{\ell,\alpha}[f](a,\underline{b},s) = F_{\ell,\alpha}(a,\underline{b},s) &= \ < \psi^{a,\underline{b},s}_{\ell,\alpha}, f > \\
&= \int_{\mathbb{R}^m} \big(\psi^{a,\underline{b},s}_{\ell,\alpha}(\underline{x})\big)^{\dagger}\ f(\underline{x})\ dV(\underline{x})
\end{aligned}$$

where, still for $\alpha > -m/2$ and $\ell > 0$, the continuous family of wavelets $\psi^{a,\underline{b},s}_{\ell,\alpha}$ is given by

$$\psi^{a,\underline{b},s}_{\ell,\alpha}(\underline{x}) = \frac{1}{a^{m/2}}\ s\ \psi_{\ell,\alpha}\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\ \overline{s}\quad ,$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$ and $s \in \mathrm{Spin}(m)$.
In order for the CLCWT $T_{\ell,\alpha}$ to be an isometry, the wavelet $\psi_{\ell,\alpha}$ must satisfy the conditions (see Definition 4.1):

$$\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi})\ \big(\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi})\big)^{\dagger}$$

has to be scalar-valued and

$$C_{\ell,\alpha} = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi}) \, \left(\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) < \infty \quad . \tag{7.12}$$

By means of Lemma 7.1, the first condition is easily verified. Indeed,

$$\begin{aligned}
&\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi}) \, \left(\mathcal{F}[\psi_{\ell,\alpha}](\underline{\xi})\right)^\dagger \\
&= \frac{(\Gamma(\alpha+2\ell+m))^2}{4} \, \frac{|\underline{\xi}|^{2\ell+2-m}}{(1+|\underline{\xi}|^2)^{\alpha+2\ell+m/2+1}} \left\{ \left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 \right. \\
&\qquad \left. + \ (\alpha+2\ell+m)^2 \left(P^{-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 \right\}
\end{aligned}$$

is clearly scalar-valued.
Moreover, the admissibility constant (7.12) takes the form

$$\begin{aligned}
C_{\ell,\alpha} &= (2\pi)^m \, \frac{(\Gamma(\alpha+2\ell+m))^2}{4} \\
&\quad \int_{\mathbb{R}^m} \frac{|\underline{\xi}|^{2\ell+2-2m}}{(1+|\underline{\xi}|^2)^{\alpha+2\ell+m/2+1}} \left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 dV(\underline{\xi}) \\
&+ (2\pi)^m \, \frac{(\Gamma(\alpha+2\ell+m))^2}{4} \, (\alpha+2\ell+m)^2 \\
&\quad \int_{\mathbb{R}^m} \frac{|\underline{\xi}|^{2\ell+2-2m}}{(1+|\underline{\xi}|^2)^{\alpha+2\ell+m/2+1}} \left(P^{-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 dV(\underline{\xi})
\end{aligned}$$

Introducing spherical co-ordinates, the first integral defining the admissibility constant can be simplified to

$$\begin{aligned}
&\int_{\mathbb{R}^m} \frac{|\underline{\xi}|^{2\ell+2-2m}}{(1+|\underline{\xi}|^2)^{\alpha+2\ell+m/2+1}} \left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 dV(\underline{\xi}) \\
&= A_m \int_0^{+\infty} \frac{\rho^{2\ell-m+1}}{(1+\rho^2)^{\alpha+2\ell+m/2+1}} \left(P^{1-m/2}_{\alpha+2\ell+m/2}\big((1+\rho^2)^{-1/2}\big)\right)^2 d\rho
\end{aligned} \tag{7.13}$$

As the associated Legendre functions of the first kind $P^\mu_\nu(x)$ with $\mu \neq 1, 2, 3, \ldots$ have the following behaviour near the singular point $+1$ (see for e.g. [82]):

$$\frac{2^{\mu/2} \, (1-x)^{-\mu/2}}{\Gamma(1-\mu)}$$

and as

$$P_\nu^\mu(0) = 2^\mu \; \pi^{-1/2} \; \cos\left(\frac{\pi}{2}(\nu+\mu)\right) \; \frac{\Gamma\left(\frac{1}{2}+\frac{\nu+\mu}{2}\right)}{\Gamma\left(1+\frac{\nu-\mu}{2}\right)} \quad ,$$

the integral (7.13) is finite if we make the restriction $\alpha \neq -2\ell-2, -2\ell-4, -2\ell-6, \ldots$.
In an analogous manner, one obtains that the integral

$$\begin{aligned}&\int_{\mathbb{R}^m} \frac{|\underline{\xi}|^{2\ell+2-2m}}{(1+|\underline{\xi}|^2)^{\alpha+2\ell+m/2+1}} \left(P^{-m/2}_{\alpha+2\ell+m/2}\big((1+|\underline{\xi}|^2)^{-1/2}\big)\right)^2 \, dV(\underline{\xi}) \\ &\qquad = A_m \int_0^{+\infty} \frac{\rho^{2\ell-m+1}}{(1+\rho^2)^{\alpha+2\ell+m/2+1}} \left(P^{-m/2}_{\alpha+2\ell+m/2}\big((1+\rho^2)^{-1/2}\big)\right)^2 \, d\rho\end{aligned}$$

is finite, provided that $\alpha \neq -2\ell-1, -2\ell-3, -2\ell-5, \ldots$.
The above reasoning leads to the conclusion that the admissibility constant $C_{\ell,\alpha}$ is finite if we make the additional restriction $\alpha \neq -2\ell-1, -2\ell-2, -2\ell-3, -2\ell-4, \ldots$.
Summarizing, if the conditions $\ell > 0$, $\alpha > -m/2$ and $\alpha \neq -2\ell-1, -2\ell-2, -2\ell-3, -2\ell-4, \ldots$ are fulfilled, then the Clifford-Laguerre CWT $T_{\ell,\alpha}$ is an isometry between two L_2-spaces.

Chapter 8

The "Half" CCWT: general theory

In Chapter 4 we obtained the mother wavelet conditions in the orthogonal Clifford analysis setting. However, even if the candidate mother wavelets satisfy more general conditions, a Clifford CWT theory can be developed. This necessitates the use of the orthogonal decomposition of the space of square integrable functions into the Hardy space $H^2(\mathbb{R}^m)$ and its orthogonal complement. In this way a nice relationship is established between the theory of the Clifford CWT on the one hand, and the theory of Hardy spaces on the other hand.

8.1 "Half" Generalized CCWTs

Suppose that the Clifford algebra-valued function $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ does not satisfy the mother wavelet conditions of the general Clifford CWT theory established in Definition 4.1, but shows the more general property:

$$\mathcal{F}[\psi](\underline{\xi}) \, \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger = A(\underline{\xi}) + B(\underline{\xi}) \, \frac{\underline{\xi}}{|\underline{\xi}|} \tag{8.1}$$

with A and B scalar-valued functions.
As we will show in this section, we still can use ψ as kernel function for a multi-dimensional Clifford CWT. For this purpose, we are forced to decompose each square integrable function $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ into its Hardy components (see

Section 2.2)

$$f = \mathbb{P}^+[f] + \mathbb{P}^-[f]$$

with

$$\mathbb{P}^+[f] = \frac{1}{2}\,(f + H[f]) \in H^2(\mathbb{R}^m)$$

and

$$\mathbb{P}^-[f] = \frac{1}{2}\,(f - H[f]) \in H^2(\mathbb{R}^m)^\perp \quad ,$$

where H stands for the Hilbert transform.
In what follows, we briefly denote $f^\pm = \mathbb{P}^\pm[f]$.

The *"Half" Generalized CCWTs* (HGCCWTs) associated with ψ apply to functions $f^+ \in H^2(\mathbb{R}^m)$, respectively $f^- \in H^2(\mathbb{R}^m)^\perp$, by

$$\begin{aligned} T^\pm_\psi[f^\pm](a,\underline{b},s) = F^\pm_\psi(a,\underline{b},s) &= \ <\psi^{a,\underline{b},s}, f^\pm> \\ &= \int_{\mathbb{R}^m} \left(\psi^{a,\underline{b},s}(\underline{x})\right)^\dagger \, f^\pm(\underline{x})\; dV(\underline{x}) \quad , \end{aligned}$$

where, like before, the continuous family of wavelets

$$\psi^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{m/2}} \; s \; \psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right) \; \overline{s} \quad ,$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$ and $s \in \mathrm{Spin}(m)$, originates from ψ by dilation, translation and spinor-rotation.
Similarly as in Section 4.1, the definition of the HGCCWTs can be rewritten in frequency space as

$$F^\pm_\psi(a,\underline{b},s) = a^{m/2} \; s \; (2\pi)^{m/2} \; \mathcal{F}\left[\left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s} \; \mathcal{F}[f^\pm](\underline{\xi})\right](-\underline{b}) \quad . \tag{8.2}$$

Now our aim is to derive the conditions such that T^+_ψ, respectively T^-_ψ, maps the Hardy space $H^2(\mathbb{R}^m)$, respectively $H^2(\mathbb{R}^m)^\perp$, isometrically into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m)$.
To that end we calculate

$$\int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F^\pm_\psi(a,\underline{b},s)\right)^\dagger \; G^\pm_\psi(a,\underline{b},s) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \; ds \quad . \tag{8.3}$$

Similarly as in the proof of Theorem 4.1, we obtain by means of (8.2) and the Parseval formula for the Fourier transform

$$\int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a, \underline{b}, s)\right)^\dagger \; G_\psi^\pm(a, \underline{b}, s) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \; ds$$
$$= (2\pi)^m \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger \left(\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s \; \mathcal{F}[\psi](a\overline{s}\underline{\xi}s) \; \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s} \; \frac{da}{a} \; ds \right)$$
$$\mathcal{F}[g^\pm](\underline{\xi}) \; dV(\underline{\xi}) \quad . \qquad (8.4)$$

By means of the substitution

$$\underline{\xi} = \frac{t}{a} \; \underline{\eta} \quad ; \quad \underline{\eta} \in S^{m-1} \quad ,$$

the integral between brackets in (8.4) becomes

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s \; \mathcal{F}[\psi](a\overline{s}\underline{\xi}s) \; \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s} \; \frac{da}{a} \; ds$$
$$= \int_{\mathrm{Spin}(m)} \int_0^{+\infty} s \; \mathcal{F}[\psi](t\overline{s}\underline{\eta}s) \; \left(\mathcal{F}[\psi](t\overline{s}\underline{\eta}s)\right)^\dagger \; \overline{s} \; \frac{dt}{t} \; ds \quad . \qquad (8.5)$$

The assumption (8.1) yields the following expression for the integrand of the above integral:

$$\begin{aligned} s \; \mathcal{F}[\psi](t\overline{s}\underline{\eta}s) \; \left(\mathcal{F}[\psi](t\overline{s}\underline{\eta}s)\right)^\dagger \; \overline{s} \quad &= \quad s \; \left(A(t\overline{s}\underline{\eta}s) + B(t\overline{s}\underline{\eta}s) \; \overline{s}\underline{\eta}s\right) \; \overline{s} \\ &= \quad A(t\overline{s}\underline{\eta}s) + B(t\overline{s}\underline{\eta}s) \; \underline{\eta} \quad . \end{aligned}$$

Hence the integral (8.5) takes the following form

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s \; \mathcal{F}[\psi](a\overline{s}\underline{\xi}s) \; \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger \; \overline{s} \; \frac{da}{a} \; ds$$
$$= \left(\int_{\mathrm{Spin}(m)} \int_0^{+\infty} A(t\overline{s}\underline{\eta}s) \; \frac{dt}{t} \; ds \right) + \left(\int_{\mathrm{Spin}(m)} \int_0^{+\infty} B(t\overline{s}\underline{\eta}s) \; \frac{dt}{t} \; ds \right) \underline{\eta} \quad .$$

Taking into account that $\overline{s}\underline{\eta}s = \underline{\nu} \in S^{m-1}$ for all $\underline{\eta} \in S^{m-1}$, the integration over

Spin(m) turns into an integration over the unit sphere S^{m-1} of $\mathbb{R}^m$:

$$
\begin{aligned}
&\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\ \left(\mathcal{F}[\psi](a\overline{s}\underline{\xi}s)\right)^\dagger\ \overline{s}\ \frac{da}{a}\ ds \\
&= \left(\int_0^{+\infty} \int_{S^{m-1}} A(t\underline{\nu})\ dS(\underline{\nu})\ \frac{dt}{t}\right) + \left(\int_0^{+\infty} \int_{S^{m-1}} B(t\underline{\nu})\ dS(\underline{\nu})\ \frac{dt}{t}\right) \underline{\eta} \\
&= c_1 + c_2\ \underline{\eta}\ ,
\end{aligned}
$$

where we have put

$$
c_1 = \int_0^{+\infty} \int_{S^{m-1}} A(t\underline{\nu})\ dS(\underline{\nu})\ \frac{dt}{t} = \int_{\mathbb{R}^m} A(\underline{u})\ \frac{dV(\underline{u})}{|\underline{u}|^m}
$$

and

$$
c_2 = \int_0^{+\infty} \int_{S^{m-1}} B(t\underline{\nu})\ dS(\underline{\nu})\ \frac{dt}{t} = \int_{\mathbb{R}^m} B(\underline{u})\ \frac{dV(\underline{u})}{|\underline{u}|^m} \quad .
$$

Assuming that the integrals c_1 and c_2 converge, the integral (8.4) becomes

$$
\begin{aligned}
&\int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b},s)\right)^\dagger\ G_\psi^\pm(a,\underline{b},s)\ \frac{da}{a^{m+1}}\ dV(\underline{b})\ ds \\
&= (2\pi)^m \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger\ (c_1 + c_2\ \underline{\eta})\ \mathcal{F}[g^\pm](\underline{\xi})\ dV(\underline{\xi}) \\
&= (2\pi)^m\ c_1\ < f^\pm, g^\pm > + (2\pi)^m\ c_2 \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger\ \underline{\eta}\ \mathcal{F}[g^\pm](\underline{\xi})\ dV(\underline{\xi})\ ,
\end{aligned}
$$

where $\underline{\eta} = \dfrac{\underline{\xi}}{|\underline{\xi}|}$.

As

$$
\underline{\eta}\ \mathcal{F}[g^\pm](\underline{\xi}) = \underline{\eta}\ P^\pm\ \mathcal{F}[g](\underline{\xi}) = (\mp i)\ P^\pm\ \mathcal{F}[g](\underline{\xi}) = (\mp i)\ \mathcal{F}[g^\pm](\underline{\xi})
$$

with $P^\pm = \frac{1}{2}(1 \pm i\underline{\eta})$ the Clifford-Heaviside functions (see Section 2.2), this integral may be further simplified to

$$
\begin{aligned}
&\int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b},s)\right)^\dagger\ G_\psi^\pm(a,\underline{b},s)\ \frac{da}{a^{m+1}}\ dV(\underline{b})\ ds \\
&\qquad = (2\pi)^m\ c_1\ < f^\pm, g^\pm > \mp (2\pi)^m\ ic_2\ < f^\pm, g^\pm > \\
&\qquad = (2\pi)^m\ (c_1 \mp ic_2)\ < f^\pm, g^\pm > \quad .
\end{aligned}
$$

Hence we can define the following inner products on the spaces of transforms

$$[F_\psi^\pm, G_\psi^\pm] = \frac{1}{C^\pm} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b},s)\right)^\dagger \, G_\psi^\pm(a,\underline{b},s) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \, ds$$

with

$$C^\pm = (2\pi)^m \, (c_1 \mp i c_2) \quad .$$

These inner products satisfy the Parseval formulae

$$[F_\psi^\pm, G_\psi^\pm] = < f^\pm, g^\pm > \quad ,$$

which implies that the HGCCWTs $T_\psi^\pm$ are isometries, as it should be.

Conclusion 8.1 *If a Clifford algebra-valued function* $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ *satisfies in frequency space the following condition*

$$\mathcal{F}[\psi](\underline{\xi}) \, \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger = A(\underline{\xi}) + B(\underline{\xi}) \, \frac{\underline{\xi}}{|\underline{\xi}|}$$

with A *and* B *scalar-valued functions for which*

$$c_1 = \int_{\mathbb{R}^m} A(\underline{u}) \, \frac{dV(\underline{u})}{|\underline{u}|^m} \quad \textit{and} \quad c_2 = \int_{\mathbb{R}^m} B(\underline{u}) \, \frac{dV(\underline{u})}{|\underline{u}|^m}$$

are finite constants, then the corresponding HGCCWT T_ψ^+*, respectively* T_ψ^-*, is an isometry from the Hardy space* $H(\mathbb{R}^m)$*, respectively* $H(\mathbb{R}^m)^\perp$*, into the weighted* L_2*-space* $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), \frac{1}{C^+} \, a^{-(m+1)} \, da \, dV(\underline{b}) \, ds\big)$*, respectively* $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), \frac{1}{C^-} \, a^{-(m+1)} \, da \, dV(\underline{b}) \, ds\big)$*. Here we have introduced the constants* $C^\pm = (2\pi)^m(c_1 \mp i c_2)$*.*

Note that the requirement that c_1 and c_2 are finite constants, can be viewed as a *generalization of the admissibility condition* (4.3) established in Section 4.1.

Remark 8.1 The reconstruction formulae for the HGCCWTs take the form

$$g^\pm(\underline{x}) = \frac{1}{C^\pm} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \psi^{a,\underline{b},s}(\underline{x}) \, G_\psi^\pm(a,\underline{b},s) \, \frac{da}{a^{m+1}} \, dV(\underline{b}) \, ds \quad ,$$

since

$$
\begin{aligned}
&< f^{\pm}, g^{\pm} > \\
&\quad = \; [F^{\pm}_{\psi}, G^{\pm}_{\psi}] \\
&\quad = \; \frac{1}{C^{\pm}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F^{\pm}_{\psi}(a, \underline{b}, s)\right)^{\dagger} \; G^{\pm}_{\psi}(a, \underline{b}, s) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \; ds \\
&\quad = \; \frac{1}{C^{\pm}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} < f^{\pm}, \psi^{a,\underline{b},s} > \; G^{\pm}_{\psi}(a, \underline{b}, s) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \; ds \quad .
\end{aligned}
$$

The twin transforms $T^{\pm}_{\psi}$ may be combined into a *"complete" generalized CCWT* acting on $L_2(\mathbb{R}^m, dV(\underline{x}))$. Indeed, for $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ we put

$$
\begin{aligned}
T_{\psi}[f](a, \underline{b}, s) = F_{\psi}(a, \underline{b}, s) \; &= \; < \psi^{a,\underline{b},s}, f > \\
&= \; < \psi^{a,\underline{b},s}, f^{+} > + < \psi^{a,\underline{b},s}, f^{-} > \\
&= \; F^{+}_{\psi}(a, \underline{b}, s) + F^{-}_{\psi}(a, \underline{b}, s) \quad .
\end{aligned}
$$

8.2 Special case of Spin(m)-invariance: "Half" CCWTs

Let $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ be a Spin(m)-invariant Clifford algebra-valued function for which the expression $\mathcal{F}[\psi](\underline{\xi}) \; \left(\mathcal{F}[\psi](\underline{\xi})\right)^{\dagger}$ is not radial symmetric, as it should be to develop the Spin(m)-invariant CCWT theory outlined in Section 4.3. However, let us assume that ψ does satisfy the more general condition

$$
\mathcal{F}[\psi](\underline{\xi}) \; \left(\mathcal{F}[\psi](\underline{\xi})\right)^{\dagger} = A(|\underline{\xi}|) + B(|\underline{\xi}|) \; \frac{\underline{\xi}}{|\underline{\xi}|} \tag{8.6}
$$

with A and B radial symmetric functions, i.e. only depending on $|\underline{\xi}|$.

Similarly as in Section 4.3, we do not take the Spin(m)-group into consideration while defining the continuous family of wavelets stemming from ψ:

$$
\psi^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{m/2}} \; \psi\left(\frac{\underline{x} - \underline{b}}{a}\right) \quad ; \quad a \in \mathbb{R}_+ \quad , \quad \underline{b} \in \mathbb{R}^m \quad .
$$

The corresponding *"Half" CCWTs* (HCCWTs) $T^{\pm}_{\psi}$ take the form

$$
T^{\pm}_{\psi}[f^{\pm}](a, \underline{b}) = F^{\pm}_{\psi}(a, \underline{b}) = < \psi^{a,\underline{b}}, f^{\pm} > \quad ,
$$

where $f^\pm = \mathbb{P}^\pm[f]$, $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$.

Like in the previous section, we now search for the appropriate requirements which turn the above introduced HCCWTs into isometries. For that purpose, we calculate the following integral

$$\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b})\right)^\dagger \; G_\psi^\pm(a,\underline{b}) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \quad .$$

By means of, among others, the Parseval formula for the Fourier transform, we find

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b})\right)^\dagger \; G_\psi^\pm(a,\underline{b}) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \\ &= (2\pi)^m \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger \left(\int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \frac{da}{a} \right) \mathcal{F}[g^\pm](\underline{\xi}) \, dV(\underline{\xi}) \quad . \end{aligned}$$

Next, applying the substitution

$$\underline{\xi} = \frac{t}{a} \; \underline{\eta}, \quad \underline{\eta} \in S^{m-1},$$

transforms the integral between brackets into

$$\int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \; \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \; \frac{da}{a} = \int_0^{+\infty} \mathcal{F}[\psi](t\underline{\eta}) \; \left(\mathcal{F}[\psi](t\underline{\eta})\right)^\dagger \; \frac{dt}{t} \quad .$$

Taking into account the assumption (8.6) yields

$$\int_0^{+\infty} \mathcal{F}[\psi](a\underline{\xi}) \; \left(\mathcal{F}[\psi](a\underline{\xi})\right)^\dagger \; \frac{da}{a} = \int_0^{+\infty} \left(A(t) + B(t) \; \underline{\eta}\right) \; \frac{dt}{t} = c_1 + c_2 \; \underline{\eta}$$

with

$$c_1 = \int_0^{+\infty} A(t) \; \frac{dt}{t} = \frac{1}{A_m} \int_{\mathbb{R}^m} A(\underline{u}) \; \frac{dV(\underline{u})}{|\underline{u}|^m}$$

and

$$c_2 = \int_0^{+\infty} B(t) \; \frac{dt}{t} = \frac{1}{A_m} \int_{\mathbb{R}^m} B(\underline{u}) \; \frac{dV(\underline{u})}{|\underline{u}|^m} \quad .$$

If the above introduced integrals converge, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^\pm(a,\underline{b})\right)^\dagger \; G_\psi^\pm(a,\underline{b}) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \\ &= (2\pi)^m \; c_1 \; < f^\pm, g^\pm > + \; (2\pi)^m \; c_2 \int_{\mathbb{R}^m} \left(\mathcal{F}[f^\pm](\underline{\xi})\right)^\dagger \; \underline{\eta} \; \mathcal{F}[g^\pm](\underline{\xi}) \; dV(\underline{\xi}) \quad . \end{aligned}$$

As we have

$$\underline{\eta}\ \mathcal{F}[g^{\pm}](\underline{\xi}) = (\mp i)\ \mathcal{F}[g^{\pm}](\underline{\xi}) \quad ,$$

which is in fact crucial for our whole computation, the above integral becomes

$$\int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^{\pm}(a,\underline{b})\right)^\dagger\ G_\psi^{\pm}(a,\underline{b})\ \frac{da}{a^{m+1}}\ dV(\underline{b}) = (2\pi)^m\ (c_1 \mp ic_2)\ < f^{\pm}, g^{\pm} > \quad .$$

Hence, introducing the following inner products in the spaces of wavelet transforms

$$[F_\psi^{\pm}, G_\psi^{\pm}] = \frac{1}{C^{\pm}} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(F_\psi^{\pm}(a,\underline{b})\right)^\dagger\ G_\psi^{\pm}(a,\underline{b})\ \frac{da}{a^{m+1}}\ dV(\underline{b})$$

with

$$C^{\pm} = (2\pi)^m\ (c_1 \mp ic_2) \quad ,$$

the Parseval formulae

$$[F_\psi^{\pm}, G_\psi^{\pm}] = < f^{\pm}, g^{\pm} > \quad ,$$

expressing the norm-preserving character of the HCCWTs, are fulfilled.

Conclusion 8.2 *If a* Spin*(m)-invariant Clifford algebra-valued function* $\psi \in L_1(\mathbb{R}^m, dV(\underline{x})) \cap L_2(\mathbb{R}^m, dV(\underline{x}))$ *satisfies:*

$$\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger = A(|\underline{\xi}|) + B(|\underline{\xi}|)\ \frac{\underline{\xi}}{|\underline{\xi}|}$$

with A *and* B *radial symmetric functions for which the integrals*

$$c_1 = \frac{1}{A_m} \int_{\mathbb{R}^m} A(\underline{u})\ \frac{dV(\underline{u})}{|\underline{u}|^m} \quad \text{and} \quad c_2 = \frac{1}{A_m} \int_{\mathbb{R}^m} B(\underline{u})\ \frac{dV(\underline{u})}{|\underline{u}|^m}$$

converge, then the associated HCCWTs T_ψ^+ *and* T_ψ^- *are isometries from respectively* $H(\mathbb{R}^m)$ *and* $H(\mathbb{R}^m)^\perp$ *into respectively* $L_2(\mathbb{R}_+ \times \mathbb{R}^m, \frac{1}{C^+}\ a^{-(m+1)}da\ dV(\underline{b}))$ *and* $L_2(\mathbb{R}_+ \times \mathbb{R}^m, \frac{1}{C^-}\ a^{-(m+1)}da\ dV(\underline{b}))$*. The constants* $C^{\pm}$ *appearing in the weight functions are defined by* $C^{\pm} = (2\pi)^m\ (c_1 \mp ic_2)$*.*

Again, we can interprete the convergence of the integrals c_1 and c_2 as the analogue of the admissibility condition (4.7) of the general Spin(m)-invariant Clifford wavelet theory of Section 4.3.

Remark 8.2 For the sake of completeness, we mention the inversion formulae for the HCCWTs:

$$g^{\pm}(\underline{x}) = \frac{1}{C^{\pm}} \int_{\mathbb{R}^m} \int_0^{+\infty} \psi^{a,\underline{b}}(\underline{x}) \; G_{\psi}^{\pm}(a,\underline{b}) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \quad .$$

Finally, let us mention that the HCCWTs may also be combined into a *complete CCWT*:

$$\begin{aligned} T_{\psi}[f](a,\underline{b}) = F_{\psi}(a,\underline{b}) = < \psi^{a,\underline{b}}, f > &= < \psi^{a,\underline{b}}, f^{+} > + < \psi^{a,\underline{b}}, f^{-} > \\ &= F_{\psi}^{+}(a,\underline{b}) + F_{\psi}^{-}(a,\underline{b}) \end{aligned}$$

with $f^{\pm} = \mathbb{P}^{\pm}[f]$, $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$.

Chapter 9

Clifford-Jacobi polynomials and associated "Half" CCWT

Up to now, we have introduced wavelets based on a Clifford generalization of the classical Hermite, Gegenbauer and Laguerre polynomials. In this chapter, we consider the problem of constructing Clifford algebra-valued wavelet functions starting from a Clifford generalization of the remaining class of classical orthogonal polynomials, namely the Jacobi polynomials. The resulting Clifford-Jacobi wavelets (see [28]) are an application of the "Half" CCWT theory described in the previous chapter.

9.1 Introduction

On the real line the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1,\ n = 0, 1, 2, \ldots)$ may be defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{n!\ 2^n} \frac{1}{(1-x)^\alpha (1+x)^\beta} \frac{d^n}{dx^n} \left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \quad .$$

They constitute an orthogonal basis for the Hilbert space $L_2(\,]-1,+1[,$ $(1-x)^\alpha(1+x)^\beta\ dx)$ and satisfy the orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ P_m^{(\alpha,\beta)}(x)\ (1-x)^\alpha(1+x)^\beta\ dx$$
$$= \frac{\Gamma(\alpha+n+1)\ \Gamma(\beta+n+1)}{n!\ \Gamma(\alpha+\beta+n+1)} \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1}\ \delta_{n,m} \quad . \qquad (9.1)$$

Furthermore, they satisfy the recurrence relation ($n = 2, 3, 4, \ldots$)

$$2n(\alpha+\beta+n)(\alpha+\beta+2n-2)\ P_n^{(\alpha,\beta)}(x)$$
$$= [(\alpha+\beta+2n-2)_3\ x + (\alpha^2-\beta^2)(\alpha+\beta+2n-1)]\ P_{n-1}^{(\alpha,\beta)}(x)$$
$$-\ 2(\alpha+n-1)(\beta+n-1)(\alpha+\beta+2n)\ P_{n-2}^{(\alpha,\beta)}(x) \quad , \qquad (9.2)$$

with $(\alpha)_n$ Pochhammer's symbol defined by:

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad .$$

Moreover, one also has

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n\ P_n^{(\beta,\alpha)}(x) \quad .$$

The Gegenbauer polynomials ($\alpha = \beta = \lambda - 1/2$), the Chebyshev polynomials ($\alpha = \beta = \pm 1/2$) and the Legendre polynomials ($\alpha = \beta = 0$) are special cases of the Jacobi polynomials.

As a Clifford generalization of the traditional Jacobi weight, we take the Clifford algebra-valued function $F(\underline{x}) = (1+\underline{x})^\alpha(1-\underline{x})^\beta$ with $\alpha, \beta \in \mathbb{R}$. The Cauchy-Kowalewskaia extension of this weight function generates the so-called general Clifford-Jacobi polynomials (see Section 9.2). For these polynomials a recurrence relation and a Rodrigues formula are established, but no orthogonality relation could be derived. However the special case $\alpha = \beta + 1$ does yield orthogonal polynomials, which are called the special Clifford-Jacobi polynomials (see Section 9.3.1). These polynomials are the appropriate building blocks for the so-called Clifford-Jacobi wavelets (see Section 9.3.2). However, these wavelets do not satisfy the general mother wavelet conditions established in Chapter 4. In order to use them as kernel functions for a multi-dimensional Clifford CWT, we must invoke the "Half" CCWT theory established in Section 8.2 of the previous chapter. Furthermore, similar to the Clifford-Hermite, Clifford-Gegenbauer and Clifford-Laguerre case, we can generalize the so-called "Half" Clifford-Jacobi CWT by including a left solid inner spherical monogenic in the Clifford-Jacobi weight function. In this way we obtain the so-called "Half" generalized Clifford-Jacobi CWT, which yields an application of the Clifford-wavelet theory developed in Section 8.1.

9.2 The General Clifford-Jacobi polynomials

As a generalization to Clifford analysis of the classical Jacobi weight function, we take the Clifford algebra-valued function $F(\underline{x}) = (1+\underline{x})^\alpha(1-\underline{x})^\beta$ with $\alpha, \beta \in \mathbb{R}$. Keeping in mind the properties (2.8) and (2.9) of the Clifford-Heaviside functions $P^\pm$, we define the function $(1+\underline{x})^\alpha$ as follows:

$$(1+\underline{x})^\alpha := (1-ir)^\alpha \ P^+ + (1+ir)^\alpha \ P^-$$

with $r = |\underline{x}|$. Note that both terms $(1 \pm ir)^\alpha$ are well-defined for each $r \in [0, +\infty[$. Indeed, the complex-valued function $(1+z)^\alpha$ $(\alpha \in \mathbb{R})$ is defined in the whole complex plane except for a branch cut, which can be chosen along the negative real axis from -1 to $-\infty$.
Similarly, we define

$$(1-\underline{x})^\beta := (1+ir)^\beta \ P^+ + (1-ir)^\beta \ P^- \quad .$$

This method for defining the factors $(1+\underline{x})^\alpha$ and $(1-\underline{x})^\beta$ was also used in [81] and [98].
Hence, the weight function may also be written as

$$F(\underline{x}) = (1-ir)^\alpha(1+ir)^\beta \ P^+ + (1+ir)^\alpha(1-ir)^\beta \ P^- \quad .$$

Note that the second term is the complex conjugate of the first one, i.e.

$$\left((1-ir)^\alpha(1+ir)^\beta \ P^+\right)^c = (1+ir)^\alpha(1-ir)^\beta \ P^- \quad ,$$

in accordance with the fact that $F(\underline{x})$ takes its values in the real Clifford algebra $\mathbb{R}_{0,m}$.

The *general Clifford-Jacobi polynomials*, denoted by $J^+_{\ell,\alpha,\beta}(\underline{x})$ and $J^-_{\ell,\alpha,\beta}(\underline{x})$, are generated by the CK-extension $F^*(x_0, \underline{x})$ of the weight function $F(\underline{x})$:

$$\begin{aligned}
&F^*(x_0, \underline{x}) \\
&= \sum_{\ell=0}^{\infty} (-1)^\ell \, \frac{x_0^\ell}{\ell!} \, \partial_{\underline{x}}^\ell F(\underline{x}) \\
&= \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \, (1-ir)^{\alpha-\ell}(1+ir)^{\beta-\ell} r^{-\ell} \left(J^+_{\ell,\alpha,\beta}(\underline{x}) \, P^+ + J^-_{\ell,\alpha,\beta}(\underline{x}) \, P^- \right) \\
&+ \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \, (1+ir)^{\alpha-\ell}(1-ir)^{\beta-\ell} r^{-\ell} \left(\left(J^-_{\ell,\alpha,\beta}(\underline{x})\right)^c \, P^+ + \left(J^+_{\ell,\alpha,\beta}(\underline{x})\right)^c \, P^- \right) \\
&= \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \, (1+\underline{x})^{\alpha-\ell}(1-\underline{x})^{\beta-\ell} r^{-\ell} \left(J^+_{\ell,\alpha,\beta}(\underline{x}) \, P^+ + \left(J^+_{\ell,\alpha,\beta}(\underline{x})\right)^c \, P^- \right) \\
&+ \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \, (1-\underline{x})^{\alpha-\ell}(1+\underline{x})^{\beta-\ell} r^{-\ell} \left(J^-_{\ell,\alpha,\beta}(\underline{x}) \, P^- + \left(J^-_{\ell,\alpha,\beta}(\underline{x})\right)^c \, P^+ \right) \quad .
\end{aligned}$$

By definition, we have

$$F^*(0, \underline{x}) = (1+\underline{x})^\alpha (1-\underline{x})^\beta \quad ,$$

which yields

$$J^+_{0,\alpha,\beta}(\underline{x}) = 1 \quad \text{and} \quad J^-_{0,\alpha,\beta}(\underline{x}) = 0 \quad .$$

Furthermore, the monogenicity of $F^*(x_0, \underline{x})$ leads to the recurrence relation:

$$\begin{aligned}
&J^+_{\ell+1,\alpha,\beta}(\underline{x}) \, P^+ + J^-_{\ell+1,\alpha,\beta}(\underline{x}) \, P^- \\
&\quad = i\big((\alpha-\ell)(1+ir) - (\beta-\ell)(1-ir)\big) \, \underline{x} \, \left(J^+_{\ell,\alpha,\beta}(\underline{x}) \, P^+ + J^-_{\ell,\alpha,\beta}(\underline{x}) \, P^-\right) \\
&\qquad + \; i\ell \, (1+|\underline{x}|^2) \, \left(-J^+_{\ell,\alpha,\beta}(\underline{x}) \, P^+ + J^-_{\ell,\alpha,\beta}(\underline{x}) \, P^-\right) \\
&\qquad\qquad - \; (1+|\underline{x}|^2) \, r \, \partial_{\underline{x}}\left[J^+_{\ell,\alpha,\beta}(\underline{x}) \, P^+ + J^-_{\ell,\alpha,\beta}(\underline{x}) \, P^-\right] \quad ,
\end{aligned}$$

from which the general Clifford-Jacobi polynomials can be computed recursively:

$$\begin{aligned}
J^+_{1,\alpha,\beta}(\underline{x}) &= \left(-i(\alpha+\beta) + \frac{i}{2}(1-m)\right) \, \underline{x}^2 + i(\alpha-\beta) \, \underline{x} - \frac{i}{2}(1-m) \\
J^-_{1,\alpha,\beta}(\underline{x}) &= \frac{i}{2}(1-m) \, \underline{x}^2 - \frac{i}{2}(1-m)
\end{aligned}$$

$$\begin{aligned}
J^+_{2,\alpha,\beta}(\underline{x}) &= \left(-\alpha(\alpha-1)-\beta(\beta-1)-2\alpha\beta+(1-m)(\alpha+\beta-\frac{1}{2})\right)\ \underline{x}^4 \\
&\quad+\ \big(2\alpha(\alpha-1)-2\beta(\beta-1)+(1-m)(\beta-\alpha)\big)\ \underline{x}^3 \\
&\quad+\big(-\alpha(\alpha-1)-\beta(\beta-1)+2\alpha\beta-(1-m)(\alpha+\beta-1)\big)\ \underline{x}^2 \\
&\quad+(1-m)(\alpha-\beta)\ \underline{x}-\frac{1}{2}(1-m) \\
J^-_{2,\alpha,\beta}(\underline{x}) &= \frac{(1-m)}{2}\ \underline{x}^4+(m-1)\ \underline{x}^2+\frac{1}{2}(1-m)
\end{aligned}$$

etc.

Note that $J^{\pm}_{\ell,\alpha,\beta}(\underline{x})$ is a polynomial of degree 2ℓ in the variable $\underline{x}$, which is also seen by the recurrence relation.
Moreover, the following Rodrigues formula holds:

$$\begin{aligned}
&\partial^{\ell}_{\underline{x}}\big[(1+\underline{x})^{\alpha}(1-\underline{x})^{\beta}\big] \\
&\quad= (-1)^{\ell}\ r^{-\ell}\ \Bigg((1-ir)^{\alpha-\ell}(1+ir)^{\beta-\ell}\ \big(J^+_{\ell,\alpha,\beta}(\underline{x})P^+ + J^-_{\ell,\alpha,\beta}(\underline{x})\ P^-\big) \\
&\qquad\qquad +\ (1+ir)^{\alpha-\ell}(1-ir)^{\beta-\ell}\ \big(J^+_{\ell,\alpha,\beta}(\underline{x})\ P^+ + J^-_{\ell,\alpha,\beta}(\underline{x})\ P^-\big)^c\Bigg)\ .
\end{aligned}$$

Remark 9.1 Combining the above Rodrigues formula with the Clifford-Stokes theorem does not yield an orthogonality relation for the general Clifford-Jacobi polynomials w.r.t. the weight function $F(\underline{x})$. However, it is possible to construct orthogonal polynomials in the special case where $\alpha=\beta+1$ (see next section).

9.3 The Special Clifford-Jacobi polynomials and associated HCCWT

9.3.1 The Special Clifford-Jacobi polynomials

In this section we consider the special case where $\alpha=\beta+1$ ($\beta\in\mathbb{R}$), i.e. we consider the Clifford algebra-valued weight function:

$$F(\underline{x})=(1+\underline{x})^{\beta+1}\ (1-\underline{x})^{\beta}=(1+\underline{x})\ (1+|\underline{x}|^2)^{\beta}\ .$$

Natural powers of the Dirac operator acting on this weight function can be written as

$$\partial^{\ell}_{\underline{x}}\big[(1+|\underline{x}|^2)^{\beta}\ (1+\underline{x})\big]=(1+|\underline{x}|^2)^{\beta-\ell}\ (1+\underline{x})\ J_{\ell,\beta}(\underline{x})$$

with $J_{\ell,\beta}(\underline{x})$ a polynomial of degree ℓ in the vector variable $\underline{x}$. The CK-extension of $F(\underline{x})$ thus takes the form

$$F^*(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \; (1 + |\underline{x}|^2)^{\beta-\ell} \; (1 + \underline{x}) \; J_{\ell,\beta}(\underline{x}) \quad .$$

The polynomials $J_{\ell,\beta}(\underline{x})$ generated by the above CK-extension are called the *special Clifford-Jacobi polynomials.*

Remark 9.2 Note that if we take more generally $\alpha = \beta + n$ $(n \in \mathbb{N} \setminus \{1\})$, i.e. we consider the weight function

$$G(\underline{x}) = (1 + \underline{x})^n \; (1 + |\underline{x}|^2)^\beta \quad , \quad n = 2, 3, 4, \ldots \quad ,$$

then we can also write

$$\partial_{\underline{x}}^\ell \big[(1 + |\underline{x}|^2)^\beta \; (1 + \underline{x})^n \big] = (1 + |\underline{x}|^2)^{\beta-\ell} \; (1 + \underline{x}) \; J_{\ell,\beta,n}(\underline{x})$$

with $J_{\ell,\beta,n}(\underline{x})$ a polynomial of degree $n + \ell - 1$ in $\underline{x}$.
However, in order to obtain, by means of the CK-extension technique, orthogonal polynomials w.r.t. the weight function $G(\underline{x})$, we should have had a relation of the form

$$\partial_{\underline{x}}^\ell \big[(1 + |\underline{x}|^2)^\beta \; (1 + \underline{x})^n \big] = (1 + |\underline{x}|^2)^{\beta-\ell} \; (1 + \underline{x})^n \; J^*_{\ell,\beta,n}(\underline{x})$$

with $J^*_{\ell,\beta,n}(\underline{x})$ a polynomial of degree ℓ in $\underline{x}$.
Hence, the case $\alpha = \beta + 1$ is the only case where the CK-extension technique leads to orthogonal polynomials w.r.t. the weight function.

From the monogenicity relation of $F^*(x_0, \underline{x})$ we derive the recurrence relation:

$$J_{\ell+1,\beta}(\underline{x}) = 2(\ell - \beta) \; \underline{x} \; J_{\ell,\beta}(\underline{x}) + (\underline{x} - 1) \; \partial_{\underline{x}} \big[(1 + \underline{x}) \; J_{\ell,\beta}(\underline{x}) \big] \quad .$$

As it follows from

$$F^*(0, \underline{x}) = (1 + \underline{x}) \; (1 + |\underline{x}|^2)^\beta$$

that $J_{0,\beta}(\underline{x}) = 1$, we thus obtain

$$\begin{aligned}
J_{1,\beta}(\underline{x}) &= -(2\beta + m) \; \underline{x} + m \\
J_{2,\beta}(\underline{x}) &= 2\beta(2\beta + m) \; \underline{x}^2 - 4\beta \; \underline{x} - 2\beta m \\
J_{3,\beta}(\underline{x}) &= 2\beta(2\beta + m)(2 - 2\beta - m) \; \underline{x}^3 + 2\beta(-4 + 4\beta + 2\beta m + m^2) \; \underline{x}^2 \\
&\quad +2\beta(2\beta m + 4\beta - 4 + m^2) \; \underline{x} - 2\beta m(m + 2)
\end{aligned}$$

etc.

Note that $J_{\ell,\beta}(\underline{x})$ is a polynomial of degree ℓ in the variable $\underline{x}$, as it should be, with real coefficients depending on the parameter β and on the dimension m.

From the explicit formula (2.4) for the CK-extension, we obtain the Rodrigues formula:

$$\begin{aligned} J_{\ell,\beta}(\underline{x}) &= (-1)^\ell \ (1+|\underline{x}|^2)^{\ell-\beta} \ (1+\underline{x})^{-1} \ \partial_{\underline{x}}^\ell \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^\beta\big] \\ &= (-1)^\ell \ (1-\underline{x})^{\ell-\beta} \ (1+\underline{x})^{\ell-\beta-1} \ \partial_{\underline{x}}^\ell \big[(1+\underline{x})^{\beta+1} \ (1-\underline{x})^\beta\big] \end{aligned} \quad (9.3)$$

which together with the Clifford-Stokes theorem yields the following orthogonality relation.

Theorem 9.1 *Whenever $\ell < t < (-2\beta - m)/2$, one has the orthogonality relation*

$$\int_{\mathbb{R}^m} \overline{J_{\ell,\beta+\ell}}(\underline{x}) \ J_{t,\beta+t}(\underline{x}) \ (1+\underline{x}) \ (1+|\underline{x}|^2)^\beta \ dV(\underline{x}) = 0 \quad .$$

Proof. This proof is similar to the proof of Theorem 6.1; however we include it for the sake of completeness and clarity for the reader.
As $J_{\ell,\beta+\ell}(\underline{x})$ is a polynomial of degree ℓ in $\underline{x}$, it is sufficient to show that for each $\ell < t$ and $2t < -2\beta - m$:

$$\int_{\mathbb{R}^m} \underline{x}^\ell \ J_{t,\beta+t}(\underline{x}) \ (1+\underline{x}) \ (1+|\underline{x}|^2)^\beta \ dV(\underline{x}) = 0 \quad .$$

By means of the Rodrigues formula (9.3) and the Clifford-Stokes theorem we find

$$\begin{aligned} &\int_{\mathbb{R}^m} \underline{x}^\ell \ J_{t,\beta+t}(\underline{x}) \ (1+\underline{x}) \ (1+|\underline{x}|^2)^\beta \ dV(\underline{x}) \\ &\quad = (-1)^t \int_{\mathbb{R}^m} \underline{x}^\ell \ \partial_{\underline{x}}^t \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) \\ &\quad = (-1)^t \left(\int_{\partial\mathbb{R}^m} \underline{x}^\ell \ d\underline{\sigma} \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \right. \\ &\qquad \left. - \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) \right) \quad . \end{aligned}$$

The first term vanishes if the degree of homogeneity of the integrand $\underline{x}^\ell \ d\underline{\sigma} \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big]$ is strictly smaller then zero, i.e. the first term is zero if

$$\ell + (m-1) - (t-1) + 1 + 2(\beta+t) < 0 \quad .$$

The above condition is fulfilled, since by assumption ℓ is at most $t-1$ and $2t < -2\beta - m$.
We thus obtain

$$\int_{\mathbb{R}^m} \underline{x}^\ell \; J_{t,\beta+t}(\underline{x}) \; (1+\underline{x}) \; (1+|\underline{x}|^2)^\beta \; dV(\underline{x}) = (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \; \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta+t}\big] \; dV(\underline{x}) \quad .$$

Applying again the Clifford-Stokes theorem yields

$$\int_{\mathbb{R}^m} \underline{x}^\ell \; J_{t,\beta+t}(\underline{x}) \; (1+\underline{x}) \; (1+|\underline{x}|^2)^\beta \; dV(\underline{x}) = (-1)^{t+1} \Bigg(\int_{\partial\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \; d\underline{\sigma} \; \partial_{\underline{x}}^{t-2} \big[(1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta+t}\big] - \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}^2) \; \partial_{\underline{x}}^{t-2} \big[(1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta+t}\big] \; dV(\underline{x}) \Bigg) \quad .$$

It is easily seen that the integral over $\partial\mathbb{R}^m$ is again zero by means of the assumptions.
Successively repeating this argument leads to

$$\int_{\mathbb{R}^m} \underline{x}^\ell \; J_{t,\beta+t}(\underline{x}) \; (1+\underline{x}) \; (1+|\underline{x}|^2)^\beta \; dV(\underline{x}) = (-1)^{t+\ell+1} \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}^{\ell+1}) \; \partial_{\underline{x}}^{t-(\ell+1)} \big[(1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta+t}\big] \; dV(\underline{x}) = 0 \quad . \square$$

9.3.2 The Clifford-Jacobi wavelets

Note that Theorem 9.1 implies that for $0 < t < (-2\beta - m)/2$:

$$\int_{\mathbb{R}^m} J_{t,\beta+t}(\underline{x}) \; (1+\underline{x}) \; (1+|\underline{x}|^2)^\beta \; dV(\underline{x}) = 0 \quad .$$

Consequently, the $L_1 \cap L_2$-functions

$$\psi_{t,\beta}(\underline{x}) = J_{t,\beta+t}(\underline{x}) \; (1+\underline{x}) \; (1+|\underline{x}|^2)^\beta = (-1)^t \; \partial_{\underline{x}}^t \big[(1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta+t}\big]$$

have zero momentum. In Section 9.3.3 we will show that they can be used as mother wavelets; we call them the *Clifford-Jacobi wavelets*. Note that the condition $0 < t < (-2\beta - m)/2$ forces us to make the restriction $\beta < -m/2$.

It is easily seen that these Clifford-Jacobi wavelets are Spin(m)-invariant. Moreover, the wavelets $\psi_{t,\beta}(\underline{x})$ have vanishing moments if the condition $2\beta < -m-t-1$ is fulfilled, as is shown in the next proposition.

Proposition 9.1 *If $2\beta < -m-t-1$, the Clifford-Jacobi wavelet $\psi_{t,\beta}(\underline{x})$ has vanishing moments:*

$$\int_{\mathbb{R}^m} \underline{x}^j \ \psi_{t,\beta}(\underline{x}) \ dV(\underline{x}) = 0$$

for $0 \leq j < -m-t-2\beta-1$ and $j < t$.

Proof. This property follows once more from repeated applications of the Clifford-Stokes theorem:

$$\begin{aligned}\int_{\mathbb{R}^m} \underline{x}^j \ \psi_{t,\beta}(\underline{x}) \ dV(\underline{x}) &= (-1)^t \int_{\mathbb{R}^m} \underline{x}^j \ \partial_{\underline{x}}^t \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) \\ &= (-1)^t \Bigg(\int_{\partial\mathbb{R}^m} \underline{x}^j \ d\underline{\sigma} \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \\ &\quad - \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) \Bigg) \ .\end{aligned}$$

The integral over $\partial\mathbb{R}^m$ is zero by means of the assumption $j < -m-t-2\beta-1$, which implies that $2\beta < -m-t-1$.
Hence, we have

$$\int_{\mathbb{R}^m} \underline{x}^j \ \psi_{t,\beta}(\underline{x}) \ dV(\underline{x}) = (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) \ .$$

Repeating this argument yields

$$\begin{aligned}&\int_{\mathbb{R}^m} \underline{x}^j \ \psi_{t,\beta}(\underline{x}) \ dV(\underline{x}) \\ &= (-1)^{t+j+1} \int_{\mathbb{R}^m} (\underline{x}^j \partial_{\underline{x}}^{j+1}) \ \partial_{\underline{x}}^{t-(j+1)} \big[(1+\underline{x}) \ (1+|\underline{x}|^2)^{\beta+t}\big] \ dV(\underline{x}) = 0 \ . \ \square\end{aligned}$$

We now determine the Clifford-Jacobi wavelets in the frequency domain.

Lemma 9.1 *The Fourier transform of the Clifford-Jacobi wavelets takes the form:*

$$\begin{aligned}&\mathcal{F}[\psi_{t,\beta}](\underline{\xi}) \\ &= (-i)^t \ \frac{2^{\beta+t+1}}{\Gamma(-\beta-t)} \ |\underline{\xi}|^{-m/2-\beta-t-1} \ \underline{\xi}^t \Big(|\underline{\xi}| \ K_{m/2+\beta+t}(|\underline{\xi}|) - i K_{m/2+\beta+t+1}(|\underline{\xi}|) \ \underline{\xi} \Big)\end{aligned}$$

with $K_\nu(t)$ the modified Bessel function of the second kind.

Proof. By means of Proposition 2.1 we obtain

$$\mathcal{F}[\psi_{t,\beta}](\underline{\xi}) = (-i)^t \; \underline{\xi}^t \; \left(\mathcal{F}\big[(1+|\underline{x}|^2)^{\beta+t}\big](\underline{\xi}) + \mathcal{F}\big[\underline{x} \; (1+|\underline{x}|^2)^{\beta+t}\big](\underline{\xi}) \right) \quad .$$

The first Fourier transform in the above expression was calculated in the proof of Lemma 6.1 (see expression (6.9)). In a similar way, we now calculate the second one.
First, introducing spherical co-ordinates

$$\underline{x} = r\underline{\omega} \quad , \quad \underline{\xi} = \rho\underline{\eta} \quad ; \quad r = |\underline{x}| \quad , \quad \rho = |\underline{\xi}| \quad , \quad \underline{\omega}, \underline{\eta} \in S^{m-1} \quad ,$$

and applying Proposition 2.2 on the solid spherical harmonic $S_1(\underline{x}) = \underline{x}$, yields consecutively:

$$\begin{aligned} &\mathcal{F}\big[\underline{x} \; (1+|\underline{x}|^2)^{\beta+t}\big](\underline{\xi}) \\ &\quad = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \; \underline{x} \; (1+|\underline{x}|^2)^{\beta+t} \; dV(\underline{x}) \\ &\quad = \frac{1}{(2\pi)^{m/2}} \int_0^{+\infty} (1+r^2)^{\beta+t} \; r^m \; dr \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) \; \underline{\omega} \; dS(\underline{\omega}) \\ &\quad = (-i) \; \rho^{(1-m)/2} \; \underline{\eta} \int_0^{+\infty} (1+r^2)^{\beta+t} \; r^{(m+1)/2} \; J_{m/2}(\rho r) \; \rho^{1/2} \; r^{1/2} \; dr. \quad (9.4) \end{aligned}$$

Next, with $\nu = m/2 \in \mathbb{R}$ and $\mu = -\beta - t - 1 \in \mathbb{R}$, formula (6.7) becomes:

$$\begin{aligned} &\int_0^{+\infty} r^{(m+1)/2} \; (1+r^2)^{\beta+t} \; J_{m/2}(\rho r) \; \rho^{1/2} \; r^{1/2} \; dr \\ &\qquad\qquad = \frac{\rho^{-\beta-t-1/2}}{2^{-\beta-t-1} \; \Gamma(-\beta-t)} \; K_{m/2+\beta+t+1}(\rho) \end{aligned}$$

provided the condition

$$-1 < \frac{m}{2} < 2(-\beta-t-1) + \frac{3}{2} = -2\beta - 2t - \frac{1}{2}$$

is fulfilled.
Naturally we have that $-1 < m/2$ holds. Moreover, the assumption $0 < t < (-2\beta - m)/2$ made in the beginning of this section implies:

$$m < -2\beta - 2t \quad \text{or} \quad 0 < \frac{m}{2} < -2\beta - 2t - \frac{m}{2} \quad .$$

As $1/2 < m/2$, we thus indeed have

$$\frac{m}{2} < -2\beta - 2t - \frac{m}{2} < -2\beta - 2t - \frac{1}{2} \quad .$$

Consequently, expression (9.4) takes the form

$$\mathcal{F}\big[\underline{x}\ (1+|\underline{x}|^2)^{\beta+t}\big](\underline{\xi}) = (-i)\ \frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\ \rho^{-\beta-t-m/2}\ K_{m/2+\beta+t+1}(\rho)\ \underline{\eta} \quad ,$$

which finally leads to the desired expression for the Fourier transform of the Clifford-Jacobi wavelets. □

9.3.3 The "Half" Clifford-Jacobi CWT

In view of Lemma 9.1, we see that

$$\begin{aligned}
&\mathcal{F}[\psi_{t,\beta}](\underline{\xi})\ \big(\mathcal{F}[\psi_{t,\beta}](\underline{\xi})\big)^\dagger \\
&= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 |\underline{\xi}|^{-m-2\beta} \left(\big(K_{m/2+\beta+t}(|\underline{\xi})\big)^2 + \big(K_{m/2+\beta+t+1}(|\underline{\xi}|)\big)^2\right) \\
&\quad -\ 2i\ \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 |\underline{\xi}|^{-m-2\beta}\ K_{m/2+\beta+t+1}(|\underline{\xi}|)\ K_{m/2+\beta+t}(|\underline{\xi}|)\ \frac{\underline{\xi}}{|\underline{\xi}|} \quad .
\end{aligned}$$

Hence, the Clifford-Jacobi wavelets belong to the category of Spin(m)-invariant candidate mother wavelets ψ which satisfy in frequency space the condition

$$\mathcal{F}[\psi](\underline{\xi})\ \big(\mathcal{F}[\psi](\underline{\xi})\big)^\dagger = A(|\underline{\xi}|) + B(|\underline{\xi}|)\ \frac{\underline{\xi}}{|\underline{\xi}|}$$

with A and B radial symmetric functions.
Hence, as outlined in Section 8.2 (see Conclusion 8.2), we must now verify whether the integrals

$$\begin{aligned}
c_1 &= \frac{1}{A_m}\left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 \int_{\mathbb{R}^m} |\underline{u}|^{-2m-2\beta}\Big(\big(K_{m/2+\beta+t}(|\underline{u}|)\big)^2 \\
&\qquad\qquad + \big(K_{m/2+\beta+t+1}(|\underline{u}|)\big)^2\Big)\ dV(\underline{u}) \\
&= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 \int_0^{+\infty} r^{-m-2\beta-1}\ \Big(\big(K_{m/2+\beta+t}(r)\big)^2 \\
&\qquad\qquad + \big(K_{m/2+\beta+t+1}(r)\big)^2\Big)\ dr
\end{aligned}$$

and

$$
\begin{aligned}
c_2 &= -\frac{2i}{A_m}\left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 \int_{\mathbb{R}^m} |\underline{u}|^{-2m-2\beta}\ K_{m/2+\beta+t+1}(|\underline{u}|) \\
&\qquad\qquad K_{m/2+\beta+t}(|\underline{u}|)\ dV(\underline{u}) \\
&= -2i\left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 \int_0^{+\infty} r^{-m-2\beta-1}\ K_{m/2+\beta+t+1}(r)\ K_{m/2+\beta+t}(r)\ dr
\end{aligned}
$$

converge. This is indeed the case in view of the limiting behaviour of the modified Bessel functions of the second kind (see Section 6.2.3) and since moreover we consider $0 < t < (-2\beta - m)/2$.
Consequently, the so-called *"Half" Clifford-Jacobi CWT* $T^+_{t,\beta}$, respectively $T^-_{t,\beta}$, which applies to functions $f^+ \in H^2(\mathbb{R}^m)$, respectively $f^- \in H^2(\mathbb{R}^m)^\perp$, by

$$T^\pm_{t,\beta}[f^\pm](a,\underline{b}) = F^\pm_{t,\beta}(a,\underline{b}) = \int_{\mathbb{R}^m} \overline{\psi^{a,\underline{b}}_{t,\beta}}(\underline{x})\ f^\pm(\underline{x})\ dV(\underline{x}) \quad ,$$

maps the Hardy space $H(\mathbb{R}^m)$, respectively $H(\mathbb{R}^m)^\perp$, isometrically into the weighted L_2-space $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m, \frac{1}{C^+}\ a^{-(m+1)}\ da\ dV(\underline{b})\big)$, respectively $L_2\big(\mathbb{R}_+ \times \mathbb{R}^m, \frac{1}{C^-}\ a^{-(m+1)}\ da\ dV(\underline{b})\big)$. The constants $C^\pm$ are defined by $C^\pm = (2\pi)^m$ $(c_1 \mp ic_2)$.

Remark 9.3 Note that the Clifford-Jacobi wavelets $\psi_{t,\beta}(\underline{x})$ can be written in terms of the Clifford-Gegenbauer wavelets (see Section 6.2.2) which we now, for the sake of clarity, denote by $\psi^G_{t,\alpha}$:

$$\psi_{t,\beta}(\underline{x}) = \psi^G_{t,\beta}(\underline{x}) + (-1)^t\ \partial^t_{\underline{x}}\big[\underline{x}\ (1 + |\underline{x}|^2)^{\beta+t}\big] \quad .$$

For $0 < t < (-2\beta - m)/2$ the Spin(m)-invariant $L_1 \cap L_2$-functions

$$\Phi_{t,\beta}(\underline{x}) = (-1)^t \partial^t_{\underline{x}}\big[\underline{x}\ (1 + |\underline{x}|^2)^{\beta+t}\big]$$

can also be used as mother wavelets.
Indeed, as their Fourier transform takes the following form:

$$\mathcal{F}[\Phi_{t,\beta}](\underline{\xi}) = (-i)^{t+1} \frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\ |\underline{\xi}|^{-m/2-\beta-t-1}\ K_{m/2+\beta+t+1}(|\underline{\xi}|)\ \underline{\xi}^{t+1} \quad ,$$

the expression

$$\mathcal{F}[\Phi_{t,\beta}](\underline{\xi})\ \big(\mathcal{F}[\Phi_{t,\beta}](\underline{\xi})\big)^\dagger = \frac{2^{2\beta+2t+2}}{\big(\Gamma(-\beta-t)\big)^2}\ |\underline{\xi}|^{-m-2\beta}\ \big(K_{m/2+\beta+t+1}(|\underline{\xi}|)\big)^2$$

is radial symmetric.
Moreover, introducing spherical co-ordinates, we obtain

$$\begin{aligned} C^*_{t,\beta} &= \frac{(2\pi)^m}{A_m} \int_{\mathbb{R}^m} \frac{\mathcal{F}[\Phi_{t,\beta}](\underline{\xi}) \left(\mathcal{F}[\Phi_{t,\beta}](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m} \, dV(\underline{\xi}) \\ &= (2\pi)^m \frac{2^{2\beta+2t+2}}{\left(\Gamma(-\beta-t)\right)^2} \int_0^{+\infty} \rho^{-m-2\beta-1} \left(K_{m/2+\beta+t+1}(\rho)\right)^2 \, d\rho \quad . \end{aligned}$$

Under the assumption $0 < t < (-2\beta - m)/2$, the above admissibility constant is finite.
From Section 4.3 it thus follows that the Clifford CWT defined by

$$T^*_{t,\beta}[f](a, \underline{b}) = < \Phi^{a,\underline{b}}_{t,\beta}, f >$$

yields an isometry between the L_2-space $L_2(\mathbb{R}^m, dV(\underline{x}))$ and $L_2\Bigg(\mathbb{R}_+ \times \mathbb{R}^m,$ $\dfrac{1}{C^*_{t,\beta}} \; a^{-(m+1)} \; da \; dV(\underline{b})\Bigg)$.

9.4 The Generalized Special Clifford-Jacobi polynomials and associated HGCCWT

9.4.1 The Generalized Special Clifford-Jacobi polynomials

The CK-extension technique also works for the more general weight function

$$G(\underline{x}) = (1+\underline{x})^{\beta+1}(1-\underline{x})^{\beta} \; P_k(\underline{x}) = (1+\underline{x}) \; (1+|\underline{x}|^2)^{\beta} \; P_k(\underline{x}) \quad , \quad \beta \in \mathbb{R}$$

with $P_k(\underline{x})$ an arbitrary but fixed left solid inner spherical monogenic of order k. In this way, we obtain the so-called *generalized special Clifford-Jacobi polynomials* $J_{\ell,\beta,k}(\underline{x})$ $(\ell = 0, 1, 2, \ldots)$.
Indeed, natural powers of the Dirac operator acting on the weight $G(\underline{x})$ can be written as

$$\partial^\ell_{\underline{x}} \left[(1+|\underline{x}|^2)^{\beta} \; (1+\underline{x}) \; P_k(\underline{x}) \right] = (1+|\underline{x}|^2)^{\beta-\ell} \; (1+\underline{x}) \; J_{\ell,\beta,k}(\underline{x}) \; P_k(\underline{x})$$

with $J_{\ell,\beta,k}(\underline{x})$ a polynomial of degree ℓ in the vector variable $\underline{x}$. The CK-extension G^* of the weight function G thus takes the form

$$G^*(x_0, \underline{x}) = \sum_{\ell=0}^{\infty} \frac{x_0^\ell}{\ell!} \; (1+|\underline{x}|^2)^{\beta-\ell} \; (1+\underline{x}) \; J_{\ell,\beta,k}(\underline{x}) \; P_k(\underline{x}) \quad .$$

The monogenicity of $G^*(x_0, \underline{x})$ leads to the recurrence relation:

$$J_{\ell+1,\beta,k}(\underline{x})\ P_k(\underline{x}) = 2(\ell-\beta)\ \underline{x}\ J_{\ell,\beta,k}(\underline{x})\ P_k(\underline{x}) + \ (\underline{x}-1)\ \partial_{\underline{x}}\Big[J_{\ell,\beta,k}(\underline{x})\ (1+\underline{x})\ P_k(\underline{x})\Big] \quad . \tag{9.5}$$

From

$$G^*(0,\underline{x}) = (1+\underline{x})\ (1+|\underline{x}|^2)^\beta\ P_k(\underline{x}) \quad ,$$

we obtain

$$J_{0,\beta,k}(\underline{x}) = 1 \quad .$$

Hence, by means of the recurrence relation (9.5), we can compute some lower degree polynomials

$$\begin{array}{rcl} J_{1,\beta,k}(\underline{x}) & = & -(2k+m+2\beta)\ \underline{x} + 2k + m \\ J_{2,\beta,k}(\underline{x}) & = & 2\beta(2k+m+2\beta)\ \underline{x}^2 - 4\beta\ \underline{x} - 2\beta(2k+m) \\ \text{etc.} & & \end{array}$$

As expected, putting $k = 0$ yields the special Clifford-Jacobi polynomials of Section 9.3.1.

Combining the Clifford-Stokes theorem with the Rodrigues formula:

$$\begin{aligned} J_{\ell,\beta,k}(\underline{x})\ P_k(\underline{x}) & \\ = & \ (-1)^\ell\ (1+|\underline{x}|^2)^{\ell-\beta}\ (1+\underline{x})^{-1}\ \partial_{\underline{x}}^\ell\Big[(1+|\underline{x}|^2)^\beta\ (1+\underline{x})\ P_k(\underline{x})\Big] \\ = & \ (-1)^\ell\ (1-\underline{x})^{\ell-\beta}\ (1+\underline{x})^{\ell-\beta-1}\ \partial_{\underline{x}}^\ell\Big[(1+\underline{x})^{\beta+1}\ (1-\underline{x})^\beta\ P_k(\underline{x})\Big] \quad , \end{aligned}$$

enables us to prove the following orthogonality relation.

Theorem 9.2 *Whenever $\ell < t < (-2\beta - m - k_1 - k_2)/2$, one has the orthogonality relation*

$$\int_{\mathbb{R}^m} \big(P_{k_1}(\underline{x})\big)^\dagger\ \overline{J_{\ell,\beta+\ell,k_1}}(\underline{x})\ (1+|\underline{x}|^2)^\beta\ (1+\underline{x})\ J_{t,\beta+t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ dV(\underline{x}) = 0 \quad .$$

Proof. The proof runs along the same lines as the one of Theorem 6.2; we therefore omit it. □

9.4.2 The Generalized Clifford-Jacobi wavelets

By means of Theorem 9.2 we obtain, under the restriction $0 < t < (-2\beta - m - k_2)/2$, the following candidate mother wavelets

$$\begin{aligned}\psi_{t,k_2,\beta}(\underline{x}) &= (1+|\underline{x}|^2)^{\beta}\ (1+\underline{x})\ J_{t,\beta+t,k_2}(\underline{x})\ P_{k_2}(\underline{x}) \\ &= (-1)^t\ \partial_{\underline{x}}^t \Big[(1+|\underline{x}|^2)^{\beta+t}\ (1+\underline{x})\ P_{k_2}(\underline{x})\Big]\ ,\end{aligned}$$

which we call the *generalized Clifford-Jacobi wavelets.* The condition $0 < t < (-2\beta - m - k_2)/2$ implies again that we are forced to make the restriction $\beta < -m/2$.

Naturally, these wavelets are not invariant under the rotation group Spin(m). Hence this group must be taken into account when defining the associated CWT in the next section.

Furthermore, in a similar way as was done in Section 6.3.2 for the generalized Clifford-Gegenbauer wavelets, one can prove that the mother wavelet $\psi_{t,k_2,\beta}(\underline{x})$ has a number of vanishing moments.

Proposition 9.2 *If* $2\beta < -m - t - k_2 - 1$*, the generalized Clifford-Jacobi wavelet* $\psi_{t,k_2,\beta}(\underline{x})$ *has vanishing moments*

$$\int_{\mathbb{R}^m} \underline{x}^j\ \psi_{t,k_2,\beta}(\underline{x})\ dV(\underline{x}) = 0$$

for $0 \leq j < -m - t - 2\beta - k_2 - 1$ *and* $j < t$*.*

In order to verify the mother wavelet conditions, we must calculate the Fourier transform of the generalized Clifford-Jacobi wavelets.

Lemma 9.2 *The Fourier transform of the generalized Clifford-Jacobi wavelets takes the form*

$$\begin{aligned}\mathcal{F}[\psi_{t,k_2,\beta}](\underline{\xi}) = \frac{(-i)^{t+k_2} 2^{\beta+t+1}}{\Gamma(-\beta-t)}\ |\underline{\xi}|^{-m/2-\beta-t-k_2-1}\ \underline{\xi}^t\ \Big(&|\underline{\xi}|\ K_{m/2+\beta+t+k_2}(|\underline{\xi}|) \\ &-\ iK_{m/2+\beta+t+1+k_2}(|\underline{\xi}|)\ \underline{\xi}\Big)\ P_{k_2}(\underline{\xi})\end{aligned}$$

with $K_\nu(t)$ *the modified Bessel function of the second kind.*

Proof. The proof is similar to that of Lemma 9.1.
First, Proposition 2.1 yields

$$\mathcal{F}[\psi_{t,k_2,\beta}](\underline{\xi}) = (-i)^t \ \underline{\xi}^t \ \Big(\mathcal{F}\big[(1+|\underline{x}|^2)^{\beta+t} \ P_{k_2}(\underline{x})\big](\underline{\xi}) + \ \mathcal{F}\big[(1+|\underline{x}|^2)^{\beta+t} \ \underline{x} \ P_{k_2}(\underline{x})\big](\underline{\xi}) \Big) \ . \quad (9.6)$$

Thereupon, we calculate the term $\mathcal{F}\big[(1+|\underline{x}|^2)^{\beta+t} \ \underline{x} \ P_{k_2}(\underline{x})\big](\underline{\xi})$.
Introducing spherical co-ordinates, applying Proposition 2.2 on the solid spherical harmonic $S_{k_2+1}(\underline{x}) = \underline{x}P_{k_2}(\underline{x})$ and using (6.7) with $\nu = k_2 + m/2 \in \mathbb{R}$ and $\mu = -\beta - t - 1 \in \mathbb{R}$ gives consecutively:

$$\begin{aligned}
&\mathcal{F}\big[(1+|\underline{x}|^2)^{\beta+t} \ \underline{x} \ P_{k_2}(\underline{x})\big](\underline{\xi}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \ (1+|\underline{x}|^2)^{\beta+t} \ \underline{x} \ P_{k_2}(\underline{x}) \ dV(\underline{x}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_0^{+\infty} (1+r^2)^{\beta+t} \ r^{k_2+m} \ dr \\
&\qquad \int_{S^{m-1}} \exp\left(-ir\rho < \underline{\omega}, \underline{\eta} >\right) \ \underline{\omega} \ P_{k_2}(\underline{\omega}) \ dS(\underline{\omega}) \\
&= (-i)^{k_2+1} \rho^{(1-m)/2} \ \underline{\eta} \ P_{k_2}(\underline{\eta}) \int_0^{+\infty} (1+r^2)^{\beta+t} \ r^{k_2+1/2+m/2} \\
&\qquad\qquad J_{k_2+m/2}(\rho r) \ \rho^{1/2} \ r^{1/2} \ dr \\
&= (-i)^{k_2+1} \ \frac{2^{\beta+t+1}}{\Gamma(-\beta-t)} \ |\underline{\xi}|^{-\beta-t-m/2-k_2-1} \ K_{k_2+m/2+\beta+t+1}(|\underline{\xi}|) \\
&\qquad \underline{\xi} \ P_{k_2}(\underline{\xi}) \ .
\end{aligned} \quad (9.7)$$

In order to justify the application of formula (6.7) with $\nu = k_2 + m/2$ and $\mu = -\beta - t - 1$, we still have to verify the condition

$$-1 < k_2 + \frac{m}{2} < 2(-\beta - t - 1) + \frac{3}{2} = -2\beta - 2t - \frac{1}{2} \ .$$

Naturally, the restriction $-1 < k_2 + m/2$ is fulfilled, since $k_2 \in \mathbb{N}$ and $m > 1$. Moreover, the assumption $0 < t < (-2\beta - m - k_2)/2$, or equivalently $k_2 + m < -2\beta - 2t$ yields the second inequality:

$$k_2 + \frac{m}{2} < -2\beta - 2t - \frac{m}{2} < -2\beta - 2t - \frac{1}{2} \ .$$

Inserting the expressions (6.16) and (9.7) in formula (9.6) finally yields the desired result. □

9.4.3 The "Half" generalized Clifford-Jacobi CWT

If $P_{k_2}(\underline{x})\ \left(P_{k_2}(\underline{x})\right)^\dagger$ is scalar-valued, then Lemma 9.2 implies that

$$\begin{aligned}
&\mathcal{F}[\psi_{t,k_2,\beta}](\underline{\xi})\ \left(\mathcal{F}[\psi_{t,k_2,\beta}](\underline{\xi})\right)^\dagger \\
&= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\ |\underline{\xi}|^{-2\beta-m-2k_2}\Big(\left(K_{\beta+t+m/2+k_2}(|\underline{\xi}|)\right)^2 \\
&\quad + \left(K_{\beta+t+m/2+1+k_2}(|\underline{\xi}|)\right)^2\Big)\ P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger \\
&\quad -2i\ \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\ |\underline{\xi}|^{-2\beta-m-2k_2}\ K_{\beta+t+m/2+1+k_2}(|\underline{\xi}|)\ K_{\beta+t+m/2+k_2}(|\underline{\xi}|) \\
&\quad P_{k_2}(\underline{\xi})\ \left(P_{k_2}(\underline{\xi})\right)^\dagger\ \frac{\underline{\xi}}{|\underline{\xi}|}\ .
\end{aligned}$$

From the above expression it is clear that, in order to associate to the generalized Clifford-Jacobi wavelets a Clifford CWT, we have to invoke the "Half" generalized CCWT theory exposed in Section 8.1.
Hence, we must consider the following integrals:

$$\begin{aligned}
c_1 &= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\ \int_{\mathbb{R}^m} |\underline{u}|^{-2\beta-2m-2k_2}\Big(\left(K_{\beta+t+m/2+k_2}(|\underline{u}|)\right)^2 \\
&\quad + \left(K_{\beta+t+m/2+1+k_2}(|\underline{u}|)\right)^2\Big)\ P_{k_2}(\underline{u})\ \left(P_{k_2}(\underline{u})\right)^\dagger\ dV(\underline{u}) \\
&= \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\ \int_0^{+\infty} r^{-2\beta-m-1}\ \Big(\left(K_{\beta+t+m/2+k_2}(r)\right)^2 \\
&\quad + \left(K_{\beta+t+m/2+1+k_2}(r)\right)^2\Big)\ dr\ \int_{S^{m-1}} |P_{k_2}(\underline{\omega})|^2\ dS(\underline{\omega})
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= -2i\ \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2\ \int_{\mathbb{R}^m} |\underline{u}|^{-2\beta-2m-2k_2}\ K_{\beta+t+m/2+1+k_2}(|\underline{u}|) \\
&\quad K_{\beta+t+m/2+k_2}(|\underline{u}|)\ P_{k_2}(\underline{u})\ \left(P_{k_2}(\underline{u})\right)^\dagger\ dV(\underline{u})
\end{aligned}$$

$$= \quad -2i\ \left(\frac{2^{\beta+t+1}}{\Gamma(-\beta-t)}\right)^2 \int_0^{+\infty} r^{-2\beta-m-1}\ K_{\beta+t+m/2+1+k_2}(r)$$
$$K_{\beta+t+m/2+k_2}(r)\ dr \quad \int_{S^{m-1}} |P_{k_2}(\underline{\omega})|^2\ dS(\underline{\omega}) \quad .$$

Taking into account the assumption that $0 < t < (-2\beta - m - k_2)/2$, and the earlier mentioned limiting behaviour of the modified Bessel functions of the second kind, we see that c_1 and c_2 are (complex) constants.
Consequently, from Conclusion 8.1 we know that the so-called *"Half" generalized Clifford-Jacobi CWTs* defined by

$$T^{\pm}_{t,k_2,\beta}[f^{\pm}](a,\underline{b},s) = F^{\pm}_{t,k_2,\beta}(a,\underline{b},s) = \int_{\mathbb{R}^m} \left(\psi^{a,\underline{b},s}_{t,k_2,\beta}(\underline{x})\right)^{\dagger}\ f^{\pm}(\underline{x})\ dV(\underline{x})$$

with $f^{\pm} = \mathbb{P}^{\pm}[f]$, $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$, are isometries from the Hardy space $H(\mathbb{R}^m)$, respectively $H(\mathbb{R}^m)^{\perp}$ into the weighted L_2-space $L_2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m)$, $\frac{1}{C^+}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds)$, respectively $L_2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m), \frac{1}{C^-}\ a^{-(m+1)}\ da\ dV(\underline{b})\ ds)$ with $C^{\pm} = (2\pi)^m\ (c_1 \mp ic_2)$.

Chapter 10

Clifford algebra-valued orthogonal polynomials

In this chapter we present a new method, different from the CK-extension technique, for constructing Clifford algebra-valued orthogonal polynomials in Euclidean space or in the open unit ball of Euclidean space. When considering in the preceding chapters Clifford algebra-valued orthogonal polynomials, we mostly dealt with scalar-valued weight functions. Now the class of weight functions involved is enlarged to encompass Clifford algebra-valued functions. The method, developed in [24] and [25], consists in transforming the orthogonality relation on the Euclidean space or on the open unit ball into an orthogonality relation on the real axis by means of the Clifford-Heaviside functions. Consequently, appropriate orthogonal polynomials on the real axis give rise to Clifford algebra-valued orthogonal polynomials in Euclidean space or in the open unit ball of it.

10.1 Introduction

In our quest for new Clifford wavelets we came across a simple but highly efficient method for constructing Clifford algebra-valued orthogonal polynomials. Except for the special Clifford-Jacobi polynomials (see Section 9.3.1), all polynomials constructed in the foregoing chapters satisfy orthogonality relations with respect to *scalar-valued* weight functions. It should be emphasized that the class of weight functions is now enlarged with *Clifford algebra-valued* real-analytic functions. Unfortunately, we were not able yet to construct Clifford-wavelets based on the newly obtained orthogonal polynomials.

In a first section we expose our methodology. It consists, roughly speaking, of transforming the orthogonality relation on the Euclidean space or on the open unit ball into an orthogonality relation on the real axis. Crucial to this transformation are the Clifford-Heaviside functions introduced in Section 2.2. Apparently our construction method is simple, but nevertheless it should be emphasized that this is entirely due to the power of Clifford analysis and the existence of the idempotent Clifford-Heaviside functions, inexisting in complex or harmonic analysis.

The method is then applied to specific cases; in each of these cases known orthogonal polynomials on the real axis lead to orthogonal Clifford algebra-valued polynomials. In Section 10.3 three specific examples of such orthogonal polynomials in the open unit ball are discussed. The obtained Clifford-Jacobi polynomials on the unit ball (see Section 10.3.1) and shifted Clifford-Jacobi polynomials (see Section 10.3.4) are new; the Clifford-Gegenbauer polynomials on the unit ball (see Section 10.3.3) coincide, up to constants, with the generalized Gegenbauer polynomials which were already introduced in [47], albeit in a different way. A number of those multi-dimensional orthogonal polynomials are calculated and in most cases an explicit recurrence relation is established.

Next, we present Clifford algebra-valued orthogonal polynomials in Euclidean space, obtained by our new approach. The so-called Clifford-Laguerre polynomials of second type (see Section 10.4.2) and half-range Clifford-Hermite polynomials (see Section 10.4.3) are entirely new, while the Clifford-Hermite polynomials of second type constructed in Section 10.4.1 coincide, up to constants, with the radial Clifford-Hermite polynomials. Again, some lower-degree polynomials are explicitly calculated and in each case a recurrence relation is obtained.

10.2 The general construction method

In this section, we expose a general method for constructing Clifford algebra-valued polynomials of the form

$$p_n(i\underline{x}) = \sum_{k=0}^{n} a_k (i\underline{x})^k \quad , \quad a_k \in \mathbb{C} \quad , \quad k = 0, 1, 2, \ldots, n \quad ,$$

which are orthogonal with respect to a Clifford algebra-valued weight function on the Euclidean space $\mathbb{R}^m$ or on the unit ball $B(1) = \{\underline{x} \in \mathbb{R}^m \; ; \; |\underline{x}| < 1\}$.

Note that the polynomials considered take their values in $\mathbb{C}_m^0 \oplus \mathbb{C}_m^1$, i.e. a scalar plus a vector, also called paravector.

Definition 10.1 *If $W(r) = \sum_{j=0}^{\infty} b_j r^j$ $(b_j \in \mathbb{C}, j \in \mathbb{N} \cup \{0\})$ is real-analytic in the neighbourhood of the origin $r = 0$, then $W(i\underline{x})$ is defined as $W(i\underline{x}) = \sum_{j=0}^{\infty} b_j (i\underline{x})^j$.*

Proposition 10.1 *If $W(r)$ is real-analytic in $]-\rho, \rho[$, then in $B(\rho) = \{\underline{x} \in \mathbb{R}^m \; ; \; |\underline{x}| < \rho\}$ one has:*
(i) $W(i\underline{x})$ is real-analytic in the variables $(x_1, \ldots, x_m)$
(ii) $W(i\underline{x})P^+ = P^+W(i\underline{x}) = W(r)P^+$
(iii) $W(i\underline{x})P^- = P^-W(i\underline{x}) = W(-r)P^-$
(iv) $W(i\underline{x}) = W(r)P^+ + W(-r)P^-$.

Proof.
(i) Straightforward.
(ii) Applying the properties of P^+, we obtain successively

$$W(i\underline{x})P^+ = \sum_{j=0}^{\infty} b_j (i\underline{x})^j \; P^+ = \sum_{j=0}^{\infty} b_j (i\underline{x})^j \; (P^+)^j = \sum_{j=0}^{\infty} b_j (rP^+)^j = \sum_{j=0}^{\infty} b_j r^j P^+$$

and thus

$$W(i\underline{x})P^+ = W(r)P^+ \quad .$$

Moreover, we also have

$$W(i\underline{x})P^+ = P^+W(i\underline{x}) \quad .$$

(iii) Similar to (ii), we find

$$\begin{aligned} W(i\underline{x}) \; P^- &= \sum_{j=0}^{\infty} b_j (i\underline{x})^j \; P^- = \sum_{j=0}^{\infty} b_j (i\underline{x})^j \; (P^-)^j = \sum_{j=0}^{\infty} b_j (-rP^-)^j \\ &= \sum_{j=0}^{\infty} b_j (-r)^j P^- = W(-r)P^- \quad . \end{aligned}$$

(iv) The formulae in (ii) and (iii) lead to

$$W(i\underline{x}) = W(i\underline{x})P^+ + W(i\underline{x})P^- = W(r)P^+ + W(-r)P^- \quad ,$$

where we have used the fact that $P^+ + P^- = 1$. □

In what follows we will show how, by means of the above properties, integrals over the Euclidean space $\mathbb{R}^m$ or the open unit ball $B(1)$ can be rewritten in terms of integrals over the real axis. Consequently, the problem of constructing Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ which are orthogonal on $\mathbb{R}^m$ or $B(1)$ will be reduced to constructing orthogonal polynomials on the real axis.

Four types of Clifford algebra-valued orthogonal polynomials will be distinguished, two types on $B(1)$ and two on $\mathbb{R}^m$.

10.2.1 Type 1: Orthogonality on $B(1)$ with respect to the weight function $W(ix)$

First we consider the construction of Clifford algebra-valued orthogonal polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ with respect to a Clifford algebra-valued weight function $W(i\underline{x})$ in the unit ball $B(1)$, thus satisfying the orthogonality relation

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \int_{B(1)} \big(p_n(i\underline{x})\big)^\dagger \; p_{n'}(i\underline{x}) \; W(i\underline{x}) \; dV(\underline{x}) = 0 \quad ,$$

whenever $n \neq n'$.

Using the fact that $P^+ + P^- = 1$, we have

$$\begin{aligned}\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = & \int_{B(1)} \big(p_n(i\underline{x})\big)^\dagger \; p_{n'}(i\underline{x}) \; W(i\underline{x}) \; P^+ \; dV(\underline{x}) \\ & + \int_{B(1)} \big(p_n(i\underline{x})\big)^\dagger \; p_{n'}(i\underline{x}) \; W(i\underline{x}) \; P^- \; dV(\underline{x}) \quad . \end{aligned} \tag{10.1}$$

As the polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ satisfy

$$p_n(i\underline{x}) \; P^\pm = p_n(\pm r) \; P^\pm \quad \text{and} \quad \big(p_n(i\underline{x})\big)^\dagger \; P^\pm = \big(p_n(\pm r)\big)^c \; P^\pm \quad ,$$

expression (10.1) becomes:

$$\begin{aligned}\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = & \int_{B(1)} \big(p_n(r)\big)^c \; p_{n'}(r) \; W(r) \; P^+ \; dV(\underline{x}) \\ & + \int_{B(1)} \big(p_n(-r)\big)^c \; p_{n'}(-r) \; W(-r) \; P^- \; dV(\underline{x}) \quad . \end{aligned} \tag{10.2}$$

Moreover, introducing spherical co-ordinates and taking into account that (see Section 2.3)

$$\int_{S^{m-1}} \underline{\omega}\ dS(\underline{\omega}) = 0 \quad ,$$

expression (10.2) can be simplified to:

$$\begin{aligned}
&(p_n(i\underline{x}), p_{n'}(i\underline{x})) \\
&= \int_0^1 \big(p_n(r)\big)^c\ p_{n'}(r)\ W(r)\ r^{m-1}\ dr \int_{S^{m-1}} \frac{1}{2}(1+i\underline{\omega})\ dS(\underline{\omega}) \\
&\quad + \int_0^1 \big(p_n(-r)\big)^c\ p_{n'}(-r)\ W(-r)\ r^{m-1}\ dr \int_{S^{m-1}} \frac{1}{2}(1-i\underline{\omega})\ dS(\underline{\omega}) \\
&= \frac{A_m}{2} \left(\int_0^1 \big(p_n(r)\big)^c\ p_{n'}(r)\ W(r)\ r^{m-1}\ dr \right. \\
&\quad \left. + \int_0^1 \big(p_n(-r)\big)^c\ p_{n'}(-r)\ W(-r)\ r^{m-1}\ dr \right) \\
&= \frac{A_m}{2} \left(\int_0^1 \big(p_n(r)\big)^c\ p_{n'}(r)\ W(r)\ |r|^{m-1}\ dr \right. \\
&\quad \left. + \int_{-1}^0 \big(p_n(u)\big)^c\ p_{n'}(u)\ W(u)\ |u|^{m-1}\ du \right) \\
&= \frac{A_m}{2} \int_{-1}^1 \big(p_n(r)\big)^c\ p_{n'}(r)\ W(r)\ |r|^{m-1}\ dr \quad .
\end{aligned}$$

Conclusion 10.1 *We may conclude that the polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ are orthogonal on $B(1)$ with respect to $W(i\underline{x})$ if and only if the polynomials $\{p_n(r)\}_{n\geq 0}$ are orthogonal on $[-1,1]$ with respect to the weight function $W(r)|r|^{m-1}$. Herewith it is tacitly assumed that the weight functions $W(r)$ and $W(i\underline{x})$ are real-analytic in respectively $[-1,1]$ and $B(1)$ and that moreover all integrals involved are convergent.*

Remark 10.1 Note that in the special case where the dimension m is odd, the polynomials $\{p_n(r)\}_{n\geq 0}$ should be orthogonal on $[-1,1]$ with respect to the weight function $W(r)r^{m-1}$.

10.2.2 Type 2: Orthogonality on $B(1)$ with respect to the weight function $W(ix)P^+$

A second type consists of Clifford algebra-valued orthogonal polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ on $B(1)$ with respect to a Clifford algebra-valued weight function of the form $W(i\underline{x})P^+ = W(r)P^+$, thus satisfying the orthogonality relation

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \int_{B(1)} \big(p_n(i\underline{x})\big)^\dagger \, p_{n'}(i\underline{x}) \, W(i\underline{x}) \, P^+ \, dV(\underline{x}) = 0 \quad ,$$

whenever $n \neq n'$.
The above inner product can be rewritten as

$$\begin{aligned}
\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) &= \int_{B(1)} \big(p_n(r)\big)^c \, p_{n'}(r) \, W(r) \, P^+ \, dV(\underline{x}) \\
&= \int_0^1 \big(p_n(r)\big)^c \, p_{n'}(r) \, W(r) \, r^{m-1} dr \int_{S^{m-1}} \frac{1}{2}(1 + i\underline{\omega}) \, dS(\underline{\omega}) \\
&= \frac{A_m}{2} \int_0^1 \big(p_n(r)\big)^c \, p_{n'}(r) \, W(r) \, r^{m-1} \, dr \quad .
\end{aligned}$$

Conclusion 10.2 *The polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ are orthogonal on $B(1)$ with respect to $W(i\underline{x})P^+$ if and only if the polynomials $\{p_n(r)\}_{n\geq 0}$ are orthogonal on $[0,1]$ with respect to the weight function $W(r)r^{m-1}$.*

Remark 10.2 Note that when considering in $B(1)$ the weight function $W(i\underline{x})P^- = W(-r)P^-$, one obtains similarly

$$\begin{aligned}
&\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) \\
&\quad = \int_{B(1)} \big(p_n(i\underline{x})\big)^\dagger \, p_{n'}(i\underline{x}) \, W(i\underline{x}) \, P^- \, dV(\underline{x}) \\
&\quad = \int_{B(1)} \big(p_n(-r)\big)^c \, p_{n'}(-r) \, W(-r) \, P^- \, dV(\underline{x}) \\
&\quad = \int_0^1 \big(p_n(-r)\big)^c \, p_{n'}(-r) \, W(-r) \, r^{m-1} \, dr \int_{S^{m-1}} \frac{1}{2}(1 - i\underline{\omega}) \, dS(\underline{\omega}) \\
&\quad = \frac{A_m}{2} \, (-1)^{m-1} \int_{-1}^0 \big(p_n(u)\big)^c \, p_{n'}(u) \, W(u) \, u^{m-1} \, du \quad .
\end{aligned}$$

In this case, we thus need polynomials $\{p_n(r)\}_{n\geq 0}$ which are orthogonal on $[-1,0]$ with respect to the weight function $W(r)r^{m-1}$.

10.2.3 Type 3: Orthogonality on $\mathbb{R}^m$ with respect to the weight function $W(ix)$

In this section, we search for Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ which are orthogonal on $\mathbb{R}^m$ with respect to a Clifford algebra-valued weight function $W(i\underline{x})$. In other words, they should satisfy the orthogonality relation

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \int_{\mathbb{R}^m} \big(p_n(i\underline{x})\big)^\dagger \; p_{n'}(i\underline{x}) \; W(i\underline{x}) \; dV(\underline{x}) = 0 \quad ,$$

whenever $n \neq n'$.
Similarly as in Section 10.2.1, we find

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \frac{A_m}{2} \int_{-\infty}^{+\infty} \big(p_n(r)\big)^c \; p_{n'}(r) \; W(r) \; |r|^{m-1} \; dr \quad .$$

Conclusion 10.3 *The problem of constructing orthogonal polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ on $\mathbb{R}^m$ with respect to $W(i\underline{x})$ is equivalent with the construction of polynomials $\{p_n(r)\}_{n\geq 0}$ which are orthogonal on $]-\infty, +\infty[$ with respect to the weight function $W(r)|r|^{m-1}$. Again, it is tacitly assumed that the weight functions $W(r)$ and $W(i\underline{x})$ are real-analytic in respectively $]-\infty, +\infty[$ and $\mathbb{R}^m$ and that moreover all integrals involved converge.*

Remark 10.3 If the dimension m is odd, the polynomials $\{p_n(r)\}_{n\geq 0}$ should be orthogonal on $]-\infty, +\infty[$ with respect to $W(r)r^{m-1}$.

10.2.4 Type 4: Orthogonality on $\mathbb{R}^m$ with respect to the weight function $W(ix)P^+$

Finally, we consider the construction of Clifford algebra-valued polynomials $\{p_n(i\underline{x})\}_{n\geq 0}$ which satisfy the orthogonality relation

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \int_{\mathbb{R}^m} \big(p_n(i\underline{x})\big)^\dagger \; p_{n'}(i\underline{x}) \; W(i\underline{x}) \; P^+ \; dV(\underline{x}) = 0 \quad ,$$

whenever $n \neq n'$; we thus search for orthogonal polynomials in $\mathbb{R}^m$ with respect to a Clifford algebra-valued weight function of the form $W(i\underline{x})P^+ = W(r)P^+$.
Similar to Section 10.2.2, we obtain:

$$\big(p_n(i\underline{x}), p_{n'}(i\underline{x})\big) = \frac{A_m}{2} \int_{0}^{+\infty} \big(p_n(r)\big)^c \; p_{n'}(r) \; W(r) \; r^{m-1} \; dr \quad ,$$

which yields the following conclusion.

Conclusion 10.4 *The polynomials* $\{p_n(i\underline{x})\}_{n\geq 0}$ *are orthogonal on* $\mathbb{R}^m$ *with respect to the weight function* $W(i\underline{x})P^+$ *if and only if the polynomials* $\{p_n(r)\}_{n\geq 0}$ *are orthogonal on* $[0, +\infty[$ *with respect to* $W(r)r^{m-1}$.

Remark 10.4 Similarly as in Section 10.2.2, we can also consider the weight function $W(i\underline{x})P^- = W(-r)P^-$ in $\mathbb{R}^m$.
In this case, we need to look for orthogonal polynomials $\{p_n(r)\}_{n\geq 0}$ on $]-\infty, 0]$ with respect to $W(r)r^{m-1}$.

10.3 Examples in the open unit ball of Euclidean space

10.3.1 The Clifford-Jacobi polynomials on the unit ball

In this section, we focus on Clifford algebra-valued orthogonal polynomials in $B(1)$ with respect to the specific Clifford algebra-valued weight function $(1 + i\underline{x})^\alpha(1 - i\underline{x})^\beta$ $(\alpha, \beta > -1)$.
According to the general theory of Section 10.2.1, we aim at constructing orthogonal polynomials $\{p_n(r)\}_{n\geq 0}$ on $[-1, 1]$ with respect to the weight function $(1 + r)^\alpha(1 - r)^\beta|r|^{m-1}$ $(\alpha, \beta > -1)$.

In [113] Vanlessen considers the generalized Jacobi weight function

$$w(x) = (1 - x)^\alpha(1 + x)^\beta \; h(x) \; \prod_{\nu=1}^{n_0} |x - x_\nu|^{2\lambda_\nu} \quad , \quad x \in [-1, 1] \quad , \qquad (10.3)$$

where n_0 is a fixed number, with

$$\alpha > -1 \quad , \quad \beta > -1 \quad , \quad -1 < x_1 < x_2 < \ldots < x_{n_0} < 1 \quad ,$$
$$-\frac{1}{2} < \lambda_\nu < 0 \quad \text{or} \quad 0 < \lambda_\nu < +\infty$$

and with h real analytic and strictly positive on $[-1, 1]$.
This weight function leads to a sequence of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$, satisfying a recurrence relation; in [113] the asymptotic behaviour of the coefficients in this recurrence relation is studied. The generalized Jacobi weight function has been the subject of many other papers. However, neither an explicit expression, nor a Rodrigues formula, nor even an explicit recurrence relation seems to exist for the polynomials $\{P_n(x)\}_{n\geq 0}$ associated to this weight function.

Now, putting $n_0 = 1$, $x_1 = 0$, $2\lambda_1 = m-1$, $h(x) = 1$ and swapping α and β in (10.3), we obtain the specific weight function

$$w^*(x) = (1+x)^\alpha (1-x)^\beta \; |x|^{m-1} \quad , \quad x \in [-1,1] \; .$$

Consequently, the weight function $(1+r)^\alpha(1-r)^\beta |r|^{m-1}$ $(\alpha, \beta > -1)$ belongs to the class of generalized Jacobi weight functions, yielding the existence of a sequence of orthogonal polynomials $\{p_n(r)\}_{n\geq 0}$ associated to this weight function.

We now compute a number of those polynomials $\{p_n(r)\}_{n\geq 0}$ by means of the Gram-Schmidt procedure (see for e.g. [88]) for the inner product

$$< f, g > = \int_{-1}^{1} \big(f(r)\big)^c \; g(r) \; (1+r)^\alpha (1-r)^\beta |r|^{m-1} \; dr \quad , \quad \alpha, \beta > -1$$

starting of from the set $\{1, r, r^2, \ldots\}$.
This obviously yields

$$\begin{aligned} p_0(r) &= 1 \\ p_1(r) &= r - \frac{< p_0, r >}{< p_0, p_0 >} \; p_0(r) \\ p_2(r) &= r^2 - \frac{< p_0, r^2 >}{< p_0, p_0 >} \; p_0(r) - \frac{< p_1, r^2 >}{< p_1, p_1 >} \; p_1(r) \end{aligned}$$

and so forth, where the inner products appearing can be calculated using the following results (see for e.g. [64]):

$$\begin{aligned} &\int_0^1 r^n \; (1+r)^\alpha (1-r)^\beta \; dr \\ &= \frac{\Gamma(n+1) \; \Gamma(\beta+1)}{\Gamma(\beta+n+2)} \; F(-\alpha, n+1; \beta+n+2; -1) \quad ; \quad \beta > -1 \; , \; n \in \mathbb{N} \end{aligned} \tag{10.4}$$

and

$$\begin{aligned} &\int_0^1 r^n \; (1-r)^\alpha (1+r)^\beta \; dr \\ &\quad = \frac{\Gamma(n+1) \; \Gamma(\alpha+1)}{\Gamma(\alpha+n+2)} \; F(-\beta, n+1; \alpha+n+2; -1) \quad ; \quad \alpha > -1 \; , \; n \in \mathbb{N} \; . \end{aligned} \tag{10.5}$$

Here $F(a,b;c;z)$ denotes the hypergeometric function already mentioned in Section 7.3. This hypergeometric function is a single-valued analytic function of z in the whole z-plane with a branch cut along the positive real axis from one to infinity. The parameters a, b and c are arbitrary complex numbers.
By means of the formulae (10.4) and (10.5), we obtain for $n \in \mathbb{N}$ and $\alpha, \beta > -1$:

$$\begin{aligned}
&N(n)\\
&:= \int_{-1}^{1} r^n \ (1+r)^\alpha (1-r)^\beta \ |r|^{m-1} \ dr\\
&= \int_{-1}^{0} r^n \ (1+r)^\alpha (1-r)^\beta \ (-r)^{m-1} \ dr + \int_{0}^{1} r^n \ (1+r)^\alpha (1-r)^\beta \ r^{m-1} \ dr\\
&= \int_{0}^{1} (-u)^n \ (1-u)^\alpha (1+u)^\beta \ u^{m-1} \ du + \int_{0}^{1} r^n \ (1+r)^\alpha (1-r)^\beta \ r^{m-1} \ dr\\
&= (-1)^n \frac{\Gamma(n+m)\ \Gamma(\alpha+1)}{\Gamma(\alpha+n+m+1)} \ F(-\beta, n+m, \alpha+n+m+1; -1)\\
&\quad + \frac{\Gamma(n+m)\ \Gamma(\beta+1)}{\Gamma(\beta+n+m+1)} \ F(-\alpha, n+m, \beta+n+m+1; -1)\\
&= (-1)^n \ B(n+m, \alpha+1) \ F(-\beta, n+m; \alpha+n+m+1; -1)\\
&\quad + B(n+m, \beta+1) \ F(-\alpha, n+m; \beta+n+m+1; -1) \ ,
\end{aligned}$$

where $B(x,y)$ denotes the Beta function.
It follows that the orthogonal polynomials $\{p_n(r)\}_{n\geq 0}$ are given by:

$$\begin{aligned}
p_0(r) &= 1\\
p_1(r) &= r - \frac{N(1)}{N(0)}\\
p_2(r) &= r^2 - \frac{N(3)N(0) - N(1)N(2)}{N(2)N(0) - \big(N(1)\big)^2} \ r - \frac{N(2)}{N(0)}\\
&\quad + \frac{N(3)N(0) - N(1)N(2)}{N(2)N(0) - \big(N(1)\big)^2} \ \frac{N(1)}{N(0)}
\end{aligned}$$

etc.

This eventually leads to a new sequence of Clifford algebra-valued orthogonal polynomials $\{\mathcal{J}_n^{(\alpha,\beta)}(i\underline{x})\}_{n\geq 0}$ in the unit ball $B(1)$ with respect to the weight function $(1+i\underline{x})^\alpha (1-i\underline{x})^\beta$ $(\alpha, \beta > -1)$.
As a result of the above Gram-Schmidt procedure, we find, for the lower-degree

polynomials

$$\begin{aligned}
\mathcal{J}_0^{(\alpha,\beta)}(i\underline{x}) &= 1 \\
\mathcal{J}_1^{(\alpha,\beta)}(i\underline{x}) &= (i\underline{x}) - \frac{N(1)}{N(0)} \\
\mathcal{J}_2^{(\alpha,\beta)}(i\underline{x}) &= (i\underline{x})^2 - \frac{N(3)N(0)-N(1)N(2)}{N(2)N(0)-(N(1))^2}\,(i\underline{x}) - \frac{N(2)}{N(0)} \\
&\quad + \frac{N(3)N(0)-N(1)N(2)}{N(2)N(0)-(N(1))^2}\,\frac{N(1)}{N(0)}
\end{aligned}$$

and so on.

We call them the *Clifford-Jacobi polynomials on the unit ball.*

10.3.2 Special case of the Clifford-Jacobi polynomials on the unit ball

For the Clifford-Jacobi polynomials constructed in the foregoing section, we have neither an explicit formula, nor an explicit recurrence relation.
However, for the special case where $\beta = \alpha+1$, we do obtain an explicit recurrence relation.

Remark 10.5 Note that a similar situation occurred when studying the Clifford-Jacobi polynomials on the Euclidean space $\mathbb{R}^m$. These Clifford algebra-valued polynomials were constructed in Chapter 9 by means of the CK-extension procedure. There the case where in the weight function $F(\underline{x}) = (1+\underline{x})^\alpha(1-\underline{x})^\beta$ $(\alpha,\beta \in \mathbb{R})$ $\alpha = \beta+1$, is also special. Indeed, only when $\alpha = \beta+1$ it was possible to obtain, by means of the CK-extension technique, orthogonal polynomials on $\mathbb{R}^m$.

In [9], Atia explicitly gives the coefficients in the recurrence relation of a sequence of monic orthogonal polynomials $\{P_n^{(\alpha,\alpha+1)}(x,\mu)\}_{n\geq 0}$, $\mathrm{Re}(\alpha) > -1$, on $[-1,1]$ with respect to the weight function $|x|^{-\mu}(1+x)^\alpha(1-x)^{\alpha+1}$. For $n \neq n'$ these polynomials thus satisfy:

$$\int_{-1}^{1} P_n^{(\alpha,\alpha+1)}(x,\mu)\; P_{n'}^{(\alpha,\alpha+1)}(x,\mu)\; |x|^{-\mu}(1+x)^\alpha(1-x)^{\alpha+1}\, dx = 0 \quad .$$

Here μ is an arbitrary parameter with $\mathrm{Re}(-\mu) > -1$. For $\mu = 0$ one obtains the Jacobi polynomial sequence (see Section 9.1).

The polynomials $\{P_n^{(\alpha,\alpha+1)}(x,\mu)\}_{n\geq 0}$ satisfy a three-term recurrence relation

$$P_{n+2}^{(\alpha,\alpha+1)}(x,\mu) = (x-\beta_{n+1})\ P_{n+1}^{(\alpha,\alpha+1)}(x,\mu) - \gamma_{n+1}\ P_n^{(\alpha,\alpha+1)}(x,\mu) \tag{10.6}$$
$$P_0^{(\alpha,\alpha+1)}(x,\mu) = 1\ , \qquad P_1^{(\alpha,\alpha+1)}(x,\mu) = x-\beta_0\ ,$$

with

$$\begin{aligned}
\beta_0 &= -\frac{\mu-1}{\mu-2\alpha-3} \\
\beta_{n+1} &= (-1)^n\ \frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}\ (2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)} \\
\gamma_{2n+1} &= 2\ \frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2} \\
\gamma_{2n+2} &= \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2}\ .
\end{aligned}$$

Putting $\mu = 1-m$, we obtain orthogonal polynomials on $[-1,1]$ with respect to the weight function $|x|^{m-1}(1+x)^\alpha(1-x)^{\alpha+1}$, $\mathrm{Re}(\alpha) > -1$.
In this special case, the recurrence relation (10.6) takes the form

$$\begin{aligned}
P_{n+2}^{(\alpha,\alpha+1)}(x,1-m) &= (x-\widetilde{\beta}_{n+1})\ P_{n+1}^{(\alpha,\alpha+1)}(x,1-m) \\
&\quad - \widetilde{\gamma}_{n+1}\ P_n^{(\alpha,\alpha+1)}(x,1-m)
\end{aligned} \tag{10.7}$$
$$P_0^{(\alpha,\alpha+1)}(x,1-m) = 1\ , \qquad P_1^{(\alpha,\alpha+1)}(x,1-m) = x-\widetilde{\beta}_0\ ,$$

with

$$\begin{aligned}
\widetilde{\beta}_0 &= -\frac{m}{2\alpha+m+2} \\
\widetilde{\beta}_{n+1} &= (-1)^n\ \frac{(m-1)(m+2n+2\alpha+3)+(-1)^{n+1}\ (2\alpha+1)}{(2n+2\alpha+m+2)(2n+2\alpha+m+4)} \\
\widetilde{\gamma}_{2n+1} &= 2\ \frac{(n+\alpha+1)(2n+m)}{(4n+2\alpha+m+2)^2} \\
\widetilde{\gamma}_{2n+2} &= \frac{(2n+2)(2n+2\alpha+m+2)}{(4n+2\alpha+m+4)^2}\ .
\end{aligned}$$

According to Section 10.2.1, the polynomials $\{P_n^{(\alpha,\alpha+1)}(x,1-m)\}_{n\geq 0}$ lead in a straightforward manner to orthogonal Clifford algebra-valued polynomials in $B(1)$ with respect to the Clifford algebra-valued weight function $(1+i\underline{x})^\alpha$

$(1 - i\underline{x})^{\alpha+1}$, $\alpha > -1$. We denote these Clifford algebra-valued polynomials by $\{\widetilde{\mathcal{J}}_n^{(\alpha,\alpha+1)}(i\underline{x})\}_{n\geq 0}$.
From (10.7) we obtain the recurrence relation they satisfy:

$$\widetilde{\mathcal{J}}_{n+2}^{(\alpha,\alpha+1)}(i\underline{x}) = (i\underline{x} - \widetilde{\beta}_{n+1})\ \widetilde{\mathcal{J}}_{n+1}^{(\alpha,\alpha+1)}(i\underline{x}) - \widetilde{\gamma}_{n+1}\ \widetilde{\mathcal{J}}_{n}^{(\alpha,\alpha+1)}(i\underline{x}) \quad , \quad n \geq 0$$
$$\widetilde{\mathcal{J}}_{0}^{(\alpha,\alpha+1)}(i\underline{x}) = 1 \quad , \qquad \widetilde{\mathcal{J}}_{1}^{(\alpha,\alpha+1)}(i\underline{x}) = i\underline{x} - \widetilde{\beta}_0$$

and using the symbolic software Maple, we get:

$$\begin{aligned}
\widetilde{\mathcal{J}}_{0}^{(\alpha,\alpha+1)}(i\underline{x}) &= 1 \\
\widetilde{\mathcal{J}}_{1}^{(\alpha,\alpha+1)}(i\underline{x}) &= i\underline{x} + \frac{m}{2+m+2\alpha} \\
\widetilde{\mathcal{J}}_{2}^{(\alpha,\alpha+1)}(i\underline{x}) &= (i\underline{x})^2 + \frac{2}{2\alpha+4+m}\ (i\underline{x}) - \frac{m}{2\alpha+4+m} \\
\widetilde{\mathcal{J}}_{3}^{(\alpha,\alpha+1)}(i\underline{x}) &= (i\underline{x})^3 + \frac{2+m}{2\alpha+6+m}\ (i\underline{x})^2 - \frac{2+m}{2\alpha+6+m}\ (i\underline{x}) \\
&\quad - \frac{(2+m)m}{(2\alpha+4+m)(2\alpha+6+m)} \\
\widetilde{\mathcal{J}}_{4}^{(\alpha,\alpha+1)}(i\underline{x}) &= (i\underline{x})^4 + \frac{4}{2\alpha+8+m}\ (i\underline{x})^3 - \frac{2(2+m)}{2\alpha+8+m}\ (i\underline{x})^2 \\
&\quad - \frac{4(2+m)}{(2\alpha+6+m)(2\alpha+8+m)}\ (i\underline{x}) \\
&\quad + \frac{m(2+m)}{(2\alpha+6+m)(2\alpha+8+m)} \\
\widetilde{\mathcal{J}}_{5}^{(\alpha,\alpha+1)}(i\underline{x}) &= (i\underline{x})^5 + \frac{4+m}{2\alpha+10+m}\ (i\underline{x})^4 - \frac{2(4+m)}{2\alpha+10+m}\ (i\underline{x})^3 \\
&\quad - \frac{2(2+m)(4+m)}{(2\alpha+8+m)(2\alpha+10+m)}\ (i\underline{x})^2 \\
&\quad + \frac{(2+m)(4+m)}{(2\alpha+8+m)(2\alpha+10+m)}\ i\underline{x} \\
&\quad + \frac{m(4+m)(2+m)}{(2\alpha+6+m)(2\alpha+8+m)(2\alpha+10+m)}
\end{aligned}$$

etc.

It can be verified, by a tedious calculation, that indeed the above polynomials coincide with the Clifford-Jacobi polynomials of the foregoing section, where $\beta = \alpha + 1$.

10.3.3 The Clifford-Gegenbauer polynomials on the unit ball

In this section, we construct orthogonal Clifford algebra-valued polynomials in $B(1)$ with respect to the scalar-valued weight function $(1-|\underline{x}|^2)^\alpha, \alpha > -1$. These polynomials apparently are a special case of the Clifford-Jacobi polynomials of Section 10.3.1, where now $\alpha = \beta$.

Starting point are the so-called generalized Gegenbauer orthogonal polynomials $\{S_n^{(\alpha,\gamma)}(x)\}_{n\geq 0}$ on the real line (see [12]). They are orthogonal on $[-1,1]$ with respect to the weight function $(1-x^2)^\alpha|x|^{2\gamma+1}$, $\alpha, \gamma > -1, \gamma \neq -\frac{1}{2}$ and thus satisfy the orthogonality relation:

$$\int_{-1}^{1} S_n^{(\alpha,\gamma)}(x)\ (1-x^2)^\alpha |x|^{2\gamma+1}\ x^k\ dx = 0 \quad , \quad k = 0, 1, 2, \ldots, n-1 \quad .$$

These generalized Gegenbauer orthogonal polynomials belong to the class of so-called semi-classical orthogonal sequences, which is a subject of current research (see for e.g. [11]).
They satisfy the recurrence relation

$$\begin{aligned} S_{n+2}^{(\alpha,\gamma)}(x) &= x\ S_{n+1}^{(\alpha,\gamma)}(x) - \kappa_{n+1}\ S_n^{(\alpha,\gamma)}(x) \quad , \quad n \geq 0 \\ S_1^{(\alpha,\gamma)}(x) = x \quad &, \qquad S_0^{(\alpha,\gamma)}(x) = 1 \quad , \end{aligned}$$

with

$$\kappa_{n+1} = \frac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{4(n+1+\alpha+\gamma)(n+2+\alpha+\gamma)} \quad \text{and} \quad \delta_n = (2\gamma+1)\ \frac{1+(-1)^n}{2} \quad .$$

Furthermore, they may be expressed in terms of the classical Jacobi polynomials $\{P_n^{(\alpha,\gamma)}(x)\}_{n\geq 0}$ introduced in Section 9.1:

$$\begin{aligned} S_{2n}^{(\alpha,\gamma)}(x) &= \frac{n!\ \Gamma(n+\alpha+\gamma+1)}{\Gamma(2n+\alpha+\gamma+1)}\ P_n^{(\alpha,\gamma)}(2x^2-1) \\ S_{2n+1}^{(\alpha,\gamma)}(x) &= \frac{n!\ \Gamma(n+\alpha+\gamma+2)}{\Gamma(2n+\alpha+\gamma+2)}\ x\ P_n^{(\alpha,\gamma+1)}(2x^2-1) \quad . \end{aligned}$$

From the following explicit expression for the classical Jacobi polynomials

$\{P_n^{(\alpha,\gamma)}(x)\}_{n\geq 0}$ (see for e.g. [82]):

$$\begin{aligned} P_n^{(\alpha,\gamma)}(x) & \\ &= 2^{-n}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\gamma}{n-r}(x+1)^r(x-1)^{n-r} \\ &= \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\gamma+n+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\alpha+\gamma+n+r+1)}{\Gamma(\alpha+r+1)}\left(\frac{x-1}{2}\right)^r \end{aligned} \tag{10.8}$$

it follows that

$$\begin{aligned} S_{2n}^{(\alpha,\gamma)}(x) & \\ &= \frac{n!\,\Gamma(n+\alpha+\gamma+1)}{\Gamma(2n+\alpha+\gamma+1)}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\gamma}{n-r}x^{2r}\,(x^2-1)^{n-r} \\ &= \frac{\Gamma(\alpha+n+1)}{\Gamma(2n+\alpha+\gamma+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\alpha+\gamma+n+r+1)}{\Gamma(\alpha+r+1)}(x^2-1)^r \end{aligned} \tag{10.9}$$

and

$$\begin{aligned} S_{2n+1}^{(\alpha,\gamma)}(x) & \\ &= \frac{n!\,\Gamma(n+\alpha+\gamma+2)}{\Gamma(2n+\alpha+\gamma+2)}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\gamma+1}{n-r}x^{2r+1}\,(x^2-1)^{n-r} \\ &= \frac{\Gamma(\alpha+n+1)}{\Gamma(2n+\alpha+\gamma+2)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\alpha+\gamma+2+n+r)}{\Gamma(\alpha+r+1)}\,x\,(x^2-1)^r. \end{aligned} \tag{10.10}$$

For $\gamma = m/2-1 > -1$, the generalized Gegenbauer orthogonal polynomials $\{S_n^{(\alpha,m/2-1)}(x)\}_{n\geq 0}$ are orthogonal on $[-1,1]$ with respect to $(1-x^2)^\alpha|x|^{m-1}$, $\alpha > -1$.
Hence the Clifford algebra-valued polynomials $\{\mathcal{G}_n^{(\alpha)}(i\underline{x})\}_{n\geq 0}$ which are orthogonal on $B(1)$ with respect to the scalar-valued weight function $(1-|\underline{x}|^2)^\alpha$, $\alpha > -1$ satisfy the recurrence relation:

$$\mathcal{G}_{n+2}^{(\alpha)}(i\underline{x}) = (i\underline{x})\;\mathcal{G}_{n+1}^{(\alpha)}(i\underline{x}) - \widetilde{\kappa}_{n+1}\;\mathcal{G}_n^{(\alpha)}(i\underline{x}) \quad , \quad n\geq 0$$
$$\mathcal{G}_1^{(\alpha)}(i\underline{x}) = i\underline{x} \quad , \qquad \mathcal{G}_0^{(\alpha)}(i\underline{x}) = 1 \quad ,$$

with

$$\widetilde{\kappa}_{n+1} = \frac{(n+1+\widetilde{\delta}_n)(n+1+2\alpha+\widetilde{\delta}_n)}{4(n+\alpha+\frac{m}{2})(n+\alpha+\frac{m}{2}+1)} \quad \text{and} \quad \widetilde{\delta}_n = (m-1)\,\frac{1+(-1)^n}{2} \quad .$$

We call these polynomials the *Clifford-Gegenbauer polynomials on the unit ball.* They can be expressed in terms of the classical Jacobi polynomials by:

$$\mathcal{G}_{2n}^{(\alpha)}(i\underline{x}) = n!\,\frac{\Gamma(n+\alpha+\frac{m}{2})}{\Gamma(2n+\alpha+\frac{m}{2})}\,P_n^{(\alpha,m/2-1)}(2|\underline{x}|^2-1)$$

and

$$\mathcal{G}_{2n+1}^{(\alpha)}(i\underline{x}) = n!\,\frac{\Gamma(n+\alpha+\frac{m}{2}+1)}{\Gamma(2n+\alpha+\frac{m}{2}+1)}\,(i\underline{x})\,P_n^{(\alpha,m/2)}(2|\underline{x}|^2-1) \quad .$$

Moreover, from (10.9) and (10.10) we obtain an explicit expression for these Clifford-Gegenbauer polynomials:

$$\begin{aligned}\mathcal{G}_{2n}^{(\alpha)}(i\underline{x}) &= \frac{n!\,\Gamma(n+\alpha+\frac{m}{2})}{\Gamma(2n+\alpha+\frac{m}{2})}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\frac{m}{2}-1}{n-r}|\underline{x}|^{2r}\,(|\underline{x}|^2-1)^{n-r}\\ &= \frac{\Gamma(\alpha+n+1)}{\Gamma(2n+\alpha+\frac{m}{2})}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\alpha+n+r+\frac{m}{2})}{\Gamma(\alpha+r+1)}\,(|\underline{x}|^2-1)^r\end{aligned}$$

and

$$\begin{aligned}&\mathcal{G}_{2n+1}^{(\alpha)}(i\underline{x})\\ &= n!\,\frac{\Gamma(n+\alpha+\frac{m}{2}+1)}{\Gamma(2n+\alpha+\frac{m}{2}+1)}\sum_{r=0}^{n}\binom{n+\alpha}{r}\binom{n+\frac{m}{2}}{n-r}\,i\underline{x}\,|\underline{x}|^{2r}\,(|\underline{x}|^2-1)^{n-r}\\ &= \frac{\Gamma(\alpha+n+1)}{\Gamma(2n+\alpha+\frac{m}{2}+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(\alpha+n+r+\frac{m}{2}+1)}{\Gamma(\alpha+r+1)}\,(i\underline{x})\,(|\underline{x}|^2-1)^r \quad .\end{aligned}$$

Finally, we give a few examples

$$\begin{aligned}\mathcal{G}_{0}^{(\alpha)}(i\underline{x}) &= 1\\ \mathcal{G}_{1}^{(\alpha)}(i\underline{x}) &= i\underline{x}\\ \mathcal{G}_{2}^{(\alpha)}(i\underline{x}) &= (i\underline{x})^2 - \frac{m}{2\alpha+m+2}\\ \mathcal{G}_{3}^{(\alpha)}(i\underline{x}) &= (i\underline{x})^3 - \frac{m+2}{2\alpha+m+4}\,(i\underline{x})\\ \mathcal{G}_{4}^{(\alpha)}(i\underline{x}) &= (i\underline{x})^4 - \frac{m+2}{\alpha+\frac{m}{2}+3}\,(i\underline{x})^2 + \frac{(m-2)(m+4)+8}{(2\alpha+m+6)(2\alpha+m+4)}\\ \mathcal{G}_{5}^{(\alpha)}(i\underline{x}) &= (i\underline{x})^5 - \frac{m+4}{\alpha+\frac{m}{2}+4}\,(i\underline{x})^3 + \frac{(m-2)(m+8)+24}{(2\alpha+m+8)(2\alpha+m+6)}\,i\underline{x}\end{aligned}$$

etc.

It can be verified that indeed the above polynomials coincide with the Clifford-Jacobi polynomials of Section 10.3.1 where $\beta = \alpha$.

It should be noted that Clifford algebra-valued orthogonal polynomials in $B(1)$ with respect to $(1-|\underline{x}|^2)^\alpha$, $\alpha > -1$ were already introduced in [47] by Cnops. These so-called *Generalized Gegenbauer polynomials* $C^\alpha_{n,m}(\underline{x})$ were constructed by means of a Rodrigues formula.
In terms of the Jacobi polynomials on the real line, they are given by

$$\begin{aligned} C^\alpha_{2n,m}(\underline{x}) &= 2^{2n}\ (\alpha+n+1)_n\ n!\ P_n^{(m/2-1,\alpha)}(1+2\underline{x}^2) \\ C^\alpha_{2n+1,m}(\underline{x}) &= -2^{2n+1}\ (\alpha+n+1)_{n+1}\ n!\ \underline{x}\ P_n^{(m/2,\alpha)}(1+2\underline{x}^2)\ , \end{aligned} \tag{10.11}$$

which corrects a result from [47, p.57] and [55, p.294].
Recall from Section 9.1 that

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n\ P_n^{(\beta,\alpha)}(x)\ .$$

Hence, expressions (10.11) can be rewritten as

$$\begin{aligned} C^\alpha_{2n,m}(\underline{x}) &= (-1)^n\ 2^{2n}\ (\alpha+n+1)_n\ n!\ P_n^{(\alpha,m/2-1)}(2|\underline{x}|^2-1) \\ C^\alpha_{2n+1,m}(\underline{x}) &= (-1)^{n+1}\ 2^{2n+1}\ (\alpha+n+1)_{n+1}\ n!\ \underline{x}\ P_n^{(\alpha,m/2)}(2|\underline{x}|^2-1)\ . \end{aligned}$$

Consequently, our Clifford-Gegenbauer polynomials $\mathcal{G}_n^{(\alpha)}(i\underline{x})$ and the Generalized Gegenbauer polynomials $C^\alpha_{n,m}(\underline{x})$ of Cnops are related by:

$$\begin{aligned} C^\alpha_{2n,m}(\underline{x}) &= (-1)^n\ 2^{2n}\ (\alpha+n+1)_n\ \left(\alpha+n+\frac{m}{2}\right)_n\ \mathcal{G}_{2n}^{(\alpha)}(i\underline{x}) \\ C^\alpha_{2n+1,m}(\underline{x}) &= (-1)^n\ i\ 2^{2n+1}\ (\alpha+n+1)_{n+1}\ \left(\alpha+n+\frac{m}{2}+1\right)_n\ \mathcal{G}_{2n+1}^{(\alpha)}(i\underline{x})\ . \end{aligned}$$

10.3.4 The shifted Clifford-Jacobi polynomials

In order to construct Clifford algebra-valued orthogonal polynomials in $B(1)$ with respect to the Clifford algebra-valued weight function $(1-i\underline{x})^\alpha P^+$, $\alpha > -1$, we need orthogonal polynomials on $[0,1]$ with respect to $(1-r)^\alpha r^{m-1}$, $\alpha > -1$ (see Section 10.2.2). These polynomials are obtained by the change of variables $x = 2r-1$ in the classical Jacobi polynomials $\{P_n^{(\alpha,m-1)}(x)\}_{n\geq 0}$, thus satisfying the orthogonality relation (see (9.1))

$$\begin{aligned} &\int_0^1 P_n^{(\alpha,m-1)}(2r-1)\ P_{n'}^{(\alpha,m-1)}(2r-1)\ (1-r)^\alpha r^{m-1}\ dr \\ &\qquad = \frac{\Gamma(\alpha+n+1)\ \Gamma(m+n)}{n!\ \Gamma(\alpha+n+m)\ (\alpha+2n+m)}\ \delta_{n,n'}\ . \end{aligned}$$

According to the general theory of Section 10.2, this orthogonality relation leads to

$$\begin{aligned}\int_{B(1)} & P_n^{(\alpha,m-1)}(2(i\underline{x})-1)\; P_{n'}^{(\alpha,m-1)}\big(2(i\underline{x})-1\big)\;\big(1-i\underline{x}\big)^{\alpha} P^+ \; dV(\underline{x}) \\ &= \frac{A_m}{2}\; \frac{\Gamma(\alpha+n+1)\;\Gamma(m+n)}{n!\;\Gamma(\alpha+n+m)\;(\alpha+2n+m)}\;\delta_{n,n'} \quad ,\end{aligned}$$

expressing the orthogonality on $B(1)$ of the polynomials

$$Q_n^{(\alpha)}(i\underline{x}) = P_n^{(\alpha,m-1)}\big(2(i\underline{x})-1\big) \quad , \quad n \in \mathbb{N} \quad ,$$

which we call the *shifted Clifford-Jacobi polynomials.*
From the explicit expression (10.8) for the classical Jacobi polynomials we obtain an explicit expression for our shifted Clifford-Jacobi polynomials:

$$\begin{aligned} Q_n^{(\alpha)}(i\underline{x}) &= \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+m-1}{n-k}\;(i\underline{x})^k\;(i\underline{x}-1)^{n-k} \\ &= \frac{\Gamma(\alpha+n+1)}{n!\;\Gamma(\alpha+m+n)} \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+m+n+k)}{\Gamma(\alpha+k+1)}\;(i\underline{x}-1)^k \quad . \end{aligned}$$

Moreover the recurrence relation (9.2) for the Jacobi polynomials on the real line leads to the following three-term recurrence relation for the polynomials $Q_n^{(\alpha)}(i\underline{x})$:

$$\begin{aligned} & 2n\;(\alpha+m+n-1)\;(\alpha+m+2n-3)\;Q_n^{(\alpha)}(i\underline{x}) \\ &= \Big((\alpha+m+2n-3)_3\;(2\;i\underline{x}-1) + \big(\alpha^2-(m-1)^2\big)\;(\alpha+m+2n-2)\Big) \\ &\qquad Q_{n-1}^{(\alpha)}(i\underline{x}) - 2(\alpha+n-1)\;(m+n-2)\;(\alpha+m+2n-1)\;Q_{n-2}^{(\alpha)}(i\underline{x}) \quad , \end{aligned}$$

with $n = 2, 3, 4, \ldots$.
We conclude by giving a few examples of these shifted Clifford-Jacobi polynomials:

$$\begin{aligned} Q_0^{(\alpha)}(i\underline{x}) &= 1 \\ Q_1^{(\alpha)}(i\underline{x}) &= (\alpha+m+1)\;i\underline{x} - m \end{aligned}$$

$$Q_2^{(\alpha)}(i\underline{x}) = \frac{(\alpha+m+3)(\alpha+m+2)}{2}\,(i\underline{x})^2 - (m+1)(m+\alpha+2)\; i\underline{x} + \frac{m(m+1)}{2}$$

etc.

10.4 Examples in Euclidean space

10.4.1 The Clifford-Hermite polynomials of second type

In this section we focus on Clifford algebra-valued orthogonal polynomials in $\mathbb{R}^m$ with respect to the specific weight function $W(i\underline{x}) = \exp(-|i\underline{x}|^2) = \exp(-|\underline{x}|^2)$. Herewith we will follow the general theory of Section 10.2.3.

The construction is based on the monic generalized Hermite polynomials $K_n^{(\gamma)}(x)$ orthogonal on $]-\infty,+\infty[$ with respect to $|x|^\gamma \exp(-x^2)$; $\gamma > -1$ (see e.g. [44], [56] and [111]).
These polynomials $K_n^{(\gamma)}(x)$ satisfy the recurrence relation

$$K_{n+1}^{(\gamma)}(x) = x\; K_n^{(\gamma)}(x) - \widehat{a}_n\; K_{n-1}^{(\gamma)}(x) \quad , \quad n \geq 0 \quad , \tag{10.12}$$
$$K_{-1}^{(\gamma)}(x) = 0 \quad , \qquad K_0^{(\gamma)}(x) = 1$$

with

$$\widehat{a}_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+\gamma}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, they can be expressed in terms of the generalized Laguerre polynomials introduced in Section 7.1:

$$\begin{aligned} K_{2n}^{(\gamma)}(x) &= (-1)^n\; n!\; L_n^{(\gamma/2-1/2)}(x^2) \\ K_{2n+1}^{(\gamma)}(x) &= (-1)^n\; n!\; x\; L_n^{(\gamma/2+1/2)}(x^2) \quad . \end{aligned}$$

From the explicit expression for the generalized Laguerre polynomials on the real line (see for e.g. [82]):

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \quad , \tag{10.13}$$

we obtain

$$\begin{aligned} K_{2n}^{(\gamma)}(x) &= (-1)^n \; n! \; \sum_{k=0}^{n} (-1)^k \binom{n+\frac{\gamma}{2}-\frac{1}{2}}{n-k} \frac{x^{2k}}{k!} \\ K_{2n+1}^{(\gamma)}(x) &= (-1)^n \; n! \; \sum_{k=0}^{n} (-1)^k \binom{n+\frac{\gamma}{2}+\frac{1}{2}}{n-k} \frac{x^{2k+1}}{k!} \quad . \end{aligned}$$

The generalized Hermite polynomials $\{K_n^{(m-1)}(x)\}_{n\geq 0}$ are orthogonal on $]-\infty,+\infty[$ with respect to $\exp(-x^2)|x|^{m-1}$. According to Section 10.2.3, these polynomials are the desired building blocks for the Clifford algebra-valued polynomials $\{K_n(i\underline{x})\}_{n\geq 0}$, which are orthogonal on $\mathbb{R}^m$ with respect to $\exp(-|\underline{x}|^2)$. We call these polynomials the *Clifford-Hermite polynomials of second type*. Converting the above results for the generalized Hermite polynomials to the Clifford analysis setting according to the general construction method of section 10.2, we obtain the recurrence relation

$$\begin{gathered} K_{n+1}(i\underline{x}) = i\underline{x} \; K_n(i\underline{x}) - \widetilde{a}_n \; K_{n-1}(i\underline{x}) \quad , \quad n \geq 0 \quad , \\ K_{-1}(i\underline{x}) = 0 \quad , \quad K_0(i\underline{x}) = 1 \quad , \end{gathered}$$

where now

$$\widetilde{a}_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+m-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

The connection with the classical generalized Laguerre polynomials is given by

$$\begin{aligned} K_{2n}(i\underline{x}) &= (-1)^n \; n! \; L_n^{(m/2-1)}(|\underline{x}|^2) \\ K_{2n+1}(i\underline{x}) &= (-1)^n \; n! \; i\underline{x} \; L_n^{(m/2)}(|\underline{x}|^2) \end{aligned}$$

and we have the explicit expression

$$\begin{aligned} K_{2n}(i\underline{x}) &= (-1)^n \; n! \; \sum_{k=0}^{n} (-1)^k \binom{n+\frac{m}{2}-1}{n-k} \frac{(i\underline{x})^{2k}}{k!} \\ K_{2n+1}(i\underline{x}) &= (-1)^n \; n! \; \sum_{k=0}^{n} (-1)^k \binom{n+\frac{m}{2}}{n-k} \frac{(i\underline{x})^{2k+1}}{k!} \quad . \end{aligned}$$

The first Clifford-Hermite polynomials of second type are given by

$$\begin{aligned}
K_0(i\underline{x}) &= 1 \\
K_1(i\underline{x}) &= i\underline{x} \\
K_2(i\underline{x}) &= (i\underline{x})^2 - \left(\frac{m}{2}\right) \;=\; |\underline{x}|^2 - \left(\frac{m}{2}\right) \\
K_3(i\underline{x}) &= (i\underline{x})^3 - \left(\frac{m+2}{2}\right) i\underline{x} \;=\; i|\underline{x}|^2\underline{x} - \left(\frac{m+2}{2}\right) i\underline{x} \\
K_4(i\underline{x}) &= (i\underline{x})^4 - (m+2)\ (i\underline{x})^2 + \left(\frac{m+2}{2}\right)\left(\frac{m}{2}\right) \\
&= |\underline{x}|^4 - (m+2)\ |\underline{x}|^2 + \left(\frac{m+2}{2}\right)\left(\frac{m}{2}\right) \\
K_5(i\underline{x}) &= (i\underline{x})^5 - (m+4)\ (i\underline{x})^3 + \left(\frac{m+4}{2}\right)\left(\frac{m+2}{2}\right)(i\underline{x}) \\
&= i|\underline{x}|^4\underline{x} - i(m+4)\ |\underline{x}|^2\underline{x} + \left(\frac{m+4}{2}\right)\left(\frac{m+2}{2}\right)(i\underline{x})
\end{aligned}$$

etc.

Note that $K_{2n}(i\underline{x})$ is real-valued, while $K_{2n+1}(i\underline{x})$ is complex vector-valued.

Recall from Section 5.2.1 that the radial Clifford-Hermite polynomials, which were constructed by means of the CK-extension technique, are also orthogonal on $\mathbb{R}^m$ with respect to the exponential weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$.
Moreover, the radial Clifford-Hermite polynomials can be expressed in terms of the generalized Laguerre polynomials on the real line:

$$\begin{aligned}
H_{2n}(\underline{x}) &= 2^n n!\ L_n^{(m/2-1)}\left(-\frac{\underline{x}^2}{2}\right) \\
H_{2n+1}(\underline{x}) &= 2^n n!\ \underline{x}\ L_n^{(m/2)}\left(-\frac{\underline{x}^2}{2}\right) \quad ,
\end{aligned}$$

which corrects a result from [47, p.70] and [55, p.309].
Consequently we also have

$$\begin{aligned}
H_{2n}(\sqrt{2}\underline{x}) &= 2^n n!\ L_n^{(m/2-1)}(|\underline{x}|^2) \\
H_{2n+1}(\sqrt{2}\underline{x}) &= 2^n n!\ \sqrt{2}\ \underline{x}\ L_n^{(m/2)}(|\underline{x}|^2) \quad ,
\end{aligned}$$

from which we obtain the following relation between the Clifford-Hermite polynomials of second type and the radial Clifford-Hermite polynomials:

$$\begin{aligned} K_{2n}(i\underline{x}) &= (-1)^n \, 2^{-n} \, H_{2n}(\sqrt{2}\underline{x}) \\ K_{2n+1}(i\underline{x}) &= (-1)^n \, i \, 2^{-(n+1/2)} \, H_{2n+1}(\sqrt{2}\underline{x}) \quad . \end{aligned}$$

10.4.2 The Clifford-Laguerre polynomials of second type

In this section we construct Clifford algebra-valued polynomials which are orthogonal on $\mathbb{R}^m$ with respect to the Clifford algebra-valued weight function $W(i\underline{x})P^+ = \exp(-i\underline{x})(i\underline{x})^\alpha P^+$, $\alpha > -m$.
The first factor in this weight function $\exp(-i\underline{x})$ is defined by means of the real-analytic function $\exp(-r)$ on the real r-axis (see Definition 10.1). The second factor $(i\underline{x})^\alpha$, $\alpha > -m$ is defined by

$$(i\underline{x})^\alpha = r^\alpha \Big(P^+ + \exp(i\pi\alpha) \; P^- \Big)$$

(see [55, p.349] and [17, p.14]).
Note that indeed:

$$(i\underline{x})^\alpha \; P^+ = r^\alpha \Big(\big(P^+\big)^2 + \exp(i\pi\alpha) \; P^- P^+ \Big) = r^\alpha P^+$$

and hence

$$\begin{aligned} W(i\underline{x}) \; P^+ &= \exp(-i\underline{x}) \; (i\underline{x})^\alpha \; P^+ = \exp(-i\underline{x}) \; r^\alpha P^+ \\ &= \exp(-i\underline{x}) \; P^+ \; r^\alpha = \exp(-r) \; r^\alpha \; P^+ \quad . \end{aligned}$$

By the change of variable $\alpha \to \alpha + m - 1 > -1$ in the orthogonality relation (7.1) for the generalized Laguerre polynomials $\{L_n^{(\alpha)}(r)\}_{n\geq 0}$, $\alpha > -1$, we get

$$\int_0^{+\infty} \exp(-r) r^{\alpha+m-1} \; L_n^{(\alpha+m-1)}(r) \; L_{n'}^{(\alpha+m-1)}(r) \; dr = \Gamma(\alpha+m) \binom{n+\alpha+m-1}{n} \delta_{n,n'} \quad .$$

In a similar way, the explicit expression (10.13) and the recurrence relation (7.2) for the generalized Laguerre polynomials lead to

$$L_n^{(\alpha+m-1)}(r) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha+m-1}{n-k} \frac{r^k}{k!}$$

and

$$n\, L_n^{(\alpha+m-1)}(r) = (2n+\alpha+m-2-r)\, L_{n-1}^{(\alpha+m-1)}(r) - (n+\alpha+m-2)\, L_{n-2}^{(\alpha+m-1)}(r) \quad .$$

The above results for the polynomials $\{L_n^{(\alpha+m-1)}(r)\}_{n\geq 0}$ immediately give rise to the corresponding results for the Clifford algebra-valued polynomials $\{\mathcal{L}_n^{(\alpha)}(i\underline{x})\}_{n\geq 0}$ orthogonal on $\mathbb{R}^m$ with respect to $\exp(-i\underline{x})(i\underline{x})^\alpha P^+$, $\alpha > -m$, which we call the *Clifford-Laguerre polynomials of second type.* They take the explicit form

$$\mathcal{L}_n^{(\alpha)}(i\underline{x}) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha+m-1}{n-k} \frac{(i\underline{x})^k}{k!}$$

and satisfy the recurrence relation

$$n\, \mathcal{L}_n^{(\alpha)}(i\underline{x}) = (2n+\alpha+m-2-i\underline{x})\, \mathcal{L}_{n-1}^{(\alpha)}(i\underline{x}) - (n+\alpha+m-2)\, \mathcal{L}_{n-2}^{(\alpha)}(i\underline{x}) \quad .$$

The first Clifford-Laguerre polynomials of second type are calculated to be:

$$\begin{aligned}
\mathcal{L}_0^{(\alpha)}(i\underline{x}) &= 1 \\
\mathcal{L}_1^{(\alpha)}(i\underline{x}) &= -i\underline{x} + \alpha + m \\
\mathcal{L}_2^{(\alpha)}(i\underline{x}) &= \frac{1}{2}\,(i\underline{x})^2 - (\alpha+m+1)\, i\underline{x} + \frac{1}{2}(\alpha+m)(\alpha+m+1) \\
\mathcal{L}_3^{(\alpha)}(i\underline{x}) &= -\frac{1}{6}(i\underline{x})^3 + \frac{1}{2}(\alpha+m+2)(i\underline{x})^2 - \frac{1}{2}(\alpha+m+1)(\alpha+m+2)(i\underline{x}) \\
&\quad + \frac{1}{6}(\alpha+m)(\alpha+m+1)(\alpha+m+2)
\end{aligned}$$

etc.

10.4.3 The half-range Clifford-Hermite polynomials

This section contains the construction of Clifford algebra-valued orthogonal polynomials in $\mathbb{R}^m$ with respect to the Clifford algebra-valued weight function $W(i\underline{x})P^+ = \exp(-(i\underline{x})^2)P^+$; they are called the *half-range Clifford-Hermite polynomials.*
According to the general theory of Section 10.2.4, this construction is based on orthogonal polynomials on $[0, +\infty[$ with respect to the weight function $\exp(-r^2)\, r^{m-1}$.

In [10] a method is developed for calculating the coefficients in the recurrence relation for the so-called half-range generalized Hermite polynomials on the real line $\{\phi_n^\gamma(x)\}_{n\geq 0}$, $\gamma > -1$. These are monic orthogonal polynomials on the interval $[0,+\infty[$ with respect to the weight function $x^\gamma \exp(-x^2)$, $\gamma > -1$, thus satisfying

$$\int_0^{+\infty} x^\gamma \exp(-x^2)\ \phi_n^\gamma(x)\ \phi_{n'}^\gamma(x)\ dx = 0 \quad ,$$

whenever $n \neq n'$.
Note that the half-range generalized Hermite polynomials are related to the Freud polynomials which are orthogonal on $]-\infty,+\infty[$ with respect to the weight function $|x|^\alpha \exp(-x^4)$. These Freud polynomials were studied by Freud in [68] and by Nevai in [96].
The recurrence formula satisfied by the half-range generalized Hermite polynomials takes the form:

$$\phi_{n+1}^\gamma(x) = (x-\alpha_n)\ \phi_n^\gamma(x) - \beta_n\ \phi_{n-1}^\gamma(x) \quad , \quad n \geq 0 \tag{10.14}$$

with

$$\phi_{-1}^\gamma(x) = 0 \quad , \quad \phi_0^\gamma(x) = 1 \quad .$$

The coefficients α_n and β_n in their turn satisfy the recurrence relations:

$$\beta_n + \beta_{n-1} + \alpha_{n-1}^2 = \frac{2n-1+\gamma}{2} \tag{10.15}$$

$$\alpha_n \alpha_{n-1} \beta_n = \left(\frac{n+\frac{\gamma}{2}}{2} - \beta_n\right)^2 - \frac{\gamma^2}{16} \tag{10.16}$$

with starting values

$$\alpha_0 = \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \quad \text{and} \quad \beta_0 = 0 \quad .$$

The recurrence procedure for the coefficients α_n and β_n appears to be straightforward. Given α_0 and β_0 one calculates β_1 from (10.15) and then one uses (10.16) to calculate α_1, and so on.
Unfortunately, while the procedure is simple, the system is rather poorly conditioned. How this comes about and what methods can be used to overcome this problem is explained in [10].

From (10.15) and (10.16) we obtain

$$\begin{aligned}
\beta_0 &= 0 \\
\alpha_0 &= \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \\
\beta_1 &= \frac{\gamma+1}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right) \\
\alpha_1 &= \frac{\left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^3 - \frac{\gamma}{2}\,\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}}{\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2} \\
&\text{etc.}
\end{aligned}$$

Now the recurrence formula (10.14) allows us to compute recursively

$$\begin{aligned}
\phi_0^\gamma(x) &= 1 \\
\phi_1^\gamma(x) &= x - \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)} \\
\phi_2^\gamma(x) &= x^2 - \frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{2\,\Gamma\left(\frac{\gamma+1}{2}\right)\left(\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2\right)}\,x \\
&\quad + \frac{\left(\Gamma\left(\frac{\gamma}{2}+1\right)\right)^2}{2\left(\Gamma\left(\frac{\gamma+1}{2}\right)\right)^2\left(\frac{1+\gamma}{2} - \left(\frac{\Gamma\left(\frac{\gamma}{2}+1\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^2\right)} - \frac{1+\gamma}{2} \\
&\text{etc.}
\end{aligned}$$

We observe that the half-range generalized Hermite polynomials $\{\phi_n^\gamma(x)\}_{n\geq 0}$ take the form

$$\phi_n^\gamma(x) = \sum_{k=0}^{n} h_k(\gamma)\, x^k \quad \text{with } h_n(\gamma) = 1 \text{ and } h_k(\gamma) \in \mathbb{R} \ \text{ for } k = 0, 1, 2, \ldots, n.$$

In agreement with the general construction theory of Section 10.2.4, the substitutions $\gamma \to m-1$ and $x \to i\underline{x}$, yield the half-range Clifford-Hermite polynomials:

$$\phi_n(i\underline{x}) = \sum_{k=0}^{n} h_k(m-1)\ (i\underline{x})^k$$

with

$$h_n(m-1) = 1 \quad \text{and} \quad h_k(m-1) \in \mathbb{R} \ \text{ for } k = 0,1,2,\ldots,n \ \ .$$

They satisfy the recurrence relation:

$$\phi_{n+1}(i\underline{x}) = (i\underline{x} - \widetilde{\alpha}_n)\ \phi_n(i\underline{x}) - \widetilde{\beta}_n\ \phi_{n-1}(i\underline{x}) \quad , \quad n \geq 0$$

with

$$\phi_{-1}(i\underline{x}) = 0 \quad \text{and} \quad \phi_0(i\underline{x}) = 1 \ \ .$$

The coefficients $\widetilde{\alpha}_n$ and $\widetilde{\beta}_n$ in the recurrence relation can be calculated from:

$$\widetilde{\beta}_n + \widetilde{\beta}_{n-1} + \widetilde{\alpha}_{n-1}^2 = \frac{2n+m-2}{2}$$

$$\widetilde{\alpha}_n \widetilde{\alpha}_{n-1} \widetilde{\beta}_n = \left(\frac{n + \frac{m-1}{2}}{2} - \widetilde{\beta}_n \right)^2 - \frac{(m-1)^2}{16} \quad ,$$

with starting values

$$\widetilde{\alpha}_0 = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \quad \text{and} \quad \widetilde{\beta}_0 = 0 \ \ .$$

A few examples are

$$\begin{aligned}
\phi_0(i\underline{x}) &= 1 \\
\phi_1(i\underline{x}) &= i\underline{x} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \\
\phi_2(i\underline{x}) &= (i\underline{x})^2 - \frac{\Gamma\left(\frac{m+1}{2}\right)}{2\ \Gamma\left(\frac{m}{2}\right) \left(\frac{m}{2} - \left(\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \right)^2 \right)}\ i\underline{x} \\
&\quad + \frac{\left(\Gamma\left(\frac{m+1}{2}\right) \right)^2}{2 \left(\Gamma\left(\frac{m}{2}\right) \right)^2 \left(\frac{m}{2} - \left(\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \right)^2 \right)} - \frac{m}{2}
\end{aligned}$$

etc.

Part II

The Clifford-Fourier Transform

Chapter 11

The Fractional Fourier transform in the framework of Clifford analysis

The Mehler formula for the classical Hermite polynomials on the real line allows for an integral representation of the one-dimensional Fractional Fourier transform. In this chapter we introduce a multi-dimensional Fractional Fourier transform in the Clifford analysis setting. By showing that it coincides with the classical multi-dimensional Fractional Fourier transform in the tensorial approach we are able to prove Mehler's formula for the generalized Clifford-Hermite polynomials of Clifford analysis.

11.1 Introduction

The Fractional Fourier transform (abbreviated FrFT) may be considered as a fractional power of the classical Fourier transform. It has been intensely studied during the last decade, an attention it may have partially gained because of the vivid interest in time-frequency analysis methods of signal processing (see for e.g. [40, 91, 95]).

In the one-dimensional case, one obtains an integral representation for the FrFT by means of Mehler's formula for the classical Hermite polynomials (see [117]). Here, we will proceed the other way around. First we introduce a multi-dimensional FrFT in the framework of Clifford analysis making use of the generalized Clifford-Hermite polynomials introduced in Section 5.3.1. Then we show that this FrFT coincides with the classical tensorial FrFT in higher dimension. In this way we are able to prove Mehler's formula for the generalized Clifford-Hermite polynomials.

The outline of this chapter is as follows. In Section 11.2 we describe the classical FrFT. Next, we introduce the FrFT in the Clifford analysis setting (Section 11.3). First we prove some differential equations satisfied by the generalized Clifford-Hermite polynomials which will be used in the sequel (Section 11.3.1). By means of the generalized Clifford-Hermite polynomials we then construct an orthonormal basis for the space of square integrable functions of eigenfunctions of the Fourier transform (Section 11.3.2). In Section 11.3.3 we define the FrFT in Clifford analysis and show that it can be written as an operator exponential. From this operator exponential form it becomes clear that our FrFT coincides with the classical tensorial higher dimensional FrFT. This allows us to derive the so-called Mehler formula for the generalized Clifford-Hermite polynomials (Section 11.4).

11.2 The classical Fractional Fourier transform

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as in 1929 (see [50, 80, 119]). It has been rediscovered in quantum mechanics, optics and signal processing. The boom in publications started in the early years of the 1990's and it is still going on. A recent state of the art can be found in [97].

The FrFT on the real line is first defined on a basis for the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$. For this basis one uses a complete orthonormal set of eigenfunctions of the Fourier transform given by

$$\mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \exp(-i\xi x)\; f(x)\; dx \quad , \quad f \in \mathcal{S}(\mathbb{R}) \quad .$$

A possible choice for these eigenfunctions are the normalized Hermite functions:

$$\phi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}}\; \exp\left(-\frac{x^2}{2}\right)\; H_n(x) \quad ,$$

where

$$H_n(x) = (-1)^n\; \exp(x^2)\; \frac{d^n}{dx^n}\left[\exp(-x^2)\right] \tag{11.1}$$

are the Hermite polynomials associated with the weight function $\exp(-x^2)$. These eigenfunctions satisfy the orthonormality relation

$$< \phi_n, \phi_m > = \delta_{n,m} \quad \text{with} \quad < f, g > = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\ g(x)\ dx$$

and the eigenvalue equation

$$\mathcal{F}[\phi_n] = \exp\left(-in\frac{\pi}{2}\right)\ \phi_n \quad .$$

The eigenvalue for ϕ_n is thus given by $\lambda_n = \lambda^n$ with $\lambda = \exp\left(-i\frac{\pi}{2}\right)$ representing a rotation over an angle $\frac{\pi}{2}$.
The FrFT generalizes this concept of rotating over an angle in the classical Fourier transform situation. Like the classical Fourier transform corresponds to a rotation in the time-frequency plane over an angle $\alpha = \frac{\pi}{2}$, the FrFT corresponds to a rotation over an arbitrary angle $\alpha = a\frac{\pi}{2}$ with $a \in \mathbb{R}$.
Consequently the FrFT is defined by

$$\mathcal{F}^a[\phi_n] = \exp\left(-ina\frac{\pi}{2}\right)\ \phi_n = \lambda_n^a\ \phi_n = \lambda_a^n\ \phi_n \quad , \tag{11.2}$$

with $\lambda_a = \exp\left(-ia\frac{\pi}{2}\right) = \exp(-i\alpha)$. Thus the classical Fourier transform corresponds to $\mathcal{F}^1$. Note also that for $\alpha = 0$ or $a = 0$ we get the identity operator $\mathcal{F}^0 = I$ and for $\alpha = \pi$ or $a = 2$ we get the parity operator $\mathcal{F}^2[f](\xi) = f(-\xi)$.

The FrFT can be written as an operator exponential $\mathcal{F}^a = \exp(-i\alpha\mathcal{H})$, so that

$$\exp(-i\alpha\mathcal{H})\left[\exp\left(-\frac{x^2}{2}\right)\ H_n(x)\right] = \exp(-in\alpha)\ \exp\left(-\frac{x^2}{2}\right)\ H_n(x) \quad .$$

Differentiating the above relation with respect to α, setting $\alpha = 0$ and then using the differential equation

$$\frac{d^2}{dx^2}[H_n(x)] - 2x\ \frac{d}{dx}[H_n(x)] + 2n\ H_n(x) = 0 \quad ,$$

one can easily verify that the operator $\mathcal{H}$ is given by $\mathcal{H} = -\frac{1}{2}\left(\frac{d^2}{dx^2} - x^2 + 1\right)$.

As the set of normalized Hermite functions ϕ_n constitutes an orthonormal basis for $L_2(\mathbb{R}, dx)$, each function $f \in L_2(\mathbb{R}, dx)$ can be expanded in terms of these eigenfunctions ϕ_n :

$$f = \sum_{n=0}^{\infty} a_n \ \phi_n \quad , \tag{11.3}$$

where the coefficients a_n are given by

$$\begin{aligned} a_n &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi_n(x) \ f(x) \ dx \\ &= \frac{1}{\sqrt{2^n n! \ \pi \ \sqrt{2}}} \int_{-\infty}^{+\infty} H_n(x) \ \exp\left(-\frac{x^2}{2}\right) \ f(x) \ dx \quad . \end{aligned} \tag{11.4}$$

Applying the FrFT on this function yields

$$\mathcal{F}^a[f] = \sum_{n=0}^{\infty} a_n \ \exp\left(-ina\frac{\pi}{2}\right) \ \phi_n \quad . \tag{11.5}$$

The calculation of Fractional Fourier transforms by means of the series (11.5) is usually not practical. In order to obtain the integral representation of the operator $\mathcal{F}^a$, a formula due to Mehler is used:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{2^n n!} \ \exp(-in\alpha) \ H_n(\xi) \ H_n(x) \\ &\quad = \frac{1}{\sqrt{1-\exp(-2i\alpha)}} \ \exp\left(\frac{2x\xi \ \exp(-i\alpha) - \exp(-2i\alpha) \ (\xi^2+x^2)}{1-\exp(-2i\alpha)}\right) \quad . \end{aligned}$$

Inserting a_n from equation (11.4) into equation (11.5) and using Mehler's formula, one obtains

$$\begin{aligned} \mathcal{F}^a[f](\xi) = {} & \frac{1}{\sqrt{\pi}\sqrt{1-\exp(-2i\alpha)}} \\ & \int_{-\infty}^{+\infty} \exp\left(\frac{2x\xi \ \exp(-i\alpha) - \exp(-2i\alpha) \ (\xi^2+x^2)}{1-\exp(-2i\alpha)}\right) \\ & \qquad \exp\left(-\frac{\xi^2+x^2}{2}\right) \ f(x) \ dx \quad . \end{aligned} \tag{11.6}$$

Note that for $0 < |\alpha| < \pi$ this expression can also be written as

$$\mathcal{F}^a[f](\xi) = \frac{1}{\sqrt{2\pi|\sin(\alpha)|}} \exp\left(-\frac{i}{2}\left(\frac{\pi}{2}\hat{\alpha} - \alpha\right)\right) \exp\left(\frac{i}{2}\xi^2 \cot(\alpha)\right) \int_{-\infty}^{+\infty} \exp\left(-i\frac{x\xi}{\sin(\alpha)} + \frac{i}{2}x^2 \cot(\alpha)\right) f(x)\, dx \quad ,$$

where $\hat{\alpha} = \text{sgn}(\sin(\alpha))$.
It was already mentioned that $\mathcal{F}^0[f](\xi) = f(\xi)$ and $\mathcal{F}^{\pm\pi}[f](\xi) = f(-\xi)$. Furthermore, when $|\alpha| > \pi$, the definition is taken modulo 2π and reduced to the interval $[-\pi, \pi]$.

The FrFT can be extended to higher dimensions by taking tensor products. If $K_a(\xi, x)$ denotes the kernel of the one-dimensional FrFT, i.e.

$$\mathcal{F}^a[f](\xi) = \int_{-\infty}^{+\infty} K_a(\xi, x)\ f(x)\ dx \quad ,$$

then one defines the m-dimensional FrFT as follows:

$$\mathcal{F}^{a_1,\ldots,a_m}[f](\xi_1, \ldots, \xi_m) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} K_{a_1,\ldots,a_m}(\xi_1, \ldots, \xi_m\ ;\ x_1, \ldots, x_m)\ f(x_1, \ldots, x_m)\ dV(\underline{x}) \quad ,$$

where

$$K_{a_1,\ldots,a_m}(\xi_1, \ldots, \xi_m\ ;\ x_1, \ldots, x_m) = K_{a_1}(\xi_1, x_1) \ldots K_{a_m}(\xi_m, x_m) \quad .$$

11.3 The Fractional Fourier transform in the framework of Clifford analysis

11.3.1 Differential equations for the generalized Clifford-Hermite polynomials

In this section we list a few useful differential equations satisfied by the generalized Clifford-Hermite polynomials (see Section 5.3.1).

Proposition 11.1 *The generalized Clifford-Hermite polynomials satisfy*
(i)

$$\partial_{\underline{x}}\big[\underline{x}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] = a_{\ell,k}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) + \underline{x}\ \partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big]$$

(ii)

$$\partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] = -C_{\ell,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})$$

(iii)

$$\partial^2_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] - \underline{x}\ \partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] = C_{\ell,k}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) \quad ,$$

with

$$a_{\ell,k} = \begin{cases} -(m+2k) & \textit{for } \ell \textit{ even} \\ m+2k-2 & \textit{for } \ell \textit{ odd} \end{cases}$$

and

$$C_{\ell,k} = \begin{cases} \ell & \textit{for } \ell \textit{ even} \\ \ell-1+m+2k & \textit{for } \ell \textit{ odd.} \end{cases}$$

Proof.
(i) Taking into account formula (2.5), we indeed have for ℓ even, i.e. $\ell = 2p$,

$$\begin{aligned} &\partial_{\underline{x}}\big[\underline{x}\ H_{2p,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = \partial_{\underline{x}}\big[H_{2p,k}(\underline{x})\big]\ \underline{x}\ P_k(\underline{x}) + H_{2p,k}(\underline{x})\ \partial_{\underline{x}}\big[\underline{x}\ P_k(\underline{x})\big] \\ &\quad = \underline{x}\ \partial_{\underline{x}}\big[H_{2p,k}(\underline{x})\big]P_k(\underline{x}) - (m+2k)\ H_{2p,k}(\underline{x})\ P_k(\underline{x}) \\ &\quad = \underline{x}\ \partial_{\underline{x}}\big[H_{2p,k}(\underline{x})\ P_k(\underline{x})\big] - (m+2k)\ H_{2p,k}(\underline{x})\ P_k(\underline{x}) \quad . \end{aligned}$$

Now we consider the case ℓ odd, i.e. $\ell = 2p+1$. As $H_{2p+1,k}(\underline{x})$ takes the form $\underline{x}f(r)$ with f a polynomial of degree p in r^2, we obtain

$$\begin{aligned} \partial_{\underline{x}}\big[\underline{x}\ H_{2p+1,k}(\underline{x})\ P_k(\underline{x})\big] &= \partial_{\underline{x}}\big[f(r)\ \underline{x}^2\ P_k(\underline{x})\big] \\ &= \partial_{\underline{x}}\big[f(r)\big]\ \underline{x}^2 P_k(\underline{x}) + f(r)\ \partial_{\underline{x}}\big[\underline{x}^2 P_k(\underline{x})\big] \\ &= \underline{x}\ \partial_{\underline{x}}\big[f(r)\big]\ \underline{x}\ P_k(\underline{x}) - 2f(r)\ \underline{x}\ P_k(\underline{x}) \quad . \end{aligned}$$

As

$$\partial_{\underline{x}}\big[H_{2p+1,k}(\underline{x})\ P_k(\underline{x})\big] = \partial_{\underline{x}}\big[f(r)\big]\ \underline{x}\ P_k(\underline{x}) - (m+2k)\ f(r)\ P_k(\underline{x}) \quad ,$$

we obtain the desired result:

$$\begin{aligned} &\partial_{\underline{x}}\big[\underline{x}\ H_{2p+1,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\qquad = \underline{x}\ \partial_{\underline{x}}\big[H_{2p+1,k}(\underline{x})\ P_k(\underline{x})\big] + (m+2k-2)\ H_{2p+1,k}(\underline{x})\ P_k(\underline{x}) \quad . \end{aligned}$$

(ii) We prove the statement by induction.
First, for $\ell = 0$ we have

$$\partial_{\underline{x}}\big[H_{0,k}(\underline{x})\ P_k(\underline{x})\big] = \partial_{\underline{x}}\big[P_k(\underline{x})\big] = 0 \quad ,$$

while for $\ell = 1$

$$\begin{aligned} \partial_{\underline{x}}\big[H_{1,k}(\underline{x})\ P_k(\underline{x})\big] &= \partial_{\underline{x}}\big[\underline{x}\ P_k(\underline{x})\big] = -(m+2k)\ P_k(\underline{x}) \\ &= -C_{1,k}\ H_{0,k}(\underline{x})\ P_k(\underline{x}) \quad . \end{aligned}$$

Next, assume that the property holds for $\ell - 1$, i.e.

$$\partial_{\underline{x}}\big[H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] = -C_{\ell-1,k}\ H_{\ell-2,k}(\underline{x})\ P_k(\underline{x}) \quad .$$

Using (5.5) and the induction hypothesis, we obtain

$$\begin{aligned} &\partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = \partial_{\underline{x}}\big[\underline{x}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] - \partial_{\underline{x}}^2\big[H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = \partial_{\underline{x}}\big[\underline{x}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] + C_{\ell-1,k}\ \partial_{\underline{x}}\big[H_{\ell-2,k}(\underline{x})\ P_k(\underline{x})\big] \quad . \end{aligned}$$

By means of (i), the induction hypothesis and (5.5), this becomes consecutively

$$\begin{aligned} &\partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = a_{\ell-1,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) + \underline{x}\ \partial_{\underline{x}}\big[H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\qquad + C_{\ell-1,k}\ \partial_{\underline{x}}\big[H_{\ell-2,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = a_{\ell-1,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) - C_{\ell-1,k}\ (\underline{x} - \partial_{\underline{x}})\big[H_{\ell-2,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = a_{\ell-1,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) - C_{\ell-1,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) \\ &\quad = -C_{\ell,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) \quad . \end{aligned}$$

(iii) By means of (ii) and formula (5.5) we have consecutively

$$\begin{aligned} &\partial_{\underline{x}}^2\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] - \underline{x}\ \partial_{\underline{x}}\big[H_{\ell,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = -C_{\ell,k}\ \partial_{\underline{x}}\big[H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] + C_{\ell,k}\ \underline{x}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) \\ &\quad = C_{\ell,k}\ (\underline{x} - \partial_{\underline{x}})\big[H_{\ell-1,k}(\underline{x})\ P_k(\underline{x})\big] \\ &\quad = C_{\ell,k}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) \quad . \quad \square \end{aligned}$$

Formula (iii) of Proposition 11.1 generalizes the differential equation

$$\frac{d^2}{dx^2}[H_n(x)] - 2x\ \frac{d}{dx}[H_n(x)] + 2nH_n(x) = 0$$

satisfied by the classical Hermite polynomials on the real line.
Furthermore, combining formula (ii) of Proposition 11.1 and formula (5.5) yields

$$H_{\ell+1,k}(\underline{x})\ P_k(\underline{x}) - \underline{x}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) - C_{\ell,k}\ H_{\ell-1,k}(\underline{x})\ P_k(\underline{x}) = 0 \quad , \tag{11.7}$$

which is a generalization of the recurrence relation

$$H_{n+1}(x) + 2nH_{n-1}(x) - 2xH_n(x) = 0$$

satisfied by the classical Hermite polynomials.

11.3.2 Orthonormal basis for $L_2(\mathbb{R}^m, dV(x))$ of eigenfunctions of the Fourier transform

In this section we will construct an orthonormal basis for $L_2(\mathbb{R}^m, dV(\underline{x}))$ consisting of eigenfunctions of the Fourier transform.
We start with the following result.

Proposition 11.2 *The set*

$$\left\{ \frac{1}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\underline{x})\ P_k^{(j)}(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{4}\right)\ ;\quad s,k \in \mathbb{N},\ j \leq \dim(M_\ell^+(k)) \right\}$$

constitutes an orthonormal basis for $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$.

Proof. By means of (5.10), the orthonormality of the set is straightforward. Now, take $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$. This means that

$$\int_{\mathbb{R}^m} |f(\underline{x})|^2\ dV(\underline{x}) = \int_{\mathbb{R}^m} \left| f(\underline{x})\ \exp\left(\frac{|\underline{x}|^2}{4}\right)\right|^2\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ dV(\underline{x}) < \infty \quad ,$$

In other words, $f(\underline{x})\ \exp\left(\frac{|\underline{x}|^2}{4}\right) \in L_2\left(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ dV(\underline{x})\right)$.
Consequently, there exists a linear combination

$$\sum_{s=0}^{N}\sum_{k=0}^{N'}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \frac{a_{s,k,j}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}\ P_k^{(j)} \quad , \quad a_{s,k,j} \in \mathbb{C}_m$$

such that

$$\left\| f \exp\left(\frac{|\underline{x}|^2}{4}\right) - \sum_{s=0}^{N}\sum_{k=0}^{N'}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \frac{a_{s,k,j}}{(\gamma_{s,k})^{1/2}} H_{s,k} P_k^{(j)} \right\|_{L_2\left(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right) dV(\underline{x})\right)}$$

tends to zero if $N, N' \to \infty$. This also implies that

$$\left\| f \exp\left(\frac{|\underline{x}|^2}{4}\right) - \sum_{s=0}^{N}\sum_{k=0}^{N'}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \frac{a_{s,k,j}}{(\gamma_{s,k})^{1/2}}\ H_{s,k} P_k^{(j)} \right\|^2_{L_2\left(\mathbb{R}^m, \exp\left(-\frac{|\underline{x}|^2}{2}\right) dV(\underline{x})\right)}$$

tends to zero if $N, N' \to \infty$ or that

$$\int_{\mathbb{R}^m} \left| f(\underline{x})\ \exp\left(\frac{|\underline{x}|^2}{4}\right) - \sum_{s=0}^{N}\sum_{k=0}^{N'}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \frac{a_{s,k,j}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\underline{x})\ P_k^{(j)}(\underline{x}) \right|^2 \exp\left(-\frac{|\underline{x}|^2}{2}\right) dV(\underline{x})$$

tends to zero if $N, N' \to \infty$.
So finally we obtain that

$$\int_{\mathbb{R}^m} \left| f(\underline{x}) - \sum_{s=0}^{N}\sum_{k=0}^{N'}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \frac{a_{s,k,j}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\underline{x})\ P_k^{(j)}(\underline{x}) \exp\left(-\frac{|\underline{x}|^2}{4}\right) \right|^2 dV(\underline{x})$$

tends to zero if $N, N' \to \infty$, which proves the statement. $\square$

In view of (2.7), we carry out the substitution $\underline{x} \to \sqrt{2}\ \underline{x}$, which leads to the following orthonormal basis for $L_2(\mathbb{R}^m, dV(\underline{x}))$:

$$\left\{ \frac{2^{m/4}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ ;\ s,k \in \mathbb{N},\ j \leq \dim(M_\ell^+(k)) \right\} .$$

Now we will prove that this basis consists of eigenfunctions of the Fourier transform.

Proposition 11.3 *For all solid inner spherical monogenics P_k of order $k \in \mathbb{N}$ and all $s \in \mathbb{N}$ one has*

$$\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right)\ H_{s,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi}) = \exp\left(-i(s+k)\frac{\pi}{2}\right) \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ H_{s,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi}) \ .$$

Proof. By means of Proposition 2.1 and Proposition 2.3, one can easily verify that the statement holds for $s = 0$ and $s = 1$.
Assuming that it holds for s, we now prove it for s+1.
By means of the recurrence relation (11.7) we have

$$\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s+1,k}(\sqrt{2}\underline{x})P_k(\sqrt{2}\underline{x}) = \sqrt{2}\underline{x}\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{x})P_k(\sqrt{2}\underline{x}) + C_{s,k}\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s-1,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x}) \quad .$$

Taking the Fourier transform yields:

$$\begin{aligned}\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s+1,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi}) &= \sqrt{2}i\ \partial_{\underline{\xi}}\left[\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi})\right] \\ &\quad + C_{s,k}\ \mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s-1,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi}) \quad .\end{aligned}$$

In view of the induction hypothesis, this becomes

$$\begin{aligned}\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s+1,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi}) &= \sqrt{2}i\ \exp\left(-i(s+k)\frac{\pi}{2}\right)\ \partial_{\underline{\xi}}\left[\exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi})\right] \\ &\quad + C_{s,k}\ \exp\left(-i(s-1+k)\frac{\pi}{2}\right)\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s-1,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi}) \quad .\end{aligned}$$

Using formula (ii) of Proposition 11.1 we have

$$\begin{aligned}\partial_{\underline{\xi}}\left[\exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{\xi})P_k(\sqrt{2}\underline{\xi})\right] &= -\underline{\xi}\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{\xi})P_k(\sqrt{2}\underline{\xi}) \\ &\quad - \sqrt{2}\ C_{s,k}\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s-1,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi}) \quad .\end{aligned}$$

Consequently we finally obtain

$$\begin{aligned}
&\mathcal{F}\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s+1,k}(\sqrt{2}\underline{x})\ P_k(\sqrt{2}\underline{x})\right](\underline{\xi}) \\
&= \exp\left(-i(s+1+k)\frac{\pi}{2}\right)\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\ \left(\sqrt{2}\underline{\xi}\ H_{s,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi})\right. \\
&\quad \left. + C_{s,k}\ H_{s-1,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi})\right) \\
&= \exp\left(-i(s+1+k)\frac{\pi}{2}\right)\ \exp\left(-\frac{|\underline{\xi}|^2}{2}\right) H_{s+1,k}(\sqrt{2}\underline{\xi})\ P_k(\sqrt{2}\underline{\xi}) \quad . \quad \square
\end{aligned}$$

11.3.3 The multi-dimensional Fractional Fourier transform: definition and operator exponential form

In what follows, we denote for $s, k \in \mathbb{N},\ j \leq \dim(M_\ell^+(k))$:

$$\phi_{s,k,j}(\underline{x}) = \frac{2^{m/4}}{(\gamma_{s,k})^{1/2}}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \quad .$$

In the previous section (see Proposition 11.3), we have shown that

$$\mathcal{F}\left[\phi_{s,k,j}\right](\underline{\xi}) = \exp\left(-i(s+k)\frac{\pi}{2}\right)\ \phi_{s,k,j}(\underline{\xi}) \quad .$$

Corresponding to the definition on the real line (see (11.2)), we define the multi-dimensional FrFT in Clifford analysis by:

$$\begin{aligned}
\mathcal{F}_C^a\left[\phi_{s,k,j}\right](\underline{\xi}) &= \exp\left(-i(s+k)a\frac{\pi}{2}\right)\ \phi_{s,k,j}(\underline{\xi}) \quad ; \quad a \in \mathbb{R} \\
&= \exp\left(-i(s+k)\alpha\right)\ \phi_{s,k,j}(\underline{\xi})
\end{aligned}$$

with $\alpha = a\dfrac{\pi}{2}$.

Now we will show that, similar to the classical case, the FrFT $\mathcal{F}_C^a$ can be written as an operator exponential.

Proposition 11.4 *The Fractional Fourier transform $\mathcal{F}_C^a$ can be written as an operator exponential*

$$\mathcal{F}_C^a = \exp\left(-i\alpha\mathcal{H}_C\right) = \exp\left(-ia\frac{\pi}{2}\mathcal{H}_C\right) \quad ,$$

where the operator $\mathcal{H}_C$ *is given by*

$$\mathcal{H}_C = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - mI) = -\frac{1}{2}(\Delta_m - |\underline{x}|^2 + mI)$$

with I *the identity operator.*

Proof. First we note that the operator exponential $\exp(-i\alpha\mathcal{H}_C)$ is defined as the series

$$\exp(-i\alpha\mathcal{H}_C) = \sum_{n=0}^{\infty}(-i\alpha)^n \, \frac{\mathcal{H}_C^n}{n!} \quad .$$

Differentiating the relation

$$\exp(-i\alpha\mathcal{H}_C)\left[H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x}) \; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right]$$
$$= \exp\big(-i(s+k)\alpha\big) \; H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x}) \; \exp\left(-\frac{|\underline{x}|^2}{2}\right)$$

with respect to α, and setting α equal to zero, yields

$$\mathcal{H}_C\left[H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x}) \; \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right]$$
$$= (s+k) \; H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x}) \; \exp\left(-\frac{|\underline{x}|^2}{2}\right) \quad .$$

Now we will verify that the operator $\mathcal{H}_C$ is indeed given by

$$\mathcal{H}_C = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - mI) \quad .$$

We have

$$(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) \; H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x})\right]$$
$$= -\exp\left(-\frac{|\underline{x}|^2}{2}\right) \; \partial_{\underline{x}}\big[\underline{x} \; H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x})\big]$$
$$- \; \underline{x} \; \exp\left(-\frac{|\underline{x}|^2}{2}\right) \; \partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x})\big]$$
$$+ \; \exp\left(-\frac{|\underline{x}|^2}{2}\right) \; \partial_{\underline{x}}^2\big[H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x})\big]$$
$$- \; m\exp\left(-\frac{|\underline{x}|^2}{2}\right) \; H_{s,k}(\sqrt{2}\underline{x}) \; P_k^{(j)}(\sqrt{2}\underline{x}) \quad . \qquad (11.8)$$

From formula (i) of Proposition 11.1 we readily obtain

$$\partial_{\underline{x}}\big[\underline{x}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \\ = a_{s,k}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x}) + \underline{x}\ \partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \quad .$$

Consequently, (11.8) becomes

$$\begin{aligned} &(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\right] \\ &= \exp\left(-\frac{|\underline{x}|^2}{2}\right) \Big\{\partial_{\underline{x}}^2\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] - 2\underline{x}\ \partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \\ &\qquad - \ (a_{s,k} + m)\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\Big\} \quad . \end{aligned}$$

Furthermore, formula (iii) of Proposition 11.1 implies

$$\partial_{\underline{x}}^2\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] - 2\underline{x}\ \partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \\ = 2\ C_{s,k}\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x}) \quad ,$$

which finally leads to

$$\begin{aligned} &\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - mI)\left[\exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\right] \\ &\quad = \ \frac{1}{2}\ \big(2C_{s,k} - a_{s,k} - m\big)\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \\ &\quad = \ (s+k)\ H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{2}\right) \quad , \end{aligned}$$

since for all s

$$2C_{s,k} - a_{s,k} - m = 2s + 2k \quad . \quad \square$$

From Proposition 11.4 we observe that, surprisingly, our FrFT coincides with the classical tensorial higher dimensional FrFT $\mathcal{F}^{a_1,\dots,a_m}$ with $a_1 = a_2 = \ldots = a_m = a$.

11.4 The Mehler Formula for the generalized Clifford-Hermite polynomials

A Clifford algebra-valued square integrable function $f(\underline{x})$ can be expanded in terms of the eigenfunctions $\{\phi_{s,k,j}\}$ (see Section 11.3.3):

$$f(\underline{x}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \phi_{s,k,j}(\underline{x})\ a_{s,k,j} \quad , \tag{11.9}$$

where the Clifford algebra-valued coefficients $a_{s,k,j}$ are given by

$$a_{s,k,j} = < \phi_{s,k,j}, f > = \int_{\mathbb{R}^m} \left(\phi_{s,k,j}(\underline{x})\right)^\dagger\ f(\underline{x})\ dV(\underline{x}) \quad . \tag{11.10}$$

By applying the operator $\mathcal{F}_C^a$, we get

$$\begin{aligned}
\mathcal{F}_C^a[f](\underline{\xi}) &= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \mathcal{F}_C^a[\phi_{s,k,j}](\underline{\xi})\ a_{s,k,j} \\
&= \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \exp\left(-i(s+k)\alpha\right)\ \phi_{s,k,j}(\underline{\xi})\ a_{s,k,j} \quad .
\end{aligned}$$

We thus have obtained the definition of the FrFT $\mathcal{F}_C^a$ in the form of a series.

By replacing $a_{s,k,j}$ in the series by their integral expression (11.10) it is turned into

$$\mathcal{F}_C^a[f](\underline{\xi}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \exp\left(-i(s+k)\alpha\right)\frac{2^{m/4}}{(\gamma_{s,k})^{1/2}} H_{s,k}(\sqrt{2}\underline{\xi}) P_k^{(j)}(\sqrt{2}\underline{\xi})$$

$$\exp\left(-\frac{|\underline{\xi}|^2}{2}\right)\int_{\mathbb{R}^m}\frac{2^{m/4}}{(\gamma_{s,k})^{1/2}}\ \left(H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\right)^\dagger\ \exp\left(-\frac{|\underline{x}|^2}{2}\right)\ f(\underline{x})\ dV(\underline{x})$$

$$= 2^{m/2}\int_{\mathbb{R}^m}\left\{\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M_\ell^+(k))} \frac{\exp\left(-i(s+k)\alpha\right)}{\gamma_{s,k}}\ H_{s,k}(\sqrt{2}\underline{\xi})\ P_k^{(j)}(\sqrt{2}\underline{\xi})\right.$$

$$\left.\left(H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\right)^\dagger\ \right\}\exp\left(-\frac{|\underline{x}|^2+|\underline{\xi}|^2}{2}\right)\ f(\underline{x})\ dV(\underline{x}) \quad . \tag{11.11}$$

On the other hand, from the previous section we know that our FrFT $\mathcal{F}_C^a$ coincides with $\mathcal{F}^{a,\ldots,a}$. Consequently, by means of (11.6) we have

$$
\begin{aligned}
\mathcal{F}_C^a[f](\underline{\xi}) &= \int_{\mathbb{R}^m} K_a(\xi_1, x_1)\ldots K_a(\xi_m, x_m)\ f(\underline{x})\ dV(\underline{x}) \\
&= \left(\frac{1}{\sqrt{\pi}\sqrt{1-\exp(-2i\alpha)}}\right)^m \int_{\mathbb{R}^m} \exp\left(\frac{2x_1\xi_1\exp(-i\alpha) - \exp(-2i\alpha)(\xi_1^2+x_1^2)}{1-\exp(-2i\alpha)}\right) \\
&\quad \exp\left(-\frac{\xi_1^2+x_1^2}{2}\right)\ldots\ \exp\left(\frac{2x_m\xi_m\exp(-i\alpha) - \exp(-2i\alpha)(\xi_m^2+x_m^2)}{1-\exp(-2i\alpha)}\right) \\
&\quad \exp\left(-\frac{\xi_m^2+x_m^2}{2}\right)\ f(\underline{x})\ dV(\underline{x}) \\
&= \left(\frac{1}{\sqrt{\pi}\sqrt{1-\exp(-2i\alpha)}}\right)^m \int_{\mathbb{R}^m} \exp\left(\frac{2 < \underline{x}, \underline{\xi} > \exp(-i\alpha)}{1-\exp(-2i\alpha)}\right) \\
&\quad \exp\left(-\frac{(|\underline{x}|^2+|\underline{\xi}|^2)\exp(-2i\alpha)}{1-\exp(-2i\alpha)}\right)\ \exp\left(-\frac{|\underline{x}|^2+|\underline{\xi}|^2}{2}\right)\ f(\underline{x})\ dV(\underline{x}) \quad . \qquad (11.12)
\end{aligned}
$$

Compairing (11.11) and (11.12) yields

Theorem 11.1 *The Mehler formula for the generalized Clifford-Hermite polynomials takes the following form:*

$$
\begin{aligned}
&\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{\exp(-i(s+k)\alpha)}{\gamma_{s,k}}\ H_{s,k}(\sqrt{2}\underline{\xi})\ \left(\sum_{j=1}^{\dim(M_\ell^+(k))} P_k^{(j)}(\sqrt{2}\underline{\xi})\big(P_k^{(j)}(\sqrt{2}\underline{x})\big)^\dagger\right) \\
&\big(H_{s,k}(\sqrt{2}\underline{x})\big)^\dagger \quad = \quad \left(\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{1-\exp(-2i\alpha)}}\right)^m \\
&\qquad \exp\left(\frac{2 < \underline{x}, \underline{\xi} > \exp(-i\alpha) - (|\underline{x}|^2+|\underline{\xi}|^2)\exp(-2i\alpha)}{1-\exp(-2i\alpha)}\right) \quad .
\end{aligned}
$$

Chapter 12

The Clifford-Fourier Transform

Recently several generalizations to higher dimension of the Fourier transform, using Clifford algebra, have been introduced, including our Clifford-Fourier transform, which we defined in [26] as an operator exponential with a Clifford algebra-valued kernel.
In this chapter an overview is given of all these generalizations. Moreover, an in depth study of our Clifford-Fourier transform is presented. Herewith, particular attention is paid to the two-dimensional situation, since in this case we succeed in finding a closed form for the integral kernel of the Clifford-Fourier transform leading to further properties, both in the L_1 and in the L_2 context (see [30]).

12.1 Introduction

Two robust tools used in image processing and computer vision for the analysis of scalar fields are *convolution* and *Fourier transformation.* Several attempts have been made to extend these methods to two- and three-dimensional vector fields and even multi-vector fields. Let us first give an overview of those generalized Fourier transforms.

In [38] Bülow and Sommer define a so-called quaternionic Fourier transform of two-dimensional signals $f(x_1, x_2)$ taking their values in the algebra $\mathbb{H}$ of real quaternions. If, traditionally, the basis vectors in $\mathbb{H}$ are denoted by i and j, with $i^2 = j^2 = -1$, then this quaternionic Fourier transform takes the form

$$\mathcal{F}^q[f](u_1, u_2) = \int_{\mathbb{R}^2} \exp(-2\pi i u_1 x_1)\ f(x_1, x_2)\ \exp(-2\pi j u_2 x_2)\ dV(\underline{x}) \quad .$$

Due to the non-commutativity of the multiplication in $\mathbb{H}$, the convolution theorem for this quaternionic Fourier transform is rather complicated. This is also the case for its higher dimensional analogue, the so-called Clifford-Fourier transform given by

$$\mathcal{F}^{cl}[f](\underline{u}) = \int_{\mathbb{R}^m} f(\underline{x})\ \exp(-2\pi e_1 u_1 x_1)\ \ldots\ \exp(-2\pi e_m u_m x_m)\ dV(\underline{x}) \quad .$$

Note that for $m = 1$ this Clifford-Fourier transform reduces to the standard Fourier transform on the real line, while for $m = 2$ the quaternionic Fourier transform is reobtained when restricting to real signals. Finally Bülow and Sommer also introduce a so-called *commutative* hypercomplex Fourier transform given by

$$\mathcal{F}^h[f](\underline{u}) = \int_{\mathbb{R}^m} f(\underline{x})\ \exp\left(-2\pi \sum_{j=1}^{m} \tilde{e}_j u_j x_j\right)\ dV(\underline{x}) \quad ,$$

where the basis vectors $(\tilde{e}_1, \ldots, \tilde{e}_m)$ obey the *commutative* multiplication rules $\tilde{e}_j\tilde{e}_k = \tilde{e}_k\tilde{e}_j$, $j, k = 1, \ldots, m$, while still $\tilde{e}_j^2 = -1$, $j = 1, \ldots, m$. This commutative hypercomplex Fourier transform offers the advantage of a simple convolution theorem. The hypercomplex Fourier transforms $\mathcal{F}^q$, $\mathcal{F}^{cl}$ and $\mathcal{F}^h$ enable Bülow and Sommer to establish a theory of multi-dimensional signal analysis and in particular to introduce the notions of multi-dimensional analytic signal, Gabor filter (see the next chapter), instantaneous and local amplitude and phase, etc. In [67] Felsberg makes use of the low dimensional Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{3,0}$ to define his Clifford-Fourier transform as

$$\mathcal{F}[f](\underline{u}) = \int \exp(-2\pi I < \underline{u}, \underline{x} >)\ f(\underline{x})\ dV(\underline{x}) \quad ,$$

where I stands for the pseudo-scalar $e_1 e_2$ in the case of one-dimensional signals, or $e_1 e_2 e_3$ in the case of two-dimensional signals. It is used a.o. to introduce a concept of two-dimensional analytic signal.

In [60, 62] Ebling and Scheuermann study convolution and Clifford-Fourier transformation of two- and three-dimensional signals, using the respective Fourier

kernels $\exp\left(-e_1e_2(\xi_1x_1+\xi_2x_2)\right)$ and $\exp\left(-e_1e_2e_3(\xi_1x_1+\xi_2x_2+\xi_3x_3)\right)$, where again e_1e_2 and $e_1e_2e_3$ are the pseudo-scalars in the Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{3,0}$ respectively. Note that the latter Fourier kernel is also used by Mawardi and Hitzer in [90] to define their Clifford-Fourier transform of three-dimensional signals. These Clifford-Fourier transforms and the corresponding convolution theorems allow Ebling and Scheuermann for a.o. the analysis of vector-valued patterns in the frequency domain.
The above mentioned Clifford-Fourier kernel of Bülow and Sommer $\exp\left(-2\pi e_1u_1x_1\right)\ \ldots\ \exp\left(-2\pi e_mu_mx_m\right)$ was already introduced in [15] and [104] as a theoretical concept in the framework of Clifford analysis. This generalized Fourier transform was further elaborated by Sommen in [106, 107] in connection with similar generalizations of the Cauchy, Hilbert and Laplace transforms. In this context also the work of Li, McIntosh and Qian should be mentioned; in [81] they generalize the standard multi-dimensional Fourier transform of a function in $\mathbb{R}^m$, by extending the Fourier kernel $\exp\left(i<\underline{\xi},\underline{x}>\right)$ to a function which is holomorphic in $\mathbb{C}^m$ and monogenic in $\mathbb{R}^{m+1}$.

Recall that one of the most fundamental features of Clifford analysis is the factorization (2.3) of the Laplace operator. Whereas in general the square root of the Laplace operator is only a pseudo-differential operator, by embedding Euclidean space into a Clifford algebra $\sqrt{-\Delta_m}$ can be realized as the Dirac operator $\partial_{\underline{x}}$. It occurred to us that, in the same order of ideas, the classical multi-dimensional Fourier transform, should not be replaced nor improved by a Clifford analysis alternative. However, a refinement of the classical Fourier transform automatically appears within the language of Clifford analysis in much the same way as the notion of electron spin appears in the Pauli matrix formalism. It is what we call the "Clifford-Fourier" transform.
The key step in its construction is to interpret the standard Fourier transform as an operator exponential (see Section 11.3.3):

$$\mathcal{F}=\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\frac{\pi}{2}\right)^k\ \mathcal{H}^k \quad ,$$

where $\mathcal{H}$ is the scalar operator

$$\mathcal{H}=\frac{1}{2}(-\Delta_m+r^2-m) \quad .$$

Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second order operators containing the angular Dirac operator Γ, leads in a natural way to a *pair* of transforms $\mathcal{F}_{\mathcal{H}^\pm}$, the harmonic average of which is precisely the standard Fourier transform.

Moreover, the two-dimensional case of this Clifford-Fourier transform is special in that we succeed in finding a closed form for the kernel of the integral representation. This closed form enables us to generalize the well-known results for the standard Fourier transform both in the L_1 and in the L_2 context. Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension, which would be very interesting for application to e.g. 3D flow analysis.

Let us end this introductory section with an overview of the content of this chapter. In Section 12.2 we recall two alternative approaches to the classical Fourier transform. Next, we discuss two commuting operators O_1 and O_2 (Section 12.3.1) which are used to split the scalar-valued kernel operator $\mathcal{H}$ and thus are crucial to the definition of the Clifford-Fourier transform (Section 12.3.2). The eigenfunctions of this new Clifford-Fourier transform are computed and its relation with the standard Fourier transform is established. Furthermore, we develop an adequate operational calculus in Section 12.3.3. Next, we thoroughly study the Clifford-Fourier transform in the specific case of two dimensions. We start with the computation of its integral kernel (Section 12.4.1). In Section 12.4.2 we examine the two-dimensional Clifford-Fourier transform as a linear operator in respectively the space of integrable functions, the space of rapidly decreasing functions and the space of square integrable functions. Furthermore, in Section 12.4.3, we give an explicit connection between the two-dimensional Clifford-Fourier transform and the standard tensorial Fourier transform and moreover a surprising connection with the Clifford-Fourier transform of Ebling and Scheuermann. In Section 12.4.4 we calculate, as an example, the two-dimensional Clifford-Fourier transform of the box function. We end this chapter with a discussion of the new possibilities offered by the Clifford-Fourier transform for the analysis of vector field signals.

12.2 Alternative representations of the classical Fourier transform

The idea behind the definition of our Clifford-Fourier transform originates from the *operator exponential representation* of the classical Fourier transform:

$$\mathcal{F}[f] = \exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[f] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-i\frac{\pi}{2}\right)^n \mathcal{H}^n[f] \tag{12.1}$$

with $\mathcal{H}$ the scalar-valued differential operator given by

$$\mathcal{H} = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2 - m) = \frac{1}{2}(-\Delta_m + r^2 - m) \quad .$$

Note that the operators $\mathcal{H}$ and $\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)$ are Fourier invariant, i.e.

$$\mathcal{F}\big[\mathcal{H}[f]\big] = \mathcal{H}\big[\mathcal{F}[f]\big] \quad \text{and} \quad \mathcal{F}\left[\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[f]\right] = \exp\left(-i\frac{\pi}{2}\mathcal{H}\right)\big[\mathcal{F}[f]\big] \quad .$$

The equivalence of this operator exponential form (12.1) with the traditional integral form (2.6) may be proved in a rather easy way in the framework of Clifford analysis.
From the previous chapter, we know that the set

$$\left\{\psi_{s,k,j}(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\right\} \quad ,$$

$s, k \in \mathbb{N}\cup\{0\}$, $j = 1, 2, \ldots, \dim\big(M_\ell^+(k)\big)$, constitutes an *orthogonal* basis for the space $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ of square integrable functions. The basis functions $\psi_{s,k,j}$ satisfy the orthogonality relation

$$< \psi_{s,k_1,j_1} \ , \ \psi_{t,k_2,j_2} > = \frac{\gamma_{s,k_1}}{2^{m/2}}\ \delta_{s,t}\ \delta_{k_1,k_2}\ \delta_{j_1,j_2} \quad . \tag{12.2}$$

Hence a square integrable function f can be expanded as follows

$$f(\underline{x}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim\left(M_\ell^+(k)\right)} \psi_{s,k,j}(\underline{x})\ b_{s,k,j} \quad . \tag{12.3}$$

The orthogonality relation (12.2) implies that the Clifford algebra-valued coefficients $b_{s,k,j}$ are given by the integral representation

$$b_{s,k,j} = \frac{2^{m/2}}{\gamma_{s,k}} \int_{\mathbb{R}^m} \big(\psi_{s,k,j}(\underline{x})\big)^\dagger\ f(\underline{x})\ dV(\underline{x}) \quad . \tag{12.4}$$

In Section 11.3 we have shown that these $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$-basis functions $\psi_{s,k,j}$ are simultaneous eigenfunctions of the Fourier transform operator $\mathcal{F}$ in integral form and of the kernel operator $\mathcal{H}$. We thus have at the same time (see

Proposition 11.3):

$$\begin{aligned} \mathcal{F}[\psi_{s,k,j}](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \psi_{s,k,j}(\underline{x}) \, dV(\underline{x}) \\ &= \exp\left(-i(s+k)\frac{\pi}{2}\right) \psi_{s,k,j}(\underline{\xi}) \end{aligned}$$

and (see proof of Proposition 11.4)

$$\mathcal{H}\big[\psi_{s,k,j}(\underline{x})\big] = (s+k) \; \psi_{s,k,j}(\underline{x}) \quad .$$

It then follows that

$$\begin{aligned} \exp\left(-i\frac{\pi}{2}\mathcal{H}\right)[\psi_{s,k,j}] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\pi}{2}\right)^n \mathcal{H}^n[\psi_{s,k,j}] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i\frac{\pi}{2}\right)^n (s+k)^n \; \psi_{s,k,j} \\ &= \exp\left(-i\frac{\pi}{2}(s+k)\right) \psi_{s,k,j} \\ &= \mathcal{F}[\psi_{s,k,j}] \quad , \end{aligned}$$

which immediately gives rise to the desired equivalence in $L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$.

Moreover, if the function $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ is developed in terms of the basis functions $\psi_{s,k,j}$ according to (12.3), then its Fourier transform takes the *series expansion form*

$$\mathcal{F}[f](\underline{\xi}) = \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=1}^{\dim(M_\ell^+(k))} \exp\left(-i(s+k)\frac{\pi}{2}\right) \psi_{s,k,j}(\underline{\xi}) \; b_{s,k,j} \quad .$$

12.3 The Clifford-Fourier Transform

Note that due to the scalar character of the standard Fourier kernel, the Fourier spectrum inherits its Clifford algebra character from the original signal, without any interaction with the Fourier kernel. So in order to genuinely introduce the Clifford analysis character in the Fourier transform, the idea occurred to us to replace in the operator exponential (12.1), the scalar-valued operator $\mathcal{H}$ by a Clifford algebra-valued one. To that end we aim at factorizing the operator $\mathcal{H}$, making use of the factorization of the Laplace operator by the Dirac operator. This leads us to considering the operators

$$O_1 = \frac{1}{2}\,(\partial_{\underline{x}} - \underline{x})(\partial_{\underline{x}} + \underline{x}) \quad \text{and} \quad O_2 = \frac{1}{2}\,(\partial_{\underline{x}} + \underline{x})(\partial_{\underline{x}} - \underline{x}) \ ,$$

which turn out to be crucial in our approach.

12.3.1 The operators O_1 and O_2

The operators O_1 and O_2 were introduced in [31] while studying (anti-) monogenic operators in the generalized Clifford-Hermite polynomial setting. They satisfy the following properties.

Proposition 12.1 *One has*
(i)

$$O_1 = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) + \left(\Gamma - \frac{m}{2}\right) = \mathcal{H} + \Gamma$$

(ii)

$$O_2 = \frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2) - \left(\Gamma - \frac{m}{2}\right) = \mathcal{H} - \Gamma + m$$

(iii)

$$O_1 + O_2 = \partial_{\underline{x}}^2 - \underline{x}^2 = 2\left(\mathcal{H} + \frac{m}{2}\right)$$

(iv)

$$O_1 - O_2 = 2\left(\Gamma - \frac{m}{2}\right)$$

(v) O_1 and O_2 are Fourier invariant operators
(vi) O_1 and O_2 are commuting operators
(vii)

$$O_1\big[\psi_{s,k,j}(\underline{x})\big] = C_{s,k}\ \psi_{s,k,j}(\underline{x})$$

(viii)

$$O_2\big[\psi_{s,k,j}(\underline{x})\big] = C_{s+1,k}\ \psi_{s,k,j}(\underline{x}) \ .$$

Proof.
(i)(ii) Taking into account that the angular Dirac operator Γ may be written as

$$\Gamma = -\frac{1}{2}(\underline{x}\ \partial_{\underline{x}} - \partial_{\underline{x}}\ \underline{x} - m) \ ,$$

the results follow from a straightforward computation.
(iii)(iv) Trivial.
(v) This property follows directly from the Fourier invariance of the operators $\mathcal{H}$ and Γ.
(vi) As Γ commutes with the Laplace operator Δ_m and with the multiplication operator r, we have that

$$\left[\frac{1}{2}(\partial_{\underline{x}}^2 - \underline{x}^2), \Gamma\right] = \left[\frac{1}{2}(-\Delta_m + r^2), \Gamma\right] = 0$$

which, in view of (i) and (ii) yields $[O_1, O_2] = 0$.
(vii)(viii) First, we have that

$$(\partial_{\underline{x}} - \underline{x})[\psi_{s,k,j}(\underline{x})] = \exp\left(-\frac{|\underline{x}|^2}{2}\right) (\partial_{\underline{x}} - 2\underline{x})\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \quad . \tag{12.5}$$

Formula (5.5) implies that

$$(2\underline{x} - \partial_{\underline{x}})\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] = \sqrt{2}\ H_{s+1,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x}) \quad .$$

Consequently, (12.5) becomes

$$(\partial_{\underline{x}} - \underline{x})[\psi_{s,k,j}(\underline{x})] = -\sqrt{2}\ \psi_{s+1,k,j}(\underline{x}) \quad . \tag{12.6}$$

Next, it is immediately verified that

$$(\partial_{\underline{x}} + \underline{x})[\psi_{s,k,j}(\underline{x})] = \exp\left(-\frac{|\underline{x}|^2}{2}\right) \partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] \quad .$$

Moreover, from formula (ii) of Proposition 11.1 we readily obtain that

$$\partial_{\underline{x}}\big[H_{s,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x})\big] = -\sqrt{2}\ C_{s,k}\ H_{s-1,k}(\sqrt{2}\underline{x})\ P_k^{(j)}(\sqrt{2}\underline{x}) \quad .$$

Hence, we find that

$$(\partial_{\underline{x}} + \underline{x})[\psi_{s,k,j}(\underline{x})] = -\sqrt{2}\ C_{s,k}\ \psi_{s-1,k,j}(\underline{x}) \quad . \tag{12.7}$$

By combining the results (12.6) and (12.7), the basis functions $\psi_{s,k,j}$ are found to be eigenfunctions of O_1 and O_2. □

Remark 12.1 Note that $(\partial_{\underline{x}} - \underline{x})$ increases the degree of the generalized Clifford-Hermite polynomial, so that it may be qualified as a *creation operator*. In the same order of ideas, $(\partial_{\underline{x}} + \underline{x})$ is an *annihilation operator*.

12.3.2 The definition of the Clifford-Fourier Transform

In view of Proposition 12.1 (vii) and (viii), we define the Clifford-Fourier transform as the *pair* of transformations

$$\mathcal{F}_{\mathcal{H}^+} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^+\right) \quad \text{and} \quad \mathcal{F}_{\mathcal{H}^-} = \exp\left(-i\frac{\pi}{2}\mathcal{H}^-\right)$$

with the operators $\mathcal{H}^+$ and $\mathcal{H}^-$ closely linked to the operators O_1 and O_2 . As we want the classical Fourier transform to be the harmonic average of the Clifford-Fourier transform pair $\{\mathcal{F}_{\mathcal{H}^+}, \mathcal{F}_{\mathcal{H}^-}\}$, i.e.

$$\mathcal{F}^2 = \mathcal{F}_{\mathcal{H}^+}\ \mathcal{F}_{\mathcal{H}^-}$$

with $\mathcal{F}^2$ the parity operator:

$$\mathcal{F}^2[f](\underline{x}) = f(-\underline{x}) \quad ,$$

the operators $\mathcal{H}^+$ and $\mathcal{H}^-$ must satisfy

$$\mathcal{H}^+ + \mathcal{H}^- = 2\ \mathcal{H}$$

or

$$\mathcal{H}^+ + \mathcal{H}^- = \partial_{\underline{x}}^2 - \underline{x}^2 - m = O_1 + O_2 - m \quad .$$

This eventually inspires the following definition of $\mathcal{H}^+$ and $\mathcal{H}^-$.

Definition 12.1 *One puts*

$$\mathcal{H}^+ = O_1 \quad \textit{and} \quad \mathcal{H}^- = O_2 - m \quad .$$

Note that the operators $\mathcal{H}^+$ and $\mathcal{H}^-$ contain a scalar part and a bivector part. The following properties are easily proved.

Proposition 12.2 *One has*
(i)

$$\mathcal{H}^\pm = \mathcal{H} \pm \Gamma \quad .$$

(ii) $\mathcal{H}^+$ and $\mathcal{H}^-$ are Fourier invariant.
(iii)

$$\mathcal{H}^\pm\left[\psi_{s,k,j}(\underline{x})\right] = C^\pm_{s,k}\ \psi_{s,k,j}(\underline{x})$$

with

$$C^+_{s,k} := C_{s,k}$$

and

$$C^-_{s,k} := C_{s+1,k} - m = \begin{cases} s + 2k & \textit{for } s \textit{ even} \\ s + 1 - m & \textit{for } s \textit{ odd} \quad . \end{cases}$$

Corollary 12.1 *The basis functions $\psi_{s,k,j}$ are eigenfunctions of the Clifford-Fourier transform:*

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\psi_{s,k,j}\right](\underline{\xi}) = \exp\left(-i\frac{\pi}{2}C^{\pm}_{s,k}\right)\ \psi_{s,k,j}(\underline{\xi}) \quad .$$

Now if $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ is expanded w.r.t. the basis $\big\{\psi_{s,k,j}(\underline{x})\ ;$ $s,k \in \mathbb{N}\cup\{0\}, j = 1,\ldots,\dim\big(M^+_\ell(k)\big)\big\}$, the eigenvalue equation of Corollary 12.1 immediately yields the series representation of the Clifford-Fourier transform:

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) = \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=1}^{\dim(M^+_\ell(k))} \exp\left(-i\frac{\pi}{2}C^{\pm}_{s,k}\right)\ \psi_{s,k,j}(\underline{\xi})\ b_{s,k,j} \quad ,$$

the coefficients $b_{s,k,j}$ being given by (12.4).

Moreover, as the orthogonal L_2-basis $\big\{\psi_{s,k,j}(\underline{x})\ ;\ s,k \in \mathbb{N}\cup\{0\}, j = 1,\ldots,$ $\dim\big(M^+_\ell(k)\big)\big\}$ consists of eigenfunctions of both the operators $\mathcal{H}$ and Γ, one can easily verify the following properties.

Proposition 12.3
(i) The operators $\mathcal{H}$, Γ, O_1, O_2, $\mathcal{H}^+$ and $\mathcal{H}^-$ are self-adjoint, i.e. for all $f, g \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ and T any of the mentioned operators one has

$$< T[f], g > = < f, T[g] > \quad .$$

(ii) The operators $\mathcal{H}$, O_1 and O_2 are non-negative, i.e. for each $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ and T any of the mentioned operators one has

$$\big[< T[f], f >\big]_0 \geq 0 \quad .$$

Next, by means of Proposition 12.2 (i), we obtain in terms of operator exponentials

$$\begin{aligned} \mathcal{F}_{\mathcal{H}^\pm} &= \exp\left(-i\frac{\pi}{2}\left(\mathcal{H}\pm\Gamma\right)\right) = \exp\left(\mp i\frac{\pi}{2}\Gamma\right)\ \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) \\ &= \exp\left(\mp i\frac{\pi}{2}\Gamma\right)\ \mathcal{F} \quad . \end{aligned} \tag{12.8}$$

This establishes the relationship between the classical Fourier transform and the newly introduced Clifford-Fourier transform. Note that use has been made of the commuting property of the operators $\mathcal{H}$ and Γ, so that indeed

$$\begin{aligned} \exp\left(-i\frac{\pi}{2}(\mathcal{H}\pm\Gamma)\right) &= \exp\left(-i\frac{\pi}{2}\mathcal{H}\right)\ \exp\left(\mp i\frac{\pi}{2}\Gamma\right) \\ &= \exp\left(\mp i\frac{\pi}{2}\Gamma\right)\ \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) \quad . \end{aligned}$$

It thus turns out that the Clifford-Fourier transform is obtained as the composition of the classical Fourier transform with the operator exponential

$$\exp\left(\mp i\frac{\pi}{2}\Gamma\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mp i\frac{\pi}{2}\right)^k \Gamma^k \quad .$$

As an immediate consequence, we obtain an integral representation for the Clifford-Fourier transform:

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(\mp i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right) \left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] f(\underline{x})\, dV(\underline{x}) \quad .$$

Introducing the square root of the Clifford-Fourier transforms, in the sense of the Fractional Fourier Transform (see [95] and [97]), by

$$\sqrt{\mathcal{F}_{\mathcal{H}^\pm}} = \exp\left(-i\frac{\pi}{4}\ \mathcal{H}^\pm\right)$$

we also obtain that

$$\sqrt{\mathcal{F}_{\mathcal{H}^+}}\ \sqrt{\mathcal{F}_{\mathcal{H}^-}} = \sqrt{\mathcal{F}_{\mathcal{H}^-}}\ \sqrt{\mathcal{F}_{\mathcal{H}^+}} = \exp\left(-i\frac{\pi}{4}\ \left(\mathcal{H}^+ + \mathcal{H}^-\right)\right) = \exp\left(-i\frac{\pi}{2}\ \mathcal{H}\right)$$

leading to the factorization of the standard Fourier transform:

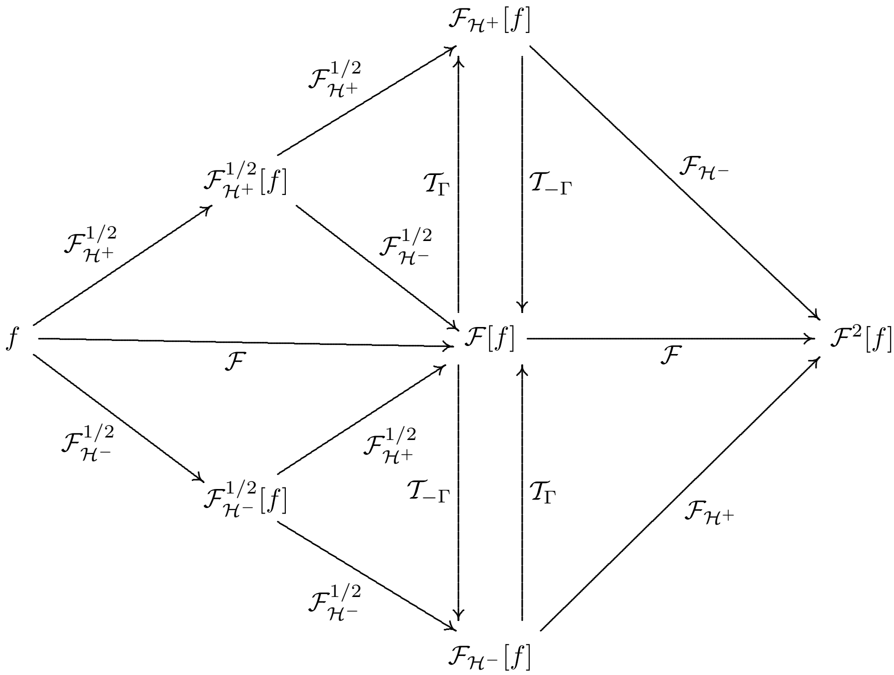
$$\mathcal{F} = \sqrt{\mathcal{F}_{\mathcal{H}^+}}\ \sqrt{\mathcal{F}_{\mathcal{H}^-}} = \sqrt{\mathcal{F}_{\mathcal{H}^-}}\ \sqrt{\mathcal{F}_{\mathcal{H}^+}} \quad .$$

Note that each operator which is (anti-) invariant under the classical Fourier transform and commutes with the angular Dirac operator Γ, is also (anti-) invariant under the Clifford-Fourier transform. For example, the operators $\Gamma - \frac{m}{2}$ and $E + \frac{m}{2}$ are respectively invariant and anti-invariant under the classical Fourier transform; as they both commute with the angular Dirac operator Γ, they show the (anti-)invariance property w.r.t the Clifford-Fourier transform.

For the inversion of the Clifford-Fourier transform, it suffices to observe that

$$\left(\mathcal{F}_{\mathcal{H}^\pm}\right)^{-1} = \exp\left(i\frac{\pi}{2}\mathcal{H}^\pm\right) = \exp\left(\pm i\frac{\pi}{2}\Gamma\right) \mathcal{F}^{-1} \quad .$$

Finally, using the notation $\mathcal{T}_T = \exp\left(-i\frac{\pi}{2}T\right)$, we can draw the following picture



12.3.3 Operational calculus

As is the case for the classical Fourier transform, an operational calculus may be based upon the Clifford-Fourier transform. The operational formulae are derived from the relation (12.8) expressing the Clifford-Fourier transform in terms of the classical Fourier transform $\mathcal{F}$.

Proposition 12.4 *The Clifford-Fourier transform satisfies:*
(i) the linearity property

$$\mathcal{F}_{\mathcal{H}^\pm}[f\lambda + g\mu] = \mathcal{F}_{\mathcal{H}^\pm}[f]\ \lambda + \mathcal{F}_{\mathcal{H}^\pm}[g]\ \mu \quad for\ \lambda, \mu \in \mathbb{C}_m$$

(ii) the change of scale property

$$\mathcal{F}_{\mathcal{H}^\pm}\left[f(a\underline{x})\right](\underline{\xi}) = \frac{1}{a^m}\ \mathcal{F}_{\mathcal{H}^\pm}\left[f(\underline{x})\right]\left(\frac{\underline{\xi}}{a}\right) \quad for\ a \in \mathbb{R}_+$$

(iii) the multiplication rule

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\underline{x} f(\underline{x})\right](\underline{\xi}) = \mp\ (\mp i)^m\ \partial_{\underline{\xi}}\ \mathcal{F}_{\mathcal{H}^\mp}\left[f(\underline{x})\right](\underline{\xi})$$

and more generally

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm}\left[\underline{x}^{2n} f(\underline{x})\right](\underline{\xi}) &= (-1)^n\ \partial_{\underline{\xi}}^{2n}\ \mathcal{F}_{\mathcal{H}^\pm}\left[f(\underline{x})\right](\underline{\xi}) \\ \mathcal{F}_{\mathcal{H}^\pm}\left[\underline{x}^{2n+1} f(\underline{x})\right](\underline{\xi}) &= \mp\ (-1)^n\ (\mp i)^m\ \partial_{\underline{\xi}}^{2n+1}\ \mathcal{F}_{\mathcal{H}^\mp}\left[f(\underline{x})\right](\underline{\xi})\end{aligned}$$

(iv) the differentiation rule

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\partial_{\underline{x}} f(\underline{x})\right](\underline{\xi}) = \mp\ (\mp i)^m\ \underline{\xi}\ \mathcal{F}_{\mathcal{H}^\mp}\left[f(\underline{x})\right](\underline{\xi})$$

and more generally

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm}\left[\partial_{\underline{x}}^{2n} f(\underline{x})\right](\underline{\xi}) &= (-1)^n\ \underline{\xi}^{2n}\ \mathcal{F}_{\mathcal{H}^\pm}\left[f(\underline{x})\right](\underline{\xi}) \\ \mathcal{F}_{\mathcal{H}^\pm}\left[\partial_{\underline{x}}^{2n+1} f(\underline{x})\right](\underline{\xi}) &= \mp\ (-1)^n\ (\mp i)^m\ \underline{\xi}^{2n+1}\ \mathcal{F}_{\mathcal{H}^\mp}\left[f(\underline{x})\right](\underline{\xi})\end{aligned}$$

(v) the mixed product rule

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm}\left[(\underline{x}\partial_{\underline{x}})^n f(\underline{x})\right](\underline{\xi}) &= (-1)^n\ (\partial_{\underline{\xi}}\underline{\xi})^n\ \mathcal{F}_{\mathcal{H}^\pm}\left[f(\underline{x})\right](\underline{\xi}) \\ \mathcal{F}_{\mathcal{H}^\pm}\left[(\partial_{\underline{x}}\underline{x})^n f(\underline{x})\right](\underline{\xi}) &= (-1)^n\ (\underline{\xi}\partial_{\underline{\xi}})^n\ \mathcal{F}_{\mathcal{H}^\pm}\left[f(\underline{x})\right](\underline{\xi}) \quad .\end{aligned}$$

Proof.
(i) Immediate. Note however that the Clifford algebra-valued coefficients λ and μ must be at the right of the functions f and g.
(ii) As the classical Fourier transform $\mathcal{F}$ satisfies the change of scale property

$$\mathcal{F}[f(a\underline{x})](\underline{\xi}) = \frac{1}{a^m}\ \mathcal{F}[f(\underline{x})]\left(\frac{\underline{\xi}}{a}\right) \quad , \quad a \in \mathbb{R}_+ \quad ,$$

we obtain

$$\mathcal{F}_{\mathcal{H}^\pm}[f(a\underline{x})](\underline{\xi}) = \frac{1}{a^m}\ \exp\left(\mp i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\ \mathcal{F}[f(\underline{x})]\left(\frac{\underline{\xi}}{a}\right) \quad .$$

The angular Dirac operator $\Gamma_{\underline{\xi}}$ is homogeneous of degree zero, since it commutes with the Euler operator E. Consequently we have

$$\Gamma_{\underline{\xi}} = \Gamma_{\underline{\xi}/a} \quad .$$

Hence the change of scale property also holds for the Clifford-Fourier transform:

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm}[f(a\underline{x})](\underline{\xi}) &= \frac{1}{a^m}\ \exp\left(\mp i\frac{\pi}{2}\Gamma_{\underline{\xi}/a}\right)\ \mathcal{F}[f(\underline{x})]\left(\frac{\underline{\xi}}{a}\right) \\ &= \frac{1}{a^m}\ \mathcal{F}_{\mathcal{H}^\pm}[f(\underline{x})]\left(\frac{\underline{\xi}}{a}\right) \quad .\end{aligned}$$

(iii) From the multiplication rule for the classical Fourier transform (see Proposition 2.1), we obtain at once that

$$\mathcal{F}_{\mathcal{H}^+}[\underline{x} f(\underline{x})](\underline{\xi}) = i \exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right) \partial_{\underline{\xi}} \mathcal{F}[f(\underline{x})](\underline{\xi}) \quad .$$

Repeated application of the commutation relation (see for e.g. [112])

$$\Gamma_{\underline{\xi}}\ \partial_{\underline{\xi}} = \partial_{\underline{\xi}}\ (m-1-\Gamma_{\underline{\xi}}) \quad ,$$

yields

$$\Gamma_{\underline{\xi}}^k\ \partial_{\underline{\xi}} = \partial_{\underline{\xi}}\ (m-1-\Gamma_{\underline{\xi}})^k \quad ,$$

which enables us to prove

$$\begin{aligned}
\exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right) \partial_{\underline{\xi}} &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-i\frac{\pi}{2}\right)^k \Gamma_{\underline{\xi}}^k\ \partial_{\underline{\xi}} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-i\frac{\pi}{2}\right)^k \partial_{\underline{\xi}}\ (m-1-\Gamma_{\underline{\xi}})^k \\
&= \partial_{\underline{\xi}}\ \exp\left(-i\frac{\pi}{2}(m-1-\Gamma_{\underline{\xi}})\right) \quad .
\end{aligned}$$

Consequently we obtain

$$\begin{aligned}
\mathcal{F}_{\mathcal{H}^+}\left[\underline{x} f(\underline{x})\right](\underline{\xi}) &= i\ \partial_{\underline{\xi}}\ \exp\left(-i\frac{\pi}{2}(m-1-\Gamma_{\underline{\xi}})\right) \mathcal{F}\left[f(\underline{x})\right](\underline{\xi}) \\
&= i\ \exp\left(-i\frac{\pi}{2}m\right) i\ \partial_{\underline{\xi}}\ \exp\left(i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right) \mathcal{F}[f(\underline{x})](\underline{\xi}) \\
&= -(-i)^m\ \partial_{\underline{\xi}} \mathcal{F}_{\mathcal{H}^-}[f(\underline{x})](\underline{\xi}) \quad .
\end{aligned}$$

The analogous result for the Clifford-Fourier transform involving the operator $\mathcal{H}^-$ is derived in a similar way.
The more general formulae are now proved by induction. For example, the formula

$$\mathcal{F}_{\mathcal{H}^+}\left[\underline{x}^{2n} f(\underline{x})\right](\underline{\xi}) = (-1)^n\ \partial_{\underline{\xi}}^{2n}\ \mathcal{F}_{\mathcal{H}^+}\left[f(\underline{x})\right](\underline{\xi})$$

holds for $n = 1$, since

$$\begin{aligned}
\mathcal{F}_{\mathcal{H}^+}\left[\underline{x}^2 f(\underline{x})\right](\underline{\xi}) &= -(-i)^m\ \partial_{\underline{\xi}}\ \mathcal{F}_{\mathcal{H}^-}\left[\underline{x} f(\underline{x})\right](\underline{\xi}) \\
&= -(-i)^m\ i^m\ \partial_{\underline{\xi}}^2\ \mathcal{F}_{\mathcal{H}^+}[f(\underline{x})](\underline{\xi}) = -\partial_{\underline{\xi}}^2\ \mathcal{F}_{\mathcal{H}^+}[f(\underline{x})](\underline{\xi}) \quad .
\end{aligned}$$

Assuming that it holds for n , we now prove it for $n+1$:

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}\left[\underline{x}^{2(n+1)} f(\underline{x})\right](\underline{\xi}) &= \mathcal{F}_{\mathcal{H}^+}\left[\underline{x}^{2n}\underline{x}^2 f(\underline{x})\right](\underline{\xi}) \\ &= (-1)^n\ \partial_{\underline{\xi}}^{2n}\ \mathcal{F}_{\mathcal{H}^+}\left[\underline{x}^2 f(\underline{x})\right](\underline{\xi}) \\ &= (-1)^{n+1}\ \partial_{\underline{\xi}}^{2(n+1)}\ \mathcal{F}_{\mathcal{H}^+}\left[f(\underline{x})\right](\underline{\xi})\ .\end{aligned}$$

(iv) By means of the differentiation rule for the classical Fourier transform and the commutation relation (see for e.g. [112])

$$\Gamma_{\underline{\xi}}\ \underline{\xi} = \underline{\xi}\ (m-1-\Gamma_{\underline{\xi}})\ ,$$

a similar argument may be developed as in the proof of the multiplication rule.
(v) This result follows by combining the multiplication and differentiation rules. For example we have

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}\left[\underline{x}\partial_{\underline{x}} f(\underline{x})\right](\underline{\xi}) &= -(-i)^m\ \partial_{\underline{\xi}}\ \mathcal{F}_{\mathcal{H}^-}\left[\partial_{\underline{x}} f(\underline{x})\right](\underline{\xi}) \\ &= -(-i)^m\ i^m\ \partial_{\underline{\xi}}\ \underline{\xi}\ \mathcal{F}_{\mathcal{H}^+}[f(\underline{x})](\underline{\xi}) \\ &= -\partial_{\underline{\xi}}\ \underline{\xi}\ \mathcal{F}_{\mathcal{H}^+}[f(\underline{x})](\underline{\xi})\end{aligned}$$

and hence, by repeated use,

$$\mathcal{F}_{\mathcal{H}^+}\left[(\underline{x}\partial_{\underline{x}})^n f(\underline{x})\right](\underline{\xi}) = (-1)^n\ (\partial_{\underline{\xi}}\underline{\xi})^n\ \mathcal{F}_{\mathcal{H}^+}\left[f(\underline{x})\right](\underline{\xi})\ . \quad \square$$

As the Fourier transform of a radial function remains radial, and the angular Dirac operator Γ does not affect radial functions, the next result follows readily.

Proposition 12.5 *For a radial function f one has*

$$\mathcal{F}_{\mathcal{H}^\pm}[f] = \mathcal{F}[f]\ .$$

12.4 The Two-Dimensional Clifford-Fourier Transform

The purpose of this section is twofold: to present an in depth study of our Clifford-Fourier transform in the specific case of two dimensions, thus providing a theoretical background for the use of this integral transformation, and to show how our two-dimensional Clifford-Fourier transform fits in the picture of all already existing Clifford-Fourier transforms sketched in Section 12.1 and in this way to promote our higher dimensional Clifford-Fourier transform as a possible tool for multi-dimensional signal analysis. The two-dimensional case of the Clifford-Fourier transform is special in that we are able to obtain a closed form for the kernel of the integral representation.

12.4.1 The integral kernel

In the sequel the following Clifford numbers play a crucial role:

$$P^{\pm} = \frac{1}{2}(1 \pm i e_{12}) \quad .$$

They are self-adjoint mutually orthogonal idempotents which, by multiplication, transform e_{12} into the imaginary unit i .

Lemma 12.1 *The Clifford numbers P^+ and P^- satisfy the following properties:*
(i)

$$P^+ + P^- = 1 \ ; \quad P^+P^- = P^-P^+ = 0 \ ; \quad (P^{\pm})^2 = P^{\pm}$$

(ii)

$$\begin{aligned} P^+(ie_{12}) &= P^+ \quad \textit{or} \quad P^+ i = P^+(-e_{12}) = (-e_{12})P^+ \ ; \\ P^-(ie_{12}) &= -P^- \quad \textit{or} \quad P^- i = P^- e_{12} = e_{12}P^- \end{aligned}$$

(iii) for $k \in \mathbb{N}$:

$$\begin{aligned} P^+(ie_{12})^k &= P^+ \quad \textit{or} \quad P^+(e_{12})^k = P^+(-i)^k \ ; \\ P^-(ie_{12})^k &= (-1)^k P^- \quad \textit{or} \quad P^-(e_{12})^k = P^- i^k \ . \end{aligned}$$

Proof.
(i) By a straightforward computation we find

$$P^+P^- = \frac{1}{4}(1 + ie_{12})(1 - ie_{12}) = \frac{1}{4}(1 - ie_{12} + ie_{12} - 1) = 0$$

and

$$(P^{\pm})^2 = \frac{1}{4}(1 \pm ie_{12})(1 \pm ie_{12}) = \frac{1}{4}(1 \pm 2\ ie_{12} + 1) = P^{\pm} \quad .$$

(ii) We easily obtain

$$P^+(ie_{12}) = \frac{1}{2}(1 + ie_{12})(ie_{12}) = \frac{1}{2}(ie_{12} + 1) = P^+ \quad ,$$

which by multiplication with $(-e_{12})$ yields

$$P^+ i = P^+(-e_{12}) \quad .$$

The result for P^- is derived similarly.
(iii) In view of the above, we have

$$\begin{aligned} P^+(ie_{12})^k &= P^+ \ (ie_{12}) \ (ie_{12}) \ \ldots \ (ie_{12}) = P^+ \ (ie_{12}) \ \ldots \ (ie_{12}) \\ &= \ldots = P^+ \end{aligned}$$

or, by multiplication with $(-i)^k$,

$$P^+(e_{12})^k = P^+(-i)^k \quad .$$

The result for P^- is proved in an analogous manner. □

A) Calculation of the kernel for $\mathcal{F}_{\mathcal{H}^+}$

We calculate the kernel of the Clifford-Fourier transform involving the operator $\mathcal{H}^+$. This kernel is given by

$$\exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] \quad .$$

By means of Lemma 12.1 (i) and $\Gamma = -e_{12}L_{12}$, we now have

$$\begin{aligned} &\exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] \\ &\qquad = P^+ \exp\left(i\frac{\pi}{2}e_{12}L_{12}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] \\ &\qquad\qquad + P^- \exp\left(i\frac{\pi}{2}e_{12}L_{12}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] \quad . \qquad (12.9) \end{aligned}$$

Moreover, again using the properties of the Clifford numbers P^+ and P^-, we obtain

$$\begin{aligned} P^+ \exp\left(i\frac{\pi}{2}e_{12}L_{12}\right) &= P^+ \sum_{k=0}^{\infty} \frac{(ie_{12})^k}{k!} \left(\frac{\pi}{2}\right)^k (L_{12})^k \\ &= P^+ \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\pi}{2}\right)^k (L_{12})^k \\ &= P^+ \exp\left(\frac{\pi}{2}L_{12}\right) \end{aligned}$$

and similarly

$$P^- \exp\left(i\frac{\pi}{2}e_{12}L_{12}\right) = P^- \exp\left(-\frac{\pi}{2}L_{12}\right) \quad .$$

Hence expression (12.9) becomes

$$\begin{aligned}\exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)&\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right]\\ &= P^+ \exp\left(\frac{\pi}{2}L_{12}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right]\\ &\quad + \ P^- \exp\left(-\frac{\pi}{2}L_{12}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] \quad . \qquad (12.10)\end{aligned}$$

Now we prove the following intermediate result.

Lemma 12.2 *Let f be a real-analytic function in $\mathbb{R}^2$, let $L_{12} = \xi_1\partial_{\xi_2} - \xi_2\partial_{\xi_1}$ be the angular momentum operator and let $R^{\pm}$ be the operator exponential given by*

$$R^{\pm} := \exp\left(\pm\frac{\pi}{2}L_{12}\right) \quad .$$

Then one has

$$R^+[f(\xi_1,\xi_2)] = f(-\xi_2,\xi_1) \quad \textit{and} \quad R^-[f(\xi_1,\xi_2)] = f(\xi_2,-\xi_1) \quad .$$

Proof. In terms of polar co-ordinates

$$\begin{cases}\xi_1 = r\cos(\theta)\\ \xi_2 = r\sin(\theta)\end{cases}$$

with $r = |\underline{\xi}| \in [0,+\infty[$ and $\theta \in [0,2\pi[$, the operator exponential R^+ takes the form

$$R^+_\theta := \exp\left(\frac{\pi}{2}\ \partial_\theta\right) \quad .$$

We have

$$R^+_\theta[f(r,\theta)] = R^+_\psi[f(r,\theta+\psi)]_{\psi=0} = \sum_{k=0}^{\infty}\frac{1}{k!}\ \left(\frac{\pi}{2}\right)^k\ \partial^k_\psi[f(r,\theta+\psi)]_{\psi=0} \quad .$$

As we assume f to be real-analytic in $\mathbb{R}^2$, this becomes

$$R^+_\theta[f(r,\theta)] = f\left(r,\theta+\frac{\pi}{2}\right) \quad ,$$

which leads to the desired result.
The result for the operator exponential R^- is proved in a similar way. □

Remark 12.2 The operator exponentials R^+ and R^- may be qualified as an anti-clockwise, respectively a clockwise, rotation by a right angle.

Returning to the calculation of the two-dimensional Clifford-Fourier kernel, we obtain by applying Lemma 12.2 to (12.10) :

$$\exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right]$$
$$= P^+ \exp\left(-i(x_2\xi_1 - x_1\xi_2)\right) + P^- \exp\left(-i(x_1\xi_2 - x_2\xi_1)\right) \quad .$$

This expression for the kernel of $\mathcal{F}_{\mathcal{H}^+}$ can be further simplified using the following result.

Lemma 12.3 *One has*

$$P^+ \exp\left(-i(x_2\xi_1 - x_1\xi_2)\right) = P^+ \exp\left(\underline{\xi} \wedge \underline{x}\right)$$

and

$$P^- \exp\left(-i(x_1\xi_2 - x_2\xi_1)\right) = P^- \exp\left(\underline{\xi} \wedge \underline{x}\right) \quad .$$

Proof. By means of Lemma 12.1 (iii), we have consecutively

$$\begin{aligned} P^+ \exp\left(-i(x_2\xi_1 - x_1\xi_2)\right) &= P^+ \sum_{k=0}^{\infty} \frac{(-i)^k}{k!}(x_2\xi_1 - x_1\xi_2)^k \\ &= P^+ \sum_{k=0}^{\infty} \frac{(e_{12})^k}{k!}(x_2\xi_1 - x_1\xi_2)^k \\ &= P^+ \exp\left(e_{12}(\xi_1 x_2 - \xi_2 x_1)\right) \\ &= P^+ \exp\left(\underline{\xi} \wedge \underline{x}\right) \quad . \end{aligned}$$

The second statement is proved similarly. □

In view of the foregoing lemma, we finally obtain a closed form for the kernel of $\mathcal{F}_{\mathcal{H}^+}$:

$$\begin{aligned} \exp\left(-i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] &= P^+ \exp\left(\underline{\xi} \wedge \underline{x}\right) + P^- \exp\left(\underline{\xi} \wedge \underline{x}\right) \\ &= \exp\left(\underline{\xi} \wedge \underline{x}\right) \quad . \end{aligned}$$

Hence the Clifford-Fourier transform involving the operator $\mathcal{H}^+$, has the following integral representation:

$$\mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\left(\underline{\xi} \wedge \underline{x}\right)\, f(\underline{x})\, dV(\underline{x}) \quad .$$

B) Calculation of the kernel for $\mathcal{F}_{\mathcal{H}^-}$

The computation of the kernel of the two-dimensional Clifford-Fourier transform involving the operator $\mathcal{H}^-$ runs along the same lines.
It is given by

$$\exp\left(i\frac{\pi}{2}\Gamma_{\underline{\xi}}\right)\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right] = \exp\left(-(\underline{\xi} \wedge \underline{x})\right) = \exp\left(\underline{x} \wedge \underline{\xi}\right) \quad .$$

The integral representation for $\mathcal{F}_{\mathcal{H}^-}$ then takes the form

$$\mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\left(\underline{x} \wedge \underline{\xi}\right)\, f(\underline{x})\, dV(\underline{x}) \quad .$$

Remark 12.3
1. Note that the two-dimensional Clifford-Fourier kernels consist of a scalar and a bivector part, i.e. they are so-called parabivectors.
2. The Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ may be qualified as a *co-axial Fourier transform*, since its integral kernel may also be rewritten as

$$\exp\left(\pm(\underline{\xi} \wedge \underline{x})\right) = \cos\left(\xi_1 x_2 - \xi_2 x_1\right) \pm e_{12}\ \sin\left(\xi_1 x_2 - \xi_2 x_1\right)$$

where $\xi_1 x_2 - \xi_2 x_1$ takes constant values on co-axial cylinders, which in two dimensions take the form of two lines parallel and symmetric w.r.t. the fixed vector $\underline{\xi}$.
Hence, in terms of the Fourier cosine and the Fourier sine transform:

$$\begin{aligned} \mathcal{F}_{\cos}[f](\underline{\xi}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \cos\left(\xi_1 x_2 - \xi_2 x_1\right)\, f(\underline{x})\, dV(\underline{x}) \quad , \\ \mathcal{F}_{\sin}[f](\underline{\xi}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \sin\left(\xi_1 x_2 - \xi_2 x_1\right)\, f(\underline{x})\, dV(\underline{x}) \quad , \end{aligned}$$

the Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ takes the form

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) = \mathcal{F}_{\cos}[f](\underline{\xi}) \pm e_{12}\ \mathcal{F}_{\sin}[f](\underline{\xi}) \quad .$$

3. More generally in the higher dimensional case the co-axial Fourier transform may be defined as

$$\mathcal{F}_{\mathrm{coax}}[f](\underline{\xi}) = \int_{\mathbb{R}^m} \exp\left(\underline{x} \wedge \underline{\xi}\right)\, f(\underline{x})\, dV(\underline{x}) \quad .$$

Writing $\underline{x} \wedge \underline{\xi} = |\underline{x} \wedge \underline{\xi}|\ I$, in view of $I^2 = -1$ we get

$$\exp(\underline{x} \wedge \underline{\xi}) = \cos(|\underline{x} \wedge \underline{\xi}|) + I \sin(|\underline{x} \wedge \underline{\xi}|) \quad .$$

Note that for $\underline{\xi}$ fixed, the "phase" $|\underline{x} \wedge \underline{\xi}|$ is constant on co-axial cylinders. Clearly the Clifford-Fourier transform and the co-axial Fourier transform reduce to the same integral transform in the two-dimensional case.

12.4.2 The two-dimensional Clifford-Fourier transform as a linear operator

1) The Clifford-Fourier transform in $L_1\big(\mathbb{R}^2, dV(\underline{x})\big)$

The two-dimensional Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^+}[f]$ is well-defined for each integrable function $f \in L_1\big(\mathbb{R}^2, dV(\underline{x})\big)$. Indeed, by means of the properties (2.1) of the Clifford norm we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{x})\ f(\underline{x})\ dV(\underline{x})\right| &\leq \int_{\mathbb{R}^2} \big|\exp(\underline{\xi} \wedge \underline{x})\ f(\underline{x})\big|\ dV(\underline{x}) \\ &\leq 4 \int_{\mathbb{R}^2} \big|\exp(\underline{\xi} \wedge \underline{x})\big|\ |f(\underline{x})|\ dV(\underline{x}) \quad .\end{aligned}$$

Furthermore, using Lemma 12.3, we obtain:

$$\begin{aligned}&\big|\exp(\underline{\xi} \wedge \underline{x})\big| \\ &= \big|P^+ \exp\big(-i(x_2\xi_1 - x_1\xi_2)\big) + P^- \exp\big(-i(x_1\xi_2 - x_2\xi_1)\big)\big| \\ &\leq 4\ \big|P^+\big|\ \big|\exp\big(-i(x_2\xi_1 - x_1\xi_2)\big)\big| + 4\ \big|P^-\big|\ \big|\exp\big(-i(x_1\xi_2 - x_2\xi_1)\big)\big| \\ &\leq 8 \ ,\end{aligned}$$

since $|P^{\pm}| \leq 1$.
Hence we have

$$\left|\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{x})\ f(\underline{x})\ dV(\underline{x})\right| \leq 32 \int_{\mathbb{R}^2} |f(\underline{x})|\ dV(\underline{x}) < \infty$$

and also

$$\exp(\underline{\xi} \wedge \underline{x})\ f(\underline{x}) \in L_1\big(\mathbb{R}^2, dV(\underline{x})\big) \quad .$$

A similar argument holds for the Clifford-Fourier transform involving the operator $\mathcal{H}^-$.

The above reasoning leads to the following theorem.

Theorem 12.1 *Let $f \in L_1\big(\mathbb{R}^2, dV(\underline{x})\big)$. Then $\mathcal{F}_{\mathcal{H}^\pm}[f] \in L_\infty\big(\mathbb{R}^2\big) \cap C_0\big(\mathbb{R}^2\big)$ and moreover*

$$\|\mathcal{F}_{\mathcal{H}^\pm}[f]\|_\infty \leq \frac{16}{\pi} \, \|f\|_1 \quad .$$

Proof. In view of the above, we have:

$$\begin{aligned} \big|\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\big| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \exp\big(\pm(\underline{\xi} \wedge \underline{x})\big) \; f(\underline{x}) \; dV(\underline{x}) \right| \\ &\leq \frac{16}{\pi} \int_{\mathbb{R}^2} |f(\underline{x})| \; dV(\underline{x}) = \frac{16}{\pi} \, \|f\|_1 \quad . \end{aligned}$$

Moreover, one can easily verify that

$$\big|\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) - \mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}')\big| \leq \frac{32}{\pi} \, \|f\|_1 \quad .$$

Hence, $|\underline{\xi} - \underline{\xi}'| \to 0$ implies

$$\big|\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) - \mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}')\big| \to 0 \quad ,$$

in other words $\mathcal{F}_{\mathcal{H}^\pm}[f]$ is continuous. □

In Section 12.3.3 some operational formulae (viz. the right $\mathbb{C}_m$-linearity, change of scale, multiplication, differentiation and mixed product rule) were derived for the Clifford-Fourier transform in arbitrary dimension. In the special two-dimensional case, the obtained closed forms for the integral kernels enable us to prove some further results for the Clifford-Fourier transform in $L_1\big(\mathbb{R}^2, dV(\underline{x})\big)$.

Proposition 12.6 *Let $f, g \in L_1\big(\mathbb{R}^2, dV(\underline{x})\big)$. The two-dimensional Clifford-Fourier transform satisfies:*
(i) shift theorem

$$\mathcal{F}_{\mathcal{H}^\pm}[\tau_{\underline{h}} f(\underline{x})](\underline{\xi}) = \exp\big(\pm(\underline{\xi} \wedge \underline{h})\big) \; \mathcal{F}_{\mathcal{H}^\pm}[f(\underline{x})](\underline{\xi}) \quad ,$$

where $\tau_{\underline{h}}$ denotes the translation by $\underline{h}$, i.e. $\tau_{\underline{h}} f(\underline{x}) = f(\underline{x} - \underline{h})$
(ii) frequency reversion

$$\mathcal{F}_{\mathcal{H}^\pm}[f](-\underline{\xi}) = \mathcal{F}_{\mathcal{H}^\mp}[f](\underline{\xi})$$

(iii) Hermitian conjugation

$$\begin{aligned}\left(\mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi})\right)^\dagger &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f^\dagger(\underline{x})\ \exp\left(\underline{x}\wedge\underline{\xi}\right)\ dV(\underline{x}) \quad , \\ \left(\mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi})\right)^\dagger &= \frac{1}{2\pi}\int_{\mathbb{R}^2} f^\dagger(\underline{x})\ \exp\left(\underline{\xi}\wedge\underline{x}\right)\ dV(\underline{x})\end{aligned}$$

(iv) modulation theorem

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\exp\left(\underline{x}\wedge\underline{h}\right)\ f(\underline{x})\right](\underline{\xi}) = \tau_{\pm\underline{h}}\mathcal{F}_{\mathcal{H}^\pm}[f(\underline{x})](\underline{\xi})$$

(v) transfer formula

$$\int_{\mathbb{R}^2}\left(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\right)^\dagger\ g(\underline{\xi})\ dV(\underline{\xi}) = \int_{\mathbb{R}^2} f^\dagger(\underline{\xi})\ \mathcal{F}_{\mathcal{H}^\pm}[g](\underline{\xi})\ dV(\underline{\xi})$$

(vi) convolution theorem

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^\pm}[f^p * g](\underline{\xi}) &= 2\pi\ \mathcal{F}_{\mathcal{H}^\pm}[f^p](\underline{\xi})\ \mathcal{F}_{\mathcal{H}^\pm}[g](\underline{\xi}) \quad , \\ \mathcal{F}_{\mathcal{H}^\pm}[\vec{f} * g](\underline{\xi}) &= 2\pi\ \mathcal{F}_{\mathcal{H}^\pm}[\vec{f}](\underline{\xi})\ \mathcal{F}_{\mathcal{H}^\mp}[g](\underline{\xi}) \quad ,\end{aligned}$$

where $*$ denotes the Clifford convolution given by

$$(f * g)(\underline{x}) = \int_{\mathbb{R}^2} f(\underline{x}-\underline{x}')\ g(\underline{x}')\ dV(\underline{x}')$$

and f^p, respectively $\vec{f}$, stands for respectively a parabivector-valued and vector-valued function, i.e.

$$f^p(\underline{x}) = f_0(\underline{x})+f_{12}(\underline{x})e_{12} \quad ; \quad \vec{f} = f_1(\underline{x})e_1+f_2(\underline{x})e_2 \quad (f_0, f_1, f_2, f_{12} : \mathbb{R}^2 \to \mathbb{C}) \ .$$

Proof. We restrict ourselves to the proofs for the Clifford-Fourier transform involving the operator $\mathcal{H}^+$, the proofs for $\mathcal{F}_{\mathcal{H}^-}$ being similar.
(i) By means of the substitution $\underline{u} = \underline{x} - \underline{h}$, we obtain

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}[\tau_{\underline{h}} f(\underline{x})](\underline{\xi}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\left(\underline{\xi}\wedge\underline{x}\right)\ f(\underline{x}-\underline{h})\ dV(\underline{x}) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\left(\underline{\xi}\wedge\underline{u}+\underline{\xi}\wedge\underline{h}\right)\ f(\underline{u})\ dV(\underline{u}) \\ &= \exp\left(\underline{\xi}\wedge\underline{h}\right)\ \frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\left(\underline{\xi}\wedge\underline{u}\right)\ f(\underline{u})\ dV(\underline{u}) \\ &= \exp\left(\underline{\xi}\wedge\underline{h}\right)\ \mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) \quad .\end{aligned}$$

Here we have used the fact that

$$\exp\left(\underline{\xi}\wedge\underline{u}+\underline{\xi}\wedge\underline{h}\right)=\exp\left(\underline{\xi}\wedge\underline{u}\right)\ \exp\left(\underline{\xi}\wedge\underline{h}\right)=\exp\left(\underline{\xi}\wedge\underline{h}\right)\ \exp\left(\underline{\xi}\wedge\underline{u}\right)\quad,$$

since the exponentials $\exp\left(\underline{\xi}\wedge\underline{u}\right)$ and $\exp\left(\underline{\xi}\wedge\underline{h}\right)$ commute.
(ii) This property follows at once from

$$\exp\big((-\underline{\xi})\wedge\underline{x}\big)=\exp\left(\underline{x}\wedge\underline{\xi}\right)\quad.$$

(iii) Taking into account that

$$\overline{\underline{\xi}\wedge\underline{x}}=\underline{x}\wedge\underline{\xi}$$

and hence also that

$$\overline{\exp\left(\underline{\xi}\wedge\underline{x}\right)}=\exp\left(\underline{x}\wedge\underline{\xi}\right)\ ,$$

the result follows immediately.
(iv) We have consecutively

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}\Big[\exp\left(\underline{x}\wedge\underline{h}\right)\ f(\underline{x})\Big](\underline{\xi}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\left(\underline{\xi}\wedge\underline{x}\right)\ \exp\left(\underline{x}\wedge\underline{h}\right)\ f(\underline{x})\ dV(\underline{x})\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\big((\underline{\xi}-\underline{h})\wedge\underline{x}\big)\ f(\underline{x})\ dV(\underline{x})\\ &= \mathcal{F}_{\mathcal{H}^+}[f(\underline{x})](\underline{\xi}-\underline{h})\quad.\end{aligned}$$

(v) First note that both integrals in formula (v) are well-defined, since $\big(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\big)^\dagger\ g(\underline{\xi})$ and $f^\dagger(\underline{\xi})\ \mathcal{F}_{\mathcal{H}^\pm}[g](\underline{\xi})$ belong to the space $L_1\big(\mathbb{R}^2,dV(\underline{x})\big)$ of integrable functions.
Moreover using property (iii) and changing the order of integration yields

$$\begin{aligned}&\int_{\mathbb{R}^2}\big(\mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi})\big)^\dagger\ g(\underline{\xi})\ dV(\underline{\xi})\\ &\qquad= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}f^\dagger(\underline{x})\ \exp\left(\underline{x}\wedge\underline{\xi}\right)\ dV(\underline{x})\right)\ g(\underline{\xi})\ dV(\underline{\xi})\\ &\qquad= \int_{\mathbb{R}^2}f^\dagger(\underline{x})\ \left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\exp\left(\underline{x}\wedge\underline{\xi}\right)\ g(\underline{\xi})\ dV(\underline{\xi})\right)\ dV(\underline{x})\\ &\qquad= \int_{\mathbb{R}^2}f^\dagger(\underline{x})\ \mathcal{F}_{\mathcal{H}^+}[g](\underline{x})\ dV(\underline{x})\quad.\end{aligned}$$

(vi) Let us first consider the case of a parabivector-valued function f^p.
Changing the order of integration and applying the substitution $\underline{u}=\underline{x}-\underline{x}'$,

yields consecutively

$$
\begin{aligned}
&\mathcal{F}_{\mathcal{H}^+}[f^p * g](\underline{\xi}) \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{x}) \left(\int_{\mathbb{R}^2} f^p(\underline{x} - \underline{x}')\ g(\underline{x}')\ dV(\underline{x}') \right) dV(\underline{x}) \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{x})\ f^p(\underline{x} - \underline{x}')\ dV(\underline{x}) \right) g(\underline{x}')\ dV(\underline{x}') \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{u})\ \exp(\underline{\xi} \wedge \underline{x}')\ f^p(\underline{u})\ dV(\underline{u}) \right) g(\underline{x}')\ dV(\underline{x}') \quad .
\end{aligned}
$$

As a parabivector-valued function f^p commutes with the Clifford-Fourier kernels, i.e.

$$
\exp\big(\pm(\underline{\xi} \wedge \underline{x})\big)\ f^p(\underline{u}) = f^p(\underline{u})\ \exp\big(\pm(\underline{\xi} \wedge \underline{x})\big) \quad ,
$$

we indeed obtain

$$
\begin{aligned}
&\mathcal{F}_{\mathcal{H}^+}[f^p * g](\underline{\xi}) \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{u})\ f^p(\underline{u})\ dV(\underline{u}) \right) \exp(\underline{\xi} \wedge \underline{x}')\ g(\underline{x}')\ dV(\underline{x}') \\
&\quad = \mathcal{F}_{\mathcal{H}^+}[f^p](\underline{\xi}) \left(\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{x}')\ g(\underline{x}')\ dV(\underline{x}') \right) \\
&\quad = 2\pi\ \mathcal{F}_{\mathcal{H}^+}[f^p](\underline{\xi})\ \mathcal{F}_{\mathcal{H}^+}[g](\underline{\xi}) \quad .
\end{aligned}
$$

On the other hand a vector-valued function $\vec{f}$, by commutation, transforms the kernel of $\mathcal{F}_{\mathcal{H}^+}$ into the kernel of $\mathcal{F}_{\mathcal{H}^-}$ and vice versa, i.e.

$$
\exp\big(\pm(\underline{\xi} \wedge \underline{x})\big)\ \vec{f}(\underline{u}) = \vec{f}(\underline{u})\ \exp\big(\mp(\underline{\xi} \wedge \underline{x})\big) \quad .
$$

Hence, for a vector-valued function $\vec{f}$ we find

$$
\begin{aligned}
&\mathcal{F}_{\mathcal{H}^+}[\vec{f} * g](\underline{\xi}) \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{u})\ \exp(\underline{\xi} \wedge \underline{x}')\ \vec{f}(\underline{u})\ dV(\underline{u}) \right) g(\underline{x}')\ dV(\underline{x}') \\
&\quad = \int_{\mathbb{R}^2} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \exp(\underline{\xi} \wedge \underline{u})\ \vec{f}(\underline{u})\ dV(\underline{u}) \right) \exp(\underline{x}' \wedge \underline{\xi})\ g(\underline{x}')\ dV(\underline{x}') \\
&\quad = \mathcal{F}_{\mathcal{H}^+}[\vec{f}](\underline{\xi}) \left(\int_{\mathbb{R}^2} \exp(\underline{x}' \wedge \underline{\xi})\ g(\underline{x}')\ dV(\underline{x}') \right) \\
&\quad = 2\pi\ \mathcal{F}_{\mathcal{H}^+}[\vec{f}](\underline{\xi})\ \mathcal{F}_{\mathcal{H}^-}[g](\underline{\xi}) \quad .
\end{aligned}
$$

Note that in case of a general $\mathbb{C}_2$-valued function $f = f^p + \vec{f}$, the convolution theorem inevitably leads to two terms:

$$\mathcal{F}_{\mathcal{H}^\pm}[f * g](\underline{\xi}) = 2\pi \ \mathcal{F}_{\mathcal{H}^\pm}[f^p](\underline{\xi}) \ \mathcal{F}_{\mathcal{H}^\pm}[g](\underline{\xi}) + 2\pi \ \mathcal{F}_{\mathcal{H}^\pm}[\vec{f}](\underline{\xi}) \ \mathcal{F}_{\mathcal{H}^\mp}[g](\underline{\xi}) \quad . \qquad \Box$$

Remark 12.4
1. Property (ii) implies that it is sufficient to compute one of the Clifford-Fourier transforms.
2. As the Clifford algebra is *not* commutative, property (iii) *can not* be rewritten in the more elegant form

$$\big(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\big)^\dagger = \mathcal{F}_{\mathcal{H}^\mp}[f^\dagger](\underline{\xi}) \quad .$$

However we have

$$\big(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\big)^\dagger = [f^\dagger]\mathcal{F}_{\mathcal{H}^\mp}(\underline{\xi}) \quad ,$$

where the notation $[f^\dagger]\mathcal{F}_{\mathcal{H}^\mp}$ means that the Clifford-Fourier transform is now acting from the right upon the function $f^\dagger$.

2) The Clifford-Fourier transform in $\mathcal{S}(\mathbb{R}^2)$

We now investigate the Clifford-Fourier transform in a dense subspace of $L_1(\mathbb{R}^2, dV(\underline{x}))$, viz. the right $\mathbb{C}_2$-module of rapidly decreasing $\mathbb{C}_2$-valued functions.

Theorem 12.2 *Let* $\varphi \in \mathcal{S}(\mathbb{R}^2)$ *. Then* $\mathcal{F}_{\mathcal{H}^\pm}[\varphi] \in \mathcal{S}(\mathbb{R}^2)$ *.*

Proof. We will show that for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$ the following inequality holds

$$\begin{aligned} p^*_{\ell,k}\big(\mathcal{F}_{\mathcal{H}^\pm}[\varphi]\big) &= \sup_{\underline{\xi}\in\mathbb{R}^2} \ \sup_{|\alpha|\le k} \ \sup_{|\beta|\le \ell} \left|\xi^\alpha \partial_\xi^\beta \big[\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{\xi})\big]\right| \\ &\le C \sup_{\underline{x}\in\mathbb{R}^2} \left((1+|\underline{x}|^2)^{\ell+2} \sum_{|\gamma|\le k} |\partial_x^\gamma \varphi(\underline{x})| \right) = C \ p_{\ell+2,k}(\varphi) \quad . \qquad (12.11) \end{aligned}$$

Here $\{p^*_{\ell,k} \ ; \ \ell, k \in \mathbb{N}\}$ and $\{p_{\ell,k} \ ; \ \ell, k \in \mathbb{N}\}$ are two equivalent systems of seminorms on $\mathcal{S}(\mathbb{R}^2)$. Moreover we use the notation $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}$ and $\partial_\xi^\beta = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}$.
One can easily verify that for every multi-index $\alpha \in \mathbb{N}^2$:

$$\partial_\xi^\alpha \big[\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{\xi})\big] = (\pm e_{12})^{\alpha_1} \ (\mp e_{12})^{\alpha_2} \ \mathcal{F}_{\mathcal{H}^\pm}[x_1^{\alpha_2} x_2^{\alpha_1} \varphi(\underline{x})](\underline{\xi})$$

and

$$\mathcal{F}_{\mathcal{H}^\pm}[\partial_x^\alpha \varphi(\underline{x})](\underline{\xi}) = (\pm e_{12})^{\alpha_1}\ (\mp e_{12})^{\alpha_2}\ \xi_1^{\alpha_2}\xi_2^{\alpha_1}\ \mathcal{F}_{\mathcal{H}^\pm}[\varphi(\underline{x})](\underline{\xi}) \quad .$$

For arbitrary $\underline{\xi} \in \mathbb{R}^2$ and multi-indices α, $\beta \in \mathbb{N}^2$ with $|\alpha| \leq k$ and $|\beta| \leq \ell$, we thus have

$$\begin{aligned} &\xi^\alpha \partial_\xi^\beta \left[\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{\xi})\right] \\ &\quad = (\pm e_{12})^{\beta_1}\ (\mp e_{12})^{\beta_2}\ \xi_1^{\alpha_1}\xi_2^{\alpha_2}\ \mathcal{F}_{\mathcal{H}^\pm}\left[x_1^{\beta_2}x_2^{\beta_1}\varphi(\underline{x})\right](\underline{\xi}) \\ &\quad = (\pm e_{12})^{\beta_1+\alpha_1}\ (\mp e_{12})^{\beta_2+\alpha_2}\ \mathcal{F}_{\mathcal{H}^\pm}\left[\partial_{x_1}^{\alpha_2}\partial_{x_2}^{\alpha_1}\left[x_1^{\beta_2}x_2^{\beta_1}\varphi(\underline{x})\right]\right](\underline{\xi}) \quad . \end{aligned}$$

By means of the properties of the Clifford norm and the estimate $|\exp\left(\pm(\underline{\xi}\wedge\underline{x})\right)| \leq 8$, we obtain

$$\begin{aligned} \left|\xi^\alpha \partial_\xi^\beta \left[\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{\xi})\right]\right| &\leq 4\left|\frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\left(\pm(\underline{\xi}\wedge\underline{x})\right)\ \partial_{x_1}^{\alpha_2}\partial_{x_2}^{\alpha_1}\left[x_1^{\beta_2}x_2^{\beta_1}\varphi(\underline{x})\right]\ dV(\underline{x})\right| \\ &\leq \frac{64}{\pi}\int_{\mathbb{R}^2}\left|\partial_{x_1}^{\alpha_2}\partial_{x_2}^{\alpha_1}[x_1^{\beta_2}x_2^{\beta_1}\varphi(\underline{x})]\right|\ dV(\underline{x}) \quad . \end{aligned} \tag{12.12}$$

Furthermore, using the estimate

$$\begin{aligned} \left|\partial_{x_1}^{\alpha_2}\partial_{x_2}^{\alpha_1}\left[x_1^{\beta_2}x_2^{\beta_1}\varphi(\underline{x})\right]\right| &\leq \sum_{|\gamma|\leq|\alpha|} C_\gamma^\alpha\ \left|\partial_{x_1}^{\alpha_2-\gamma_1}\ \partial_{x_2}^{\alpha_1-\gamma_2}[x_1^{\beta_2}x_2^{\beta_1}]\ \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\left[\varphi(\underline{x})\right]\right| \\ &\leq C'\ (1+|\underline{x}|^2)^{|\beta|}\sum_{|\gamma|\leq|\alpha|}\left|\partial_x^\gamma\varphi(\underline{x})\right| \quad , \end{aligned}$$

the inequality (12.12) still for $|\alpha| \leq k$ and $|\beta| \leq \ell$ becomes

$$\begin{aligned} &\left|\xi^\alpha \partial_\xi^\beta \left[\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{\xi})\right]\right| \\ &\quad \leq C'\ \frac{64}{\pi}\int_{\mathbb{R}^2}(1+|\underline{x}|^2)^\ell\sum_{|\gamma|\leq k}|\partial_x^\gamma\varphi(\underline{x})|\ dV(\underline{x}) \\ &\quad \leq C'\ \frac{64}{\pi}\ \sup_{\underline{x}\in\mathbb{R}^2}\left((1+|\underline{x}|^2)^{\ell+2}\sum_{|\gamma|\leq k}|\partial_x^\gamma\varphi(\underline{x})|\right)\int_{\mathbb{R}^2}\frac{1}{(1+|\underline{x}|^2)^2}\ dV(\underline{x}) \\ &\quad = C\ \sup_{\underline{x}\in\mathbb{R}^2}\left((1+|\underline{x}|^2)^{\ell+2}\sum_{|\gamma|\leq k}|\partial_x^\gamma\varphi(\underline{x})|\right) \quad . \end{aligned}$$

Hence we have proved the desired inequality (12.11). □

Inequality (12.11) immediately yields the following result, where we use the notation $\varphi_j \underset{\mathcal{S}}{\rightarrow} \varphi$ for the convergence of the sequence φ_j in $\mathcal{S}\big(\mathbb{R}^2\big)$.

Corollary 12.2 *If* $\varphi_j \underset{\mathcal{S}}{\rightarrow} \varphi$, *then* $\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j] \underset{\mathcal{S}}{\rightarrow} \mathcal{F}_{\mathcal{H}^\pm}[\varphi]$.

Remark 12.5 The two-dimensional Clifford-Fourier transform in $\mathcal{S}\big(\mathbb{R}^2\big)$ is thus a continuous operator.

For the inversion of the Clifford-Fourier transform, one has the following proposition.

Proposition 12.7 *The inverse of the two-dimensional Clifford-Fourier transform takes the form*

$$(\mathcal{F}_{\mathcal{H}^\pm})^{-1} = \mathcal{F}_{\mathcal{H}^\pm} \quad .$$

Proof. Recall from Section 12.3.2 that

$$(\mathcal{F}_{\mathcal{H}^\pm})^{-1} = \exp\left(\pm i \frac{\pi}{2} \Gamma\right) \mathcal{F}^{-1}$$

or in integral form

$$(\mathcal{F}_{\mathcal{H}^\pm})^{-1}[f](\underline{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(\pm i \frac{\pi}{2} \Gamma_{\underline{\xi}}\right) \Big[\exp\left(i < \underline{x}, \underline{\xi} >\right)\Big] \; f(\underline{x}) \; dV(\underline{x}) \quad .$$

Similarly as in Section 12.4.1, one can prove that

$$\exp\left(\pm i \frac{\pi}{2} \Gamma_{\underline{\xi}}\right) \Big[\exp\left(i < \underline{x}, \underline{\xi} >\right)\Big] = \exp\left(\pm(\underline{\xi} \wedge \underline{x})\right) \quad ,$$

from which the desired result follows. □

We can now state the following result.

Proposition 12.8 *For each* $\varphi \in \mathcal{S}\big(\mathbb{R}^2\big)$ *, one has*

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\mathcal{F}_{\mathcal{H}^\pm}[\varphi]\right] = \varphi \quad ,$$

or in other words

$$(\mathcal{F}_{\mathcal{H}^\pm})^2 = I_{\mathcal{S}\left(\mathbb{R}^2\right)} \quad .$$

By Corollary 12.2 and Proposition 12.8 we finally obtain the following fundamental result.

Theorem 12.3 *The two-dimensional Clifford-Fourier transform* $\mathcal{F}_{\mathcal{H}^\pm}$ *is a homeomorphism of the right* $\mathbb{C}_2$*-module* $\mathcal{S}\big(\mathbb{R}^2\big)$.

3) The Clifford-Fourier transform in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$

The definition of the Clifford-Fourier transform in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ follows classical lines and makes use of the results obtained in the foregoing section on the Clifford-Fourier transform in the dense subspace $\mathcal{S}\big(\mathbb{R}^2\big)$. We start with the following lemma.

Lemma 12.4 *For all* $\varphi,\ \psi \in \mathcal{S}\big(\mathbb{R}^2\big)$*, one has*

$$< \varphi, \psi > = \ \langle \mathcal{F}_{\mathcal{H}^\pm}[\varphi], \mathcal{F}_{\mathcal{H}^\pm}[\psi] \rangle \quad .$$

Proof. Taking into account Proposition 12.8 and the transfer formula (see Proposition 12.6 (v)), we have consecutively

$$\begin{aligned} < \varphi, \psi > &= \int_{\mathbb{R}^2} \big(\varphi(\underline{x})\big)^\dagger\ \psi(\underline{x})\ dV(\underline{x}) \\ &= \int_{\mathbb{R}^2} \big(\mathcal{F}_{\mathcal{H}^\pm}\big[\mathcal{F}_{\mathcal{H}^\pm}[\varphi]\big](\underline{x})\big)^\dagger\ \psi(\underline{x})\ dV(\underline{x}) \\ &= \int_{\mathbb{R}^2} \big(\mathcal{F}_{\mathcal{H}^\pm}[\varphi](\underline{x})\big)^\dagger\ \mathcal{F}_{\mathcal{H}^\pm}[\psi](\underline{x})\ dV(\underline{x}) = \big\langle \mathcal{F}_{\mathcal{H}^\pm}[\varphi], \mathcal{F}_{\mathcal{H}^\pm}[\psi] \big\rangle \quad . \qquad \square \end{aligned}$$

In particular we have the following result.

Lemma 12.5 *For each* $\varphi \in \mathcal{S}\big(\mathbb{R}^2\big)$*, one has*

$$\|\varphi\|_2 = \|\mathcal{F}_{\mathcal{H}^\pm}[\varphi]\|_2 \quad .$$

This implies that the operator $\mathcal{F}_{\mathcal{H}^\pm}$ can be uniquely extended to $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$. Indeed, take $f \in L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$. By means of the density of $\mathcal{S}\big(\mathbb{R}^2\big)$ in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$, there exists a sequence $(\varphi_j)_{j\in\mathbb{N}} \in \mathcal{S}\big(\mathbb{R}^2\big)$ which converges in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ to f . Hence $(\varphi_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$. In view of Lemma 12.5, $(\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j])_{j\in\mathbb{N}}$ is also a Cauchy sequence in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ and thus convergent in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$. The limit of $(\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j])_{j\in\mathbb{N}}$, which is independent of the chosen sequence $(\varphi_j)_{j\in\mathbb{N}}$, is called the Clifford-Fourier transform of f. We denote it by $\mathbb{F}_{\mathcal{H}^\pm}[f]$.

The above introduced Clifford-Fourier transform in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ has the following properties.

Proposition 12.9 *The two-dimensional Clifford-Fourier transform* $\mathbb{F}_{\mathcal{H}^\pm}$ *in* $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ *satisfies the following properties:*
(i) $\mathbb{F}_{\mathcal{H}^\pm}$ *is right* $\mathbb{C}_2$*-linear, i.e. for all* $f, g \in L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ *and for all* $\lambda, \mu \in \mathbb{C}_2$ *one has*

$$\mathbb{F}_{\mathcal{H}^\pm}[f\lambda + g\mu] = \mathbb{F}_{\mathcal{H}^\pm}[f]\ \lambda + \mathbb{F}_{\mathcal{H}^\pm}[g]\ \mu \quad .$$

(ii) The restriction of $\mathbb{F}_{\mathcal{H}^\pm}$ *to* $\mathcal{S}\big(\mathbb{R}^2\big)$ *is* $\mathcal{F}_{\mathcal{H}^\pm}$ *.*
(iii) $\mathbb{F}_{\mathcal{H}^\pm}$ *is bounded on* $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ *.*
(iv) The inverse of $\mathbb{F}_{\mathcal{H}^\pm}$ *is precisely* $\mathbb{F}_{\mathcal{H}^\pm}$*, or*

$$(\mathbb{F}_{\mathcal{H}^\pm})^2 = I_{L_2\left(\mathbb{R}^2, dV(\underline{x})\right)} \quad .$$

(v) The adjoint $(\mathbb{F}_{\mathcal{H}^\pm})^*$ *is given by*

$$(\mathbb{F}_{\mathcal{H}^\pm})^* = \mathbb{F}_{\mathcal{H}^\pm} = (\mathbb{F}_{\mathcal{H}^\pm})^{-1} \quad .$$

Proof.
(i) Trivial.
(ii) Immediate.
(iii) Take $f \in L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ and a sequence $\varphi_j \in \mathcal{S}\big(\mathbb{R}^2\big)$ which converges in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ to f, i.e.

$$\|\varphi_j\|_2 \underset{\mathbb{R}}{\rightarrow} \|f\|_2 \quad .$$

Furthermore, by definition

$$\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j] \underset{L_2}{\rightarrow} \mathbb{F}_{\mathcal{H}^\pm}[f] \quad ,$$

i.e.

$$\|\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j]\|_2 \underset{\mathbb{R}}{\rightarrow} \|\mathbb{F}_{\mathcal{H}^\pm}[f]\|_2 \quad .$$

Taking into account Lemma 12.5, we thus obtain

$$\|f\|_2 = \|\mathbb{F}_{\mathcal{H}^\pm}[f]\|_2 \quad ,$$

which proves the statement.
(iv) Take again $f \in L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ and a sequence $\varphi_j \in \mathcal{S}\big(\mathbb{R}^2\big)$ which converges in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ to f . By definition we have consecutively

$$\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j] \underset{L_2}{\rightarrow} \mathbb{F}_{\mathcal{H}^\pm}[f]$$

and

$$\mathcal{F}_{\mathcal{H}^\pm}\left[\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j]\right] \underset{L_2}{\longrightarrow} \mathbb{F}_{\mathcal{H}^\pm}\left[\mathbb{F}_{\mathcal{H}^\pm}[f]\right] \quad .$$

Hence by means of Proposition 12.8 we find

$$\varphi_j \underset{L_2}{\longrightarrow} \mathbb{F}_{\mathcal{H}^\pm}\left[\mathbb{F}_{\mathcal{H}^\pm}[f]\right] \quad ,$$

which leads to

$$f = \mathbb{F}_{\mathcal{H}^\pm}\left[\mathbb{F}_{\mathcal{H}^\pm}[f]\right] \quad .$$

(v) To compute the adjoint $(\mathbb{F}_{\mathcal{H}^\pm})^*$, we start with $f, g \in L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ and sequences φ_j and ψ_j in $\mathcal{S}\big(\mathbb{R}^2\big)$ which converge in $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ to f and g respectively. By definition we have

$$\mathcal{F}_{\mathcal{H}^\pm}[\varphi_j] \underset{L_2}{\longrightarrow} \mathbb{F}_{\mathcal{H}^\pm}[f] \quad \text{and} \quad \mathcal{F}_{\mathcal{H}^\pm}[\psi_j] \underset{L_2}{\longrightarrow} \mathbb{F}_{\mathcal{H}^\pm}[g] \quad .$$

In terms of the inner product on $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$ it follows that

$$< \varphi_j, \psi_j > \underset{\mathbb{C}_2}{\longrightarrow} < f, g > \quad \text{and} \quad \big\langle \mathcal{F}_{\mathcal{H}^\pm}[\varphi_j], \mathcal{F}_{\mathcal{H}^\pm}[\psi_j] \big\rangle \underset{\mathbb{C}_2}{\longrightarrow} \big\langle \mathbb{F}_{\mathcal{H}^\pm}[f], \mathbb{F}_{\mathcal{H}^\pm}[g] \big\rangle \quad .$$

Moreover applying Lemma 12.4 gives

$$< f, g > = \big\langle \mathbb{F}_{\mathcal{H}^\pm}[f], \mathbb{F}_{\mathcal{H}^\pm}[g] \big\rangle$$

and hence also

$$\big\langle \mathbb{F}_{\mathcal{H}^\pm}[f], g \big\rangle = \Big\langle \mathbb{F}_{\mathcal{H}^\pm}\left[\mathbb{F}_{\mathcal{H}^\pm}[f]\right], \mathbb{F}_{\mathcal{H}^\pm}[g] \Big\rangle = \big\langle f, \mathbb{F}_{\mathcal{H}^\pm}[g] \big\rangle \quad . \qquad \square$$

Summarizing we obtain the following fundamental result.

Theorem 12.4 *The two-dimensional Clifford-Fourier transform $\mathbb{F}_{\mathcal{H}^\pm}$ is a unitary operator on the right $\mathbb{C}_2$-module $L_2\big(\mathbb{R}^2, dV(\underline{x})\big)$.*

12.4.3 Connection with the classical Fourier transform and the Clifford-Fourier transform of Ebling and Scheuermann

In this section we first derive an explicit connection between the two-dimensional Clifford-Fourier transform pair $\{\mathcal{F}_{\mathcal{H}^+}, \mathcal{F}_{\mathcal{H}^-}\}$ and the classical tensorial Fourier transform $\mathcal{F}$ in the plane.

By means of Lemma 12.3 we can rewrite the Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^+}$ as:

$$\begin{aligned}\mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) = P^+ \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}^2} \exp\left(-i(\xi_1 x_2 - \xi_2 x_1)\right)\, f(\underline{x})\, dV(\underline{x}) \\ +\ P^- \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}^2} \exp\left(-i(\xi_2 x_1 - \xi_1 x_2)\right)\, f(\underline{x})\, dV(\underline{x}) \quad .\end{aligned}$$

Furthermore one easily verifies that

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(-i(\xi_1 x_2 - \xi_2 x_1)\right)\, f(\underline{x})\, dV(\underline{x}) = \mathcal{F}[f](e_{12}\underline{\xi})$$

and similarly

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(-i(\xi_2 x_1 - \xi_1 x_2)\right)\, f(\underline{x})\, dV(\underline{x}) = \mathcal{F}[f](-e_{12}\underline{\xi}) \quad .$$

This yields the following relation between the Clifford-Fourier transform involving the operator $\mathcal{H}^+$ and the standard Fourier transform:

$$\mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) = P^+\ \mathcal{F}[f](e_{12}\underline{\xi}) + P^-\ \mathcal{F}[f](-e_{12}\underline{\xi}) \quad . \tag{12.13}$$

Similarly one obtains

$$\mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi}) = P^+\ \mathcal{F}[f](-e_{12}\underline{\xi}) + P^-\ \mathcal{F}[f](e_{12}\underline{\xi}) \quad . \tag{12.14}$$

Remark 12.6
1. The transformations $\underline{\xi} \to e_{12}\ \underline{\xi}$ and $\underline{\xi} \to -e_{12}\ \underline{\xi}$ represent an anti-clockwise, respectively a clockwise, rotation by a right angle.
2. For a radial function f, expressions (12.13) and (12.14) reduce to

$$\mathcal{F}_{\mathcal{H}^\pm}[f] = \mathcal{F}[f] \quad ,$$

since the Fourier transform of a radial function remains radial.
Note that we already obtained this result for the Clifford-Fourier transform in arbitrary dimension (see Proposition 12.5*).*

Moreover, the Clifford-Fourier transform of Ebling and Scheuermann (see Section 12.1)

$$\mathcal{F}^e[f](\underline{\xi}) = \int_{\mathbb{R}^2} \exp\left(-e_{12}(x_1\xi_1 + x_2\xi_2)\right)\, f(\underline{x})\, dV(\underline{x})$$

can be expressed in terms of the Clifford-Fourier transform:

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^{\pm}}[f](\xi_1, \xi_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(\pm e_{12}(\xi_1 x_2 - \xi_2 x_1)\right) f(\underline{x})\, dV(\underline{x}) \quad .$$

Indeed we have

$$\mathcal{F}^e[f](\underline{\xi}) = 2\pi\ \mathcal{F}_{\mathcal{H}^{\pm}}[f](\mp\xi_2, \pm\xi_1) = 2\pi\ \mathcal{F}_{\mathcal{H}^{\pm}}[f](\pm e_{12}\underline{\xi}) \quad ,$$

taking into account that under the isomorphism between the Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{0,2}$, both pseudoscalars e_{12} are isomorphic images of each other. Note that in [62] the Fourier kernel is at the right hand side of the function f instead of the left.

12.4.4 Example: the box function

In this section we calculate, as an illustration, the Clifford-Fourier transform pair of the box function:

$$f(x_1, x_2) = \begin{cases} A & \text{if } a \leq x_1 \leq b \text{ and } c \leq x_2 \leq d \\ 0 & \text{if otherwise} \end{cases}$$

with A a constant.
Its classical Fourier transform reads

$$\begin{aligned} \mathcal{F}[f](\underline{\xi}) &= -\left(\frac{A}{2\pi}\right) \frac{1}{\xi_1 \xi_2} \Big(\exp(-ib\xi_1) - \exp(-ia\xi_1)\Big) \\ &\qquad \Big(\exp(-id\xi_2) - \exp(-ic\xi_2)\Big) \quad . \end{aligned}$$

In view of relation (12.13), this yields the following expression for the Clifford-Fourier transform involving the operator $\mathcal{H}^+$:

$$\begin{aligned} \mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) &= P^+ \left(\frac{A}{2\pi}\right) \frac{1}{\xi_1 \xi_2} \Big(\exp(ib\xi_2) - \exp(ia\xi_2)\Big) \\ &\quad \Big(\exp(-id\xi_1) - \exp(-ic\xi_1)\Big) \\ &\quad + P^- \left(\frac{A}{2\pi}\right) \frac{1}{\xi_1 \xi_2} \Big(\exp(-ib\xi_2) - \exp(-ia\xi_2)\Big) \\ &\quad \Big(\exp(id\xi_1) - \exp(ic\xi_1)\Big) \quad . \end{aligned} \tag{12.15}$$

By means of, inter alia,

$$\begin{aligned} P^+ \exp(ib\xi_2) &= P^+ \sum_{k=0}^{\infty} \frac{i^k}{k!}(b\xi_2)^k = P^+ \sum_{k=0}^{\infty} \frac{(-e_{12})^k}{k!}(b\xi_2)^k \\ &= P^+ \exp(-e_{12}b\xi_2) \ , \end{aligned}$$

expression (12.15) can be simplified to

$$\begin{aligned} \mathcal{F}_{\mathcal{H}^+}[f](\underline{\xi}) &= \left(\frac{A}{2\pi}\right)\frac{1}{\xi_1\xi_2}\Big[P^+ \Big(\exp(-e_{12}b\xi_2) - \exp(-e_{12}a\xi_2)\Big) \\ &\quad \Big(\exp(e_{12}d\xi_1) - \exp(e_{12}c\xi_1)\Big) \\ &\quad + P^- \Big(\exp(-e_{12}b\xi_2) - \exp(-e_{12}a\xi_2)\Big) \\ &\quad \Big(\exp(e_{12}d\xi_1) - \exp(e_{12}c\xi_1)\Big)\Big] \\ &= \left(\frac{A}{2\pi}\right)\frac{1}{\xi_1\xi_2}\Big(\exp(-e_{12}b\xi_2) - \exp(-e_{12}a\xi_2)\Big) \\ &\quad \Big(\exp(e_{12}d\xi_1) - \exp(e_{12}c\xi_1)\Big) \ . \end{aligned}$$

By a similar compution, one obtains

$$\begin{aligned} \mathcal{F}_{\mathcal{H}^-}[f](\underline{\xi}) &= \left(\frac{A}{2\pi}\right)\frac{1}{\xi_1\xi_2}\Big(\exp(e_{12}b\xi_2) - \exp(e_{12}a\xi_2)\Big) \\ &\quad \Big(\exp(-e_{12}d\xi_1) - \exp(-e_{12}c\xi_1)\Big) \ . \end{aligned}$$

12.4.5 Analysis of vector field signals

For the sake of clarity, let us state that we call a signal $f(\underline{x})$ real if it takes its values in a real Clifford algebra, that we call it m-dimensional if it is defined for $\underline{x} \in \mathbb{R}^m$, and that we call it an n-vector field signal if it consists of n components which are distinguished from each other by attaching them to different basis vectors in the Clifford algebra under consideration.

The spectrum of a real two-dimensional signal under the two-dimensional Clifford-Fourier transform, will consist of a scalar part and a bivector part, which is similar to its complex structure under the standard Fourier transform, up to a

rotation by a right angle of the frequency variable, as is illustrated by example 12.4.4.
A real two-dimensional 3-vector field signal can be written in several equivalent ways such as

$$f(\underline{x}) = f_0(x_1, x_2) + e_1 f_1(x_1, x_2) + e_2 f_2(x_1, x_2) \quad .$$

Its spectrum will have the following structure

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f](\underline{\xi}) = F_0(\xi_1, \xi_2) + e_1 F_1(\xi_1, \xi_2) + e_2 F_2(\xi_1, \xi_2) + e_{12} F_{12}(\xi_1, \xi_2)$$

with

$$\begin{array}{rcl} F_0 & = & \mathcal{F}_{\cos}[f_0] \\ F_1 & = & \mathcal{F}_{\cos}[f_1] \mp \mathcal{F}_{\sin}[f_2] \\ F_2 & = & \mathcal{F}_{\cos}[f_2] \pm \mathcal{F}_{\sin}[f_1] \\ F_{12} & = & \pm \mathcal{F}_{\sin}[f_0] \quad . \end{array}$$

Similar formulae hold for the structure of the spectrum of n-vector field signals, but it is clear that the Clifford-Fourier transform and the co-axial Fourier transform will provide genuinely new possibilities for analysing vector field signals only from dimension 3 on.

Chapter 13

Clifford-Filters for early vision

Among the mathematical models suggested for the receptive field profiles of the human visual system, the Gabor model is well-known and widely used. Another, less used, model that agrees with the Gaussian derivative model for human vision is the Hermite model, which is based on analysis filters of the Hermite transform. It offers some advantages like being an orthogonal basis and having better match to experimental physiological data.
In this chapter we expand the filter functions of the classical Hermite transform into the generalized Clifford-Hermite polynomials. Moreover, we construct a new multi-dimensional Hermite transform within Clifford analysis using the radial and generalized Clifford-Hermite polynomials and we compare this newly introduced Clifford-Hermite transform with the Clifford-Hermite continuous wavelet transform. Next, we introduce Gabor filters in the Clifford analysis setting. These so-called Clifford-Gabor filters are based on the Clifford-Fourier transform discussed in the foregoing chapter. Finally, we present further properties of the Clifford-Gabor filters, such as their relationship with other types of Gabor filters and their localization in the spatial and in the frequency domain formalized by the uncertainty principle.

13.1 Introduction

Image processing has been much inspired by the human vision, in particular with regard to early vision. The latter refers to the earliest stage of visual processing responsible for the measurement of local structures such as points, lines, edges and textures in order to facilitate subsequent interpretation of these structures in higher stages (known as high level vision) of the human visual system. This low level visual computation is carried out by cells of the primary visual cortex. The receptive field profiles of these cells can be interpreted as the impulse responses of the cells, which are then considered as filters. According to the Gaussian derivative theory (see [121]), the receptive field profiles of the human visual system can be approximated quite well by derivatives of Gaussians. Two mathematical models suggested for these receptive field profiles are one the Gabor model and two the Hermite model which is based on analysis filters of the Hermite transform. The Hermite filters are derivatives of Gaussians, while Gabor filters, which are defined as harmonic modulations of Gaussians, provide a good approximation to these derivatives. It is important to note that, even if the Gabor model is more widely used than the Hermite model, the latter offers some advantages like being an orthogonal basis and having better match to experimental physiological data.

In this chapter we establish the construction of the Hermite and Gabor filters both in the classical and in the Clifford analysis setting.

We start with describing the classical Hermite transform, both in the one-dimensional and in the multi-dimensional case (see Section 13.2.1). Next, in Section 13.2.2, we expand the filter functions of the classical multi-dimensional Hermite transform into the generalized Clifford-Hermite polynomials introduced in Section 5.3.1. Furthermore, we construct a new higher dimensional Hermite transform within the framework of Clifford analysis. The building blocks for this construction are the radial Clifford-Hermite polynomials (see Section 13.2.3) and the generalized Clifford-Hermite polynomials (see Section 13.2.4). In Section 13.2.5 we compare both Clifford-Hermite transforms with the Clifford-Hermite CWT of Chapter 5. Finally, we test a three-dimensional Clifford-Hermite filter function upon its selectivity for pointwise analysis of signals by calculating the corresponding Clifford-Hermite coefficients of typical scalar benchmark signals.

The topic of Section 13.3 is Gabor filters, a prominent tool for local spectral image processing and analysis. First, in Section 13.3.1, we discuss classical one-dimensional complex and real Gabor filters. They are closely related to Fourier analysis, since the impulse response of a complex Gabor filter is the conjugated integral kernel of the complex Fourier transform at a certain frequency, multiplied by a Gaussian. In Section 13.3.2 we take a look at two non-classical two-dimensional Gabor filters. First, we briefly describe the so-called quaternionic Gabor filters of Bülow and Sommer, followed by the Clifford-Gabor filters of Ebling and Scheuermann. Next, we proceed with developing our two-dimensional Clifford-Gabor filters (see Section 13.3.3). This new type of Gabor filters arises quite naturally from our study of the two-dimensional Clifford-Fourier transform. Indeed, using the conjugated kernels of the two-dimensional Clifford-Fourier transform modulated by a Gaussian gives rise to the so-called two-dimensional Clifford-Gabor filters. We also give an explicit connection between these new filters and the standard complex Gabor filters and moreover the Clifford-Gabor filters of Ebling and Scheuermann. An often cited property of Gabor filters is their optimal simultaneous localization in the spatial and in the frequency domain, which is formalized by the uncertainty principle. This property makes Gabor filters suitable for local frequency analysis. We end this chapter by showing that our two-dimensional Clifford-Gabor filters also exhibit the best possible joint localization in position and frequency space.

13.2 Hermite filters

13.2.1 The classical Hermite transform

The Hermite transform was introduced in [87] as a signal expansion technique in which a signal is windowed by a Gaussian at equidistant positions and is locally described by a weighted sum of polynomials.

In [57] an image compression scheme based on an orientation-adaptive steered Hermite transform is presented. Comparison with other compression techniques show that the proposed scheme performs very well at high compression ratios, not only in terms of peak-signal-to-noise ratio (the common used objective measure for the quality of coded images) but also in terms of perceptual image quality.

Moreover, in [86], it is demonstrated how the Hermite transform can be used for image coding and analysis. In the image coding application, the relation with existing pyramid coders is described. A new coding scheme, based on local one-dimensional image approximations, is introduced. In the image analysis application, the relation between the Hermite transform and existing line/edge detection schemes is described.

An algorithm based on the Hermite transform was applied to astronomical images in [114]. Hermite transforms have also been used in applications such as image deblurring (see [85]), noise reduction (see [66]) and estimation of perceived noise and blur (see [78]).

A) The classical one-dimensional Hermite transform

The one-dimensional Hermite transform first localizes the original signal $f(x)$ by multiplying it by a Gaussian window function

$$\widetilde{V}^{\sigma}(x) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad .$$

In order to have a complete description of the signal $f(x)$, the localization process should be repeated at a sufficient number of window positions, the spacing between the windows being chosen equidistant. In this way the following expansion of the original signal $f(x)$ is obtained:

$$f(x) = \frac{1}{\widetilde{W}^{\sigma}(x)} \sum_{k=-\infty}^{+\infty} f(x)\ \widetilde{V}^{\sigma}(x-kT) \quad , \tag{13.1}$$

where

$$\widetilde{W}^{\sigma}(x) = \sum_{k=-\infty}^{+\infty} \widetilde{V}^{\sigma}(x-kT) \quad ,$$

is the so-called *weight function*, which is positive for all x.

The next step in the Hermite transform is the decomposition of the localized signal $f(x)\ \widetilde{V}^{\sigma}(x-kT)$ into a series of orthonormal functions.
Fundamental for this expansion are the polynomials $G_n^{\sigma}(x)$ which are orthonormal with respect to $(\widetilde{V}^{\sigma}(x))^2$, i.e.

$$\int_{-\infty}^{+\infty} (\widetilde{V}^{\sigma}(x))^2\ G_m^{\sigma}(x)\ G_n^{\sigma}(x)\ dx = \delta_{m,n} \quad .$$

These uniquely determined polynomials have the following form:

$$G_n^{\sigma}(x) = \frac{1}{\sqrt{2^n n!}}\ H_n\left(\frac{x}{\sigma}\right) \quad ,$$

where H_n is the standard Hermite polynomial of order n ($n = 0, 1, 2, \ldots$) associated with the weight function $\exp(-x^2)$.
Under very general conditions (see [13]) for the original signal $f(x)$, we get the

following decomposition of the localized signal into the orthonormal functions $K_n^\sigma(x) = \widetilde{V}^\sigma(x)\ G_n^\sigma(x)$:

$$\widetilde{V}^\sigma(x-kT)\ f(x) = \sum_{n=0}^{\infty} c_n^\sigma(kT)\ K_n^\sigma(x-kT) \quad , \tag{13.2}$$

where

$$c_n^\sigma(kT) = \int_{-\infty}^{\infty} f(x)\ G_n^\sigma(x-kT)\ (\widetilde{V}^\sigma(x-kT))^2\ dx \tag{13.3}$$

are the so-called *Hermite coefficients*.
As the Hermite transform is locally, i.e. within each window function, a unitary transformation, we have the following generalization of Parseval's theorem:

$$\int_{-\infty}^{+\infty} \left(f(x)\right)^2\ (\widetilde{V}^\sigma(x-kT))^2\ dx = \sum_{n=0}^{\infty} (c_n^\sigma(kT))^2 \quad .$$

In other words, the energy of each local signal can be expressed in terms of the Hermite coefficients $c_n^\sigma(kT)$ of the expansion.

The defining relation (13.3) of the Hermite coefficients $c_n^\sigma(kT)$ can be rewritten as the convolution of the original signal $f(x)$ with the functions

$$D_n^\sigma(x) = G_n^\sigma(-x)\ (\widetilde{V}^\sigma(-x))^2 = \frac{(-1)^n}{\sqrt{2^n n!}} \frac{1}{\sigma\sqrt{\pi}}\ H_n\left(\frac{x}{\sigma}\right)\ \exp\left(-\frac{x^2}{\sigma^2}\right) \quad ,$$

followed by a downsampling by a factor T.
The functions $D_n^\sigma(x)$ are called the *Hermite filters*. By means of the Rodrigues formula (11.1) for the classical Hermite polynomials, they can be expressed as Gaussian derivatives:

$$D_n^\sigma(x) = \frac{\sigma^n}{\sqrt{2^n n!}}\ \frac{d^n}{dx^n}\left[\frac{1}{\sigma\sqrt{\pi}}\ \exp\left(-\frac{x^2}{\sigma^2}\right)\right] \quad .$$

Hence, as explained in Section 13.1, the Hermite transform models the information analysis carried out by the cortical visual receptive fields. Because these receptive fields occur in varying size, each field is suited for detecting the presence of a specific spatial frequency. With the Hermite transform, field sizes can be modelled by varying the standard deviation σ of the Gaussian envelope, while orientation selectivity can be obtained by rotation of the Hermite filters. Note that the Fourier transform of the Hermite filter D_n^σ takes the form of a Gaussian modulated by a monomial of degree n :

$$\mathcal{F}[D_n^\sigma](\xi) = \frac{\sigma^n}{\sqrt{2^n n!}}\ (i\xi)^n\ \exp\left(-\frac{\sigma^2\xi^2}{4}\right) \quad .$$

The mapping from the original signal $f(x)$ to the Hermite coefficients $c_n^\sigma(kT)$ is called the *forward Hermite transform.* The signal reconstruction from the Hermite coefficients is called the *inverse Hermite transform.*
Combining (13.1) and (13.2), we get the expansion of the complete signal into the so-called *pattern functions* Q_n^σ :

$$f(x) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{+\infty} c_n^\sigma(kT)\; Q_n^\sigma(x - kT) \quad ,$$

where

$$Q_n^\sigma(x) = \frac{K_n^\sigma(x)}{\widetilde{W}^\sigma(x)} \quad .$$

This formula implies that the inverse Hermite transform consists of interpolating the Hermite coefficients $\{c_n^\sigma(kT); k \text{ integer}\}$ with the pattern function $Q_n^\sigma(x)$ and summing up over all orders n.

In practice, the Hermite transform will often be limited to the first few terms which introduces effects of filtering and aliasing.
In order for the finite Hermite transform to describe the signal adequately, σ must be properly selected. On the one hand, we want σ to be as large as possible because integrating over large areas improves the output signal-to-noise ratio as well as the efficiency of our signal representation. On the other hand, σ can not be too large because then the signal can not be described accurately by the first few terms in the Hermite expansion.
The important problem of selecting the right value of σ is the main topic of [86].

Remark 13.1 The Hermite transform provides the connection between the derivatives of Gaussians and the so-called *Hermite functions.* In the Hermite transform, the analysis functions D_n^σ are the derivatives of Gaussians, whereas the reconstruction functions K_n^σ are the Hermite functions. The difference between the Hermite function and the derivative of the Gaussian of order n is the scale of the Gaussian in relation to the scale of the Hermite polynomial. In case of the Hermite functions the Hermite polynomials grow as fast as the exponential decays and hence the maxima of the Hermite functions have all about the same height, giving them the shape of a truncated sine/cosine wave.
The Hermite functions have two interesting properties. First, they maximize the uncertainty principle (see Section 13.3.3) and second, as for the Gaussian, their Fourier transform has the same functional form as the function itself.

B) The classical multi-dimensional Hermite transform

The Hermite transform is generalized to higher dimension in a tensorial manner.

Start from the Gaussian window function

$$\widetilde{V}^{\sigma}(\underline{x}) = \left(\frac{1}{\sqrt{\sqrt{\pi}\sigma}}\right)^{m} \exp\left(-\frac{x_1^2 + \ldots + x_m^2}{2\sigma^2}\right) \quad .$$

This window function is separable, i.e. $\widetilde{V}^{\sigma}(\underline{x}) = \widetilde{V}^{\sigma}(x_1)\ldots\widetilde{V}^{\sigma}(x_m)$.
Naturally, the polynomials

$$\begin{aligned} G^{\sigma}_{i_1-i_2,i_2-i_3,\ldots,i_m}(\underline{x}) &= \frac{1}{\sqrt{2^{i_1}(i_1-i_2)!(i_2-i_3)!\ldots i_m!}} H_{i_1-i_2}\left(\frac{x_1}{\sigma}\right) \\ &\qquad H_{i_2-i_3}\left(\frac{x_2}{\sigma}\right) \ldots H_{i_m}\left(\frac{x_m}{\sigma}\right) \\ &= G^{\sigma}_{i_1-i_2}(x_1)\; G^{\sigma}_{i_2-i_3}(x_2)\; \ldots\; G^{\sigma}_{i_m}(x_m) \end{aligned}$$

are orthonormal with respect to $(\widetilde{V}^{\sigma}(\underline{x}))^2$, i.e.

$$\delta_{i_1,\ell_1}\ldots\delta_{i_m,\ell_m} = \int_{\mathbb{R}^m} (\widetilde{V}^{\sigma}(\underline{x}))^2\; G^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x})\; G^{\sigma}_{\ell_1-\ell_2,\ldots,\ell_m}(\underline{x})\; dV(\underline{x})$$

for $i_1, \ell_1 = 0,\ldots,\infty$; $i_2 = 0,\ldots,i_1$; $\ell_2 = 0,\ldots,\ell_1$; $\ldots$; $i_m = 0,\ldots,i_{m-1}$; $\ell_m = 0,\ldots,\ell_{m-1}$.

In a similar way as in the one-dimensional case, we obtain the following decomposition of a signal $f(\underline{x})$ into the pattern functions $Q^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x})$:

$$\begin{aligned} f(\underline{x}) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{i_1}\cdots\sum_{i_m=0}^{i_{m-1}}\; \sum_{(p_{x_1},\ldots,p_{x_m})\in P} & c^{\sigma}_{i_1-i_2,\ldots,i_m}(p_{x_1},\ldots,p_{x_m}) \\ & Q^{\sigma}_{i_1-i_2,\ldots,i_m}(x_1-p_{x_1},\ldots,x_m-p_{x_m}) \quad , \end{aligned}$$

where $(p_{x_1},\ldots,p_{x_m})$ ranges over all co-ordinates in a square sampling grid P. So the reconstruction of the signal consists again of interpolating the Hermite coefficients $c^{\sigma}_{i_1-i_2,\ldots,i_m}(p_{x_1},\ldots,p_{x_m})$ with the pattern functions

$$Q^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x}) = \frac{\widetilde{V}^{\sigma}(\underline{x})\; G^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x})}{\widetilde{W}^{\sigma}(\underline{x})} = Q^{\sigma}_{i_1-i_2}(x_1)\;\ldots\; Q^{\sigma}_{i_m}(x_m)$$

where

$$\widetilde{W}^\sigma(\underline{x}) = \sum_{(p_{x_1},\ldots,p_{x_m})\in P} \widetilde{V}^\sigma(x_1 - p_{x_1},\ldots,x_m - p_{x_m})$$

is the positive weight function.

The Hermite coefficients $c^\sigma_{i_1-i_2,\ldots,i_m}(p_{x_1},\ldots,p_{x_m})$ are again obtained by convolving the original signal $f(\underline{x})$ with the filter functions

$$\begin{aligned} D^\sigma_{i_1-i_2,\ldots,i_m}(\underline{x}) &= (\widetilde{V}^\sigma(-\underline{x}))^2\ G^\sigma_{i_1-i_2,\ldots,i_m}(-\underline{x}) \\ &= \frac{(-1)^{i_1}}{\sqrt{2^{i_1}(i_1-i_2)!\ldots i_m!}} \left(\frac{1}{\sqrt{\pi}\sigma}\right)^m H_{i_1-i_2}\left(\frac{x_1}{\sigma}\right)\ \ldots \\ &\qquad H_{i_m}\left(\frac{x_m}{\sigma}\right)\ \exp\left(-\frac{x_1^2+\ldots+x_m^2}{\sigma^2}\right) \\ &= D^\sigma_{i_1-i_2}(x_1)\ \ldots\ D^\sigma_{i_m}(x_m) \end{aligned}$$

followed by a downsampling along the grid P.
These filter functions can be written as derivatives of a Gaussian:

$$D^\sigma_{i_1-i_2,\ldots,i_m}(\underline{x}) = \frac{\sigma^{i_1}}{\sqrt{2^{i_1}(i_1-i_2)!\ldots i_m!}} \left(\frac{1}{\sqrt{\pi}\sigma}\right)^m \frac{\partial^{i_1-i_2}}{\partial x_1^{i_1-i_2}} \frac{\partial^{i_2-i_3}}{\partial x_2^{i_2-i_3}} \ldots \frac{\partial^{i_m}}{\partial x_m^{i_m}} \left[\exp\left(-\frac{x_1^2+\ldots+x_m^2}{\sigma^2}\right)\right] \ .$$

Now we aim at constructing a *generating function* of these *filter functions.* Putting

$$\begin{aligned} &\widetilde{F}^\sigma_{i_1}(\underline{u};\underline{x}) \\ &= \frac{1}{i_1!}\left(u_1\frac{\partial}{\partial(x_1/\sigma)} + u_2\frac{\partial}{\partial(x_2/\sigma)} + \ldots + u_m\frac{\partial}{\partial(x_m/\sigma)}\right)^{i_1} \left(\widetilde{V}^\sigma(\underline{x})\right)^2 \\ &= \left(\frac{1}{\sqrt{\pi}\sigma}\right)^m \sum_{k_2=0}^{i_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_m=0}^{k_{m-1}} \frac{u_1^{i_1-k_2}u_2^{k_2-k_3}\ldots u_m^{k_m}}{(i_1-k_2)!(k_2-k_3)!\ldots k_m!} \frac{\partial^{i_1-k_2}}{\partial(x_1/\sigma)^{i_1-k_2}} \\ &\qquad \frac{\partial^{k_2-k_3}}{\partial(x_2/\sigma)^{k_2-k_3}} \ \cdots\ \frac{\partial^{k_m}}{\partial(x_m/\sigma)^{k_m}}\ \exp\left(-\frac{x_1^2+\ldots+x_m^2}{\sigma^2}\right) \\ &= 2^{i_1/2}\sum_{k_2=0}^{i_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_m=0}^{k_{m-1}} \frac{1}{\sqrt{(i_1-k_2)!(k_2-k_3)!\ldots k_m!}}\ u_1^{i_1-k_2}\ u_2^{k_2-k_3} \\ &\qquad \ldots\ u_m^{k_m}\ D^\sigma_{i_1-k_2,\ldots,k_m}(\underline{x}) \ , \end{aligned}$$

we obtain

$$D^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x}) = \frac{1}{2^{i_1/2}} \; \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \; \frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} \widetilde{F}^{\sigma}_{i_1}(\underline{u};\underline{x}) \quad .$$

Hence the function

$$\begin{aligned}
&\widetilde{F}^{\sigma}(\underline{u};\underline{x}) \\
&= \sum_{i_1=0}^{\infty} \frac{1}{2^{i_1/2}} \, \frac{1}{i_1!} \left(u_1 \frac{\partial}{\partial(x_1/\sigma)} + \ldots + u_m \frac{\partial}{\partial(x_m/\sigma)} \right)^{i_1} \left(\widetilde{V}^{\sigma}(\underline{x}) \right)^2 \\
&= \sum_{i_1=0}^{\infty} \sum_{k_2=0}^{i_1} \cdots \sum_{k_m=0}^{k_{m-1}} \frac{1}{\sqrt{(i_1-k_2)!\ldots k_m!}} \; u_1^{i_1-k_2} \; \ldots \; u_m^{k_m} \; D^{\sigma}_{i_1-k_2,\ldots,k_m}(\underline{x})
\end{aligned}$$

generates the filter functions, since

$$D^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x}) = \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \left(\frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} \widetilde{F}^{\sigma}(\underline{u};\underline{x}) \right)_{\underline{u}=\underline{0}} \quad .$$

Moreover we have that

$$\begin{aligned}
&c^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{t}) \\
&= (f * D^{\sigma}_{i_1-i_2,\ldots,i_m})(\underline{t}) \\
&= \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \left(\frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} (f(\underline{x}) * \widetilde{F}^{\sigma}(\underline{u};\underline{x}))(\underline{t}) \right)_{\underline{u}=\underline{0}} \quad .
\end{aligned}$$

So, if we define

$$c^{\sigma}(\underline{u};\underline{t}) = (f(\underline{x}) * \widetilde{F}^{\sigma}(\underline{u};\underline{x}))(\underline{t}) \quad ,$$

we have obtained the *generating function of the Hermite coefficients*:

$$\begin{aligned}
\widetilde{c}^{\sigma}(\underline{u};\underline{t}) &= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \cdots \sum_{i_m=0}^{i_{m-1}} \frac{1}{(i_1-i_2)!\ldots i_m!} \; u_1^{i_1-i_2} \; \ldots \; u_m^{i_m} \\
&\quad \left(\frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} c^{\sigma}(\underline{u};\underline{t}) \right)_{\underline{u}=\underline{0}} \\
&= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \cdots \sum_{i_m=0}^{i_{m-1}} \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \; u_1^{i_1-i_2} \; \ldots u_m^{i_m} \; c^{\sigma}_{i_1-i_2,\ldots,i_m}(\underline{t}) \; .
\end{aligned}$$

13.2.2 Expansion of the classical multi-dimensional Hermite filters into the generalized Clifford-Hermite polynomials

In this section we will show how the classical filter functions of the multi-dimensional Hermite transform can be expressed in terms of Clifford analysis. Starting point is the general term of the defining series of the generating function of the filter functions (see Section 13.2.1), which, up to constants, may be rewritten as

$$F_k(\underline{u};\underline{x}) = \frac{1}{k!} < \underline{u}, \partial_{\underline{x}} >^k V(\underline{x}) \quad ,$$

where we have put

$$V(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2}\right) \quad .$$

Now the Fischer decomposition leads to the following expansion (see [109]):

$$\frac{1}{k!} < \underline{u}, \underline{x} >^k = \sum_{s=0}^{k} \underline{u}^s \; Z_{k,s}(\underline{u}, \underline{x}) \; \underline{x}^s \quad ,$$

where the functions $Z_{k,s}(\underline{u}, \underline{x})$ are the so-called zonal monogenics. These zonal monogenics are homogeneous of degree $k - s$ in $\underline{u}$ and $\underline{x}$, left-monogenic in $\underline{u}$ and right-monogenic in $\underline{x}$ and have the form:

$$Z_{k,s}(\underline{u}, \underline{x}) = \frac{1}{B_{s,k-s} \dots B_{1,k-s}} \; Z_{k-s}(\underline{u}, \underline{x}) \quad ,$$

with

$$B_{2s,k} = -2s \quad , \quad B_{2s+1,k} = -(2s + 2k + m)$$

and

$$Z_k(\underline{u}, \underline{x}) = \frac{\Gamma(\frac{m}{2} - 1)}{2^{k+1}\Gamma(k + \frac{m}{2})} \; (|\underline{u}||\underline{x}|)^k \left((k + m - 2)\; C_k^{m/2-1}(t) + (m - 2)\; \frac{\underline{u} \wedge \underline{x}}{|\underline{u}||\underline{x}|} \; C_{k-1}^{m/2}(t)\right) \quad .$$

Here C_k^ℓ stands for the classical Gegenbauer polynomial with variable $t = \frac{<\underline{u},\underline{x}>}{|\underline{u}||\underline{x}|}$.

Hence, we obtain that

$$\begin{aligned} F_k(\underline{u};\underline{x}) &= \sum_{s=0}^{k} \underline{u}^s \; Z_{k,s}(\underline{u},\partial_{\underline{x}}) \; \partial_{\underline{x}}^s \; V(\underline{x}) \\ &= \sum_{s=0}^{k} \underline{u}^s \left[Z_{k,s}(\underline{u},\partial_{\underline{x}}) \; V(\underline{x})\right] \partial_{\underline{x}}^s \quad . \end{aligned} \tag{13.4}$$

Next, Proposition 2.3 leads to:

$$F_k(\underline{u};\underline{x}) = \sum_{s=0}^{k} (-1)^{k-s} \; \underline{u}^s \; \left[Z_{k,s}(\underline{u},\underline{x}) \; V(\underline{x})\right]\partial_{\underline{x}}^s \quad .$$

Moreover, as $Z_{k,s}(\underline{u},\underline{x})$ is a homogeneous right-monogenic polynomial of degree $k-s$ in $\underline{x}$, the Rodrigues formula (5.9) for the generalized Clifford-Hermite polynomials yields:

$$\left[Z_{k,s}(\underline{u},\underline{x}) \; V(\underline{x})\right]\partial_{\underline{x}}^s = Z_{k,s}(\underline{u},\underline{x}) \; \overline{H}_{s,k-s}(\underline{x}) \; V(\underline{x}) \quad .$$

Hence, we finally obtain the decomposition

$$F_k(\underline{u};\underline{x}) = \frac{1}{k!} < \underline{u},\partial_{\underline{x}} >^k V(\underline{x}) = \sum_{s=0}^{k}(-1)^{k-s} \; \underline{u}^s \; Z_{k,s}(\underline{u},\underline{x}) \; \overline{H}_{s,k-s}(\underline{x}) \; V(\underline{x}) \quad .$$

Substituting $\dfrac{\underline{x}}{\sigma}$ for $\underline{x}$ and putting

$$V^\sigma(\underline{x}) = V\left(\frac{\underline{x}}{\sigma}\right) = \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) \quad ,$$

we obtain

$$\begin{aligned} F_k\left(\underline{u};\frac{\underline{x}}{\sigma}\right) &= \sigma^k \frac{1}{k!} < \underline{u},\partial_{\underline{x}} >^k V^\sigma(\underline{x}) \\ &= \frac{1}{\sigma^k} \sum_{s=0}^{k} (-1)^{k-s} \; \sigma^s \; \underline{u}^s \; Z_{k,s}(\underline{u},\underline{x}) \; \overline{H}_{s,k-s}\left(\frac{\underline{x}}{\sigma}\right) \; V^\sigma(\underline{x}) \quad . \end{aligned}$$

As

$$\frac{1}{i_1!} < \underline{u},\partial_{\underline{x}} >^{i_1} V^\sigma(\underline{x}) = \frac{\pi^{m/2}\sigma^{m-i_1}}{2^{(i_1-m)/2}} \; \widetilde{F}_{i_1}^{\sqrt{2}\sigma}(\underline{u};\underline{x}) \quad ,$$

we have finally obtained the following decomposition of the classical filter functions into the generalized Clifford-Hermite polynomials:

$$\begin{aligned} D^{\sqrt{2}\sigma}_{i_1-i_2,\ldots,i_m}(\underline{x}) &= \frac{1}{2^{i_1/2}} \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} \widetilde{F}^{\sqrt{2}\sigma}_{i_1}(\underline{u};\underline{x}) \\ &= \frac{1}{(2\pi)^{m/2}} \frac{1}{\sigma^{m+i_1}} \frac{1}{\sqrt{(i_1-i_2)!\ldots i_m!}} \sum_{s=0}^{i_1} (-1)^{i_1-s}\, \sigma^s \\ &\quad \frac{\partial^{i_1-i_2}}{\partial u_1^{i_1-i_2}} \cdots \frac{\partial^{i_m}}{\partial u_m^{i_m}} [\underline{u}^s\ Z_{i_1,s}(\underline{u},\underline{x})]\ \overline{H}_{s,i_1-s}\left(\frac{\underline{x}}{\sigma}\right)\ V^\sigma(\underline{x}) \quad . \end{aligned}$$

13.2.3 The Clifford-Hermite transform

In this section we develop a first new multi-dimensional Hermite transform directly in Clifford analysis; we call it the *Clifford-Hermite transform.*

First, using the Gaussian window function

$$V^\sigma(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) \quad ,$$

we get the following decomposition of the original real-valued signal $f(\underline{x})$:

$$f(\underline{x}) = \frac{1}{W^\sigma(\underline{x})} \sum_{\underline{p}\in P} f(\underline{x})\ V^\sigma(\underline{x}-\underline{p}) \tag{13.5}$$

with

$$W^\sigma(\underline{x}) = \sum_{\underline{p}\in P} V^\sigma(\underline{x}-\underline{p})$$

the positive weight function and P a sampling grid in $\mathbb{R}^m$.

For the decomposition of the localized signal $f(\underline{x})\ V^\sigma(\underline{x}-\underline{p})$, we use the radial Clifford-Hermite polynomials $H_n(\underline{x})$, which satisfy the orthogonality relation (see (5.4))

$$\int_{\mathbb{R}^m} \overline{H}_n\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right)\ H_{n'}\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right)\ (V^\sigma(\underline{x}))^2\ dV(\underline{x}) = \frac{\sigma^m}{2^{m/2}}\ \gamma_n\ \delta_{n,n'} \quad .$$

The above orthogonality relation leads to the following decomposition of the filtered localized signal into the orthogonal Clifford-Hermite functions $K_n^\sigma(\underline{x}) = V^\sigma(\underline{x})\ H_n\left(\dfrac{\sqrt{2}\underline{x}}{\sigma}\right)$:

$$\left(V^\sigma(\underline{x}-\underline{p})\ f(\underline{x})\right)_{\mathrm{Fil}} = \sum_{j=0}^{\infty} c_j^\sigma(\underline{p})\ \overline{K}_j^\sigma(\underline{x}-\underline{p}) \quad , \tag{13.6}$$

where we have put

$$\begin{aligned} c_n^\sigma(\underline{p}) &= \frac{2^{m/2}}{\sigma^m \gamma_n} \int_{\mathbb{R}^m} f(\underline{x})\ H_n\left(\frac{\sqrt{2}(\underline{x}-\underline{p})}{\sigma}\right)\ (V^\sigma(\underline{x}-\underline{p}))^2\ dV(\underline{x}) \\ &= \frac{2^{m/2}}{\sigma^m \gamma_n} \int_{\mathbb{R}^m} f(\underline{x})\ K_n^\sigma(\underline{x}-\underline{p})\ V^\sigma(\underline{x}-\underline{p})\ dV(\underline{x}) \quad . \end{aligned}$$

We call $c_n^\sigma(\underline{p})$ the *Clifford-Hermite coefficients.*

Combining the formulae (13.5) and (13.6), we obtain the following decomposition of the filtered signal:

$$f_{\mathrm{Fil}}(\underline{x}) = \sum_{j=0}^{\infty} \sum_{\underline{p}\in P} c_j^\sigma(\underline{p})\ Q_j^\sigma(\underline{x}-\underline{p}) \quad ,$$

where we have introduced the pattern functions

$$Q_j^\sigma(\underline{x}) = \frac{\overline{K}_j^\sigma(\underline{x})}{W^\sigma(\underline{x})} = \frac{\overline{H}_j\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right)\ V^\sigma(\underline{x})}{W^\sigma(\underline{x})} \quad .$$

Note that the Clifford-Hermite coefficients may be expressed as the convolution of the original signal $f(\underline{x})$ with the *Clifford-Hermite filter functions* D_n^σ:

$$c_n^\sigma(\underline{p}) = \left(f * D_n^\sigma\right)(\underline{p})$$

with

$$\begin{aligned} D_n^\sigma(\underline{x}) &= \frac{2^{m/2}}{\sigma^m \gamma_n}\ H_n\left(-\frac{\sqrt{2}\underline{x}}{\sigma}\right)\ (V^\sigma(-\underline{x}))^2 \\ &= \frac{(-1)^n 2^{m/2}}{\sigma^m \gamma_n}\ H_n\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right)\ \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\ &= \frac{2^{(m-n)/2}\sigma^{n-m}}{\gamma_n}\ \partial_{\underline{x}}^n \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \quad . \end{aligned} \tag{13.7}$$

The last expression in (13.7) is obtained by using the Rodrigues formula (5.3) of the radial Clifford-Hermite polynomials. Note that the parameters of the alternatively scalar- or vector-valued Clifford-Hermite filters are the scale σ of the Gaussian and the derivative order n. Moreover, their Fourier transform in spherical co-ordinates $\underline{\xi} = \rho\underline{\eta}$, $\rho = |\underline{\xi}|$, $\underline{\eta} \in S^{m-1}$, is given by

$$\mathcal{F}[D_n^\sigma](\underline{\xi}) = \frac{(i\sigma)^n}{\gamma_n 2^{n/2}} \; \underline{\eta}^n \; \rho^n \; \exp\left(-\frac{\sigma^2\rho^2}{4}\right) \quad .$$

Hence the Clifford-Hermite filters are polar separable, i.e. their Fourier transform is expressed as the product of a spatial frequency tuning function and an orientation tuning function. Daugman has demonstrated the importance of polar separable filters in [52].

Remark 13.2 The Clifford-Hermite transform introduced above, was also constructed in the framework of quaternionic analysis (see [21]).

13.2.4 The generalized Clifford-Hermite transform

The Clifford-Hermite transform introduced in Section 13.2.3 may be generalized at once by using the generalized Clifford-Hermite polynomials.
As the development of this *generalized Clifford-Hermite transform* runs along the same lines as in Section 13.2.3, we give only the significant results.

The orthogonal functions used for the expansion of the localized signal are the generalized Clifford-Hermite functions (see also Section 11.3.3):

$$K_{n,k}^\sigma(\underline{x}) = V^\sigma(\underline{x}) \; G_{n,k}^\sigma(\underline{x}) \quad ,$$

with

$$G_{n,k}^\sigma(\underline{x}) = H_{n,k}\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right) \; P_k(\underline{x}) \quad .$$

For the coefficients in the decomposition of the filtered localized signal

$$\big(f(\underline{x}) \; V^\sigma(\underline{x} - \underline{p})\big)_{\text{Fil}} = \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} c_{j,\ell}^\sigma(\underline{p}) \; \overline{K}_{j,\ell}^\sigma(\underline{x} - \underline{p})$$

we obtain

$$
\begin{aligned}
& c^{\sigma}_{n,k}(\underline{p}) \\
&= \frac{2^{(2k+m)/2}}{\sigma^{2k+m}\gamma_{n,k}} \int_{\mathbb{R}^m} f(\underline{x})\ H_{n,k}\left(\frac{\sqrt{2}}{\sigma}(\underline{x}-\underline{p})\right)\ P_k(\underline{x}-\underline{p})\ (V^{\sigma}(\underline{x}-\underline{p}))^2\ dV(\underline{x}) \\
&= \frac{2^{(2k+m)/2}}{\sigma^{2k+m}\gamma_{n,k}} \int_{\mathbb{R}^m} f(\underline{x})\ K^{\sigma}_{n,k}(\underline{x}-\underline{p})\ V^{\sigma}(\underline{x}-\underline{p})\ dV(\underline{x}) \quad .
\end{aligned}
$$

We call $c^{\sigma}_{n,k}(\underline{p})$ the *generalized Clifford-Hermite coefficients*.

The filter functions, from which we can calculate the generalized Clifford-Hermite coefficients similarly as in section 13.2.3, now take the form

$$
\begin{aligned}
D^{\sigma}_{n,k}(\underline{x}) &= \frac{2^{(2k+m)/2}}{\sigma^{2k+m}\gamma_{n,k}}\ H_{n,k}\left(-\frac{\sqrt{2}}{\sigma}\underline{x}\right)\ P_k(-\underline{x})\ (V^{\sigma}(-\underline{x}))^2 \\
&= \frac{(-1)^{n+k}2^{(2k+m)/2}}{\sigma^{2k+m}\gamma_{n,k}}\ H_{n,k}\left(\frac{\sqrt{2}}{\sigma}\underline{x}\right)\ P_k(\underline{x})\ \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\
&= \frac{2^{(2k+m-n)/2}\sigma^{n-m-2k}}{\gamma_{n,k}}\ (-1)^k\ \partial^n_{\underline{x}}\left[\exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right)\ P_k(\underline{x})\right] \quad .
\end{aligned}
$$

These filter functions are a generalization of the filter functions in the previous section, in the sense that $D^{\sigma}_{n,k=0}(\underline{x}) = D^{\sigma}_{n}(\underline{x})$.
Their Fourier transform is given by:

$$
\mathcal{F}[D^{\sigma}_{n,k}](\underline{\xi}) = \frac{i^{k+n}\sigma^n}{2^{n/2}\ \gamma_{n,k}}\ \underline{\xi}^n\ P_k(\underline{\xi})\ \exp\left(-\frac{\sigma^2|\underline{\xi}|^2}{4}\right) \quad .
$$

13.2.5 Connection with the Clifford-Hermite CWT

Recall from Section 5.2.2 that the Clifford-Hermite CWT is defined by

$$
\begin{aligned}
F_n(a,\underline{b}) &= \int_{\mathbb{R}^m} \overline{\psi^{a,\underline{b}}_n(\underline{x})}\ f(\underline{x})\ dV(\underline{x}) \\
&= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x}-\underline{b}|^2}{2a^2}\right)\ \overline{H_n\left(\frac{\underline{x}-\underline{b}}{a}\right)}\ f(\underline{x})\ dV(\underline{x}) \qquad (13.8)
\end{aligned}
$$

with $f \in L_2\big(\mathbb{R}^m, dV(\underline{x})\big)$ the signal to be analyzed, ψ_n the Clifford-Hermite mother wavelets, $a \in \mathbb{R}_+$ the dilation parameter and $\underline{b} \in \mathbb{R}^m$ the translation

parameter.
The original signal may be reconstructed from its transform $F_n(a, \underline{b})$ by the inverse transformation (see Corollary 4.4):

$$f(\underline{x}) = \frac{1}{C_n} \int_{\mathbb{R}^m} \int_0^{+\infty} \psi_n^{a,\underline{b}}(\underline{x})\ F_n(a, \underline{b})\ \frac{da}{a^{m+1}}\ dV(\underline{b}) \tag{13.9}$$

with C_n the admissibility constant.

Putting $\sigma = \sqrt{2}\mu$ in the Clifford-Hermite coefficients of Section 13.2.3

$$\begin{aligned} c_n^\sigma(\underline{p}) &= \int_{\mathbb{R}^m} D_n^\sigma(\underline{p} - \underline{x})\ f(\underline{x})\ dV(\underline{x}) \\ &= \frac{2^{m/2}}{\sigma^m \gamma_n} \int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x} - \underline{p}|^2}{\sigma^2}\right) H_n\left(\frac{\sqrt{2}}{\sigma}(\underline{x} - \underline{p})\right)\ f(\underline{x})\ dV(\underline{x}) \ , \end{aligned}$$

we obtain

$$c_n^{\sqrt{2}\mu}(\underline{p}) = \frac{1}{\mu^m \gamma_n} \int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{x} - \underline{p}|^2}{2\mu^2}\right) H_n\left(\frac{\underline{x} - \underline{p}}{\mu}\right)\ f(\underline{x})\ dV(\underline{x}) \ . \tag{13.10}$$

Comparing (13.8) and (13.10), the connection between the Clifford-Hermite transform and the Clifford-Hermite CWT is clear: σ plays the role of the dilation parameter a, $\underline{p}$ plays the role of the translation parameter $\underline{b}$ and the translated filter functions $D_n^\sigma(\underline{p} - \underline{x})$ play the role of the wavelets $\psi_n^{a,\underline{b}}$.

There are however some differences between the two transforms. The wavelet translation parameter $\underline{b}$ is continuous, whereas the parameter $\underline{p}$ in the Hermite transform is discrete. Furthermore, the reconstruction of the filtered signal in the Hermite transform

$$f_{\text{Fil}}(\underline{x}) = \sum_{j=0}^{\infty} \sum_{\underline{p} \in P} c_j^\sigma(\underline{p})\ Q_j^\sigma(\underline{x} - \underline{p}) \tag{13.11}$$

depends on σ, which is not the case for the CWT. As mentioned before, σ is an important parameter for practical applications.
In the CWT, the quality of the signal reconstruction (13.9) depends on the mother wavelet $\psi_n(\underline{x})$; in other words it depends on the order of the derivative of the Gaussian. In the inverse Hermite transform (13.11) we sum over all these orders.

An analogous comparison can be drawn between the generalized Clifford-Hermite transform and the generalized Clifford-Hermite CWT introduced in Section 5.3.2. If we leave the spinor-rotations in the generalized Clifford-Hermite CWT out of consideration, the same connection as above can be established.

13.2.6 Benchmarking of a three-dimensional filter function

In the Hermite transform it is the filter functions which determine what kind of information is made explicit in the Hermite coefficients; therefore the main properties of the Hermite transform are determined by these filter functions.
In this section, we demonstrate that the filter functions of the Clifford-Hermite transform are selective at detecting pointwise singularities of the signal.

We test this capacity of the 3-D filter function

$$\begin{aligned} D_1^\sigma(\underline{x}) &= \frac{2^{3/2}}{\sigma^3\gamma_1}\, H_1\left(-\frac{\sqrt{2}}{\sigma}\underline{x}\right) \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \\ &= \frac{4}{\sigma^3\gamma_1}\left(-\frac{\underline{x}}{\sigma}\right) \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \quad . \end{aligned}$$

The Clifford-Hermite coefficients of the real-valued signal $f(\underline{x})$ that correspond with this filter function are

$$\begin{aligned} c_1^\sigma(\underline{p}) &= \int_{\mathbb{R}^3} f(\underline{x})\, D_1^\sigma(\underline{p}-\underline{x})\, dV(\underline{x}) \\ &= \frac{4}{\sigma^3\gamma_1}\int_{\mathbb{R}^3} f(\underline{x})\, \frac{\underline{x}-\underline{p}}{\sigma} \exp\left(-\frac{|\underline{x}-\underline{p}|^2}{\sigma^2}\right) dV(\underline{x}) \quad . \end{aligned}$$

The tests are carried out on scalar benchmark signals: the infinite rod, the semi-infinite rod and the rod of finite length.

First, the filter function is tested on the scalar benchmark signal consisting of an infinite rod, laying along the x_1-axis, modelled as

$$f^{(1)}(\underline{x}) = \delta(x_2)\ \delta(x_3) \quad ,$$

where δ stands for the delta distribution.
Its Clifford-Hermite coefficients are

$$c_1^\sigma(\underline{p}) = -\frac{4\sqrt{\pi}}{\sigma^3\gamma_1} \exp\left(-\frac{p_2^2}{\sigma^2}\right) \exp\left(-\frac{p_3^2}{\sigma^2}\right) (p_2 e_2 + p_3 e_3) \quad .$$

We observe that the coefficients decay rapidly to zero for $(p_2, p_3) \to \infty$ and that they are zero in the sampling points on the x_1-axis ($p_2 = p_3 = 0$), where there are no discontinuities.

In a second test, we apply the filter function D_1^σ on the scalar benchmark signal consisting of a thin plate in the (x_1, x_2)-plane, modelled as

$$f^{(2)}(\underline{x}) = \delta(x_3) \quad .$$

Its Clifford-Hermite coefficients

$$c_1^\sigma(\underline{p}) = -\frac{4\pi}{\sigma^2\gamma_1} \ \exp\left(-\frac{p_3^2}{\sigma^2}\right) \, p_3 e_3$$

are zero in the sampling points of the (x_1, x_2)-plane and vanish for $p_3 \to \infty$.

A third test is carried out on the scalar benchmark signal

$$f^{(3)}(\underline{x}) = \delta(x_2) \; \delta(x_3) \; Y(x_1) \quad ,$$

where Y stands for the Heaviside step function. This signal consists of a semi-infinite rod laying along the positive x_1-axis.
The corresponding Clifford-Hermite coefficients are

$$c_1^\sigma(\underline{p}) = e_1 \left(\frac{2}{\sigma^2\gamma_1} \ \exp\left(-\frac{|\underline{p}|^2}{\sigma^2}\right)\right) - \frac{2\sqrt{\pi}}{\sigma^3\gamma_1} \ \left(1 + \operatorname{erf}\left(\frac{p_1}{\sigma}\right)\right)$$
$$\exp\left(-\frac{p_2^2}{\sigma^2}\right) \ \exp\left(-\frac{p_3^2}{\sigma^2}\right) \ (p_2 e_2 + p_3 e_3) \quad ,$$

with

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) \, dt \quad .$$

In the sampling points on the x_1-axis, this means for $p_2 = p_3 = 0$, the Clifford-Hermite coefficients reduce to

$$c_1^\sigma(p_1 e_1) = e_1 \left(\frac{2}{\sigma^2\gamma_1} \ \exp\left(-\frac{p_1^2}{\sigma^2}\right)\right) \quad .$$

These coefficients show a peak at the origin, which is the more pronounced, the smaller σ is chosen (see Figure 13.1). This demonstrates that the filter function is selective at detecting the discontinuity at the origin.
Furthermore, as

$$\lim_{x\to\pm\infty} \operatorname{erf}(x) = \pm 1 \quad ,$$

it is easily seen that for $p_1 \to -\infty$, independently of p_2 and p_3, the coefficients vanish, while for $p_1 \to +\infty$ they become perpendicular to the x_1-axis.

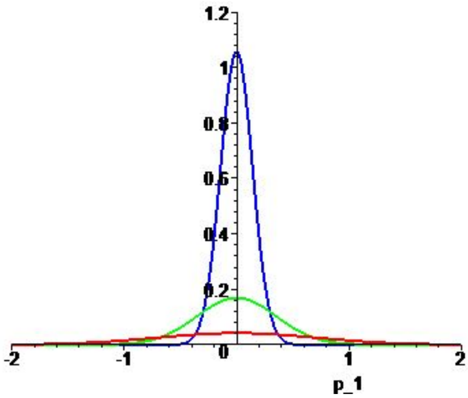


Figure 13.1: The absolute value of the Clifford-Hermite coefficients $c_1^{(\sigma)}(p_1 e_1)$ of the semi-infinite rod $f^{(3)}(\underline{x})$ for the standard deviation values $\sigma = 1$ (red), $\sigma = 0.5$ (green) and $\sigma = 0.2$ (blue).

Finally, we test the filter function on a thin rod of finite length laying on the interval $[-1, 1]$ of the x_1-axis, modelled as

$$f^{(4)}(\underline{x}) = \delta(x_2)\ \delta(x_3)\ \chi_{[-1,1]}(x_1) \quad ,$$

with $\chi_{[-1,1]}$ the characteristic function on the interval $[-1, 1]$.
Its Clifford-Hermite coefficients are

$$\begin{aligned} c_1^{\sigma}(\underline{p}) = {} & e_1 \frac{2}{\sigma^2 \gamma_1} \exp\left(-\frac{p_2^2 + p_3^2}{\sigma^2}\right) \left(\exp\left(-\frac{(1+p_1)^2}{\sigma^2}\right) - \exp\left(-\frac{(1-p_1)^2}{\sigma^2}\right)\right) \\ & - (p_2 e_2 + p_3 e_3)\ \frac{2\sqrt{\pi}}{\sigma^3 \gamma_1}\ \left(\operatorname{erf}\left(\frac{1+p_1}{\sigma}\right) - \operatorname{erf}\left(\frac{-1+p_1}{\sigma}\right)\right) \\ & \exp\left(-\frac{p_2^2}{\sigma^2}\right)\ \exp\left(-\frac{p_3^2}{\sigma^2}\right) \quad . \end{aligned}$$

First note that these coefficients vanish, independently from p_2 and p_3, for $p_1 \to \pm\infty$.
For a sampling point $p_1 e_1$ on the x_1-axis, the Clifford-Hermite coefficients become

$$c_1^{\sigma}(p_1 e_1) = e_1\ \frac{2}{\sigma^2 \gamma_1}\ \left(\exp\left(-\frac{(1+p_1)^2}{\sigma^2}\right) - \exp\left(-\frac{(1-p_1)^2}{\sigma^2}\right)\right) \quad .$$

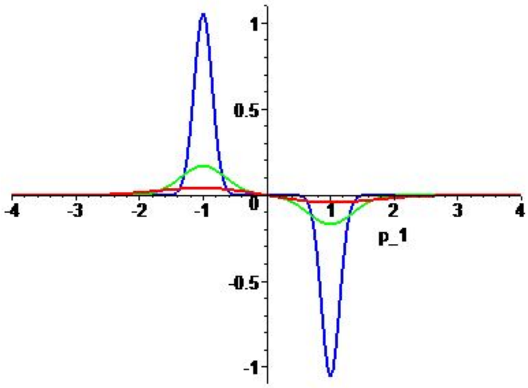


Figure 13.2: The absolute value of the Clifford-Hermite coefficients $c_1^{(\sigma)}(p_1 e_1)$ of the finite rod $f^{(4)}(\underline{x})$ for the standard deviation values $\sigma = 1$ (red), $\sigma = 0.5$ (green) and $\sigma = 0.2$ (blue).

They achieve an extreme value for $p_1 = 1$ and for $p_1 = -1$ (see Figure 13.2). So the filter function is again selective at detecting the pointwise singularities.

These four tests amply demonstrate that the filter function D_1^σ of the Hermite transform indeed shows all the characteristics needed to qualify it as an efficient tool for a pointwise analysis of signals in three dimensions.

13.3 Gabor filters

13.3.1 The classical one-dimensional Gabor filters

One of the most prominent tools for local spectral image processing and analysis are Gabor filters. They were first introduced in the field of one-dimensional signal processing by Gabor in [69] for a joint time-frequency analysis. Gabor filters have the main advantage of being simultaneously optimally localized in the spatial and in the frequency domain. Hence spatial and frequency properties are optimally analysed at the same time by Gabor filters (see also Section 13.3.3). Gabor filters also give access to the local phase of a signal. It has been shown that there is a close correspondence between the local structure of a signal and its local phase. Furthermore, certain regions in the human visual cortex can be modelled as Gabor filters (see Section 13.1). Hence, Gabor filters conform well to the human visual system's capabilities. Gabor filters have been successfully applied to different image processing and analysis tasks such as texture segmentation (see for e.g. [36, 118]), edge detection and local phase and frequency estimation for image matching.

A) One-dimensional complex Gabor filters

In this section we consider the classical Fourier transform with the angular frequency in the kernel function:

$$\mathcal{F}[f](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\ \exp(-i2\pi u x)\ dx \quad .$$

Complex Gabor filters are closely related to Fourier analysis in the following way. They are linear shift invariant (abbreviated LSI) filters, hence they can be applied by simply convolving the signal with the impulse response of the filter. The *impulse response* h of a complex Gabor filter is the complex conjugated integral kernel of the classical Fourier transform $\mathcal{F}$ of some frequency u^* multiplied with a Gaussian g centered at the origin, i.e.

$$h(x) = g(x)\ \exp(i2\pi u^* x)$$

with

$$g(x) = N \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad . \tag{13.12}$$

While Gabor filters analyse the local spectral properties of a signal, the Fourier transform decomposes a signal into its global spectral components. Hence the Fourier transform can be considered the basis upon which Gabor filters were introduced.

The parameters of the complex Gabor filter are the *normalization constant* N, the *center frequency* u^* and the *variance* or *standard deviation* σ of the Gaussian. Normally, N is chosen such that the Gaussian is amplitude normalized, i.e.

$$\|g(x)\|_1 = \int_{-\infty}^{+\infty} g(x)\ dx = 1 \quad ,$$

which implies

$$N = \frac{1}{\sqrt{2\pi}\sigma} \quad .$$

Sometimes other parameterizations of the complex Gabor filter than the one given above are used, viz.

$$\begin{aligned} h(x) &= g(x)\ \exp(i\xi^* x) \\ &= g(x)\ \exp\left(\frac{icx}{\sigma}\right) \quad . \end{aligned}$$

Here $\xi^* = 2\pi u^*$ is the *angular frequency* and $c = \xi^*\sigma$ is the *oscillation parameter.* The *transfer function* of a complex Gabor filter is a shifted Gaussian:

$$H(u) := \mathcal{F}[h](u) = \frac{1}{\sqrt{2\pi}} \exp\left(-2\pi^2\sigma^2(u-u^*)^2\right) \quad .$$

In terms of the angular frequency $\xi = 2\pi u$ this becomes

$$H(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2}(\xi-\xi^*)^2\right) \quad .$$

Hence, Gabor filters are *bandpass* filters. The main amount of energy of the Gabor filter is centered around the frequency u^* in the positive half of the frequency domain.

Analogously, the definition of the two-dimensional complex Gabor filters is based on the classical two-dimensional Fourier transform.

B) One-dimensional real Gabor filters

Besides complex Gabor filters, also real Gabor filters appear in the literature (see e.g. [101, 102]). Naturally they are obtained as the real and imaginary part of the complex Gabor filters introduced above. Hence the *impulse responses* of these real Gabor filters take the form

$$h_c(x) = g(x)\ \cos(2\pi u^* x) = g(x)\ \cos(\xi^* x) = g(x)\ \cos\left(\frac{cx}{\sigma}\right)$$

and

$$h_s(x) = g(x)\ \sin(2\pi u^* x) = g(x)\ \sin(\xi^* x) = g(x)\ \sin\left(\frac{cx}{\sigma}\right)$$

with g the Gaussian given by (13.12).
Their associated *transfer functions* are

$$\begin{aligned} H_{c/s}(u) &:= \mathcal{F}[h_{c/s}](u) \\ &= N\ \frac{\sigma}{2}\left[\exp(-\sigma^2 2\pi^2(u+u^*)^2) \pm \exp(-\sigma^2 2\pi^2(u-u^*)^2)\right] \end{aligned}$$

where the plus sign, respectively the minus sign, corresponds with the cosine, respectively the sine, Gabor filter. The above expression can be rewritten in terms of the angular frequency:

$$H_{c/s}(\xi) = N\ \frac{\sigma}{2}\left[\exp\left(-\sigma^2\frac{(\xi+\xi^*)^2}{2}\right) \pm \exp\left(-\sigma^2\frac{(\xi-\xi^*)^2}{2}\right)\right] \quad .$$

13.3.2 Different types of two-dimensional Gabor filters

A) Quaternionic Gabor filters

In [36, 37] Bülow and Sommer construct so-called *quaternionic Gabor filters* and apply them to the problems of disparity estimation and texture segmentation. The impulse response h^q of a quaternionic Gabor filter is a Gaussian windowed kernel function of the quaternionic Fourier transform mentioned in Section 12.1:

$$h^q(\underline{x}) = g(\underline{x})\ \exp\left(i2\pi u_1^* x_1\right)\ \exp\left(j2\pi u_2^* x_2\right)$$

with

$$g(\underline{x}) = N \exp\left(-\frac{x_1^2+(\epsilon x_2)^2}{2\sigma^2}\right) \quad . \tag{13.13}$$

The parameter ϵ is the so-called *aspect ratio.*
In the quaternionic frequency domain, these Gabor filters are shifted Gaussians:

$$H^q(\underline{u}) := \mathcal{F}^q[h^q](\underline{u}) = \exp\left(-2\pi^2\sigma^2\left((u_1-u_1^*)^2 + \frac{(u_2-u_2^*)^2}{\epsilon^2}\right)\right) \quad .$$

B) Gabor filters of Ebling and Scheuermann

In [61] Ebling and Scheuermann introduce two- and three-dimensional Gabor filters based on their Clifford-Fourier transforms (see Section 12.1) and use them for the description of local patterns in flow fields. The impulse responses h^e of their two-dimensional Gabor filters take the form

$$h^e(\underline{x}) = g(\underline{x}')\ \exp\left(e_{12}(\xi_1^* x_1 + \xi_2^* x_2)\right)$$

with $\underline{x}'$ a rotated version of $\underline{x}$ and g the Gaussian given by (13.13).

13.3.3 The two-dimensional Clifford-Gabor filters

A) Definition

As we dispose of a closed form for the integral kernel of our two-dimensional Clifford-Fourier transform (see Section 12.4.1), we are now able to define a new type of two-dimensional Gabor filters.

Definition 13.1 *The two-dimensional Clifford-Gabor filters $\mathcal{G}^{\pm}$ are linear shift-invariant filters with impulse response given by*

$$\begin{aligned} h^{\pm}(\underline{x}) &= g(\underline{x}) \ \exp\left(\pm(\underline{x} \wedge \underline{\xi}^*)\right) \\ &= g(\underline{x}) \ \cos\left(x_1\xi_2^* - x_2\xi_1^*\right) \pm e_{12} \ g(\underline{x}) \ \sin\left(x_1\xi_2^* - x_2\xi_1^*\right) \ , \end{aligned}$$

where g is the Gaussian given by

$$g(\underline{x}) = \frac{1}{2\pi\sigma^2} \ \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) \ .$$

The parameters of the Clifford-Gabor filters are the angular frequency $\underline{\xi}^*$ and the variance σ which determines the scale of the Gaussian envelope.

It turns out that both types of Clifford-Gabor filters $\mathcal{G}^{\pm}$ have the same transfer function.

Proposition 13.1 *The transfer function of the Clifford-Gabor filter $\mathcal{G}^{\pm}$ is given by*

$$H^{\pm}(\underline{\xi}) := \mathcal{F}_{\mathcal{H}^{\pm}}[h^{\pm}](\underline{\xi}) = \frac{1}{2\pi} \exp\left(-\frac{\sigma^2}{2}|\underline{\xi} - \underline{\xi}^*|^2\right) \ .$$

Proof. By means of the modulation theorem of the two-dimensional Clifford-Fourier transform (see Proposition 12.6 (iv)) we have

$$\begin{aligned} H^{\pm}(\underline{\xi}) &= \mathcal{F}_{\mathcal{H}^{\pm}}\left[\exp\left(\underline{x} \wedge (\pm\underline{\xi}^*)\right) \ g(\underline{x})\right](\underline{\xi}) = \mathcal{F}_{\mathcal{H}^{\pm}}[g(\underline{x})](\underline{\xi} - \underline{\xi}^*) \\ &= \mathcal{F}[g(\underline{x})](\underline{\xi} - \underline{\xi}^*) \ , \end{aligned}$$

since the Gaussian $g(\underline{x})$ is a radial function.
The desired result now follows from

$$\mathcal{F}[g(\underline{x})](\underline{\xi}) = \frac{1}{2\pi} \exp\left(-\sigma^2 \frac{|\underline{\xi}|^2}{2}\right) \ . \qquad \square$$

Note that, similar to the classical case, the transfer functions $H^{\pm}(\underline{\xi})$ are shifted Gaussians, which implies that the Clifford-Gabor filters $\mathcal{G}^{\pm}$ are bandpass filters.

Remark 13.3 In a similar way one obtains

$$\mathcal{F}_{\mathcal{H}^{\pm}}[h^{\mp}](\underline{\xi}) = \frac{1}{2\pi} \exp\left(-\frac{\sigma^2}{2} |\underline{\xi} + \underline{\xi}^*|^2 \right) \quad .$$

B) Relationship with other Gabor filters

Using the properties of the Clifford numbers $P^{\pm}$ introduced in Section 12.4.1, we can derive an explicit connection between the two-dimensional Clifford-Gabor filters $\mathcal{G}^{\pm}$ and the classical complex Gabor filters (see Section 13.3.1).
By a straightforward computation, we obtain

$$h^{\pm}(\underline{x}) = P^{+}\ h(\mp e_{12}\underline{x}) + P^{-}\ h(\pm e_{12}\underline{x}) \quad ,$$

where

$$h(x_1, x_2) = g(\underline{x})\ \exp\left(i(\xi_1^* x_1 + \xi_2^* x_2)\right)$$

is the classical two-dimensional Gabor filter in case of a symmetric Gaussian.

Naturally the Clifford-Gabor filters $\mathcal{G}^{\pm}$ can also be expressed in terms of classical one-dimensional Gabor filters which, for the sake of clarity, are now denoted with a subindex specifying the angular frequency:

$$h^{\xi^*}(x) = g(x)\ \exp(i\xi^* x) \quad .$$

It is easily seen that

$$h^{+}(\underline{x}) = P^{+}\ h^{-\xi_2^*}(x_1)\ h^{\xi_1^*}(x_2) + P^{-}\ h^{\xi_2^*}(x_1)\ h^{-\xi_1^*}(x_2) \quad .$$

A similar result holds for the impulse response of $\mathcal{G}^{-}$.

Finally, let us look for a relationship between the Clifford-Gabor filters $\mathcal{G}^{\pm}$ and the Gabor filters of Ebling and Scheuermann in case of a symmetric Gaussian (see Section 13.3.2).
We have

$$h^{e}(\underline{x}) = g(\underline{x})\ \exp\left(e_{12}(\xi_1^* x_1 + \xi_2^* x_2)\right) = h^{\pm}(\mp e_{12}\underline{x}) \quad .$$

C) Localization in the spatial and in the frequency domain

An often cited property of Gabor filters is their optimal simultaneous localization in the spatial and the frequency domain. This makes them suitable for local frequency analysis. The notion "optimal simultaneous localization" is formalized by the *uncertainty principle*, which in its most cited form states that a nonzero function and its Fourier transform can not both be sharply localized.

The uncertainty principle appeared in 1927 under the name "*Heisenberg inequality*" in the field of quantum mechanics in Heisenberg's paper [73].
However, it also has a useful interpretation in classical physics, namely it expresses a limitation on the extent to which a signal can be both time-limited and band-limited. This aspect of the uncertainty principle was already expounded by Wiener in a lecture in Göttingen in 1925. Unfortunately, no written record of this lecture seems to have survived, apart from the non-technical account in Wiener's autobiography [120]. The uncertainty principle became really fundamental in the field of signal processing after the publication of Gabor's famous article [69].

For a one-dimensional complex-valued signal f the uncertainty principle takes the form:

$$\Delta x \; \Delta\xi \geq \frac{1}{2} \quad . \tag{13.14}$$

Here Δx denotes the *width* or *spatial uncertainty* of f, defined as the square root of the variance of the energy distribution of f :

$$(\Delta x)^2 = \frac{\int_{-\infty}^{+\infty} x^2 f(x) \; f^c(x) \; dx}{\int_{-\infty}^{+\infty} f(x) \; f^c(x) \; dx} \quad .$$

Analogously, the *bandwidth* $\Delta\xi$ is given by

$$(\Delta\xi)^2 = \frac{\int_{-\infty}^{+\infty} \xi^2 \; \mathcal{F}[f](\xi) \; \left(\mathcal{F}[f](\xi)\right)^c \; d\xi}{\int_{-\infty}^{+\infty} \mathcal{F}[f](\xi) \; \left(\mathcal{F}[f](\xi)\right)^c \; d\xi} \quad .$$

The functions that minimize the inequality (13.14) are the complex Gabor filters. Hence, depending on the parameter σ, the Gabor filters are better localized in position or frequency space but they always exhibit the best possible joint localization.

Daugman extended the uncertainty principle to two-dimensional complex-valued filters or signals (see [53]):

$$\Delta x_1 \; \Delta x_2 \; \Delta\xi_1 \; \Delta\xi_2 \geq \frac{1}{4} \quad , \tag{13.15}$$

where Δx_1 is defined by

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2 \; f(x_1, x_2) \; f^c(x_1, x_2) \; dV(\underline{x})}{\int_{\mathbb{R}^2} f(x_1, x_2) \; f^c(x_1, x_2) \; dV(\underline{x})} \quad .$$

The uncertainties Δx_2, $\Delta\xi_1$ and $\Delta\xi_2$ are defined analogously.
It can be shown that two-dimensional complex Gabor filters achieve the minimum product of uncertainties, i.e.

$$\Delta x_1\ \Delta x_2\ \Delta\xi_1\ \Delta\xi_2 = \frac{1}{4} \quad .$$

Let us now consider two-dimensional Clifford algebra-valued functions:

$$\begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{C}_2 \\ \underline{x} = (x_1, x_2) & \longrightarrow & f(\underline{x}) = f(x_1, x_2) = f_0(\underline{x}) + f_1(\underline{x})e_1 + f_2(\underline{x})e_2 + f_{12}(\underline{x})e_{12} \end{array}$$

with

$$f_i : \mathbb{R}^2 \longrightarrow \mathbb{C} \quad , \quad i = 0, 1, 2, 12 \quad .$$

First we extend the definition of the uncertainties to these Clifford algebra-valued functions:

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2\ [f(\underline{x})\ f^\dagger(\underline{x})]_0\ dV(\underline{x})}{\int_{\mathbb{R}^2} [f(\underline{x})\ f^\dagger(\underline{x})]_0\ dV(\underline{x})}$$

and

$$(\Delta\xi_1)^2 = \frac{\int_{\mathbb{R}^2} \xi_1^2\ [\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\ \left(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\right)^\dagger]_0\ dV(\underline{\xi})}{\int_{\mathbb{R}^2} [\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\ \left(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi})\right)^\dagger]_0\ dV(\underline{\xi})} \quad .$$

Analogous definitions hold for Δx_2 and $\Delta\xi_2$.
For complex-valued signals, the real-valued energy distribution is given by $|f|^2 = f f^c$. For Clifford algebra-valued signals, this is given by

$$|f(\underline{x})|^2 = [f(\underline{x})\ f^\dagger(\underline{x})]_0 = |f_0(\underline{x})|^2 + |f_1(\underline{x})|^2 + |f_2(\underline{x})|^2 + |f_{12}(\underline{x})|^2 \quad .$$

Hence, the uncertainty relation for two-dimensional Clifford algebra-valued signals is identical to Daugman's relation (13.15).

In case of the Clifford-Gabor filters $\mathcal{G}^\pm$ we have

$$\begin{aligned} |h^\pm(\underline{x})|^2 &= \left[g(\underline{x})\ \exp\left(\pm(\underline{x} \wedge \underline{\xi}^*)\right)\ g(\underline{x})\ \exp\left(\mp(\underline{x} \wedge \underline{\xi}^*)\right)\right]_0 = [(g(\underline{x}))^2]_0 \\ &= \frac{1}{4\pi^2\sigma^4} \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \quad , \end{aligned}$$

where we have used the fact that

$$\left(\exp\left(\pm(\underline{x} \wedge \underline{\xi}^*)\right)\right)^\dagger = \exp\left(\mp(\underline{x} \wedge \underline{\xi}^*)\right) \quad .$$

Hence, for $\mathcal{G}^{\pm}$ we obtain

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2 \ \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \, dV(\underline{x})}{\int_{\mathbb{R}^2} \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) \, dV(\underline{x})} = \frac{\sigma^2}{2} \quad .$$

Furthermore, we have

$$\left|\mathcal{F}_{\mathcal{H}^{\pm}}[h^{\pm}](\underline{\xi})\right|^2 = \frac{1}{4\pi^2} \exp\left(-\sigma^2 |\underline{\xi} - \underline{\xi}^*|^2\right) \quad ,$$

which yields

$$(\Delta \xi_1)^2 = \frac{\int_{\mathbb{R}^2} \xi_1^2 \exp\left(-\sigma^2 |\underline{\xi} - \underline{\xi}^*|^2\right) \, dV(\underline{\xi})}{\int_{\mathbb{R}^2} \exp\left(-\sigma^2 |\underline{\xi} - \underline{\xi}^*|^2\right) \, dV(\underline{\xi})} = \frac{1}{2\sigma^2} \quad .$$

Summarizing, the uncertainties of the Clifford-Gabor filters $\mathcal{G}^{\pm}$ are given by

$$\Delta x_1 = \Delta x_2 = \frac{\sigma}{\sqrt{2}} \quad \text{and} \quad \Delta \xi_1 = \Delta \xi_2 = \frac{1}{\sqrt{2}\sigma} \quad ,$$

which implies

$$\Delta x_1 \ \Delta x_2 \ \Delta \xi_1 \ \Delta \xi_2 = \frac{1}{4} \quad .$$

Hence, as in the classical setting, the Clifford-Gabor filters $\mathcal{G}^{\pm}$ are jointly optimally localized in the spatial and in the frequency domain.

Part III

Metric Dependent Clifford Analysis and Anisotropic Clifford-Hermite Wavelets

Chapter 14

The metric dependent Clifford toolbox

In this chapter we present the idea, developed in [29], of a metric dependent Clifford analysis which offers the possibility of adjusting the co-ordinate system to preferential, not necessarily orthogonal, directions. This is achieved by considering functions taking their values in a Clifford algebra which is constructed over $\mathbb{R}^m$ by means of a symmetric bilinear form such that the scalar product of a vector $\underline{x} = \sum_{j=1}^{m} x^j e_j$ with itself now takes the form

$$< \underline{x}, \underline{x} > = \sum_{j=1}^{m} \sum_{k=1}^{m} g_{jk} \; x^j x^k \quad . \tag{14.1}$$

We refer to the tensor g_{jk} as the *metric tensor* of the Clifford algebra considered, and it is assumed that this metric tensor is real, symmetric and positive definite. This idea is in fact not completely new since Clifford analysis on manifolds with local metric tensors were already considered in e.g. [49], [70] and [83], while in [67] a specific three-dimensional tensor leaving the third dimension unaltered was introduced for analyzing two-dimensional signals and textures. What is new, is the detailed development of this Clifford analysis in a global metric dependent setting. It should be clear that this opens a new area in Clifford analysis offering a framework for a new kind of applications. In this context, we mention the metric dependent multi-dimensional Hilbert transform introduced in [14] and the anisotropic Clifford-Hermite wavelets which will be discussed thoroughly in Chapter 16.

The outline of this chapter is as follows. We start with constructing, by means of Grassmann generators, two bases: a covariant one ($e_j \ : \ j = 1, \ldots, m$) and a contravariant one ($e^j \ : \ j = 1, \ldots, m$), satisfying the general Clifford algebra multiplication rules:

$$e_j e_k + e_k e_j = -2g_{jk} \quad \text{and} \quad e^j e^k + e^k e^j = -2g^{jk} \quad , \quad 1 \leq j, k \leq m$$

with g_{jk} the metric tensor. The above multiplication rules lead in a natural way to the substitution for the classical scalar product

$$< \underline{x}, \underline{x} > = \sum_{j=1}^{m} (x^j)^2$$

of a vector $\underline{x} = \sum_{j=1}^{m} x^j e_j$ with itself by the symmetric bilinear form (14.1). Next, we generalize all necessary definitions and results of orthogonal Clifford analysis to this metric dependent setting. In this new context we introduce for e.g. the concepts of Fischer inner product, Fischer duality, monogenicity and spherical monogenics. Similar to the orthogonal case, also in the metric dependent Clifford analysis we can decompose each homogeneous polynomial into spherical monogenics, which is referred to as the *monogenic decomposition*. After a thorough investigation in the metric dependent context of the Euler and angular Dirac operators, we proceed with the introduction of the notions of harmonicity and spherical harmonics. Furthermore, we verify the orthogonal decomposition of homogeneous polynomials into harmonic ones, the so-called *harmonic decomposition*. We end this chapter with the definition and study of the so-called *anisotropic Fourier transform*, the metric dependent analogue of the classical Fourier transform.

14.1 Tensors

Let us start by recalling a few concepts concerning tensors.
Assume in Euclidean space two co-ordinate systems $(x^1, x^2, \ldots, x^N)$ and $(\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^N)$ given.

Definition 14.1 *If $(A^1, A^2, \ldots, A^N)$ in co-ordinate system $(x^1, x^2, \ldots, x^N)$ and $(\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^N)$ in co-ordinate system $(\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^N)$ are related by the transformation equations:*

$$\widetilde{A}^j = \sum_{k=1}^{N} \frac{\partial \widetilde{x}^j}{\partial x^k} A^k \quad , \quad j = 1, \ldots, N \quad ,$$

then they are said to be components of a contravariant vector.

Definition 14.2 *If* $(A_1, A_2, \ldots, A_N)$ *in co-ordinate system* $(x^1, x^2, \ldots, x^N)$ *and* $(\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_N)$ *in co-ordinate system* $(\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^N)$ *are related by the transformation equations:*

$$\widetilde{A}_j = \sum_{k=1}^{N} \frac{\partial x^k}{\partial \widetilde{x}^j} A_k \quad , \quad j = 1, \ldots, N \quad ,$$

then they are said to be components of a covariant vector.

Example 14.1 The sets of differentials $\{dx^1, \ldots, dx^N\}$ and $\{d\widetilde{x}^1, \ldots, d\widetilde{x}^N\}$ transform according to the chain rule:

$$d\widetilde{x}^j = \sum_{k=1}^{N} \frac{\partial \widetilde{x}^j}{\partial x^k} dx^k \quad , \quad j = 1, \ldots, N \quad .$$

Hence $(dx^1, \ldots, dx^N)$ is a contravariant vector.

Example 14.2 Consider the co-ordinate transformation

$$\left(\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^N\right) = \left(x^1, x^2, \ldots, x^N\right) \; A$$

with $A = (a^j_k)$ an $(N \times N)$-matrix.
We have

$$\widetilde{x}^j = \sum_{k=1}^{N} x^k a^j_k \quad \text{or equivalently} \quad \widetilde{x}^j = \sum_{k=1}^{N} \frac{\partial \widetilde{x}^j}{\partial x^k} x^k \quad ,$$

which implies that $(x^1, \ldots, x^N)$ is a contravariant vector.

Definition 14.3 *The* outer tensorial product *of two vectors is a tensor of rank 2. There are three possibilities:*
- the outer product of two contravariant vectors $(A^1, \ldots, A^N)$ *and* $(B^1, \ldots, B^N)$ *is a contravariant tensor of rank 2:*

$$C^{jk} = A^j B^k$$

- the outer product of a covariant vector $(A_1, \ldots, A_N)$ *and a contravariant vector* $(B^1, \ldots, B^N)$ *is a mixed tensor of rank 2:*

$$C^k_j = A_j B^k$$

- the outer product of two covariant vectors $(A_1, \ldots, A_N)$ *and* $(B_1, \ldots, B_N)$ *is a covariant tensor of rank 2:*

$$C_{jk} = A_j B_k \quad .$$

Example 14.3 The Kronecker-delta

$$\delta_j^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

is a mixed tensor of rank 2.

Example 14.4 The transformation matrix

$$a_k^j = \frac{\partial \widetilde{x}^j}{\partial x^k} \qquad , \quad j, k = 1, \ldots, N$$

is also a mixed tensor of rank 2.

Remark 14.1 In view of Definitions 14.1 and 14.2 it is easily seen that the transformation formulae for tensors of rank 2 take the following form:

$$\begin{aligned} \widetilde{C}^{jk} &= \widetilde{A}^j \widetilde{B}^k = \sum_{i=1}^N \frac{\partial \widetilde{x}^j}{\partial x^i} A^i \sum_{\ell=1}^N \frac{\partial \widetilde{x}^k}{\partial x^\ell} B^\ell \\ &= \sum_{i=1}^N \sum_{\ell=1}^N \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\partial \widetilde{x}^k}{\partial x^\ell} A^i B^\ell = \sum_{i=1}^N \sum_{\ell=1}^N \frac{\partial \widetilde{x}^j}{\partial x^i} \frac{\partial \widetilde{x}^k}{\partial x^\ell} C^{i\ell} \end{aligned}$$

and similarly

$$\widetilde{C}_j^k = \sum_{i=1}^N \sum_{\ell=1}^N \frac{\partial x^i}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^k}{\partial x^\ell} C_i^\ell \quad , \quad \widetilde{C}_{jk} = \sum_{i=1}^N \sum_{\ell=1}^N \frac{\partial x^i}{\partial \widetilde{x}^j} \frac{\partial x^\ell}{\partial \widetilde{x}^k} C_{i\ell} \quad .$$

Definition 14.4 *The tensorial contraction of a tensor of rank p is a tensor of rank* $(p-2)$ *which one obtains by summation over a common contravariant and covariant index.*

Example 14.5 The tensorial contraction of the mixed tensor C_k^j of rank 2 is a tensor of rank 0, i.e. a scalar:

$$\sum_{j=1}^N C_j^j = D \quad ,$$

while the tensorial contraction of the mixed tensor C_i^{jk} of rank 3 yields a contravariant vector:

$$\sum_{j=1}^{N} C_j^{jk} = D^k \quad .$$

Definition 14.5 *The inner tensorial product of two vectors is their outer product followed by contraction.*

Example 14.6 The inner tensorial product of the covariant vector $(A_1, \ldots, A_N)$ and the contravariant vector $(B^1, \ldots, B^N)$ is the tensor of rank 0, i.e. the scalar given by:

$$\sum_{j=1}^{N} A_j B^j = C \quad .$$

14.2 From Grassmann to Clifford

We consider the Grassmann algebra Λ generated by the basis elements $(\mathfrak{f}_j,\ j = 1, \ldots, m)$ satisfying the relations

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = 0 \quad 1 \leq j, k \leq m \quad .$$

These basis elements form a covariant tensor $(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_m)$ of rank 1.
Next we consider the dual Grassmann algebra Λ^+ generated by the dual basis $(\mathfrak{f}^{+j}, j = 1, \ldots, m)$, forming a contravariant tensor of rank 1 and satisfying the Grassmann identities:

$$\mathfrak{f}^{+j} \mathfrak{f}^{+k} + \mathfrak{f}^{+k} \mathfrak{f}^{+j} = 0 \quad 1 \leq j, k \leq m \quad . \tag{14.2}$$

Duality between both Grassmann algebras is expressed by:

$$\mathfrak{f}_j \mathfrak{f}^{+k} + \mathfrak{f}^{+k} \mathfrak{f}_j = \delta_j^k \quad . \tag{14.3}$$

Note that both the left and right hand side of the above equation is a mixed tensor of rank 2.

Now we introduce the fundamental covariant tensor g_{jk} of rank 2. It is assumed to have real entries, to be positive definite and symmetric:

$$g_{jk} = g_{kj} \quad , \quad 1 \leq j, k \leq m \quad .$$

Definition 14.6 *The real, positive definite and symmetric tensor g_{jk} is called the metric tensor.*

Its reciprocal tensor (a contravariant one) is given by

$$g^{jk} = \frac{1}{\det(g_{jk})}\, G^{jk} \quad ,$$

where G^{jk} denotes the cofactor of g_{jk}. It thus satisfies

$$\sum_{\ell=1}^{m} g^{j\ell} g_{\ell k} = \delta^j_k \quad .$$

In what follows we will use the Einstein summation convention, i.e. summation over equal contravariant and covariant indices is tacitly understood.
With this convention the above equation expressing reciprocity is written as

$$g^{j\ell} g_{\ell k} = \delta^j_k \quad .$$

Definition 14.7 *The covariant basis $(\mathfrak{f}^+_j, j = 1, \ldots, m)$ for the Grassmann algebra Λ^+ is given by*

$$\mathfrak{f}^+_j = g_{jk} \mathfrak{f}^{+k} \quad .$$

This covariant basis shows the following properties.

Proposition 14.1 *One has*

$$\mathfrak{f}^+_j \mathfrak{f}^+_k + \mathfrak{f}^+_k \mathfrak{f}^+_j = 0 \quad \textit{and} \quad \mathfrak{f}_j \mathfrak{f}^+_k + \mathfrak{f}^+_k \mathfrak{f}_j = g_{jk} \ ; \quad 1 \leq j,k \leq m \quad .$$

Proof. By means of respectively (14.2) and (14.3), we find

$$\mathfrak{f}^+_j \mathfrak{f}^+_k + \mathfrak{f}^+_k \mathfrak{f}^+_j = g_{j\ell} \mathfrak{f}^{+\ell} g_{kt} \mathfrak{f}^{+t} + g_{kt} \mathfrak{f}^{+t} g_{j\ell} \mathfrak{f}^{+\ell} = g_{j\ell} g_{kt} (\mathfrak{f}^{+\ell} \mathfrak{f}^{+t} + \mathfrak{f}^{+t} \mathfrak{f}^{+\ell}) = 0$$

and

$$\begin{aligned} \mathfrak{f}_j \mathfrak{f}^+_k + \mathfrak{f}^+_k \mathfrak{f}_j &= \mathfrak{f}_j g_{kt} \mathfrak{f}^{+t} + g_{kt} \mathfrak{f}^{+t} \mathfrak{f}_j \;=\; g_{kt} (\mathfrak{f}_j \mathfrak{f}^{+t} + \mathfrak{f}^{+t} \mathfrak{f}_j) \\ &= g_{kt} \delta^t_j \;=\; g_{kj} \;=\; g_{jk} \quad . \qquad \square \end{aligned}$$

Remark 14.2 By reciprocity one has:

$$\mathfrak{f}^{+k} = g^{kj} \mathfrak{f}^+_j \quad .$$

Definition 14.8 *The contravariant basis* ($\mathfrak{f}^j,\ j = 1,\ldots,m$) *for the Grassmann algebra* Λ *is given by*

$$\mathfrak{f}^j = g^{jk}\mathfrak{f}_k \quad .$$

It shows the following properties.

Proposition 14.2 *One has*

$$\mathfrak{f}^j\mathfrak{f}^k + \mathfrak{f}^k\mathfrak{f}^j = 0 \quad \textit{and} \quad \mathfrak{f}^j\mathfrak{f}^{+k} + \mathfrak{f}^{+k}\mathfrak{f}^j = g^{jk} \quad ; \quad 1 \le j,k \le m \quad .$$

Proof. A straightforward computation yields

$$\mathfrak{f}^j\mathfrak{f}^k + \mathfrak{f}^k\mathfrak{f}^j = g^{j\ell}\mathfrak{f}_\ell g^{kt}\mathfrak{f}_t + g^{kt}\mathfrak{f}_t g^{j\ell}\mathfrak{f}_\ell = g^{j\ell}g^{kt}(\mathfrak{f}_\ell\mathfrak{f}_t + \mathfrak{f}_t\mathfrak{f}_\ell) = 0$$

and

$$\begin{aligned}\mathfrak{f}^j\mathfrak{f}^{+k} + \mathfrak{f}^{+k}\mathfrak{f}^j &= g^{j\ell}\mathfrak{f}_\ell\mathfrak{f}^{+k} + \mathfrak{f}^{+k}g^{j\ell}\mathfrak{f}_\ell \;=\; g^{j\ell}(\mathfrak{f}_\ell\mathfrak{f}^{+k} + \mathfrak{f}^{+k}\mathfrak{f}_\ell) \\ &= g^{j\ell}\delta_\ell^k \;=\; g^{jk} \quad . \qquad \square\end{aligned}$$

Remark 14.3 By reciprocity one has:

$$\mathfrak{f}_k = g_{kj}\mathfrak{f}^j \quad .$$

Now we consider in the direct sum

$$\mathrm{span}_{\mathbb{C}}\{\mathfrak{f}_1,\ldots,\mathfrak{f}_m\} \oplus \mathrm{span}_{\mathbb{C}}\{\mathfrak{f}_1^+,\ldots,\mathfrak{f}_m^+\}$$

two subspaces, namely $\mathrm{span}_{\mathbb{C}}\{e_1,\ldots,e_m\}$ and $\mathrm{span}_{\mathbb{C}}\{e_{m+1},\ldots,e_{2m}\}$ where the new covariant basis elements are defined by

$$\left\{\begin{array}{lcll} e_j &=& \mathfrak{f}_j - \mathfrak{f}_j^+ \ , & j = 1,\ldots,m \\ e_{m+j} &=& i(\mathfrak{f}_j + \mathfrak{f}_j^+) \ , & j = 1,\ldots,m \quad . \end{array}\right.$$

Similarly, we consider the contravariant reciprocal subspaces $\mathrm{span}_{\mathbb{C}}\{e^1,\ldots,e^m\}$ and $\mathrm{span}_{\mathbb{C}}\{e^{m+1},\ldots,e^{2m}\}$ given by

$$\left\{\begin{array}{lcll} e^j &=& \mathfrak{f}^j - \mathfrak{f}^{+j} \ , & j = 1,\ldots,m \\ e^{m+j} &=& i(\mathfrak{f}^j + \mathfrak{f}^{+j}) \ , & j = 1,\ldots,m \quad . \end{array}\right.$$

The covariant basis shows the following properties.

Proposition 14.3 *For* $1 \leq j, k \leq m$ *one has:*
(i) $e_j e_k + e_k e_j = -2g_{jk}$
(ii) $e_{m+j} e_{m+k} + e_{m+k} e_{m+j} = -2g_{jk}$
(iii) $e_j e_{m+k} + e_{m+k} e_j = 0$.

Proof. A straightforward computation leads to
(i)

$$\begin{aligned} e_j e_k + e_k e_j &= (\mathfrak{f}_j - \mathfrak{f}_j^+)(\mathfrak{f}_k - \mathfrak{f}_k^+) + (\mathfrak{f}_k - \mathfrak{f}_k^+)(\mathfrak{f}_j - \mathfrak{f}_j^+) \\ &= \mathfrak{f}_j \mathfrak{f}_k - \mathfrak{f}_j \mathfrak{f}_k^+ - \mathfrak{f}_j^+ \mathfrak{f}_k + \mathfrak{f}_j^+ \mathfrak{f}_k^+ + \mathfrak{f}_k \mathfrak{f}_j - \mathfrak{f}_k \mathfrak{f}_j^+ - \mathfrak{f}_k^+ \mathfrak{f}_j + \mathfrak{f}_k^+ \mathfrak{f}_j^+ \\ &= (\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j) + (\mathfrak{f}_j^+ \mathfrak{f}_k^+ + \mathfrak{f}_k^+ \mathfrak{f}_j^+) - (\mathfrak{f}_j \mathfrak{f}_k^+ + \mathfrak{f}_k^+ \mathfrak{f}_j) - (\mathfrak{f}_j^+ \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j^+) \\ &= -g_{jk} - g_{kj} \;=\; -2g_{jk} \quad , \end{aligned}$$

(ii)

$$\begin{aligned} &e_{m+j} e_{m+k} + e_{m+k} e_{m+j} \\ &\quad= -(\mathfrak{f}_j + \mathfrak{f}_j^+)(\mathfrak{f}_k + \mathfrak{f}_k^+) - (\mathfrak{f}_k + \mathfrak{f}_k^+)(\mathfrak{f}_j + \mathfrak{f}_j^+) \\ &\quad= -\mathfrak{f}_j \mathfrak{f}_k - \mathfrak{f}_j \mathfrak{f}_k^+ - \mathfrak{f}_j^+ \mathfrak{f}_k - \mathfrak{f}_j^+ \mathfrak{f}_k^+ - \mathfrak{f}_k \mathfrak{f}_j - \mathfrak{f}_k \mathfrak{f}_j^+ - \mathfrak{f}_k^+ \mathfrak{f}_j - \mathfrak{f}_k^+ \mathfrak{f}_j^+ \\ &\quad= -(\mathfrak{f}_j \mathfrak{f}_k^+ + \mathfrak{f}_k^+ \mathfrak{f}_j) - (\mathfrak{f}_j^+ \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j^+) \\ &\quad= -g_{jk} - g_{kj} \;=\; -2g_{jk} \end{aligned}$$

and
(iii)

$$\begin{aligned} &e_j e_{m+k} + e_{m+k} e_j \\ &\quad= i(\mathfrak{f}_j - \mathfrak{f}_j^+)(\mathfrak{f}_k + \mathfrak{f}_k^+) + i(\mathfrak{f}_k + \mathfrak{f}_k^+)(\mathfrak{f}_j - \mathfrak{f}_j^+) \\ &\quad= i\mathfrak{f}_j \mathfrak{f}_k + i\mathfrak{f}_j \mathfrak{f}_k^+ - i\mathfrak{f}_j^+ \mathfrak{f}_k - i\mathfrak{f}_j^+ \mathfrak{f}_k^+ + i\mathfrak{f}_k \mathfrak{f}_j - i\mathfrak{f}_k \mathfrak{f}_j^+ + i\mathfrak{f}_k^+ \mathfrak{f}_j - i\mathfrak{f}_k^+ \mathfrak{f}_j^+ \\ &\quad= i(\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j) - i(\mathfrak{f}_j^+ \mathfrak{f}_k^+ + \mathfrak{f}_k^+ \mathfrak{f}_j^+) + i(\mathfrak{f}_j \mathfrak{f}_k^+ + \mathfrak{f}_k^+ \mathfrak{f}_j) - i(\mathfrak{f}_j^+ \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j^+) \\ &\quad= ig_{jk} - ig_{kj} \;=\; 0 \quad . \qquad \square \end{aligned}$$

As expected, both e-bases are linked to each other by means of the metric tensor g_{jk} and its reciprocal g^{jk}.

Proposition 14.4 *For* $j = 1, \ldots, m$ *one has*
(i)

$$e^j = g^{jk} e_k \quad \textit{and} \quad e_k = g_{kj} e^j$$

(ii)

$$e^{m+j} = g^{jk} e_{m+k} \quad \textit{and} \quad e_{m+k} = g_{kj} e^{m+j} \quad .$$

Proof.
(i)

$$e^j = \mathfrak{f}^j - \mathfrak{f}^{+j} = g^{jk}\mathfrak{f}_k - g^{jk}\mathfrak{f}_k^+ = g^{jk}(\mathfrak{f}_k - \mathfrak{f}_k^+) = g^{jk}e_k$$

(ii)

$$e^{m+j} = i(\mathfrak{f}^j + \mathfrak{f}^{+j}) \;=\; ig^{jk}(\mathfrak{f}_k + \mathfrak{f}_k^+) = g^{jk}e_{m+k} \quad . \qquad \square$$

By combining Propositions 14.3 and 14.4 we obtain the following properties of the contravariant e-basis.

Proposition 14.5 *For $1 \leq j, k \leq m$ one has:*
(i) $e^j e^k + e^k e^j = -2g^{jk}$
(ii) $e^{m+j}e^{m+k} + e^{m+k}e^{m+j} = -2g^{jk}$
(iii) $e^j e^{m+k} + e^{m+k}e^j = 0$.

The basis $(e_1, \ldots, e_m)$ and the dual basis $(e^1, \ldots, e^m)$ are also linked to each other by the following relations.

Proposition 14.6 *For $1 \leq j, k \leq m$ one has:*
(i) $e_j e^k + e^k e_j = -2\delta_j^k$
(ii) $e_{m+j}e^{m+k} + e^{m+k}e_{m+j} = -2\delta_j^k$
(iii) $e_j e^{m+k} + e^{m+k}e_j = 0$
(iv) $\sum_j e_j e^j = \sum_j e^j e_j = -m$
(v) $\frac{1}{2}\sum_j \{e_j, e^j\} = -m$
(vi) $\sum_j [e_j, e^j] = 0$
(vii) $\sum_j e_{m+j}e^{m+j} = \sum_j e^{m+j}e_{m+j} = -m$
(viii) $\frac{1}{2}\sum_j \{e_{m+j}, e^{m+j}\} = -m$
(ix) $\sum_j [e_{m+j}, e^{m+j}] = 0$.

Proof. A straightforward computation yields
(i)

$$e_j e^k + e^k e_j = e_j g^{kt} e_t + g^{kt} e_t e_j = g^{kt}(e_j e_t + e_t e_j) = g^{kt}(-2g_{jt}) = -2\delta_j^k$$

(ii)

$$\begin{aligned} e_{m+j}e^{m+k} + e^{m+k}e_{m+j} &= e_{m+j}g^{kt}e_{m+t} + g^{kt}e_{m+t}e_{m+j} \\ &= g^{kt}(e_{m+j}e_{m+t} + e_{m+t}e_{m+j}) \\ &= g^{kt}(-2g_{jt}) \;=\; -2\delta_j^k \end{aligned}$$

(iii)

$$\begin{aligned} e_j e^{m+k} + e^{m+k} e_j &= e_j g^{kt} e_{m+t} + g^{kt} e_{m+t} e_j \\ &= g^{kt}(e_j e_{m+t} + e_{m+t} e_j) \ = \ 0 \end{aligned}$$

(iv)

$$\begin{aligned} \sum_{j=1}^{m} e_j e^j &= \sum_{j=1}^{m} e_j \sum_{i=1}^{m} g^{ji} e_i \ = \ \sum_{i,j} g^{ij} e_j e_i \\ &= \frac{1}{2}\sum_{i,j} g^{ij} e_j e_i + \frac{1}{2}\sum_{i,j} g^{ij} e_j e_i \ = \ \frac{1}{2}\sum_{i,j} g^{ij} e_j e_i + \frac{1}{2}\sum_{j,i} g^{ji} e_i e_j \\ &= \frac{1}{2}\sum_{i,j} g^{ij} e_j e_i + \frac{1}{2}\sum_{i,j} g^{ij}(-e_j e_i - 2g_{ij}) \ = \ -\sum_{i,j} g^{ij} g_{ij} \\ &= -\sum_{i} \delta_i^i \ = \ -(1 + \ldots + 1) \ = \ -m \quad . \end{aligned}$$

By means of (i) we also have

$$\sum_{j=1}^{m} e^j e_j = \sum_{j=1}^{m} (-e_j e^j - 2) = m - 2m = -m \quad .$$

(v) Follows directly from (iv).
(vi) Follows directly from (iv).
(vii) Similar to (iv).
(viii) Follows directly from (vii).
(ix) Follows directly from (vii). □

Finally we consider the algebra generated by either the covariant basis $(e_j \ : \ j = 1, \ldots, m)$ or the contravariant basis $(e^j \ : \ j = 1, \ldots, m)$ and we observe that the elements of both bases satisfy the general multiplication rules of the complex Clifford algebra $\mathbb{C}_m$:

$$e_j e_k + e_k e_j = -2g_{jk} \quad , \quad 1 \leq j, k \leq m$$

and

$$e^j e^k + e^k e^j = -2g^{jk} \quad , \quad 1 \leq j, k \leq m \quad .$$

A covariant basis for $\mathbb{C}_m$ consists of the elements $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$ where $A = (i_1, i_2, \ldots, i_h) \subset \{1, \ldots, m\} = M$ is such that $1 \leq i_1 < i_2 < \ldots < i_h \leq m$.

Similarly, a contravariant basis for $\mathbb{C}_m$ consists of the elements $e^A = e^{i_1} e^{i_2} \dots e^{i_h}$ where again $A = (i_1, i_2, \dots, i_h) \subset M$ is such that $1 \le i_1 < i_2 < \dots < i_h \le m$. In both cases, taking $A = \emptyset$, yields the identity element, i.e. $e_\emptyset = e^\emptyset = 1$. Hence, any element $\lambda \in \mathbb{C}_m$ may be written as

$$\lambda = \sum_A \lambda_A e_A \quad \text{or as} \quad \lambda = \sum_A \lambda_A e^A \quad \text{with} \quad \lambda_A \in \mathbb{C} \quad .$$

In particular, the space spanned by the covariant basis $(e_j \ : \ j = 1, \dots, m)$ or the contravariant basis $(e^j \ : \ j = 1, \dots, m)$ is called the *subspace of Clifford-vectors.*

Remark 14.4 Note that the real Clifford-vector $\underline{\alpha} = \alpha^j e_j$ may be considered as the inner tensorial product of a contravariant vector α^j with real elements with a covariant vector e_j with Clifford numbers as elements, which yields a tensor of rank 0. So the Clifford-vector $\underline{\alpha}$ is a tensor of rank 0; in fact it is a Clifford number.

Similar to the orthogonal Clifford setting, the *Hermitian conjugation* is defined as the anti-involution for which

$$e_j^\dagger = -e_j \quad , \quad j = 1, \dots, m$$

and

$$\lambda^\dagger = \Big(\sum_A \lambda_A e_A\Big)^\dagger = \sum_A \lambda_A^c e_A^\dagger \quad .$$

Note that in particular for a real Clifford-vector $\underline{\alpha} = \alpha^j e_j$: $\underline{\alpha}^\dagger = -\underline{\alpha}$.

The *Hermitian inner product* on $\mathbb{C}_m$ is defined by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0 \in \mathbb{C} \quad ,$$

where $[\lambda]_0$ denotes the scalar part of the Clifford number λ.
It follows that for $\lambda \in \mathbb{C}_m$ its *Clifford norm* $|\lambda|$ is given by

$$|\lambda|^2 = (\lambda, \lambda) = [\lambda^\dagger \lambda]_0 \quad .$$

Finally let us examine the Clifford product of the two Clifford-vectors $\underline{\alpha} = \alpha^j e_j$ and $\underline{\beta} = \beta^j e_j$:

$$\begin{aligned}
\underline{\alpha}\,\underline{\beta} &= \alpha^j e_j \beta^k e_k \;=\; \alpha^j \beta^k e_j e_k \;=\; \frac{1}{2}\alpha^j \beta^k e_j e_k + \frac{1}{2}\alpha^j \beta^k(-e_k e_j - 2g_{jk}) \\
&= -g_{jk}\alpha^j \beta^k + \frac{1}{2}\alpha^j \beta^k (e_j e_k - e_k e_j) \quad .
\end{aligned}$$

It is found that this product splits up into a scalar part and a so-called bivector part:

$$\underline{\alpha}\,\underline{\beta} = \underline{\alpha} \bullet \underline{\beta} + \underline{\alpha} \wedge \underline{\beta}$$

with

$$\underline{\alpha} \bullet \underline{\beta} = -g_{jk}\alpha^j\beta^k = \frac{1}{2}(\underline{\alpha}\,\underline{\beta} + \underline{\beta}\,\underline{\alpha}) = \frac{1}{2}\{\underline{\alpha}, \underline{\beta}\}$$

the so-called *inner product* and

$$\underline{\alpha} \wedge \underline{\beta} = \frac{1}{2}\alpha^j\beta^k(e_je_k - e_ke_j) = \frac{1}{2}\alpha^j\beta^k[e_j, e_k] = \frac{1}{2}[\underline{\alpha}, \underline{\beta}]$$

the so-called *outer product*.
In particular we have that

$$\begin{aligned} e_j \bullet e_k &= -g_{jk} \quad , \quad 1 \leq j,k \leq m \\ e_j \wedge e_j &= 0 \quad , \quad j = 1, \ldots, m \end{aligned}$$

and similarly for the contravariant basis elements:

$$\begin{aligned} e^j \bullet e^k &= -g^{jk} \quad , \quad 1 \leq j,k \leq m \\ e^j \wedge e^j &= 0 \quad , \quad j = 1, \ldots, m \quad . \end{aligned}$$

The outer product of k different basis vectors is defined recursively

$$e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} = \frac{1}{2}\big(e_{i_1}(e_{i_2} \wedge \ldots \wedge e_{i_k}) + (-1)^{k-1}(e_{i_2} \wedge \ldots \wedge e_{i_k})e_{i_1}\big) \quad .$$

For $k = 0, 1, \ldots, m$ fixed, we then call

$$\mathbb{C}_m^k = \left\{\lambda \in \mathbb{C}_m \ : \ \lambda = \sum_{|A|=k} \lambda_A \ e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k} \ , \ A = (i_1, i_2, \ldots, i_k)\right\}$$

the *subspace of k-vectors*; i.e. the space spanned by the outer products of k different basis vectors. Note that the 0-vectors and 1-vectors are simply the scalars and Clifford-vectors; the 2-vectors are also called *bivectors*.

14.3 Embeddings of $\mathbb{R}^m$

By identifying the point $(x^1, \ldots, x^m) \in \mathbb{R}^m$ with the 1-vector $\underline{x}$ given by

$$\underline{x} = x^j e_j \quad ,$$

the space $\mathbb{R}^m$ is embedded in the Clifford algebra $\mathbb{C}_m$ as the subspace of 1-vectors $\mathbb{R}^1_{0,m}$ of the real Clifford algebra $\mathbb{R}_{0,m}$. In the same order of ideas, a point $(x^0, x^1, \ldots, x^m) \in \mathbb{R}^{m+1}$ is identified with a paravector $x = x^0 + \underline{x}$ in $\mathbb{R}^0_{0,m} \oplus \mathbb{R}^1_{0,m}$.

We will equip $\mathbb{R}^m$ with a metric by defining the scalar product of two basis vectors through their dot product:

$$< e_j, e_k > = -e_j \bullet e_k = g_{jk} \quad , \quad 1 \leq j, k \leq m$$

and by putting for two vectors $\underline{x}$ and $\underline{y}$:

$$< \underline{x}, \underline{y} > = < e_j x^j, e_k y^k > = < e_j, e_k > x^j y^k = g_{jk} x^j y^k \quad .$$

Note that in this way $< \ . \ , \ . \ >$ is indeed a scalar product, since the tensor g_{jk} is symmetric and positive definite. Also note that in particular

$$\|\underline{x}\|^2 = < \underline{x}, \underline{x} > = g_{jk} x^j x^k = -\underline{x} \bullet \underline{x} = -\underline{x}^2 = |\underline{x}|^2 \quad .$$

We also introduce in this metric dependent setting *spherical co-ordinates* in $\mathbb{R}^m$ by:

$$\underline{x} = r \underline{\omega}$$

with

$$r = \|\underline{x}\| = \left(g_{jk} x^j x^k\right)^{1/2} \in [0, +\infty[\quad \text{and} \quad \underline{\omega} \in S^{m-1} \quad ,$$

where S^{m-1} denotes the *unit sphere* in $\mathbb{R}^m$:

$$S^{m-1} = \left\{\underline{\omega} \in \mathbb{R}^1_{0,m} \ ; \ \|\underline{\omega}\|^2 = -\underline{\omega}^2 = g_{jk} \omega^j \omega^k = 1\right\} \quad .$$

Moreover, the *anisotropic Spin-group* consists of products of an even number of elements of the unit sphere:

$$\mathrm{Spin}(m) = \left\{s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \ ; \ \underline{\omega}_j \in S^{m-1} \ , \ j = 1, 2, \ldots, 2\ell\right\} \quad .$$

Such a $\mathrm{Spin}(m)$-element $s = \underline{\omega}_1 \ldots \underline{\omega}_{2\ell}$ satisfies

$$\begin{aligned} s s^\dagger &= \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \, \underline{\omega}_{2\ell}^\dagger \ldots \underline{\omega}_1^\dagger \ = \ (-1)^{2\ell} \underline{\omega}_1 \ldots \underline{\omega}_{2\ell} \, \underline{\omega}_{2\ell} \ldots \underline{\omega}_1 \\ &= (-1)^{2\ell} (-1)^{2\ell} \ = \ 1 \end{aligned}$$

and similarly $s^\dagger s = 1$.

Now we introduce a new basis for $\mathbb{R}^m \cong \mathbb{R}^1_{0,m}$ consisting of eigenvectors of the matrix $G = (g_{jk})$, associated to the metric tensor g_{jk}. As (g_{jk}) is real-symmetric, there exists an orthogonal matrix $A \in \mathrm{O}(m)$ such that

$$A^T G A = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$$

with $\lambda_1, \ldots, \lambda_m$ the positive eigenvalues of G. We put

$$(E_j) = (e_j)A \quad .$$

We expect the basis $(E_j \ : \ j = 1, \ldots, m)$ to be orthogonal, since E_j $(j = 1, \ldots, m)$ are eigenvectors of the matrix G.
If (x^j) and (X^j) are the column matrices representing the co-ordinates of $\underline{x}$ w.r.t. the bases (e_j) and (E_j) respectively, then we have

$$\underline{x} = (e_j)(x^j) = (E_j)(X^j) = (e_j)A(X^j)$$

and

$$(x^j) = A(X^j) \quad \text{or} \quad (X^j) = A^T(x^j) \quad .$$

Hence

$$\begin{aligned} < \underline{x}, \underline{y} > = (x^j)^T G (y^j) \quad &= \quad (X^j)^T A^T G A (Y^j) \\ &= \quad (X^j)^T \ \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \ (Y^j) \ = \ \sum_j \lambda_j X^j Y^j \quad . \end{aligned}$$

In particular we find, as expected:

$$< E_j, E_k > = 0 \quad , \quad j \neq k \quad \text{and} \quad < E_j, E_j > = \lambda_j \quad , \quad j = 1, \ldots, m \quad .$$

Involving the Clifford product we obtain

$$\{E_j, E_k\} = 2E_j \bullet E_k = -2 < E_j, E_k > = 0 \quad , \quad j \neq k$$

and

$$\{E_j, E_j\} = 2E_j \bullet E_j = -2 < E_j, E_j > = -2\lambda_j \quad , \quad j = 1, \ldots, m$$

and so for all $1 \leq j, k \leq m$

$$E_j E_k + E_k E_j = -2\lambda_j \delta_{jk} \quad .$$

Finally, if we put

$$\underline{\eta}_j = \frac{E_j}{\sqrt{\lambda_j}} \quad (j = 1, \ldots, m) \quad ,$$

we obtain an orthonormal frame in the metric dependent setting:

$$< \underline{\eta}_j, \underline{\eta}_k > = \delta_{jk} \quad ; \quad 1 \leq j, k \leq m \quad .$$

In what follows, (x'^j) denotes the column matrix containing the co-ordinates of $\underline{x}$ with respect to this orthonormal frame, i.e.

$$\underline{x} = \sum_j x'^j \; \underline{\eta}_j = \sum_j < \underline{\eta}_j, \underline{x} > \; \underline{\eta}_j \quad .$$

It is clear that the co-ordinate sets (X^j) and (x'^j) are related as follows:

$$(X^j) = P^{-1}(x'^j) \quad \text{or} \quad (x'^j) = P(X^j)$$

with

$$P = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}) \quad .$$

Hence we also have

$$(x^j) = AP^{-1}(x'^j) \quad \text{and} \quad (x'^j) = PA^T(x^j) \quad .$$

Finally, note that in the x'^j-co-ordinates, the inner product takes the following form

$$< \underline{x}, \underline{y} > = \sum_j x'^j y'^j \quad .$$

14.4 Fischer duality and Fischer decomposition

We consider the algebra $\mathcal{P}$ of Clifford algebra-valued polynomials, generated by $\{x^1, \ldots, x^m \; ; \; e_1, \ldots, e_m; i\}$. A natural inner product on $\mathcal{P}$ is the so-called *Fischer inner product*

$$\big(R(\underline{x}), S(\underline{x})\big) = \left[\left\{ R^\dagger \big(g^{1j}\partial_{x^j}, g^{2j}\partial_{x^j}, \ldots, g^{mj}\partial_{x^j}\big)[S(\underline{x})] \right\}_{\underline{x}=\underline{0}} \right]_0 \quad .$$

Note that in order to obtain the differential operator $R^\dagger(g^{1j}\partial_{x^j}, \ldots, g^{mj}\partial_{x^j})$, one first takes the Hermitian conjugate $R^\dagger$ of the polynomial R, followed by the substitution $x^k \to g^{kj}\partial_{x^j}$. These two operations

$$\left\{ \begin{array}{lcll} e_k & \stackrel{F}{\longrightarrow} & -e_k & \text{(Hermitian conjugation)} \\ x^k & \stackrel{F}{\longrightarrow} & g^{kj}\partial_{x^j} & \end{array} \right.$$

are known as *Fischer duality.*
Now we express this Fischer duality in terms of the new basis $(E_j : j = 1, \ldots, m)$ and the corresponding new co-ordinates (X^j) introduced in the foregoing section:

(i)

$$(E_j) = (e_j)A \stackrel{F}{\longrightarrow} (-e_j)A = -(e_j)A = -(E_j) \qquad \text{(Hermitian conjugation)}$$

(ii)

$$\begin{aligned} X^j = A^T_{jk}x^k \stackrel{F}{\longrightarrow} A^T_{jk}g^{ik}\partial_{x^i} &= A^T_{jk}g^{ik}\frac{\partial X^\ell}{\partial x^i}\partial_{X^\ell} = A^T_{jk}g^{ik}A^T_{\ell i}\partial_{X^\ell} \\ &= A^T_{jk}g^{ik}A_{i\ell}\,\partial_{X^\ell} = (A^T G^{-1} A)_{j\ell}\,\partial_{X^\ell} \\ &= \left(\operatorname{diag}\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_m}\right)\right)_{j\ell} \partial_{X^\ell} \\ &= \frac{1}{\lambda_j}\partial_{X^j} \quad . \end{aligned} \tag{14.4}$$

Proposition 14.7 *The basis* $\left(E_A \underline{X}^{\underline{\alpha}} = E_A (X^1)^{\alpha_1} \ldots (X^m)^{\alpha_m} \; : \; A \subset M \;,\; \underline{\alpha} \in \mathbb{N}^m\right)$ *of the space* $\mathcal{P}$ *of Clifford polynomials is orthogonal with respect to the Fischer inner product.*

Proof. We have consecutively

$$\begin{aligned} \left(E_A\underline{X}^{\underline{\alpha}}, E_B\underline{X}^{\underline{\beta}}\right) &= \left[E_A^\dagger \left(\frac{1}{\lambda_1}\partial_{X^1}\right)^{\alpha_1} \ldots \left(\frac{1}{\lambda_m}\partial_{X^m}\right)^{\alpha_m} [E_B \underline{X}^{\underline{\beta}}]\right]_0 /_{\underline{X}=\underline{0}} \\ &= \left[E_A^\dagger E_B \left(\frac{1}{\lambda_1}\right)^{\alpha_1} \ldots \left(\frac{1}{\lambda_m}\right)^{\alpha_m} (\partial_{X^1})^{\alpha_1} \ldots (\partial_{X^m})^{\alpha_m} \right. \\ &\qquad \left. (X^1)^{\beta_1} \ldots (X^m)^{\beta_m}\right]_0 /_{\underline{X}=\underline{0}} \\ &= \left(\frac{1}{\lambda_1}\right)^{\alpha_1} \ldots \left(\frac{1}{\lambda_m}\right)^{\alpha_m} \underline{\alpha}!\; \delta_{\underline{\alpha},\underline{\beta}}\; [E_A^\dagger E_B]_0 \end{aligned}$$

with $\underline{\alpha}! = \alpha_1! \ \ldots \ \alpha_m!$.
Moreover, for

$$E_A = E_{i_1} E_{i_2} \ldots E_{i_h} \quad \text{and} \quad E_B = E_{j_1} E_{j_2} \ldots E_{j_k} \ ,$$

we find

$$E_A^\dagger E_B = (-1)^{|A|} \ E_{i_h} \ldots E_{i_2} E_{i_1} E_{j_1} E_{j_2} \ldots E_{j_k} \ .$$

As

$$E_j E_k + E_k E_j = -2\lambda_j \delta_{jk} \ , \quad 1 \leq j, k \leq m \ ,$$

we have

$$[E_A^\dagger E_B]_0 = (-1)^{|A|} \ (-\lambda_{i_1})(-\lambda_{i_2}) \ldots (-\lambda_{i_h}) \ \delta_{A,B} = \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_h} \ \delta_{A,B} \ .$$

Summarizing, we have found that for $A = (i_1, i_2, \ldots, i_h)$

$$(E_A \underline{X}^{\underline{\alpha}}, E_B \underline{X}^{\underline{\beta}}) = \underline{\alpha}! \ \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_h} \left(\frac{1}{\lambda_1}\right)^{\alpha_1} \ldots \left(\frac{1}{\lambda_m}\right)^{\alpha_m} \ \delta_{A,B} \ \delta_{\underline{\alpha},\underline{\beta}} \ . \qquad \square$$

Proposition 14.8 *The Fischer inner product is positive definite.*

Proof. This follows in a straightforward way from the fact that the Fischer inner product of a basis polynomial $E_A \underline{X}^{\underline{\alpha}}$ with itself is always positive:

$$\left(E_A \underline{X}^{\underline{\alpha}}, E_A \underline{X}^{\underline{\alpha}}\right) = \underline{\alpha}! \ \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_h} \left(\frac{1}{\lambda_1}\right)^{\alpha_1} \ldots \left(\frac{1}{\lambda_m}\right)^{\alpha_m} > 0 \ . \qquad \square$$

Let $\mathcal{P}_k$ denote the subspace of $\mathcal{P}$ consisting of the homogeneous Clifford polynomials of degree k :

$$\mathcal{P}_k = \{R_k(\underline{x}) \in \mathcal{P} \ : \ R_k(t\underline{x}) = t^k R_k(\underline{x}) \ , \ t \in \mathbb{R}\} \ .$$

It follows from Proposition 14.7 that the spaces $\mathcal{P}_k$ are orthogonal w.r.t. the Fischer inner product. With a view to the *anisotropic Fischer decomposition* of the homogeneous Clifford polynomials, we now introduce the notion of *g-monogenicity*. It is defined by means of the so-called *anisotropic Dirac-operator* $\partial_{\underline{x}}$ which we introduce as the Fischer dual of the vector variable $\underline{x}$:

$$\underline{x} = x^j e_j \ \xrightarrow{F} \ -g^{jk} \partial_{x^k} e_j = -e^k \partial_{x^k} = -\partial_{\underline{x}} \ .$$

This anisotropic Dirac operator factorizes the *anisotropic Laplace operator* Δ_g, which we obtain as the Fischer dual of the scalar function $\underline{x}^2 = -\|\underline{x}\|^2$:

$$\begin{aligned}\underline{x}^2 = - < \underline{x}, \underline{x} > = -g_{jk}x^j x^k \stackrel{F}{\longrightarrow} -g_{jk}g^{ji}\partial_{x^i}g^{k\ell}\partial_{x^\ell} &= -g_{jk}g^{ji}g^{k\ell}\partial_{x^i}\partial_{x^\ell} \\ &= -\delta^i_k g^{k\ell}\partial_{x^i}\partial_{x^\ell} \\ &= -g^{i\ell}\partial^2_{x^i x^\ell} \\ &= -\Delta_g \quad ,\end{aligned}$$

where we have defined the anisotropic Laplace operator as to be

$$\Delta_g = g^{jk}\partial^2_{x^j x^k} \quad .$$

Then we have indeed that

$$\begin{aligned}\partial^2_{\underline{x}} &= \partial_{x^j}e^j\partial_{x^k}e^k = \frac{1}{2}\partial_{x^j}\partial_{x^k}e^j e^k + \frac{1}{2}\partial_{x^k}\partial_{x^j}e^k e^j \\ &= \frac{1}{2}\partial_{x^j}\partial_{x^k}e^j e^k + \frac{1}{2}\partial_{x^j}\partial_{x^k}(-e^j e^k - 2g^{jk}) \\ &= \frac{1}{2}\partial_{x^j}\partial_{x^k}e^j e^k - \frac{1}{2}\partial_{x^j}\partial_{x^k}e^j e^k - g^{jk}\partial_{x^j}\partial_{x^k} = -\Delta_g \quad .\end{aligned}$$

We also mention the expression for the anisotropic Dirac operator in the orthonormal frame introduced at the end of Section 14.3.

Lemma 14.1 *With respect to the orthonormal frame* $\underline{\eta}_j$ *(*$j = 1, \ldots, m$*), the anisotropic Dirac operator takes the form*

$$\partial_{\underline{x}} = \sum_j \partial_{x'^j}\underline{\eta}_j \quad .$$

Proof. For the sake of clearness, we do not use the Einstein summation convention.
We have consecutively

$$\begin{aligned}\partial_{\underline{x}} &= \sum_j \partial_{x^j}e^j = \sum_{j,k}\partial_{x^j}g^{jk}e_k = \sum_{j,k,t}\partial_{x^j}g^{jk}E_t A^T_{tk} \\ &= \sum_t \left(\sum_{j,k} A^T_{tk}g^{jk}\partial_{x^j}\right)E_t \quad .\end{aligned}$$

By means of (14.4) this becomes

$$\partial_{\underline{x}} = \sum_t \frac{1}{\lambda_t} \partial_{X^t} E_t = \sum_t \partial_{x'^t} \underline{\eta}_t \quad . \qquad \square$$

Definition 14.9
(i) A $\mathbb{C}_m$-valued function $F(x^1, \ldots, x^m)$ is called left g-monogenic in an open region of $\mathbb{R}^m$ if in that region

$$\partial_{\underline{x}}[F] = 0 \quad .$$

(ii) A $\mathbb{C}_m$-valued function $F(x^0, x^1, \ldots, x^m)$ is called left g-monogenic in an open region of $\mathbb{R}^{m+1}$ if in that region

$$(\partial_{x^0} + \partial_{\underline{x}})[F] = 0 \quad .$$

Here $\partial_{x^0} + \partial_{\underline{x}}$ is the so-called *anisotropic Cauchy-Riemann operator*.
The notion of *right g-monogenicity* is defined in a similar way by letting act the anisotropic Dirac operator or the anisotropic Cauchy-Riemann operator from the right.
Note that if a Clifford algebra-valued function F is left g-monogenic, then its Hermitian conjugate $F^\dagger$ is right g-monogenic, since $\partial_{\underline{x}}^\dagger = -\partial_{\underline{x}}$.

Definition 14.10
(i) A (left/right) g-monogenic homogeneous polynomial $P_k \in \mathcal{P}_k$ is called a (left/right) solid inner g-spherical monogenic of order k.
(ii) A (left/right) g-monogenic homogeneous function Q_k of degree $-(k+m-1)$ in $\mathbb{R}^m \setminus \{0\}$ is called a (left/right) solid outer g-spherical monogenic of order k.
(iii) The restriction to S^{m-1} of a (left/right) solid inner g-spherical monogenic P_k is called a (left/right) inner g-spherical monogenic, while the restriction to S^{m-1} of a (left/right) solid outer g-spherical monogenic Q_k is called a (left/right) outer g-spherical monogenic.

The set of all left, respectively right, solid inner g-spherical monogenics of order k will be denoted by $M_\ell^+(k)$, respectively $M_r^+(k)$, while the set of all left, respectively right, solid outer g-spherical monogenics of order k will be denoted by $M_\ell^-(k)$, respectively $M_r^-(k)$. Moreover, the space of left, respectively right, inner g-spherical monogenics of order k is denoted by $\mathcal{M}_\ell^+(k)$, respectively $\mathcal{M}_r^+(k)$, while the space of left, respectively right, outer g-spherical monogenics is denoted by $\mathcal{M}_\ell^-(k)$, respectively $\mathcal{M}_r^-(k)$.

Theorem 14.1 (anisotropic Fischer decomposition)
(i) Any $R_k \in \mathcal{P}_k$ has a unique orthogonal decomposition of the form

$$R_k(\underline{x}) = P_k(\underline{x}) + \underline{x} R_{k-1}(\underline{x})$$

with $P_k \in M_\ell^+(k)$ and $R_{k-1} \in \mathcal{P}_{k-1}$.
(ii) The space of homogeneous polynomials $\mathcal{P}_k$ admits the orthogonal decomposition: $\mathcal{P}_k = M_\ell^+(k) \oplus_\perp \underline{x}\mathcal{P}_{k-1}$.

Proof. This orthogonal decomposition follows from the observation that for $R_{k-1} \in \mathcal{P}_{k-1}$ and $S_k \in \mathcal{P}_k$:

$$(\underline{x} R_{k-1}(\underline{x}), S_k(\underline{x})) = -(R_{k-1}(\underline{x}), \partial_{\underline{x}}[S_k(\underline{x})]) \quad . \tag{14.5}$$

Indeed, as $-\partial_{\underline{x}}$ is the Fischer dual of $\underline{x}$, we have

$$\begin{aligned} (\underline{x} R_{k-1}(\underline{x}), S_k(\underline{x})) &= -\Big[R_{k-1}^\dagger \left(g^{1j}\partial_{x^j}, \ldots, g^{mj}\partial_{x^j} \right) \partial_{\underline{x}}[S_k(\underline{x})] \Big]_0 \\ &= -\left(R_{k-1}(\underline{x}), \partial_{\underline{x}}[S_k(\underline{x})] \right) \quad . \end{aligned}$$

Next, if for some $S_k \in \mathcal{P}_k$ and for all $R_{k-1} \in \mathcal{P}_{k-1}$

$$(\underline{x} R_{k-1}, S_k) = 0 \quad ,$$

then so will

$$(R_{k-1}, \partial_{\underline{x}}[S_k]) = 0 \quad .$$

Hence $\partial_{\underline{x}}[S_k] = 0$, which means that the orthogonal complement of $\underline{x}\mathcal{P}_{k-1}$ is a subspace of $M_\ell^+(k)$.
But if $P_k = \underline{x} R_{k-1} \in M_\ell^+(k) \cap \underline{x}\mathcal{P}_{k-1}$, then

$$(P_k, P_k) = (\underline{x} R_{k-1}, P_k) = -(R_{k-1}, \partial_{\underline{x}}[P_k]) = 0$$

and thus $P_k = 0$. So any $S_k \in \mathcal{P}_k$ may be uniquely decomposed as

$$S_k = P_k + \underline{x} R_{k-1} \quad \text{with} \quad P_k \in M_\ell^+(k) \quad \text{and} \quad R_{k-1} \in \mathcal{P}_{k-1} \quad . \qquad \Box$$

Theorem 14.2 (g-monogenic decomposition) *Any $R_k \in \mathcal{P}_k$ has a unique orthogonal decomposition of the form*

$$R_k(\underline{x}) = \sum_{s=0}^{k} \underline{x}^s P_{k-s}(\underline{x}) \quad , \quad \text{with} \quad P_{k-s} \in M_\ell^+(k-s) \quad .$$

Proof. This result follows by recursive application of the anisotropic Fischer decomposition. $\Box$

14.5 The Euler and anisotropic angular Dirac operators

The Euler and anisotropic angular Dirac operators arise quite naturally when considering the operator $\underline{x}\partial_{\underline{x}}$; in fact they are the respective scalar and bivector part of it.

Definition 14.11 *The Euler operator is the operator defined by*

$$E = x^j \partial_{x^j} \quad ;$$

the anisotropic angular Dirac operator is defined by

$$\Gamma = -\frac{1}{2} x^j \partial_{x^k} (e_j e^k - e^k e_j) = -x^j \partial_{x^k} e_j \wedge e^k \quad .$$

Proposition 14.9 *One has*

$$\underline{x}\partial_{\underline{x}} = -E - \Gamma \tag{14.6}$$

or in other words

$$-E = \underline{x} \bullet \partial_{\underline{x}} = [\underline{x}\partial_{\underline{x}}]_0 \quad \textit{and} \quad -\Gamma = \underline{x} \wedge \partial_{\underline{x}} = [\underline{x}\partial_{\underline{x}}]_2 \quad ,$$

where $[\lambda]_2$ denotes the bivector part of the Clifford number λ.

Proof. One easily finds

$$\begin{aligned} \underline{x}\partial_{\underline{x}} &= x^j e_j \partial_{x^k} e^k = \frac{1}{2} x^j \partial_{x^k} e_j e^k + \frac{1}{2} x^j \partial_{x^k} e_j e^k \\ &= \frac{1}{2} x^j \partial_{x^k} e_j e^k + \frac{1}{2} x^j \partial_{x^k} (-e^k e_j - 2\delta_j^k) \\ &= -x^j \partial_{x^j} + \frac{1}{2} x^j \partial_{x^k} (e_j e^k - e^k e_j) = -E - \Gamma \quad . \qquad \square \end{aligned}$$

As is well known, the Euler operator measures the degree of homogeneity of polynomials. Furthermore, the anisotropic angular Dirac operator measures the degree of g-monogenicity. This is expressed by the following eigenvalue equations.

Proposition 14.10
(i) For $R_k \in \mathcal{P}_k$ one has

$$E[R_k] = kR_k \quad .$$

(ii) For $P_k \in M_\ell^+(k)$ one has

$$\Gamma[P_k] = -kP_k \quad .$$

Proof.
(i) A homogeneous polynomial R_k of degree k can be written as

$$R_k(\underline{x}) = \sum_A e_A \; R_{k,A}(\underline{x})$$

with $R_{k,A}$ a scalar-valued homogeneous polynomial of degree k. Hence

$$E[R_k] = \sum_A e_A \; E[R_{k,A}] = \sum_A e_A k R_{k,A} = k R_k \quad .$$

(ii) Using (14.6) it is easily seen that

$$\Gamma[P_k] = (-E - \underline{x}\partial_{\underline{x}})[P_k] = -kP_k \quad . \qquad \square$$

Remark 14.5 A Clifford polynomial operator is an element of End($\mathcal{P}$); it transforms a Clifford polynomial into another one. Such a Clifford polynomial operator A_ℓ is called homogeneous of degree ℓ if

$$A_\ell[\mathcal{P}_k] \subset \mathcal{P}_{k+\ell} \quad .$$

The Euler operator also measures the degree of homogeneity of Clifford polynomial operators.

Proposition 14.11 *The Clifford polynomial operator A_ℓ is homogeneous of degree ℓ if and only if*

$$[E, A_\ell] = \ell A_\ell \quad .$$

Proof. Suppose that A_ℓ is homogeneous of degree ℓ. We then have

$$(EA_\ell - A_\ell E)[\mathcal{P}_k] = E[\mathcal{P}_{k+\ell}] - kA_\ell[\mathcal{P}_k] = (k+\ell)\mathcal{P}_{k+\ell} - k\mathcal{P}_{k+\ell} = \ell A_\ell[\mathcal{P}_k] \quad .$$

Conversely, assume that the Clifford polynomial operator B satisfies

$$[E, B] = \lambda B \quad .$$

For an arbitrary $R_k \in \mathcal{P}_k$ we then have

$$E(B[R_k]) = (BE + \lambda B)[R_k] = kB[R_k] + \lambda B[R_k] = (k+\lambda)B[R_k] \quad .$$

In other words, $B[R_k] \in \mathcal{P}_{k+\lambda}$ and hence B is homogeneous of degree λ. $\square$

In a series of lemmata and propositions we now establish the basic formulae and operator identities needed in the sequel.

Lemma 14.2 *One has*

$$\partial_{\underline{x}}[\underline{x}] = -m \quad .$$

Proof. By means of Proposition 14.6, we have immediately

$$\partial_{\underline{x}}[\underline{x}] = \sum_{k,j} e^k \partial_{x^k}(x^j) e_j = \sum_j e^j e_j = -m \quad . \qquad \square$$

Proposition 14.12 *One has*

$$\underline{x}\partial_{\underline{x}} + \partial_{\underline{x}}\underline{x} = -2E - m \quad \textit{and} \quad \partial_{\underline{x}}\underline{x} = -E - m + \Gamma \quad . \tag{14.7}$$

Proof. On the one hand we have

$$\underline{x}\partial_{\underline{x}} = \sum_{j,k} x^j e_j \partial_{x^k} e^k \quad ,$$

while on the other hand

$$\partial_{\underline{x}}\underline{x} = \partial_{\underline{x}}[\underline{x}] + \dot{\partial}_{\underline{x}}\underline{x} = -m + \sum_{j,k} \dot{\partial}_{x^k} e^k x^j e_j = -m + \sum_{j,k} x^j \partial_{x^k} e^k e_j \quad ,$$

where the dot-notation $\dot{\partial}_{\underline{x}}$ means that the anisotropic Dirac operator does not act on the function $\underline{x}$, but on the function at the right side of it.
Adding both equalities yields

$$\begin{aligned} \underline{x}\partial_{\underline{x}} + \partial_{\underline{x}}\underline{x} &= -m + \sum_{j,k} x^j \partial_{x^k}(e_j e^k + e^k e_j) \\ &= -m - 2\sum_{j,k} x^j \partial_{x^k} \delta_j^k = -m - 2E \quad . \end{aligned}$$

Using (14.6) this becomes

$$\partial_{\underline{x}}\underline{x} = -\underline{x}\partial_{\underline{x}} - 2E - m = E + \Gamma - 2E - m = \Gamma - E - m \quad . \ \square$$

Propositions 14.9 and 14.12 yield some nice additional results. The first corollary focuses on an interesting factorization of the anisotropic Laplace operator.

Corollary 14.1 *One has*

$$\Delta_g = (E + m - \Gamma)\ \frac{1}{||\underline{x}||^2}\ (E + \Gamma) \quad .$$

Proof. By means of (14.6) and (14.7), we obtain

$$\begin{aligned}\Delta_g &= -\partial_{\underline{x}}^2 = \partial_{\underline{x}} \frac{\underline{x}^2}{||\underline{x}||^2} \partial_{\underline{x}} = \partial_{\underline{x}}\ \underline{x}\ \frac{1}{||\underline{x}||^2}\ \underline{x}\ \partial_{\underline{x}} \\ &= (E + m - \Gamma) \frac{1}{||\underline{x}||^2} (E + \Gamma) \quad . \qquad \square\end{aligned}$$

Also the polynomials $\underline{x}P_k$ are eigenfunctions of the anisotropic angular Dirac operator.

Corollary 14.2 *For any $P_k \in M_\ell^+(k)$ one has:*

$$\Gamma[\underline{x}P_k] = (k + m - 1)\underline{x}P_k \quad .$$

Proof. By means of (14.7) we obtain

$$\underline{x}\partial_{\underline{x}}[\underline{x}P_k] = \underline{x}(-E - m + \Gamma)[P_k] = \underline{x}(-k - m - k)P_k = (-2k - m)\underline{x}P_k \quad .$$

Using (14.6) and the fact that $\underline{x}P_k \in \mathcal{P}_{k+1}$ gives

$$(-2k - m)\underline{x}P_k = -(E + \Gamma)[\underline{x}P_k] = -(k + 1)\underline{x}P_k - \Gamma[\underline{x}P_k] \quad .$$

Hence

$$\Gamma[\underline{x}P_k] = (k + m - 1)\underline{x}P_k \quad . \qquad \square$$

Lemma 14.3 *One has*

(i)

$$\partial_{x^j}[r] = \frac{1}{r} g_{jk} x^k \quad , \quad j = 1, \ldots, m$$

(ii)

$$\partial_{\underline{x}}[r] = \frac{\underline{x}}{r} \quad .$$

Proof. A straightforward computation yields

(i)

$$\begin{aligned}\partial_{x^j}[r] &= \partial_{x^j}\left[\left(g_{ik}x^i x^k\right)^{1/2}\right] = \frac{1}{2}\left(g_{ik}x^i x^k\right)^{-1/2}\left(g_{jk}x^k + g_{ij}x^i\right) \\ &= \frac{1}{2}\frac{1}{r} 2 g_{jk} x^k = \frac{1}{r} g_{jk} x^k\end{aligned}$$

(ii)

$$\partial_{\underline{x}}[r] = e^j \partial_{x^j}[r] = \frac{1}{r} e^j g_{jk} x^k = \frac{1}{r} e_k x^k = \frac{\underline{x}}{r} \quad . \ \square$$

Now it is easily proved that the anisotropic angular Dirac operator only acts on the angular co-ordinates, whence its name.

Lemma 14.4 *One has*
(i) $E[r] = r$
(ii) $\underline{x}\partial_{\underline{x}}[r] = -r$
(iii) $\Gamma[r] = 0$.

Proof. By means of the previous Lemma we easily find
(i)

$$E[r] = x^j \partial_{x^j}[r] = x^j \frac{1}{r} g_{jk} x^k = \frac{1}{r} r^2 = r$$

(ii)

$$\underline{x}\partial_{\underline{x}}[r] = \underline{x}\frac{\underline{x}}{r} = -\frac{r^2}{r} = -r$$

(iii)

$$\Gamma[r] = -(\underline{x}\partial_{\underline{x}} + E)[r] = -(-r + r) = 0 \quad . \qquad \square$$

This enables us to prove the following eigenvalue equations.

Theorem 14.3 *One has*
(i) $E[\underline{x}^s P_k] = (s+k)\underline{x}^s P_k$
(ii) $\Gamma[\underline{x}^{2s} P_k] = -k\underline{x}^{2s} P_k$
(iii) $\Gamma[\underline{x}^{2s+1} P_k] = (k+m-1)\underline{x}^{2s+1} P_k$.

Proof.
(i) Follows immediately from the fact that $\underline{x}^s P_k \in \mathcal{P}_{s+k}$.
(ii) By means of the previous Lemma we find

$$\Gamma[\underline{x}^{2s} P_k] = \underline{x}^{2s}\Gamma[P_k] = -k\underline{x}^{2s} P_k \quad .$$

(iii) Similarly we have

$$\Gamma[\underline{x}^{2s+1} P_k] = \underline{x}^{2s}\Gamma[\underline{x} P_k] = (k+m-1)\underline{x}^{2s+1} P_k \quad . \qquad \square$$

The previous Theorem combined with (14.7) yields the following result.

Theorem 14.4 *One has*

$$\partial_{\underline{x}}[\underline{x}^s P_k] = B_{s,k}\ \underline{x}^{s-1} P_k$$

with

$$B_{s,k} = \begin{cases} -s & \textit{for } s \textit{ even,} \\ -(s-1+2k+m) & \textit{for } s \textit{ odd.} \end{cases}$$

Proof. We have consecutively

$$\begin{aligned}\partial_{\underline{x}}[\underline{x}^{2s}P_k] &= \partial_{\underline{x}}\underline{x}[\underline{x}^{2s-1}P_k] = (-E-m+\Gamma)[\underline{x}^{2s-1}P_k]\\ &= -(2s+k-1)\underline{x}^{2s-1}P_k - m\underline{x}^{2s-1}P_k + (k+m-1)\underline{x}^{2s-1}P_k\\ &= -2s\underline{x}^{2s-1}P_k\end{aligned}$$

and similarly

$$\begin{aligned}\partial_{\underline{x}}[\underline{x}^{2s+1}P_k] &= \partial_{\underline{x}}\underline{x}[\underline{x}^{2s}P_k] = (-E-m+\Gamma)[\underline{x}^{2s}P_k]\\ &= -(2s+k)\underline{x}^{2s}P_k - m\underline{x}^{2s}P_k - k\underline{x}^{2s}P_k\\ &= -(2s+2k+m)\underline{x}^{2s}P_k \quad . \qquad \square\end{aligned}$$

Next, we prove the following proposition.

Proposition 14.13 *If P_k is a solid inner g-spherical monogenic of degree k, then*

$$Q_k(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|^m}\ P_k\left(\frac{\underline{x}}{\|\underline{x}\|^2}\right) = \frac{\underline{x}}{\|\underline{x}\|^{2k+m}}\ P_k(\underline{x})$$

is a g-monogenic homogeneous function of degree $-(k+m-1)$ in $\mathbb{R}^m\setminus\{0\}$, i.e. a solid outer g-spherical monogenic of degree k.
Conversely, if $\widetilde{Q}_k$ is a solid outer g-spherical monogenic of degree k, then

$$\widetilde{P}_k(\underline{x}) = \frac{\underline{x}}{\|\underline{x}\|^m}\ \widetilde{Q}_k\left(\frac{\underline{x}}{\|\underline{x}\|^2}\right) = \underline{x}\,\|\underline{x}\|^{2k+m-2}\ \widetilde{Q}_k(\underline{x})$$

is a solid inner g-spherical monogenic of degree k.

Proof. Clearly Q_k is homogeneous of degree $-(k+m-1)$, since

$$Q_k(t\underline{x}) = \frac{t^{k+1}\underline{x}P_k(\underline{x})}{t^{2k+m}||\underline{x}||^{2k+m}} = t^{-(k+m-1)}Q_k(\underline{x}) \quad .$$

Moreover, in $\mathbb{R}^m\setminus\{0\}$:

$$\begin{aligned}\partial_{\underline{x}}[Q_k(\underline{x})] &= \partial_{\underline{x}}\left[\frac{1}{||\underline{x}||^{2k+m}}\right]\underline{x}P_k + \frac{1}{||\underline{x}||^{2k+m}}\partial_{\underline{x}}[\underline{x}P_k]\\ &= -(2k+m)\frac{\underline{x}}{||\underline{x}||^{2k+m+2}}\underline{x}P_k - (2k+m)\frac{1}{||\underline{x}||^{2k+m}}P_k\\ &= \frac{(2k+m)}{||\underline{x}||^{2k+m}}P_k - \frac{(2k+m)}{||\underline{x}||^{2k+m}}P_k = 0 \quad ,\end{aligned}$$

where we have used

$$\begin{aligned}\partial_{\underline{x}}\left[\frac{1}{r^{2k+m}}\right] &= -(2k+m)\frac{1}{r^{2k+m+1}}\partial_{\underline{x}}[r] = -\frac{(2k+m)}{r^{2k+m+1}}\frac{\underline{x}}{r} \\ &= -(2k+m)\frac{\underline{x}}{r^{2k+m+2}} \quad .\end{aligned}$$

The converse result is proved in a similar way. □

The previous proposition immediately yields the following result.

Corollary 14.3 *Inner and outer g-spherical monogenics are related as follows:*

$$P_k(\underline{\omega}) \in \mathcal{M}_\ell^+(k) \iff \underline{\omega} P_k(\underline{\omega}) \in \mathcal{M}_\ell^-(k) \quad , \quad \underline{\omega} \in S^{m-1} \quad .$$

Proposition 14.14 *In terms of spherical co-ordinates the Euler operator E takes the form*

$$E = r\partial_r \quad .$$

Proof. In this proof we do not use the Einstein summation convention.
The transformation formulae from cartesian to spherical co-ordinates in $\mathbb{R}^m$ yield

$$\partial_{x^j} = \frac{\partial r}{\partial x^j}\partial_r + \sum_{k=1}^{m}\frac{\partial \omega^k}{\partial x^j}\partial_{\omega^k} \quad . \tag{14.8}$$

In view of Lemma 14.3 we have for each j fixed:

$$\frac{\partial r}{\partial x^j} = \frac{1}{r}\sum_t g_{jt}x^t \quad ,$$

and

$$\begin{aligned}\frac{\partial \omega^j}{\partial x^j} &= \frac{\partial}{\partial x^j}\left(\frac{x^j}{r}\right) = \frac{1}{r} - x^j\frac{1}{r^2}\frac{\partial r}{\partial x^j} \\ &= \frac{1}{r} - \frac{1}{r^3}x^j\sum_t g_{jt}x^t = \frac{1}{r}\big(1 - \omega^j\sum_t g_{jt}\omega^t\big) \quad ,\end{aligned}$$

while for $k \neq j$, we find

$$\frac{\partial \omega^k}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{x^k}{r}\right) = -x^k\frac{1}{r^2}\frac{\partial r}{\partial x^j} = -x^k\frac{1}{r^3}\sum_t g_{jt}x^t = -\omega^k\frac{1}{r}\sum_t g_{jt}\omega^t \quad .$$

Hence equation (14.8) becomes

$$\begin{aligned}
\partial_{x^j} \\
&= \frac{1}{r}\big(\sum_t g_{jt}x^t\big)\partial_r + \frac{1}{r}\big(1-\omega^j(\sum_t g_{jt}\omega^t)\big)\partial_{\omega^j} - \sum_{k=1,\ k\neq j}^{m} \omega^k \frac{1}{r}(\sum_t g_{jt}\omega^t)\partial_{\omega^k} \\
&= \frac{1}{r}\big(\sum_t g_{jt}x^t\big)\partial_r + \frac{1}{r}\partial_{\omega^j} - \frac{1}{r}\sum_{k=1}^{m}\omega^k(\sum_t g_{jt}\omega^t)\partial_{\omega^k} \quad . \qquad (14.9)
\end{aligned}$$

Thus we finally obtain

$$\begin{aligned}
E &= \sum_j x^j\partial_{x^j} \\
&= \frac{1}{r}\sum_j x^j\big(\sum_t g_{jt}x^t\big)\partial_r + \frac{1}{r}\sum_j x^j\partial_{\omega^j} - \frac{1}{r}\sum_j x^j\sum_k \omega^k(\sum_t g_{jt}\omega^t)\partial_{\omega^k} \\
&= \frac{1}{r}r^2\partial_r + \sum_j \omega^j\partial_{\omega^j} - \sum_j \omega^j\sum_k \omega^k(\sum_t g_{jt}\omega^t)\partial_{\omega^k} \\
&= r\partial_r + \sum_j \omega^j\partial_{\omega^j} - \sum_k \omega^k\sum_{j,t} g_{jt}\omega^j\omega^t\partial_{\omega^k} \\
&= r\partial_r + \sum_j \omega^j\partial_{\omega^j} - \sum_k \omega^k\partial_{\omega^k} \;=\; r\partial_r \quad . \qquad \Box
\end{aligned}$$

The foregoing proposition enables us to express also the anisotropic Dirac operator in spherical co-ordinates.

Proposition 14.15 *In terms of spherical co-ordinates the anisotropic Dirac operator $\partial_{\underline{x}}$ takes the form*

$$\partial_{\underline{x}} = \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma\right) \quad .$$

Proof. By means of (14.6) and Proposition 14.14 we find

$$r\underline{\omega}\partial_{\underline{x}} = -(r\partial_r + \Gamma) \quad \text{or} \quad \underline{\omega}\partial_{\underline{x}} = -\left(\partial_r + \frac{\Gamma}{r}\right) \quad .$$

Multiplying both sides of the above equation by $-\underline{\omega}$ yields the desired result.
$\Box$

Lemma 14.5 *One has*

$$\Gamma[\underline{\omega}] = (m-1)\underline{\omega} \quad , \quad \underline{\omega} \in S^{m-1} \quad .$$

Proof. Combining Lemma 14.2 with the previous Proposition 14.15 yields

$$\underline{\omega}\left(\partial_r + \frac{\Gamma}{r}\right)[r\underline{\omega}] = -m \quad .$$

Hence we also have

$$\underline{\omega}(\underline{\omega} + \Gamma[\underline{\omega}]) = -m \quad \text{or} \quad \underline{\omega}\Gamma[\underline{\omega}] = -(m-1) \quad .$$

Multiplying both sides of the above equation by $-\underline{\omega}$ yields the desired result. □

We end this section with establishing some commutation relations involving the anisotropic angular Dirac operator.

Proposition 14.16 *One has*
(i) $\Gamma\underline{\omega} = \underline{\omega}(m-1-\Gamma)$
(ii) $\Gamma\underline{x} = \underline{x}(m-1-\Gamma)$
(iii) $\Gamma\partial_{\underline{x}} = \partial_{\underline{x}}(m-1-\Gamma)$.

Proof.
(i) We have

$$(\Gamma\underline{\omega})f = \Gamma[\underline{\omega}f] = \Gamma[\underline{\omega}]f + \dot{\Gamma}\underline{\omega}\dot{f} = (m-1)\underline{\omega}f + \dot{\Gamma}\underline{\omega}\dot{f} \quad .$$

Using (14.9) yields

$$\begin{aligned}
\partial_{\underline{x}} &= \sum_j e^j \partial_{x^j} \\
&= \sum_j e^j (\sum_t g_{jt}\omega^t)\partial_r + \frac{1}{r}\sum_j e^j \partial_{\omega^j} - \frac{1}{r}\sum_j e^j \sum_k \omega^k (\sum_t g_{jt}\omega^t)\partial_{\omega^k} \\
&= (\sum_t e_t\omega^t)\partial_r - \frac{\underline{\omega}}{r}\underline{\omega}\sum_j e^j\partial_{\omega^j} - \frac{1}{r}\sum_k \omega^k(\sum_t e_t\omega^t)\partial_{\omega^k} \\
&= \underline{\omega}\partial_r - \frac{\underline{\omega}}{r}\underline{\omega}\sum_j e^j\partial_{\omega^j} - \frac{\underline{\omega}}{r}\sum_k \omega^k\partial_{\omega^k} \quad .
\end{aligned}$$

Next, keeping in mind the expression of the anisotropic Dirac operator in spherical co-ordinates, the following expression for the anisotropic angular Dirac operator Γ is obtained:

$$\Gamma = -\underline{\omega}\sum_j e^j\partial_{\omega^j} - \sum_k \omega^k\partial_{\omega^k} \quad .$$

Consequently

$$\begin{aligned}\dot{\Gamma}\underline{\omega}\dot{f} &= -\underline{\omega}\sum_k e^k \dot{\partial}_{\omega^k}\underline{\omega}\dot{f} - \sum_k \omega^k \dot{\partial}_{\omega^k}\underline{\omega}\dot{f} \\ &= -\underline{\omega}\sum_k e^k \underline{\omega}\frac{\partial f}{\partial \omega^k} - \underline{\omega}\sum_k \omega^k \frac{\partial f}{\partial \omega^k} \quad .\end{aligned}$$

From

$$e^k\underline{\omega} + \underline{\omega}e^k = e^k \sum_j \omega^j e_j + \sum_j \omega^j e_j e^k = \sum_j \omega^j (e^k e_j + e_j e^k) = -2\omega^k \quad ,$$

we infer

$$\begin{aligned}\dot{\Gamma}\underline{\omega}\dot{f} &= -\underline{\omega}\sum_k (-\underline{\omega}e^k - 2\omega^k)\frac{\partial f}{\partial \omega^k} - \underline{\omega}\sum_k \omega^k \frac{\partial f}{\partial \omega^k} \\ &= \underline{\omega}\underline{\omega}\sum_k e^k \frac{\partial f}{\partial \omega^k} + \underline{\omega}\sum_k \omega^k \frac{\partial f}{\partial \omega^k} \\ &= \underline{\omega}\left(\underline{\omega}\sum_k e^k \frac{\partial}{\partial \omega^k} + \sum_k \omega^k \frac{\partial}{\partial \omega^k}\right) f \;=\; -\underline{\omega}\Gamma f \quad .\end{aligned}$$

Hence, the following commutation relation holds

$$\Gamma\underline{\omega} = (m-1)\underline{\omega} - \underline{\omega}\Gamma \quad .$$

(ii) As $[\Gamma, r] = 0$, the above result (i) also yields

$$\Gamma\underline{x} = \underline{x}(m-1-\Gamma) \quad .$$

(iii) We now easily obtain

$$\begin{aligned}\Gamma\partial_{\underline{x}} &= \Gamma\underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma\right) \;=\; \underline{\omega}(m-1-\Gamma)\left(\partial_r + \frac{1}{r}\Gamma\right) \\ &= \underline{\omega}\left(\partial_r + \frac{1}{r}\Gamma\right)(m-1-\Gamma) \;=\; \partial_{\underline{x}}(m-1-\Gamma) \quad . \qquad \square\end{aligned}$$

14.6 Solid g-spherical harmonics

In Section 14.4 we have introduced the anisotropic Laplace operator

$$\Delta_g = g^{jk}\partial^2_{x^j x^k}$$

and shown that it is factorized by the anisotropic Dirac operator:

$$\partial_{\underline{x}}^2 = -\Delta_g \quad .$$

This leads inevitably to a relationship between the concepts of g-harmonicity and g-monogenicity.

Definition 14.12
(i) A g-harmonic homogeneous polynomial S_k of degree k in $\mathbb{R}^m$:

$$S_k(\underline{x}) \in \mathcal{P}_k \quad \textit{and} \quad \Delta_g[S_k(\underline{x})] = 0 \quad ,$$

is called a solid g-spherical harmonic of degree k.
(ii) The restriction to S^{m-1} of a solid g-spherical harmonic S_k is called a g-spherical harmonic.

The space of solid g-spherical harmonics of degree k is denoted by $H(k)$, while the space of g-spherical harmonics of degree k is denoted by $\mathcal{H}(k)$.

Obviously we have that

$$M_\ell^+(k) \subset H(k) \quad \text{and} \quad M_r^+(k) \subset H(k) \quad .$$

Note also that for $P_{k-1} \in M_\ell^+(k-1)$

$$\underline{x} P_{k-1} \in H(k) \quad ,$$

since, by Theorem 14.4,

$$\Delta_g[\underline{x} P_{k-1}] = (-\partial_{\underline{x}}^2)[\underline{x} P_{k-1}] = \partial_{\underline{x}}[(2k-2+m)P_{k-1}] = 0 \quad .$$

The following decomposition nicely illustrates in this metric dependent setting that Clifford analysis is a refinement of harmonic analysis.

Proposition 14.17 *The space of solid g-spherical harmonics can be decomposed as*

$$H(k) = M_\ell^+(k) \oplus \underline{x} M_\ell^+(k-1) \quad .$$

Proof. For each $S_k \in H(k)$ we put

$$P_{k-1} = \partial_{\underline{x}}[S_k] \quad .$$

Naturally $P_{k-1} \in M_\ell^+(k-1)$, and by Theorem 14.4,

$$\partial_{\underline{x}}[\underline{x} P_{k-1}] = -(m+2k-2)P_{k-1} \quad .$$

Hence

$$R_k = S_k + \frac{1}{m+2k-2}\,\underline{x}P_{k-1}$$

is left g-monogenic, which proves the statement. □

This g-monogenic decomposition now allows for an easy proof of the, coarser, g-harmonic decomposition of homogeneous polynomials.

Theorem 14.5 (g-harmonic (Fischer) decomposition) *Any $R_k \in \mathcal{P}_k$ has a unique decomposition of the form*

$$R_k(\underline{x}) = \sum_{2s\leq k} \|\underline{x}\|^{2s}\, S_{k-2s}(\underline{x}) \quad ,$$

where the S_{k-2s} are solid g-spherical harmonics.

Proof. Take $R_{k-2} \in \mathcal{P}_{k-2}$ and $S_k \in H(k)$.
As $-\Delta_g$ is the Fischer dual of $\underline{x}^2$, we have

$$\begin{aligned}\left(\underline{x}^2 R_{k-2}(\underline{x}), S_k(\underline{x})\right) &= -\left[R^{\dagger}_{k-2}\left(g^{1j}\partial_{x^j},\ldots,g^{mj}\partial_{x^j}\right)\Delta_g[S_k(\underline{x})]\right]_0 \\ &= -\left(R_{k-2}(\underline{x}), \Delta_g[S_k(\underline{x})]\right) \quad .\end{aligned}$$

Similarly as in the proof of Theorem 14.1, this result implies that any $R_k \in \mathcal{P}_k$ may be uniquely decomposed as

$$R_k = S_k + \|\underline{x}\|^2 R_{k-2} \quad ; \quad S_k \in H(k) \quad , \quad R_{k-2} \in \mathcal{P}_{k-2} \quad .$$

Recursive application of the above result then indeed yields:

$$R_k(\underline{x}) = \sum_{2s\leq k} \|\underline{x}\|^{2s}\, S_{k-2s}(\underline{x}) \quad \text{with} \quad S_{k-2s} \in H(k-2s) \quad . \qquad \square$$

We end this section with the introduction of the so-called *anisotropic Laplace-Beltrami-operator.*
We start from the decomposition of the anisotropic Laplace operator established

in Corollary 14.1 and pass on to spherical co-ordinates:

$$\begin{aligned}
\Delta_g &= (r\partial_r + m - \Gamma)\ \frac{1}{r^2}\ (r\partial_r + \Gamma) \\
&= r\partial_r \frac{1}{r^2}(r\partial_r + \Gamma) + \frac{m}{r^2}\ (r\partial_r + \Gamma) - \Gamma\ \frac{1}{r^2}\ (r\partial_r + \Gamma) \\
&= r(-2)\frac{1}{r^3}(r\partial_r + \Gamma) + \frac{1}{r}\partial_r(r\partial_r + \Gamma) + \frac{m}{r}\partial_r + \frac{m}{r^2}\Gamma - \Gamma\frac{1}{r}\partial_r - \frac{1}{r^2}\Gamma^2 \\
&= -\frac{2}{r}\partial_r - \frac{2}{r^2}\Gamma + \frac{1}{r}\partial_r + \partial_r^2 + \frac{1}{r}\partial_r\Gamma + \frac{m}{r}\partial_r + \frac{m}{r^2}\Gamma - \frac{1}{r}\partial_r\Gamma - \frac{1}{r^2}\Gamma^2 \\
&= \partial_r^2 + \frac{(m-1)}{r}\partial_r + \frac{1}{r^2}[(m-2)\Gamma - \Gamma^2]
\end{aligned}$$

or finally

$$\Delta_g = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta_g^* \quad ,$$

where we have put

$$\Delta_g^* = \big(m - 2 - \Gamma\big)\Gamma \quad .$$

The operator Δ_g^* is called the anisotropic Laplace-Beltrami-operator. Note that this operator commutes with the anisotropic angular Dirac operator:

$$[\Gamma, \Delta_g^*] = 0 \quad .$$

It follows that also the anisotropic Laplace operator commutes with the anisotropic angular Dirac operator:

$$[\Gamma, \Delta_g] = 0 \quad .$$

14.7 The anisotropic Fourier transform

We now define in the present anisotropic setting, a Fourier transform on $L_1(\mathbb{R}^m, dV(\underline{x}))$-functions, which by a classical argument may be extended to $L_2(\mathbb{R}^m, dV(\underline{x}))$.
This so-called *anisotropic Fourier transform* of f, denoted by $\mathcal{F}_g[f]$, is defined by:

$$\begin{aligned}
\mathcal{F}_g[f](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ f(\underline{x})\ dV(\underline{x}) \\
&= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i g_{jk} x^j \xi^k\right)\ f(x^1, \ldots, x^m)\ dV(\underline{x}) \quad .
\end{aligned}$$

To show that this definition is meaningful in this metric dependent context, it is immediately checked how this anisotropic Fourier transform behaves w.r.t. multiplication with the variable $\underline{x}$ and, by duality, w.r.t. the action of the anisotropic Dirac operator.

Proposition 14.18 *The anisotropic Fourier transform $\mathcal{F}_g$ satisfies:*
(i) the multiplication rule:

$$\mathcal{F}_g[\underline{x} f(\underline{x})](\underline{\xi}) = i\partial_{\underline{\xi}}\ \mathcal{F}_g[f(\underline{x})](\underline{\xi}) \quad ;$$

(ii) the differentiation rule:

$$\mathcal{F}_g[\partial_{\underline{x}} f(\underline{x})](\underline{\xi}) = i\underline{\xi}\ \mathcal{F}_g[f(\underline{x})](\underline{\xi}) \quad .$$

Proof. We have consecutively
(i)

$$\begin{aligned}
\mathcal{F}_g[\underline{x} f(\underline{x})](\underline{\xi}) &= \sum_j e_j \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\, x^j\ f(\underline{x})\ dV(\underline{x}) \\
&= \sum_{j,k} g_{jk} e^k \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\, x^j\ f(\underline{x})\ dV(\underline{x}) \\
&= \sum_k e^k \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ \sum_j g_{jk} x^j\ f(\underline{x}) dV(\underline{x}) \\
&= \sum_k e^k \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} i\partial_{\xi^k}\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right]\ f(\underline{x})\ dV(\underline{x}) \\
&= i\partial_{\underline{\xi}}\ \mathcal{F}_g[f(\underline{x})](\underline{\xi}) \quad .
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{F}_g[\partial_{\underline{x}} f(\underline{x})](\underline{\xi}) &= \sum_j e^j \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\, \partial_{x^j}[f(\underline{x})]\ dV(\underline{x}) \\
&= \sum_j e^j \frac{1}{(2\pi)^{m/2}} \Bigg(\int_{\mathbb{R}^m} \partial_{x^j}\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\ f(\underline{x})\right]\ dV(\underline{x}) \\
&\quad - \int_{\mathbb{R}^m} \partial_{x^j}\left[\exp\left(-i < \underline{x}, \underline{\xi} >\right)\right]\ f(\underline{x})\ dV(\underline{x}) \Bigg) \\
&= i \sum_j e^j \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ \sum_k g_{jk} \xi^k\ f(\underline{x}) dV(\underline{x})
\end{aligned}$$

$$\begin{aligned} &= i\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \sum_k e_k \xi^k \ f(\underline{x})\ dV(\underline{x}) \\ &= i\underline{\xi}\ \mathcal{F}_g[f(\underline{x})](\underline{\xi}) \quad . \qquad \square \end{aligned}$$

Another interesting question raised at once is a possible relationship with the classical Fourier transform $\mathcal{F}$.
To this end we will use again the basis $(E_j : j = 1, \ldots, m)$ consisting of eigenvectors of the matrix $G = (g_{jk})$ and the orthonormal frame $(\underline{\eta}_j : j = 1, \ldots, m)$ (see Section 14.3).

Proposition 14.19 *One has*

$$\mathcal{F}_g[f(x^j)](\xi^j) = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}}\ \mathcal{F}[f(AP^{-1}(x'^j))](PA^T(\xi^j))$$

with $\lambda_1, \ldots, \lambda_m$ *the positive eigenvalues of the matrix* $G = (g_{jk})$.

Proof. By definition we have

$$\begin{aligned} \mathcal{F}_g[f(x^j)](\xi^j) &= \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} \exp\left(-i((x^j)^T G(\xi^j))\right)\ f(x^j)\ dV(\underline{x}) \\ &= \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} \exp\left(-i((x^j)^T GAP^{-1}(\xi'^j))\right)\ f(x^j)\ dV(\underline{x}) \quad . \end{aligned}$$

By means of the substitution $(x^j) = AP^{-1}(x'^j)$ for which

$$dV(\underline{x}) = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}}\ dV(\underline{x}') \quad ,$$

this becomes

$$\begin{aligned} &\mathcal{F}_g[f(x^j)](\xi^j) \\ &\quad = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}}\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} \exp\left(-i(x'^j)^T(\xi'^j)\right)\ f(AP^{-1}(x'^j))\ dV(\underline{x}') \\ &\quad = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}}\ \mathcal{F}[f(AP^{-1}(x'^j))](PA^T(\xi^j)) \quad . \qquad \square \end{aligned}$$

This result now allows us to prove a Parseval formula for the anisotropic Fourier transform.

Theorem 14.6 *The anisotropic Fourier transform $\mathcal{F}_g$ satisfies the Parseval formula*

$$< f, h > = \lambda_1 \ldots \lambda_m \;\; < \mathcal{F}_g[f], \mathcal{F}_g[h] > \quad , \quad f, h \in L_2(\mathbb{R}^m, dV(\underline{x})) \qquad (14.10)$$

with $\lambda_1, \ldots, \lambda_m$ the positive eigenvalues of the matrix $G = (g_{jk})$.

Proof. Applying Proposition 14.19 results into

$$\begin{aligned} < \mathcal{F}_g[f(x^j)], \mathcal{F}_g[h(x^j)] > &= \int_{\mathbb{R}^m} \left(\mathcal{F}_g[f(x^j)](\xi^j)\right)^\dagger \; \mathcal{F}_g[h(x^j)](\xi^j) \; dV(\underline{\xi}) \\ &= \frac{1}{\lambda_1 \ldots \lambda_m} \int_{\mathbb{R}^m} \left(\mathcal{F}[f(AP^{-1}(x'^j))](PA^T(\xi^j))\right)^\dagger \\ &\qquad \mathcal{F}[h(AP^{-1}(x'^j))](PA^T(\xi^j)) \; dV(\underline{\xi}) \quad . \end{aligned}$$

By means of the substitution $(z^j) = PA^T(\xi^j)$ or equivalently $(\xi^j) = AP^{-1}(z^j)$ for which

$$dV(\underline{\xi}) = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \; dV(\underline{z}) \quad ,$$

this becomes

$$\begin{aligned} < \mathcal{F}_g[f(x^j)], \mathcal{F}_g[h(x^j)] > = \frac{1}{\lambda_1 \ldots \lambda_m} \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \int_{\mathbb{R}^m} \left(\mathcal{F}[f(AP^{-1}(x'^j))](z^j)\right)^\dagger \\ \mathcal{F}[h(AP^{-1}(x'^j))](z^j) \; dV(\underline{z}) \quad . \end{aligned}$$

Next, applying the Parseval formula for the classical Fourier transform $\mathcal{F}$ (see Theorem 2.1) yields

$$\begin{aligned} &< \mathcal{F}_g[f(x^j)], \mathcal{F}_g[h(x^j)] > \\ &\qquad = \frac{1}{\lambda_1 \ldots \lambda_m} \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \int_{\mathbb{R}^m} f^\dagger(AP^{-1}(x'^j)) \; h(AP^{-1}(x'^j)) \; dV(\underline{x}') \quad . \end{aligned}$$

Finally, the substitution $(u^j) = AP^{-1}(x'^j)$ or equivalently $(x'^j) = PA^T(u^j)$ for which

$$dV(\underline{x}') = \sqrt{\lambda_1 \ldots \lambda_m} \; dV(\underline{u})$$

leads to the desired result:

$$\begin{aligned} < \mathcal{F}_g[f(x^j)], \mathcal{F}_g[h(x^j)] > &= \frac{1}{\lambda_1 \ldots \lambda_m} \int_{\mathbb{R}^m} f^\dagger(u^j) \; h(u^j) \; dV(\underline{u}) \\ &= \frac{1}{\lambda_1 \ldots \lambda_m} < f, h > \quad . \qquad \square \end{aligned}$$

Remark 14.6 Note that the classical Fourier transform and the anisotropic Fourier transform are also related by:

$$\mathcal{F}_g[f(x^j)](\xi^j) = \mathcal{F}[f(x^j)](G(\xi^j))$$

allowing for a shorter proof of the Parseval formula (14.10).
Indeed, by means of the substitution $(u^j) = G(\xi^j)$ or equivalently $(\xi^j) = G^{-1}(u^j)$ for which

$$dV(\underline{\xi}) = \frac{dV(\underline{u})}{|\det(G)|} = \frac{1}{\lambda_1 \ldots \lambda_m}\, dV(\underline{u}) \quad ,$$

we find

$$\begin{aligned} &< \mathcal{F}_g[f(x^j)], \mathcal{F}_g[h(x^j)] > \\ &\qquad = \int_{\mathbb{R}^m} \left(\mathcal{F}[f(x^j)](G(\xi^j))\right)^\dagger \; \mathcal{F}[h(x^j)](G(\xi^j))\; dV(\underline{\xi}) \\ &\qquad = \frac{1}{\lambda_1 \ldots \lambda_m} \int_{\mathbb{R}^m} \left(\mathcal{F}[f(x^j)](u^j)\right)^\dagger \; \mathcal{F}[h(x^j)](u^j)\; dV(\underline{u}) \\ &\qquad = \frac{1}{\lambda_1 \ldots \lambda_m} < f, h > \quad . \end{aligned}$$

Chapter 15

Integration in metric dependent Clifford analysis

With a view to integration on hypersurfaces in the metric dependent Clifford setting established in the previous chapter, we now invoke the theory of differential forms. For a standard introduction to differential forms we may refer to e.g. [89, 115] and for an overview of Clifford differential forms in particular, we recommend [63].
Traditionally, the exterior multiplication of differentials is denoted by a $\wedge$-sign. However, in order to avoid any confusion with the Clifford outer product introduced in Section 14.2, all $\wedge$-signs indicating the exterior multiplication of differentials will be omitted in the sequel.

The outline of this chapter is as follows. In Section 15.1 we list some basic definitions and properties concerning Clifford differential forms in the metric dependent setting. We discuss e.g. fundamental operators such as the exterior derivative and the basic contraction operators; we also state Stokes's theorem, a really fundamental result in mathematical analysis. Special attention is paid to the properties of the Leray and sigma differential forms, since they both play a crucial role in establishing orthogonality relations between g-spherical monogenics on the unit sphere, the topic of Section 15.2

15.1 The basic language of Clifford differential forms

Next to the contravariant vector of *co-ordinates* $(x^1, \ldots, x^m)$ we now also consider the contravariant tensor of *basic differentials* $(dx^1, \ldots, dx^m)$.
By means of the defining relations:

$$\begin{array}{rcl} dx^j dx^j & = & 0 \ , \quad j = 1, \ldots, m \\ dx^j dx^k & = & -dx^k dx^j \ , \quad j \neq k \ , \quad j, k = 1, \ldots, m \ , \end{array}$$

where as already mentioned the $\wedge$-signs are omitted, these basic differentials generate the Grassmann algebra or exterior algebra over $\mathbb{R}^m$:

$$\Lambda \mathbb{R}^m = \sum_{r=0}^{m} \oplus \Lambda^r \mathbb{R}^m \quad .$$

Here $\Lambda^r \mathbb{R}^m$ denotes the space of real-valued r-forms.
The basic toolkit for Clifford analysis is now extended to the "algebra of Clifford forms"

$$\Phi = \mathrm{Alg}\{x^1, \ldots, x^m; dx^1, \ldots, dx^m; e_1, \ldots, e_m; i\} \quad .$$

A Clifford form $F \in \Phi$ thus takes the form

$$F = \sum_A F_A \ dx^A$$

with F_A a Clifford algebra-valued function:

$$F_A = \sum_B f_{A,B}(\underline{x}) \ e_B \quad , \quad f_{A,B} : \mathbb{R}^m \to \mathbb{C}$$

and dx^A a differential form:

$$dx^A = dx^{i_1} dx^{i_2} \ldots dx^{i_r} \quad , \quad A = (i_1, i_2, \ldots, i_r) \quad .$$

Equivalently, we can also write

$$F = \sum_B G_B \ e_B$$

with

$$G_B = \sum_A f_{A,B}(\underline{x}) \ dx^A \quad .$$

The *exterior derivative* d is defined by means of the following axioms: for a C_1-form $F \in \Phi$ one has
(A1) $d(x^j F) = dx^j F + x^j\ dF$
(A2) $d(dx^j F) = -dx^j dF$
(A3) $d(e_j F) = e_j\ dF$.
We also have the property $d^2 = 0$ and w.r.t. the local co-ordinate system, d may be expressed as:

$$d = \partial_{x^j}\ dx^j \quad .$$

Hence for a C_1-form $F = \sum_A F_A\ dx^A$ we have

$$dF = \sum_A \sum_j \partial_{x^j}[F_A]\ dx^j dx^A \quad .$$

We also mention the following basic result, perhaps the most fundamental and most intriguing theorem in mathematical analysis.

Theorem 15.1 (Stokes's theorem) *Let Σ be an oriented k-surface in $\mathbb{R}^m$ and F a continuously differentiable $(k-1)$-form. Then one has:*

$$\int_{\partial\Sigma} F = \int_\Sigma dF \quad .$$

Next we define the *basic contraction operators*, determined by the relations
(C1) $\partial_{x^j}\rfloor(x^k F) = x^k\ \partial_{x^j}\rfloor F$
(C2) $\partial_{x^j}\rfloor(dx^k F) = \delta_j^k\ F - dx^k \partial_{x^j}\rfloor F$.
Note that, due to (C2), the contraction operators $\partial_{x^j}\rfloor$ are in fact a kind of derivative with respect to the generators dx^j. They are sometimes called *fermionic derivatives.*
The *anisotropic Dirac contraction operator* is given by

$$\partial_{\underline{x}}\rfloor = e^j \partial_{x^j}\rfloor \quad .$$

Before discussing the notion of directional derivative and Lie derivative, we must point out a subtle difference between *vector fields* and *Clifford vector fields.* In differential geometry an operator of the form

$$v = v^j(\underline{x})\ \partial_{x^j} \quad , \quad v^j\ :\ \mathbb{R}^m \to \mathbb{C} \tag{15.1}$$

is called a "vector field with components v^j", whereas a Clifford vector field has the form

$$\underline{v} = v^j(\underline{x})\ e_j \quad , \quad v^j\ :\ \mathbb{R}^m \to \mathbb{C} \quad .$$

For a survey paper on differential forms versus multi-vector functions, see [16]. Moreover, in Clifford analysis one may even consider vector fields (15.1) with $\mathbb{C}_m$-valued components v^j, which leads to several new interesting possibilities. First of all, given a Clifford vector field

$$v = v^j(\underline{x})\ \partial_{x^j} \quad , \quad v^j\ :\ \mathbb{R}^m \to \mathbb{C}_m \quad ,$$

one may consider for any differential form $F = \sum_A F_A\ dx^A \in \Phi$, the *directional derivative* (or Levi-Civita connection)

$$v[F] = v^j(\underline{x})\ \partial_{x^j}[F] = \sum_A v[F_A]\ dx^A$$

and contraction

$$v\rfloor\ F = v^j\ \partial_{x^j}\rfloor F \quad .$$

One may also consider the *Lie derivative in the direction of* v

$$\mathcal{L}_v F = (v\rfloor\ d + d\ v\rfloor)F$$

that satisfies

$$\mathcal{L}_v\ d = d\ \mathcal{L}_v \quad .$$

The *Leray form* $L(\underline{x}, d\underline{x})$ is defined as

$$L(\underline{x}, d\underline{x}) = E\rfloor dx^M = \sum_j (-1)^{j+1} x^j dx^{M\setminus\{j\}}$$

with

$$E\rfloor = x^j \partial_{x^j}\rfloor \quad \text{and} \quad dx^M = dx^1 \ldots dx^m$$

and finally

$$dx^{M\setminus\{j\}} = dx^1 \ldots [dx^j] \ldots dx^m \quad , \quad j = 1, 2, \ldots, m \quad .$$

This Leray form $L(\underline{x}, d\underline{x})$, which we will use in the sequel, is naturally closely related to the elementary volume element form dx^M. We have two formulae as to that issue.

Proposition 15.1 *One has*
(i)

$$dL(\underline{x}, d\underline{x}) = m\ dx^M$$

(ii)

$$dx^M = \frac{1}{r}\ dr\ L(\underline{x}, d\underline{x}) \quad .$$

Proof. (i) In a straightforward way we obtain

$$dL(\underline{x}, d\underline{x}) = \sum_{j=1}^{m} (-1)^{j+1} \, dx^j dx^{M\setminus\{j\}} = \sum_{j=1}^{m} dx^M = m \, dx^M \quad .$$

(ii) Differentiating the relation

$$r^2 = \sum_{j,k} g_{jk} x^j x^k = \sum_j g_{jj} (x^j)^2 + 2 \sum_{j<k} g_{jk} x^j x^k \quad ,$$

yields

$$2r \, dr = 2 \sum_j g_{jj} \, x^j dx^j + 2 \sum_{j<k} g_{jk} \, dx^j x^k + 2 \sum_{j<k} g_{jk} \, x^j dx^k \quad .$$

Hence we have

$$\begin{aligned}
& r \, dr \, L(\underline{x}, d\underline{x}) \\
= \quad & \sum_j g_{jj} \, x^j \, dx^j \left(\sum_i (-1)^{i+1} x^i dx^{M\setminus\{i\}} \right) \\
& + \sum_{j<k} g_{jk} \, x^k \, dx^j \left(\sum_i (-1)^{i+1} x^i \, dx^{M\setminus\{i\}} \right) \\
& + \sum_{j<k} g_{jk} \, x^j \, dx^k \left(\sum_i (-1)^{i+1} x^i dx^{M\setminus\{i\}} \right) \\
= \quad & \sum_j g_{jj} \, (x^j)^2 \, (-1)^{j+1} dx^j dx^{M\setminus\{j\}} + \sum_{j<k} g_{jk} \, x^k x^j \, (-1)^{j+1} \, dx^j dx^{M\setminus\{j\}} \\
& + \sum_{j<k} g_{jk} \, x^j x^k \, (-1)^{k+1} \, dx^k dx^{M\setminus\{k\}} \\
= \quad & \sum_j g_{jj} \, (x^j)^2 dx^M + \sum_{j<k} g_{jk} \, x^k x^j dx^M + \sum_{j<k} g_{jk} \, x^j x^k dx^M \\
= \quad & \left(\sum_j g_{jj} \, (x^j)^2 + 2 \sum_{j<k} g_{jk} \, x^j x^k \right) \, dx^M \;\; = \;\; r^2 \, dx^M \quad ,
\end{aligned}$$

which proves the statement. □

It is clear that the Leray form is homogeneous of degree m, i.e. it transforms like

$$L\big(\lambda \underline{x}, d(\lambda \underline{x})\big) = L(\lambda \underline{x}, \lambda d\underline{x} + \underline{x} d\lambda) = \lambda^m \, L(\underline{x}, d\underline{x}) \quad .$$

Hence, we define the associated Leray form of degree 0 :

$$L(\underline{\omega}, d\underline{\omega}) := \frac{L(\underline{x}, d\underline{x})}{r^m} \quad , \quad \underline{\omega} \in S^{m-1} \quad ,$$

so that, by Proposition 15.1

$$dx^M = r^{m-1} \; dr \; L(\underline{\omega}, d\underline{\omega}) \quad .$$

The last differential form needed, is the so-called *sigma form* $\underline{\sigma}(\underline{x}, d\underline{x})$, defined as

$$\underline{\sigma}(\underline{x}, d\underline{x}) = \partial_{\underline{x}} \rfloor \; dx^M = \sum_j (-1)^{j+1} e^j dx^{M\setminus\{j\}}$$

and we shall see that also in this metric dependent context of Clifford analysis, it can be interpreted as the oriented surface element on a hypersurface.

A first auxiliary property reads as follows.

Lemma 15.1 *One has*

$$d\underline{\sigma}(\underline{x}, d\underline{x}) = 0 \quad .$$

Proof. As $d^2 = 0$, we immediately have

$$d\underline{\sigma}(\underline{x}, d\underline{x}) = \sum_j (-1)^{j+1} \; e^j \; d(dx^{M\setminus\{j\}}) = 0 \quad . \qquad \square$$

Next we search for a relationship between the Leray and the sigma form.

Theorem 15.2 *In the whole of* $\mathbb{R}^m$ *one has*

$$\underline{x} \; \underline{\sigma}(\underline{x}, d\underline{x}) + \underline{\sigma}(\underline{x}, d\underline{x}) \; \underline{x} = -2L(\underline{x}, d\underline{x}) \quad .$$

Proof. As

$$e_k e^j + e^j e_k = -2\delta_k^j \quad ,$$

we easily find by a direct computation

$$\begin{aligned}
&\underline{x} \; \underline{\sigma}(\underline{x}, d\underline{x}) + \underline{\sigma}(\underline{x}, d\underline{x}) \; \underline{x} \\
&\quad = \sum_k x^k e_k \sum_j (-1)^{j+1} e^j dx^{M\setminus\{j\}} + \sum_j (-1)^{j+1} e^j dx^{M\setminus\{j\}} \sum_k x^k e_k \\
&\quad = \sum_{j,k} x^k (-1)^{j+1} dx^{M\setminus\{j\}} (e_k e^j + e^j e_k) \\
&\quad = -2 \sum_j x^j (-1)^{j+1} dx^{M\setminus\{j\}} \; = \; -2L(\underline{x}, d\underline{x}) \quad . \qquad \square
\end{aligned}$$

Remark 15.1 The relationship established in Theorem 15.2 may also be written as

$$\{\underline{x}\ ,\ \sigma(\underline{x}, d\underline{x})\} = -2L(\underline{x}, d\underline{x}) \quad \text{or} \quad \underline{x} \bullet \sigma(\underline{x}, d\underline{x}) = -L(\underline{x}, d\underline{x}) \ .$$

Finally we look for an expression of the sigma form in terms of the classical elementary hypersurface element. To that end consider a smooth hypersurface Σ in $\mathbb{R}^m$. Let $\underline{x}$ be an arbitrary point on Σ; then we call $\underline{n}(\underline{x})$ a unit Clifford vector along the surface normal to Σ at the point $\underline{x}$. By definition $\underline{n}(\underline{x})$ is orthogonal to the tangent space of Σ at $\underline{x}$, and so, putting $d\underline{x} = \sum_{j=1}^m e_j dx^j$:

$$< \underline{n}, d\underline{x} > = 0$$

or

$$\sum_k \sum_\ell g_{k\ell}\ n^k dx^\ell = 0 \ . \tag{15.2}$$

If e.g. Σ is given by the cartesian equation $\varphi(\underline{x}) = 0$, then a Clifford normal vector at $\underline{x}$ is given by

$$\underline{N}(\underline{x}) = \sum_k e_k N^k = \sum_k e_k \sum_j \partial_{x^j}[\varphi]\ g^{jk} = \sum_k e_k\ \left(\vec{\nabla}\varphi\right)\ (g_{jk})^{-1}$$

for which indeed

$$\begin{aligned}
< \underline{N}(\underline{x}), d\underline{x} > &= \sum_k \sum_\ell g_{k\ell}\ N^k dx^\ell = \sum_k \sum_\ell \sum_j g_{k\ell}\ \partial_{x^j}[\varphi]\ g^{jk}\ dx^\ell \\
&= \sum_\ell \sum_j \delta^j_\ell\ \partial_{x^j}[\varphi]\ dx^\ell = \sum_j \partial_{x^j}[\varphi]\ dx^j = d\varphi = 0 \ .
\end{aligned}$$

Note that $\underline{N}(\underline{x})$ is not a unit vector since

$$\begin{aligned}
\|\underline{N}(\underline{x})\|^2 &= < \underline{N}(\underline{x}), \underline{N}(\underline{x}) > = \sum_j \sum_k g_{jk}\ N^j N^k \\
&= \sum_j \sum_k \sum_t \sum_s g_{jk}\ \partial_{x^t}[\varphi]\ g^{tj}\ \partial_{x^s}[\varphi]\ g^{sk} \\
&= \sum_k \sum_t \sum_s \delta^t_k\ \partial_{x^t}[\varphi]\ \partial_{x^s}[\varphi]\ g^{sk} = \sum_t \sum_s \partial_{x^t}[\varphi]\ \partial_{x^s}[\varphi]\ g^{st}
\end{aligned}$$

and hence

$$\underline{n}(\underline{x}) = \frac{\sum_k \sum_j e_k\ \partial_{x^j}[\varphi]\ g^{jk}}{\left(\sum_k \sum_j \partial_{x^k}[\varphi]\ \partial_{x^j}[\varphi]\ g^{jk}\right)^{1/2}} \ .$$

Now it may be proved that the sigma form is also orthogonal to the tangent space or, equivalently, that the sigma form lies along the surface normal at the point considered.

Proposition 15.2 *Let Σ be a smooth hypersurface and $\underline{n}(\underline{x})$ a unit normal Clifford vector at $\underline{x} \in \Sigma$, then*

$$\underline{n}(\underline{x}) \wedge \underline{\sigma}(\underline{x}, d\underline{x}) = 0 \quad .$$

Proof. By definition we have

$$\begin{aligned}
\underline{n}(\underline{x}) \wedge \underline{\sigma}(\underline{x}, d\underline{x}) &= \left(\sum_k e_k n^k\right) \wedge \left(\sum_j (-1)^{j+1} e^j dx^{M\setminus\{j\}}\right) \\
&= \left(\sum_{k,t} g_{kt}\ e^t n^k\right) \wedge \left(\sum_j (-1)^{j+1} e^j dx^{M\setminus\{j\}}\right) \\
&= \sum_k n^k \sum_{t,j} (-1)^{j+1} g_{kt}\ e^t \wedge e^j\ dx^{M\setminus\{j\}} \quad .
\end{aligned} \tag{15.3}$$

Note that the above expression is a Clifford bivector; furthermore, as $e^j \wedge e^j = 0$, only the terms with $t \neq j$ remain.
Take such a t and j fixed with, for example, $t < j$ and consider the coefficient of the bivector $e^t \wedge e^j$.
In view of the anti-commutativity of the Clifford outer product

$$e^t \wedge e^j = -e^j \wedge e^t \quad ,$$

this coefficient takes the form

$$\sum_k n^k\ (-1)^{j+1} g_{kt}\ dx^{M\setminus\{j\}} - \sum_k n^k\ (-1)^{t+1} g_{kj}\ dx^{M\setminus\{t\}} \quad .$$

As, by assumption, $t < j$, the above expression can be rewritten as

$$\begin{aligned}
&\sum_k n^k\ (-1)^{j+1}\ g_{kt}\ (-1)^{t-1}\ dx^t dx^{M\setminus\{t,j\}} \\
&- \sum_k n^k\ (-1)^{t+1}\ g_{kj}\ (-1)^{j-2}\ dx^j dx^{M\setminus\{t,j\}} \\
&\qquad = (-1)^{j+t} \left(\sum_k n^k\ g_{kt}\ dx^t + \sum_k n^k\ g_{kj}\ dx^j\right) dx^{M\setminus\{t,j\}} \quad .
\end{aligned}$$

Finally, by means of (15.2) we have

$$\left(\sum_{\ell,k} n^k g_{k\ell}\ dx^\ell\right) dx^{M\setminus\{t,j\}} = 0$$

or equivalently

$$\left(\sum_k n^k g_{kt}\ dx^t + \sum_k n^k g_{kj}\ dx^j\right) dx^{M\setminus\{t,j\}} = 0 \quad .$$

Hence, the coefficient of each bivector $e^t \wedge e^j$ in (15.3) is zero, which proves the statement. □

Corollary 15.1 *For any point* $\underline{x}$ *on the unit sphere* S^{m-1} *one has*

$$\underline{x} \wedge \underline{\sigma}(\underline{x}, d\underline{x}) = 0 \quad .$$

Proof. It suffices to note that for any point $\underline{x} \in S^{m-1}$, a unit normal vector is precisely given by $\underline{n}(\underline{x}) = \underline{x}$. □
Combining Theorem 15.2 and the previous Corollary 15.1 yields the following additional result on the unit sphere.

Corollary 15.2 *For each* $\underline{x} \in S^{m-1}$, *one has*

$$\underline{x}\ \underline{\sigma}(\underline{x}, d\underline{x}) = \underline{\sigma}(\underline{x}, d\underline{x})\ \underline{x} = -L(\underline{x}, d\underline{x})$$

and

$$\underline{\sigma}(\underline{x}, d\underline{x}) = \underline{x}\ L(\underline{x}, d\underline{x}) = L(\underline{x}, d\underline{x})\ \underline{x} \quad .$$

15.2 Orthogonal g-spherical monogenics

The aim of this section is to establish in the metric dependent setting of Clifford analysis, orthogonality relations between the inner and outer spherical monogenics on the unit sphere.
Starting point for the derivation of these orthogonality relations is the anisotropic equivalent of the Clifford-Stokes theorem.

Theorem 15.3 (anisotropic Clifford-Stokes theorem) *Let* Ω *be a compact orientable* m*-dimensional manifold with boundary* $\partial\Omega$ *and* $f, g \in C_1(\Omega)$. *Then one has:*

$$\int_{\partial\Omega} f\ \underline{\sigma}(\underline{x}, d\underline{x})\ g = \int_\Omega \big((f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)\big)\ dx^M$$

and in particular, for $f \equiv 1$ *:*

$$\int_{\partial\Omega} \underline{\sigma}(\underline{x}, d\underline{x})\ g = \int_{\Omega} \partial_{\underline{x}}[g]\ dx^M \quad .$$

Proof. Taking into account that the sigma form is an $(m-1)$-form, for which moreover $d\underline{\sigma}(\underline{x}, d\underline{x}) = 0$ (see Lemma 15.1), we have

$$\begin{aligned} d(f\ \underline{\sigma}(\underline{x}, d\underline{x})\ g) &= d\big(f\ \underline{\sigma}(\underline{x}, d\underline{x})\big)\ g + (-1)^{m-1} f\ \underline{\sigma}(\underline{x}, d\underline{x})\ dg \\ &= df\ \underline{\sigma}(\underline{x}, d\underline{x})\ g + (-1)^{m-1} f\ \underline{\sigma}(\underline{x}, d\underline{x})\ dg \\ &= \sum_j \partial_{x^j}[f]\ dx^j\ \underline{\sigma}(\underline{x}, d\underline{x})\ g \\ &\quad + (-1)^{m-1}\ f\ \underline{\sigma}(\underline{x}, d\underline{x}) \left(\sum_j dx^j\ \partial_{x^j}[g] \right) \quad . \end{aligned} \tag{15.4}$$

Now for each $j = 1, 2, \ldots, m$ fixed, we obtain

$$\begin{aligned} dx^j\ \underline{\sigma}(\underline{x}, d\underline{x}) &= dx^j \left(\sum_k (-1)^{k+1} e^k dx^{M\setminus\{k\}} \right) = \sum_k (-1)^{k+1} e^k dx^j dx^{M\setminus\{k\}} \\ &= \sum_k (-1)^{k+1} e^k (-1)^{j-1} dx^M \delta_{j,k} = e^j dx^M \end{aligned}$$

and similarly

$$\begin{aligned} \underline{\sigma}(\underline{x}, d\underline{x})\ dx^j &= \sum_k (-1)^{k+1} e^k dx^{M\setminus\{k\}} dx^j \\ &= \sum_k (-1)^{k+1} e^k (-1)^{m-j} dx^M \delta_{j,k} = (-1)^{m+1} e^j dx^M \quad . \end{aligned}$$

Hence, (15.4) becomes

$$\begin{aligned} d(f\ \underline{\sigma}(\underline{x}, d\underline{x})\ g) &= \sum_j \partial_{x^j}[f]\ e^j g\ dx^M + f \sum_j e^j \partial_{x^j}[g]\ dx^M \\ &= (f\partial_{\underline{x}})g\ dx^M + f(\partial_{\underline{x}} g)\ dx^M \quad . \end{aligned}$$

Consequently, by means of Stokes's theorem (Theorem 15.1) we indeed obtain

$$\int_{\partial\Omega} f\ \underline{\sigma}(\underline{x}, d\underline{x})\ g = \int_{\Omega} d(f\ \underline{\sigma}(\underline{x}, d\underline{x})\ g) = \int_{\Omega} [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}} g)]\ dx^M \quad . \quad \square$$

The anisotropic Clifford-Stokes theorem immediately yields the following fundamental result.

Corollary 15.3 (anisotropic Cauchy theorem) *Let Ω be a compact orientable m-dimensional manifold with boundary $\partial\Omega$. If f is right g-monogenic in Ω and g is left g-monogenic in Ω, one has:*

$$\int_{\partial\Omega} f \ \underline{\sigma}(\underline{x}, d\underline{x}) \ g = 0 \quad .$$

We now have all the necessary definitions and results of anisotropic Clifford analysis at our disposal in order to prove the orthogonality of the inner and outer g-spherical monogenics.

Theorem 15.4 (Orthogonality of g-spherical monogenics)
(i) Any inner and outer g-spherical monogenic are orthogonal, i.e. for all k and t one has

$$\int_{S^{m-1}} Q_t^\dagger(\underline{x}) \ P_k(\underline{x}) \ L(\underline{x}, d\underline{x}) = 0 \quad .$$

(ii) Inner g-spherical monogenics of different degree are orthogonal, i.e. for $t \neq k$ one has

$$\int_{S^{m-1}} P_t^\dagger(\underline{x}) \ P_k(\underline{x}) \ L(\underline{x}, d\underline{x}) = 0 \quad .$$

(iii) Outer g-spherical monogenics of different degree are orthogonal, i.e. for $t \neq k$ one has

$$\int_{S^{m-1}} Q_t^\dagger(\underline{x}) \ Q_k(\underline{x}) \ L(\underline{x}, d\underline{x}) = 0 \quad .$$

Proof. The proof is based on the anisotropic Cauchy theorem (Corollary 15.3) with Ω the closed unit ball

$$\Omega = B(1) = \{\underline{x} \in \mathbb{R}^1_{0,m} \ : \ \|\underline{x}\| = (g_{jk}x^j x^k)^{1/2} \leq 1\}$$

and $\partial\Omega$ the unit sphere:

$$\partial\Omega = S^{m-1} = \{\underline{x} \in \mathbb{R}^1_{0,m} \ : \ \underline{x}^2 = - \|\underline{x}\|^2 = -g_{jk}x^j x^k = -1\} \quad .$$

(i) Take two arbitrary g-spherical monogenics $P_k \in \mathcal{M}_\ell^+(k)$ and $Q_t \in \mathcal{M}_\ell^-(t)$. By means of Corollary 15.2 we have

$$\begin{aligned} \int_{S^{m-1}} Q_t^\dagger(\underline{x}) \ P_k(\underline{x}) \ L(\underline{x}, d\underline{x}) &= \int_{S^{m-1}} Q_t^\dagger(\underline{x}) \ (-\underline{x} \ \underline{\sigma}(\underline{x}, d\underline{x})) \ P_k(\underline{x}) \\ &= \int_{S^{m-1}} (\underline{x} \ Q_t(\underline{x}))^\dagger \ \underline{\sigma}(\underline{x}, d\underline{x}) \ P_k(\underline{x}). \quad (15.5) \end{aligned}$$

As $Q_t \in \mathcal{M}_\ell^-(t)$, there exists $P_t \in \mathcal{M}_\ell^+(t)$ such that (see Corollary 14.3)

$$P_t(\underline{x}) = \underline{x}\ Q_t(\underline{x}) \quad , \quad \underline{x} \in S^{m-1} \quad .$$

Hence equation (15.5) becomes

$$\int_{S^{m-1}} Q_t^\dagger(\underline{x})\ P_k(\underline{x})\ L(\underline{x}, d\underline{x}) = \int_{S^{m-1}} P_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) \quad .$$

Moreover, as $P_t^\dagger$ is right g-monogenic in $B(1)$, while P_k is left g-monogenic in $B(1)$, the anisotropic Cauchy theorem yields

$$\int_{S^{m-1}} P_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) = 0 \quad .$$

(ii) Take $P_t \in \mathcal{M}_\ell^+(t)$ and $P_k \in \mathcal{M}_\ell^+(k)$ arbitrarily with $t \neq k$.
For the sake of clarity, we now use the notation $\underline{\omega}$ to denote an arbitrary element of the unit sphere S^{m-1}.
Similar to (i), we have consecutively

$$\begin{aligned} \int_{S^{m-1}} P_t^\dagger(\underline{\omega})\ P_k(\underline{\omega})\ L(\underline{\omega}, d\underline{\omega}) &= \int_{S^{m-1}} (\underline{\omega} P_t(\underline{\omega}))^\dagger\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) \\ &= \int_{S^{m-1}} Q_t^\dagger(\underline{\omega})\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) \end{aligned}$$

with $Q_t \in \mathcal{M}_\ell^-(t)$ such that

$$Q_t(\underline{\omega}) = \underline{\omega}\ P_t(\underline{\omega}) \quad .$$

By definition, there also exist $Q_t \in M_\ell^-(t)$ and $P_k \in M_\ell^+(k)$ such that

$$Q_t(\underline{x})/_{S^{m-1}} = Q_t(\underline{\omega}) \quad \text{and} \quad P_k(\underline{x})/_{S^{m-1}} = P_k(\underline{\omega}) \quad .$$

Next, consider the integral

$$\int_{\partial \overline{B}(\rho)} Q_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) \quad ,$$

where $\overline{B}(\rho)$ denotes the closed ball with radius ρ

$$\overline{B}(\rho) = \{\underline{x} \in \mathbb{R}^1_{0,m}\ :\ \|\underline{x}\| = \left(g_{jk} x^j x^k\right)^{1/2} \leq \rho\} \quad .$$

This integral is independent of the radius ρ. Indeed, applying the anisotropic Cauchy theorem to the compact manifold $\Omega = \overline{B}(\rho) \setminus B(\widetilde{\rho})$ with boundary $\partial\Omega = \partial B(\rho) \cup \partial B(\widetilde{\rho})$ yields

$$\int_{\partial\Omega} Q_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) = 0$$

or, taking into account the orientation of $\underline{\sigma}(\underline{x}, d\underline{x})$,

$$\int_{\partial B(\rho)} Q_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) = \int_{\partial B(\widetilde{\rho})} Q_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) \quad .$$

Hence the following equality holds for all $\rho \in [0, \infty[$:

$$\int_{S^{m-1}} Q_t^\dagger(\underline{\omega})\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) = \int_{\partial\overline{B}(\rho)} Q_t^\dagger(\underline{x})\ \underline{\sigma}(\underline{x}, d\underline{x})\ P_k(\underline{x}) \quad .$$

Passing on to spherical co-ordinates $\underline{x} = \rho\underline{\omega}$, $\underline{\omega} \in S^{m-1}$ for all $\underline{x} \in \partial\overline{B}(\rho)$ gives

$$\begin{aligned} \int_{S^{m-1}} Q_t^\dagger(\underline{\omega})\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) &= \int_{S^{m-1}} Q_t^\dagger(\rho\underline{\omega})\ \underline{\sigma}(\rho\underline{\omega}, d(\rho\underline{\omega}))\ P_k(\rho\underline{\omega}) \\ &= \rho^{k-t} \int_{S^{m-1}} Q_t^\dagger(\underline{\omega})\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) \quad . \end{aligned}$$

As, by assumption, $t \neq k$ the above equation implies

$$\int_{S^{m-1}} Q_t^\dagger(\underline{\omega})\ \underline{\sigma}(\underline{\omega}, d\underline{\omega})\ P_k(\underline{\omega}) = 0$$

which proves the statement.
(iii) Similar to (ii). $\square$

Chapter 16

Anisotropic Clifford-Hermite wavelets

In this chapter we generalize the Clifford-Hermite wavelets of Brackx and Sommen (see Sections 5.2 and 5.3) to the metric dependent setting of Clifford analysis. The resulting wavelets, which we call *the anisotropic Clifford-Hermite wavelets*, offer the advantage of being adaptable to preferential, not necessarily orthogonal, directions in the signals or textures to be analyzed. Hence, they are particularly suited for analyzing multi-dimensional textures or signals which show more or less constant features in preferential directions. The orientation of the fundamental $(e_1, \ldots, e_m)$-frame will then be adapted to these directions, resulting in an associated metric tensor which will leave these directions unaltered.

In the first section of this chapter we introduce the anisotropic equivalent of the radial Clifford-Hermite polynomials and corresponding CCWT of orthogonal Clifford analysis. Next, in Section 16.2 we proceed with the construction of the metric dependent analogue of the generalized Clifford-Hermite polynomials and associated wavelets.

16.1 The radial anisotropic Clifford-Hermite polynomials and associated CCWT

16.1.1 The radial anisotropic Clifford-Hermite polynomials

Similar to the classical situation (see (5.1)) and to the orthogonal Clifford analysis case (see (5.3)), the so-called *radial anisotropic Clifford-Hermite polynomials* are defined by the Rodrigues formula

$$H_\ell(\underline{x}) = (-1)^\ell \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^\ell \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \quad , \quad \ell = 0, 1, 2, \ldots \quad , \tag{16.1}$$

where off course now $\partial_{\underline{x}}$ denotes the anisotropic Dirac operator introduced in Section 14.4.
They satisfy the following recurrence relation.

Proposition 16.1 *The radial anisotropic Clifford-Hermite polynomials satisfy the recurrence relation*

$$H_{\ell+1}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})[H_\ell(\underline{x})] \quad . \tag{16.2}$$

Proof. By means of Theorem 14.4, it is easily seen that

$$\begin{aligned} \partial_{\underline{x}}\left[\exp\left(\frac{\|\underline{x}\|^2}{2}\right)\right] &= \partial_{\underline{x}}\left[\exp\left(-\frac{\underline{x}^2}{2}\right)\right] = \exp\left(-\frac{\underline{x}^2}{2}\right)\left(-\frac{1}{2}\right)(-2\underline{x}) \\ &= \underline{x}\exp\left(\frac{\|\underline{x}\|^2}{2}\right) \quad . \end{aligned} \tag{16.3}$$

Hence, we have consecutively

$$\begin{aligned} \partial_{\underline{x}}[H_\ell(\underline{x})] &= (-1)^\ell\ \partial_{\underline{x}}\left[\exp\left(\frac{\|\underline{x}\|^2}{2}\right)\right] \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \\ &\quad + (-1)^\ell \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^{\ell+1}\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \\ &= \underline{x}\ (-1)^\ell \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] - H_{\ell+1}(\underline{x}) \\ &= \underline{x}\ H_\ell(\underline{x}) - H_{\ell+1}(\underline{x}) \end{aligned}$$

which proves the statement. □

The recurrence relation (16.2) allows for a straightforward computation of some lower dimensional anisotropic Clifford-Hermite polynomials:

$$\begin{aligned} H_0(\underline{x}) &= 1 \\ H_1(\underline{x}) &= \underline{x} \\ H_2(\underline{x}) &= \underline{x}^2 + m = -\|\underline{x}\|^2 + m \\ H_3(\underline{x}) &= \underline{x}^3 + (m+2)\underline{x} = \underline{x}\ (-\|\underline{x}\|^2 + m + 2) \\ H_4(\underline{x}) &= \underline{x}^4 + 2(m+2)\underline{x}^2 + m(m+2) = \|\underline{x}\|^4 - 2(m+2)\ \|\underline{x}\|^2 \\ &\quad + m(m+2) \end{aligned}$$

etc.

Note that "formally" these polynomials look exactly the same as their orthogonal equivalents discussed in Section 5.2.1. However, if we write them explicitly, the metric appears. For example, we have

$$H_2(\underline{x}) = -g_{jk}x^j x^k + m \quad .$$

As a matter of fact, this remark holds for the whole metric dependent Clifford theory developed in this and the foregoing two chapters.

By means of the Rodrigues formula (16.1) and the anisotropic Clifford-Stokes theorem (Theorem 15.3), we obtain the following orthogonality relation.

Theorem 16.1 *The radial anisotropic Clifford-Hermite polynomials* $H_\ell(\underline{x})$ *are mutually orthogonal in* $\mathbb{R}^m$ *with respect to the weight function* $\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)$, *i.e. for* $\ell \neq t$ *one has*

$$\int_{\mathbb{R}^m} H_\ell^\dagger(\underline{x})\ H_t(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) = 0 \quad .$$

Proof. This proof is the translation to the anisotropic Clifford framework of the proof of the mutual orthogonality of the radial Clifford-Hermite polynomials (see [108]).

Suppose that $\ell < t$; the case $\ell > t$ follows by Hermitian conjugation.

As $H_\ell(\underline{x})$ is a polynomial of degree ℓ in $\underline{x}$, it is sufficient to show that for each $t \in \mathbb{N}$ and $\ell < t$:

$$\int_{\mathbb{R}^m} \underline{x}^\ell\ H_t(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) = 0 \quad .$$

We prove this by induction on ℓ. For $\ell = 0$ we have for each $t > 0$:

$$\begin{aligned}
&\int_{\mathbb{R}^m} H_t(\underline{x}) \ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \, dV(\underline{x}) \\
&\qquad = \ (-1)^t \int_{\mathbb{R}^m} \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \, dV(\underline{x}) \\
&\qquad = \ (-1)^t \lim_{\rho\to\infty} \left(\int_{\partial\overline{B}(\rho)} \underline{\sigma}(\underline{x}, d\underline{x}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right]\right) = 0 \quad ,
\end{aligned}$$

where we have used the Rodrigues formula (16.1) and the anisotropic Clifford-Stokes theorem. Assume that orthogonality holds for $(\ell - 1)$ and $t > (\ell - 1)$. Take $t > \ell$. Again by means of the Rodrigues formula and the anisotropic Clifford-Stokes theorem, we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^m} \underline{x}^\ell \ H_t(\underline{x}) \ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \, dV(\underline{x}) \\
&\qquad = \ (-)^t \int_{\mathbb{R}^m} \underline{x}^\ell \ \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \, dV(\underline{x}) \\
&\qquad = \ (-1)^t \left(\lim_{\rho\to\infty} \int_{\partial\overline{B}(\rho)} \underline{x}^\ell \ \underline{\sigma}(\underline{x}, d\underline{x}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right]\right. \\
&\qquad\qquad \left. - \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \, dV(\underline{x})\right) \\
&\qquad = \ (-1)^{t+1} \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right] \, dV(\underline{x}) \\
&\qquad = \ \int_{\mathbb{R}^m} (\underline{x}^\ell \partial_{\underline{x}}) \ H_{t-1}(\underline{x}) \ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \, dV(\underline{x}) \quad .
\end{aligned}$$

From Theorem 14.4 we obtain in particular

$$\partial_{\underline{x}}[\underline{x}^\ell] = \begin{cases} -\ell \ \underline{x}^{\ell-1} & \text{for } \ell \text{ even} \\ -(\ell - 1 + m) \ \underline{x}^{\ell-1} & \text{for } \ell \text{ odd.} \end{cases}$$

Hence, by Hermitian conjugation we find

$$[\underline{x}^\ell]\partial_{\underline{x}} = \begin{cases} -\ell \ \underline{x}^{\ell-1} & \text{for } \ell \text{ even} \\ -(\ell - 1 + m) \ \underline{x}^{\ell-1} & \text{for } \ell \text{ odd.} \end{cases}$$

Summarizing: $[\underline{x}^\ell]\partial_{\underline{x}} \approx \underline{x}^{\ell-1}$, so that in view of the induction hypothesis

$$\int_{\mathbb{R}^m} \underline{x}^\ell \; H_t(\underline{x}) \; \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \; dV(\underline{x}) \approx \int_{\mathbb{R}^m} \underline{x}^{\ell-1} \; H_{t-1}(\underline{x}) \; \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \; dV(\underline{x}) \; = \; 0 \; . \; \Box$$

16.1.2 The anisotropic Clifford-Hermite wavelets

Similar to the orthogonal Clifford setting, we now show that the radial anisotropic Clifford-Hermite polynomials constructed in the foregoing section are the building blocks for multi-dimensional Clifford mother wavelet functions.

For $t > 0$ Theorem 16.1 implies that

$$\int_{\mathbb{R}^m} H_t(\underline{x}) \; \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \; dV(\underline{x}) = 0 \quad .$$

In terms of wavelet theory this means that the alternatively scalar- or vector-valued $L_1 \cap L_2$-functions

$$\psi_t(\underline{x}) = H_t(\underline{x}) \; \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) = (-1)^t \; \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right]$$

have zero momentum and are a good candidate for mother wavelets in $\mathbb{R}^m$, if at least they satisfy an appropriate admissibility condition (see Section 16.1.3). We call them the *anisotropic Clifford-Hermite wavelets*.

The orthogonality relation of the radial anisotropic Clifford-Hermite polynomials implies that the anisotropic Clifford-Hermite wavelet ψ_t has vanishing moments up to order $(t-1)$:

$$\int_{\mathbb{R}^m} \underline{x}^\ell \; \psi_t(\underline{x}) \; dV(\underline{x}) = 0 \quad , \quad \ell = 0, 1, \ldots, t-1 \quad .$$

Recall that the capacity of wavelets for detecting singularities in a signal is related to their number of vanishing moments. Hence, the anisotropic Clifford-Hermite wavelet ψ_t is particularly appropriate for pointwise signal analysis whereby the corresponding CWT will filter out polynomial behaviour of the signal up to degree $(t-1)$.

Next, we compute the anisotropic Fourier transform of the anisotropic Clifford-Hermite wavelets.

Lemma 16.1 *The anisotropic Fourier transform of the anisotropic Clifford-Hermite wavelets takes the form:*

$$\mathcal{F}_g[\psi_t(\underline{x})](\underline{\xi}) = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ (-i)^t \ \underline{\xi}^t \ \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right)$$

with $\lambda_1, \ldots, \lambda_m$ *the positive eigenvalues of the matrix* $G = (g_{jk})$.

Proof. First, Proposition 14.18 yields

$$\mathcal{F}_g[\psi_t(\underline{x})](\underline{\xi}) = (-i)^t \underline{\xi}^t \ \mathcal{F}_g\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right](\underline{\xi}) \ .$$

Moreover, by means of Proposition 14.19 and the fact that the function $\exp\left(-\frac{(x'^j)^T(x'^j)}{2}\right)$ is an eigenfunction of the classical Fourier transform (see (2.7)), we obtain consecutively

$$\begin{aligned}
&\mathcal{F}_g\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right](\underline{\xi}) \\
&= \mathcal{F}_g\left[\exp\left(-\frac{<\underline{x},\underline{x}>}{2}\right)\right](\underline{\xi}) \ = \ \mathcal{F}_g\left[\exp\left(-\frac{(x^j)^T G (x^j)}{2}\right)\right](\xi^j) \\
&= \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \mathcal{F}\left[\exp\left(-\frac{(AP^{-1}(x'^j))^T \ G \ (AP^{-1}(x'^j))}{2}\right)\right](PA^T(\xi^j)) \\
&= \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \mathcal{F}\left[\exp\left(-\frac{(x'^j)^T(x'^j)}{2}\right)\right](PA^T(\xi^j)) \\
&= \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \exp\left(-\frac{(PA^T(\xi^j))^T \ PA^T(\xi^j)}{2}\right) \\
&= \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \exp\left(-\frac{(\xi^j)^T G (\xi^j)}{2}\right) \ = \ \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \exp\left(-\frac{<\underline{\xi},\underline{\xi}>}{2}\right) \\
&= \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \ .
\end{aligned} \tag{16.4}$$

Hence, we indeed have

$$\mathcal{F}_g[\psi_t(\underline{x})](\underline{\xi}) = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_m}} \ (-i)^t \ \underline{\xi}^t \ \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \ . \qquad \square$$

16.1.3 The anisotropic Clifford-Hermite CWT

In order to introduce the corresponding *anisotropic Clifford-Hermite CWT*, we consider, still for $t > 0$, the continuous family of wavelets

$$\psi_t^{a,\underline{b}}(\underline{x}) = \frac{1}{a^{m/2}}\ \psi_t\left(\frac{\underline{x}-\underline{b}}{a}\right) \quad ,$$

with $a \in \mathbb{R}_+$ the dilation parameter and $\underline{b} \in \mathbb{R}^m$ the translation parameter. The anisotropic Clifford-Hermite CWT (g-CHCWT) applies to functions $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ by

$$\begin{aligned} T_t[f](a,\underline{b}) &= F_t(a,\underline{b}) \ = < \psi_t^{a,\underline{b}}, f > = \int_{\mathbb{R}^m} \left(\psi_t^{a,\underline{b}}(\underline{x})\right)^\dagger\ f(\underline{x})\ dV(\underline{x}) \\ &= \frac{1}{a^{m/2}} \int_{\mathbb{R}^m} \exp\left(-\frac{\|\underline{x}-\underline{b}\|^2}{2a^2}\right) \left(H_t\left(\frac{\underline{x}-\underline{b}}{a}\right)\right)^\dagger\ f(\underline{x})\ dV(\underline{x}) \quad . \end{aligned}$$

This definition can be rewritten in the frequency domain as

$$\begin{aligned} F_t(a,\underline{b}) &= \lambda_1 \ldots \lambda_m \ < \mathcal{F}_g[\psi_t^{a,\underline{b}}], \mathcal{F}_g[f] > \\ &= \lambda_1 \ldots \lambda_m \int_{\mathbb{R}^m} \left(\mathcal{F}_g[\psi_t^{a,\underline{b}}](\underline{\xi})\right)^\dagger\ \mathcal{F}_g[f](\underline{\xi})\ dV(\underline{\xi}) \quad . \end{aligned} \tag{16.5}$$

The anisotropic Fourier transform of the continuous family of wavelets:

$$\mathcal{F}_g[\psi_t^{a,\underline{b}}](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right)\ \frac{1}{a^{m/2}}\ \psi_t\left(\frac{\underline{x}-\underline{b}}{a}\right)\ dV(\underline{x}) \quad ,$$

can be calculated by means of the successive substitutions $\underline{t} = \underline{x} - \underline{b}$ and $\underline{u} = \dfrac{\underline{t}}{a}$ yielding

$$\mathcal{F}_g[\psi_t^{a,\underline{b}}](\underline{\xi}) = a^{m/2}\ \exp\left(-i < \underline{b}, \underline{\xi} >\right)\ \mathcal{F}_g[\psi_t](a\underline{\xi}) \quad .$$

Consequently (16.5) becomes

$$F_t(a,\underline{b}) = \lambda_1 \ldots \lambda_m\ a^{m/2}\ (2\pi)^{m/2}\ \mathcal{F}_g\left[\left(\mathcal{F}_g[\psi_t](a\underline{\xi})\right)^\dagger\ \mathcal{F}_g[f](\underline{\xi})\right](-\underline{b}). \tag{16.6}$$

It is clear that the g-CHCWT will map $L_2(\mathbb{R}^m, dV(\underline{x}))$ into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m$ for some weight function still to be determined. This weight function has to be chosen in such a way that the CWT is an isometry, or in

other words that the Parseval formula should hold.
Introducing the inner product

$$[F_t, Q_t] = \frac{1}{C_t} \int_{\mathbb{R}^m} \int_0^{+\infty} (F_t(a, \underline{b}))^\dagger \ Q_t(a, \underline{b}) \ \frac{da}{a^{m+1}} \ dV(\underline{b}) \quad ,$$

we thus search for the constant C_t in order to have the Parseval formula

$$[F_t, Q_t] = < f, q >$$

fulfilled. This runs along the same lines as the proof of the isometry property of a Spin(m)-invariant CCWT in the orthogonal Clifford setting (see Theorem 4.2).
By means of (16.6) and the Parseval formula for the anisotropic Fourier transform (Theorem 14.6), we have consecutively

$$\begin{aligned}
&[F_t, Q_t] \\
&= \frac{(2\pi)^m}{C_t} (\lambda_1 \ldots \lambda_m)^2 \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}_g \left[(\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \mathcal{F}_g[f](\underline{\xi}) \right] (-\underline{b})\right)^\dagger \\
&\quad \mathcal{F}_g \left[(\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \right] (-\underline{b}) \ \frac{da}{a} \ dV(\underline{b}) \\
&= \frac{(2\pi)^m}{C_t} (\lambda_1 \ldots \lambda_m)^2 \\
&\quad \int_0^{+\infty} \left\langle \mathcal{F}_g \left[(\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \mathcal{F}_g[f](\underline{\xi}) \right], \mathcal{F}_g \left[(\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \right] \right\rangle \ \frac{da}{a} \\
&= \frac{(2\pi)^m}{C_t} \lambda_1 \ldots \lambda_m \\
&\quad \int_0^{+\infty} \left\langle \left(\mathcal{F}_g[\psi_t](a\underline{\xi})\right)^\dagger \mathcal{F}_g[f](\underline{\xi}), \left(\mathcal{F}_g[\psi_t](a\underline{\xi})\right)^\dagger \mathcal{F}_g[q](\underline{\xi}) \right\rangle \ \frac{da}{a} \\
&= \frac{(2\pi)^m}{C_t} \lambda_1 \ldots \lambda_m \\
&\quad \int_{\mathbb{R}^m} \int_0^{+\infty} (\mathcal{F}_g[f](\underline{\xi}))^\dagger \ \mathcal{F}_g[\psi_t](a\underline{\xi}) \ (\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \ \frac{da}{a} \ dV(\underline{\xi}) \\
&= \frac{(2\pi)^m}{C_t} \lambda_1 \ldots \lambda_m \\
&\quad \int_{\mathbb{R}^m} (\mathcal{F}_g[f](\underline{\xi}))^\dagger \left(\int_0^{+\infty} \mathcal{F}_g[\psi_t](a\underline{\xi}) \ (\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger \ \frac{da}{a} \right) \mathcal{F}_g[q](\underline{\xi}) \ dV(\underline{\xi}) \quad .
\end{aligned}$$

By means of the substitution

$$\underline{\xi} = \frac{t}{a}\,\underline{\eta} \quad , \quad \underline{\eta} \in S^{m-1} \quad \text{for which} \quad \frac{da}{a} = \frac{dt}{t} \quad ,$$

the integral between brackets becomes

$$\int_0^{+\infty} \mathcal{F}_g[\psi_t](a\underline{\xi})\ (\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger\ \frac{da}{a} = \int_0^{+\infty} \mathcal{F}_g[\psi_t](t\underline{\eta})\ \left(\mathcal{F}_g[\psi_t](t\underline{\eta})\right)^\dagger\ \frac{dt}{t} \quad .$$

As

$$\mathcal{F}_g[\psi_t](\underline{\xi})\left(\mathcal{F}_g[\psi_t](\underline{\xi})\right)^\dagger = \frac{1}{\lambda_1 \ldots \lambda_m}\ \|\underline{\xi}\|^{2t}\ \exp\left(-\|\underline{\xi}\|^2\right)$$

is radial symmetric (i.e. only depending on $\|\underline{\xi}\|$), the integral between brackets can be further simplified to

$$\int_0^{+\infty} \mathcal{F}_g[\psi_t](a\underline{\xi})\ (\mathcal{F}_g[\psi_t](a\underline{\xi}))^\dagger\ \frac{da}{a} = \frac{1}{A_m}\int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_t](\underline{\xi})\left(\mathcal{F}_g[\psi_t](\underline{\xi})\right)^\dagger}{\|\underline{\xi}\|^m}\ dV(\underline{\xi}) \quad .$$

Consequently, if we put

$$\begin{aligned} C_t &= \frac{(2\pi)^m}{A_m}\int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_t](\underline{\xi})\left(\mathcal{F}_g[\psi_t](\underline{\xi})\right)^\dagger}{\|\underline{\xi}\|^m}\ dV(\underline{\xi}) \\ &= \frac{(2\pi)^m}{\lambda_1\ldots\lambda_m}\ \frac{(t-1)!}{2} \quad , \end{aligned} \tag{16.7}$$

the desired Parseval formula follows:

$$\begin{aligned} [F_t, Q_t] &= \lambda_1 \ldots \lambda_m \int_{\mathbb{R}^m} \left(\mathcal{F}_g[f](\underline{\xi})\right)^\dagger\ \mathcal{F}_g[q](\underline{\xi})\ dV(\underline{\xi}) \\ &= \lambda_1 \ldots \lambda_m\ < \mathcal{F}_g[f], \mathcal{F}_g[q] > = < f, q > \quad . \end{aligned}$$

Hence we have proved that the anisotropic Clifford-Hermite wavelets satisfy the *admissibility condition*

$$C_t = \frac{(2\pi)^m}{A_m}\int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_t](\underline{\xi})\ \left(\mathcal{F}_g[\psi_t](\underline{\xi})\right)^\dagger}{\left\|\underline{\xi}\right\|^m}\ dV(\underline{\xi}) < \infty \quad .$$

The anisotropic Clifford-Hermite CWT is thus an isometry between the spaces $L_2(\mathbb{R}^m, dV(\underline{x}))$ and $L_2(\mathbb{R}_+ \times \mathbb{R}^m, C_t^{-1} a^{-(m+1)}\ da\ dV(\underline{b}))$.

16.2 The generalized anisotropic Clifford-Hermite polynomials and associated CCWT

16.2.1 The generalized anisotropic Clifford-Hermite polynomials

Taking into account the orthogonal Clifford analysis case (see formula (5.9)), it is natural to define the so-called *generalized anisotropic Clifford-Hermite polynomials* by the Rodrigues formula:

$$H_{\ell,k}(\underline{x})\ P_k(\underline{x}) = (-1)^\ell \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^\ell \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_k(\underline{x})\right] \quad , \tag{16.8}$$

$\ell = 0, 1, 2, \ldots$, with $\partial_{\underline{x}}$ the anisotropic Dirac operator and $P_k(\underline{x})$ an arbitrary but fixed left solid inner g-spherical monogenic of order k. Note that for $k = 0$ these polynomials reduce to the radial anisotropic Clifford-Hermite polynomials of Section 16.1.1
They satisfy the following relation.

Lemma 16.2 *The generalized anisotropic Clifford-Hermite polynomials satisfy the relation*

$$H_{\ell+1,k}(\underline{x})\ P_k(\underline{x}) = (\underline{x} - \partial_{\underline{x}})[H_{\ell,k}(\underline{x})\ P_k(\underline{x})] \quad . \tag{16.9}$$

Proof. Using (16.3), we find consecutively

$$\begin{aligned}
&\partial_{\underline{x}}[H_{\ell,k}(\underline{x})\ P_k(\underline{x})] \\
&= (-1)^\ell\ \partial_{\underline{x}}\left[\exp\left(\frac{\|\underline{x}\|^2}{2}\right)\right] \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_k(\underline{x})\right] \\
&\quad + (-1)^\ell \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^{\ell+1}\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_k(\underline{x})\right] \\
&= (-1)^\ell\ \underline{x}\ \exp\left(\frac{\|\underline{x}\|^2}{2}\right) \partial_{\underline{x}}^\ell\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_k(\underline{x})\right] - H_{\ell+1,k}(\underline{x})\ P_k(\underline{x}) \\
&= \underline{x}\ H_{\ell,k}(\underline{x})\ P_k(\underline{x}) - H_{\ell+1,k}(\underline{x})\ P_k(\underline{x})
\end{aligned}$$

which proves the statement. □

In view of $H_{0,k}(\underline{x}) = 1$ and Theorem 14.4, the previous lemma implies that $H_{2\ell,k}(\underline{x})$ only contains even powers of $\underline{x}$ and is hence scalar-valued, while

$H_{2\ell+1,k}(\underline{x})$ only contains odd powers and is hence vector-valued.
From the equation (16.9) we now derive recurrence relations for the generalized anisotropic Clifford-Hermite polynomials.

Proposition 16.2 *The generalized anisotropic Clifford-Hermite polynomials satisfy the recurrence relations*

$$H_{2\ell+1,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})[H_{2\ell,k}(\underline{x})] \tag{16.10}$$

and

$$H_{2\ell+2,k}(\underline{x}) = (\underline{x} - \partial_{\underline{x}})[H_{2\ell+1,k}(\underline{x})] - 2k\frac{\underline{x}}{\|\underline{x}\|^2}H_{2\ell+1,k}(\underline{x}) \quad . \tag{16.11}$$

Proof. As the basic formulae and operator identities of anisotropic Clifford analysis are "formally" identical to the ones of orthogonal Clifford analysis (see Section 14.5), we can adopt the proof of the recurrence relations (5.6) and (5.7) for the generalized Clifford-Hermite polynomials (see [108]).
From (16.9) we obtain

$$\begin{aligned} H_{2\ell+1,k}(\underline{x})\ P_k(\underline{x}) &= (\underline{x} - \partial_{\underline{x}})[H_{2\ell,k}(\underline{x})\ P_k(\underline{x})] \\ &= \underline{x}\ H_{2\ell,k}(\underline{x})\ P_k(\underline{x}) - \partial_{\underline{x}}[H_{2\ell,k}(\underline{x})]\ P_k(\underline{x}) \end{aligned}$$

which proves (16.10), whereas

$$H_{2\ell+2,k}(\underline{x})\ P_k(\underline{x}) = \underline{x}\ H_{2\ell+1,k}(\underline{x})\ P_k(\underline{x}) - \partial_{\underline{x}}[H_{2\ell+1,k}(\underline{x})\ P_k(\underline{x})] \quad .$$

Moreover, by means of Proposition 14.13, we have

$$\begin{aligned} \partial_{\underline{x}}[H_{2\ell+1,k}(\underline{x})\ P_k(\underline{x})] &= -\partial_{\underline{x}}\left[r^{2k+m-1}\ \underline{\omega}\ H_{2\ell+1,k}(\underline{x})\ \underline{\omega}\ \frac{P_k(\underline{x})}{r^{2k+m-1}}\right] \\ &= -\partial_{\underline{x}}[r^{2k+m-1}\ \underline{\omega}\ H_{2\ell+1,k}(\underline{x})]\ \underline{\omega}\ \frac{P_k(\underline{x})}{r^{2k+m-1}} \quad . \end{aligned}$$

Next, taking into account the expression of the anisotropic Dirac operator in spherical co-ordinates (Proposition 14.15) and the fact that the anisotropic angular Dirac operator only acts on the angular co-ordinates (Lemma 14.4), we find

$$\begin{aligned} &\partial_{\underline{x}}[r^{2k+m-1}\underline{\omega}H_{2\ell+1,k}(\underline{x})] \\ &\quad = \underline{\omega}\partial_r[r^{2k+m-1}\ \underline{\omega}\ H_{2\ell+1,k}(\underline{x})] \\ &\quad = -(2k+m-1)\ r^{2k+m-2}H_{2\ell+1,k}(\underline{x}) + \underline{\omega}\ r^{2k+m-1}\ \underline{\omega}\ \partial_r[H_{2\ell+1,k}(\underline{x})] \quad . \end{aligned}$$

Furthermore, as $\Gamma[\underline{x}] = (m-1)\ \underline{x}$ (Lemma 14.5), we also obtain

$$\begin{aligned}\underline{\omega}\partial_r[H_{2\ell+1,k}(\underline{x})] &= \left(\partial_{\underline{x}} - \frac{\underline{\omega}}{r}\Gamma\right)[H_{2\ell+1,k}(\underline{x})] \\ &= \left(\partial_{\underline{x}} - (m-1)\frac{\underline{\omega}}{r}\right)[H_{2\ell+1,k}(\underline{x})] \quad ,\end{aligned}$$

which finally yields

$$\begin{aligned}H_{2\ell+2,k}(\underline{x}) &= \underline{x}H_{2\ell+1,k}(\underline{x}) - (2k+m-1)r^{2k+m-2}H_{2\ell+1,k}(\underline{x})\ \frac{\underline{\omega}}{r^{2k+m-1}} \\ &\quad + \underline{\omega}\ r^{2k+m-1}\left(\partial_{\underline{x}} - (m-1)\frac{\underline{\omega}}{r}\right)[H_{2\ell+1,k}(\underline{x})]\ \frac{\underline{\omega}}{r^{2k+m-1}} \\ &= \underline{x}H_{2\ell+1,k}(\underline{x}) - 2kH_{2\ell+1,k}(\underline{x})\ \frac{\underline{x}}{\|\underline{x}\|^2} - \partial_{\underline{x}}[H_{2\ell+1,k}(\underline{x})] \quad . \qquad \square\end{aligned}$$

The recurrence relations (16.10) and (16.11) combined with Theorem 14.4 allow us to compute recursively

$$\begin{aligned}H_{1,k}(\underline{x}) &= (\underline{x} - \partial_{\underline{x}})[H_{0,k}(\underline{x})] \\ &= \underline{x} \\ H_{2,k}(\underline{x}) &= (\underline{x} - \partial_{\underline{x}})[H_{1,k}(\underline{x})] - 2k\frac{\underline{x}}{\|\underline{x}\|^2}H_{1,k}(\underline{x}) \\ &= \underline{x}^2 + m + 2k = -\|\underline{x}\|^2 + m + 2k \\ H_{3,k}(\underline{x}) &= (\underline{x} - \partial_{\underline{x}})[H_{2,k}(\underline{x})] \\ &= \underline{x}^3 + (m+2k+2)\underline{x} = \underline{x}\ (-\|\underline{x}\|^2 + m + 2k + 2) \\ H_{4,k}(\underline{x}) &= (\underline{x} - \partial_{\underline{x}})[H_{3,k}(\underline{x})] - 2k\frac{\underline{x}}{\|\underline{x}\|^2}H_{3,k}(\underline{x}) \\ &= \underline{x}^4 + 2(m+2k+2)\underline{x}^2 + (m+2k+2)(2k+m) \\ &= \|\underline{x}\|^4 - 2(m+2k+2)\|\underline{x}\|^2 + (m+2k+2)(2k+m)\end{aligned}$$

etc.

Note that again these polynomials are "formally" identical to their orthogonal equivalents (see (5.8)). However, the metric is hidden within the powers of the vector variable $\underline{x}$.
The Rodrigues formula (16.8) combined with the anisotropic Clifford-Stokes theorem lead to the following orthogonality relation.

Theorem 16.2 *The generalized anisotropic Clifford-Hermite polynomials satisfy the orthogonality relation*

$$\int_{\mathbb{R}^m} (H_{\ell,k_1}(\underline{x})\ P_{k_1}(\underline{x}))^\dagger\ \ H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) = 0$$

whenever $\ell \neq t$ *or* $k_1 \neq k_2$*.*

Proof. Again, this proof is the metric dependent equivalent of the proof of the mutual orthogonality of the generalized Clifford-Hermite polynomials of orthogonal Clifford analysis (see [108]).
If $\ell = t$ and $k_1 \neq k_2$ we have

$$\begin{aligned}\int_{\mathbb{R}^m} & (P_{k_1}(\underline{x}))^\dagger\ \ (H_{\ell,k_1}(\underline{x}))^\dagger\ \ H_{\ell,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) \\ &= \int_0^{+\infty} r^{k_1+k_2+m-1}\left((H_{\ell,k_1}(\underline{x}))^\dagger\ H_{\ell,k_2}(\underline{x})\right)(r)\ \exp\left(-\frac{r^2}{2}\right)\ dr \\ &\qquad\qquad \int_{S^{m-1}} (P_{k_1}(\underline{\omega}))^\dagger\ \ P_{k_2}(\underline{\omega})\ L(\underline{\omega}, d\underline{\omega})\ =\ 0\ \ ,\end{aligned}$$

since $(H_{\ell,k_1}(\underline{x}))^\dagger\, H_{\ell,k_2}(\underline{x})$ is scalar-valued and inner g-spherical monogenics of different degree are orthogonal (see Theorem 15.4).
Take now $\ell \neq t$. Suppose that $\ell < t$; the case $\ell > t$ follows by Hermitian conjugation.
As $H_{\ell,k_1}(\underline{x})$ is a polynomial of degree ℓ in $\underline{x}$, it is sufficient to show that for each $t \in \mathbb{N}$ and $\ell < t$:

$$\int_{\mathbb{R}^m} (P_{k_1}(\underline{x}))^\dagger\ \ \underline{x}^\ell\ H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) = 0\ \ . \tag{16.12}$$

We prove this by induction on ℓ. For $\ell = 0$ we have by means of the Rodrigues formula (16.8) and the anisotropic Clifford-Stokes theorem that for $t > 0$:

$$\begin{aligned}\int_{\mathbb{R}^m} & (P_{k_1}(\underline{x}))^\dagger\ \ H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ dV(\underline{x}) \\ &= \ (-1)^t \int_{\mathbb{R}^m} (P_{k_1}(\underline{x}))^\dagger\ \ \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\ P_{k_2}(\underline{x})\right]\ dV(\underline{x})\end{aligned}$$

$$= (-1)^t \left(\lim_{\rho\to\infty} \int_{\partial\overline{B}(\rho)} (P_{k_1}(\underline{x}))^\dagger \ \underline{\sigma}(\underline{x}, d\underline{x}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x}) \right] \right.$$

$$\left. - \int_{\mathbb{R}^m} \left[(P_{k_1}(\underline{x}))^\dagger \right] \partial_{\underline{x}} \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x}) \right] \ dV(\underline{x}) \right) = 0 \ ,$$

since

$$\left[(P_{k_1}(\underline{x}))^\dagger \right] \partial_{\underline{x}} = 0 \ .$$

Assume that the orthogonality (16.12) holds for $(\ell - 1)$ and $t > (\ell - 1)$ and let $\ell < t$.
Then, again by means of the Rodrigues formula (16.8) and the anisotropic Clifford-Stokes theorem, we find

$$\int_{\mathbb{R}^m} (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \ H_{t,k_2}(\underline{x}) \ P_{k_2}(\underline{x}) \ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x})$$

$$= (-1)^t \int_{\mathbb{R}^m} (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \ \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x}) \right] dV(\underline{x})$$

$$= (-1)^t \left(\lim_{\rho\to+\infty} \int_{\partial\overline{B}(\rho)} (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \ \underline{\sigma}(\underline{x}, d\underline{x}) \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x}) \right] \right.$$

$$\left. - \int_{\mathbb{R}^m} \left[(P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \right] \partial_{\underline{x}} \ \partial_{\underline{x}}^{t-1} \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x}) \right] dV(\underline{x}) \right)$$

$$= \int_{\mathbb{R}^m} \left[(P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \right] \partial_{\underline{x}} \ H_{t-1,k_2}(\underline{x}) \ P_{k_2}(\underline{x}) \ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x}) \ . \quad (16.13)$$

By Hermitian conjugation we obtain from Theorem 14.4 that

$$\left[(P_{k_1}(\underline{x}))^\dagger \ \underline{x}^\ell \right] \partial_{\underline{x}} = \begin{cases} -\ell \ (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^{\ell-1} & \text{for } \ell \text{ even} \\ -(\ell - 1 + 2k_1 + m) \ (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^{\ell-1} & \text{for } \ell \text{ odd.} \end{cases}$$

This implies that (16.13) equals, up to a scalar,

$$\int_{\mathbb{R}^m} (P_{k_1}(\underline{x}))^\dagger \ \underline{x}^{\ell-1} \ H_{t-1,k_2}(\underline{x}) \ P_{k_2}(\underline{x}) \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x}) \ ,$$

which is zero in view of the induction hypothesis. □

We end this section with the following result, which was proved in the framework of orthogonal Clifford analysis in [108].

Theorem 16.3 *The set of functions*

$$\left\{ H_{\ell,k}(\underline{x})\ P_k(\underline{x})\ ;\ \ell, k \in \mathbb{N} \right\} \quad ,$$

$P_k(\underline{x})$ *being a solid inner g-spherical monogenic of order* k*, constitutes a basis for* $L_2\left(\mathbb{R}^m, \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x})\right)$.

Proof. In view of the g-monogenic decomposition (Theorem 14.2) it is clear that the set

$$\left\{ H_{\ell,k}(\underline{x})\ P_k(\underline{x})\ ;\ \ell, k \in \mathbb{N} \right\}$$

constitutes a basis for the set of $\mathbb{C}_m$-valued polynomials in $\mathbb{R}^m$.
The theorem now follows from the fact that the set of $\mathbb{C}_m$-valued polynomials in $\mathbb{R}^m$ is dense in the space $L_2\left(\mathbb{R}^m, \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x})\right)$. $\square$

16.2.2 The generalized anisotropic Clifford-Hermite wavelets

For $(t, k_2) \neq (0, 0)$ Theorem 16.2 implies that

$$\int_{\mathbb{R}^m} H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x})\ \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) dV(\underline{x}) \quad .$$

Hence the $L_1 \cap L_2$-functions

$$\begin{aligned} \psi_{t,k_2}(\underline{x}) &= \exp\left(-\frac{\|\underline{x}\|^2}{2}\right) H_{t,k_2}(\underline{x})\ P_{k_2}(\underline{x}) \\ &= (-1)^t\ \partial_{\underline{x}}^t \left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) P_{k_2}(\underline{x})\right] \end{aligned}$$

have zero momentum. Consequently, they can be used as mother wavelets in $\mathbb{R}^m$, if at least an appropriate admissibility condition, turning the corresponding CCWT into an isometry, is found (see Section 16.2.3). We call them the *generalized anisotropic Clifford-Hermite wavelets.*

Their anisotropic Fourier transform is again a product of the Gaussian function with a polynomial of degree $(t + k_2)$.

Lemma 16.3 *The anisotropic Fourier transform of the generalized anisotropic Clifford-Hermite wavelets takes the form:*

$$\mathcal{F}_g[\psi_{t,k_2}(\underline{x})](\underline{\xi}) = \frac{(-i)^{t+k_2}}{\sqrt{\lambda_1 \ldots \lambda_m}} \; \underline{\xi}^t \; P_{k_2}(\underline{\xi}) \; \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \quad .$$

Proof. By means of Proposition 14.18 and equation (16.4) we find

$$\begin{aligned}
\mathcal{F}_g[\psi_{t,k_2}(\underline{x})](\underline{\xi}) &= (-i)^t \; \underline{\xi}^t \; \mathcal{F}_g\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right) \; P_{k_2}(\underline{x})\right](\underline{\xi}) \\
&= (-i)^t \; \underline{\xi}^t \; P_{k_2}(i\partial_{\underline{\xi}}) \; \mathcal{F}_g\left[\exp\left(-\frac{\|\underline{x}\|^2}{2}\right)\right](\underline{\xi}) \\
&= \frac{(-i)^t i^{k_2}}{\sqrt{\lambda_1 \ldots \lambda_m}} \; \underline{\xi}^t \; P_{k_2}(\partial_{\underline{\xi}})\left[\exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right)\right] \quad .
\end{aligned}$$

As

$$\partial_{\underline{\xi}}\left[\exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right)\right] = -\underline{\xi} \; \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \quad ,$$

we also have

$$\begin{aligned}
P_{k_2}(\partial_{\underline{\xi}})\left[\exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right)\right] &= P_{k_2}(-\underline{\xi}) \; \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \\
&= (-1)^{k_2} P_{k_2}(\underline{\xi}) \; \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \quad ,
\end{aligned}$$

which finally yields the desired result:

$$\mathcal{F}_g[\psi_{t,k_2}(\underline{x})](\underline{\xi}) = \frac{(-i)^{t+k_2}}{\sqrt{\lambda_1 \ldots \lambda_m}} \; \underline{\xi}^t \; P_{k_2}(\underline{\xi}) \; \exp\left(-\frac{\|\underline{\xi}\|^2}{2}\right) \quad . \qquad \Box$$

Moreover, formula (16.12) implies that the generalized anisotropic Clifford-Hermite wavelet has a number of vanishing moments.

Proposition 16.3 *The generalized anisotropic Clifford-Hermite wavelet* ψ_{t,k_2} *has vanishing moments up to order* $(t-1)$ *:*

$$\int_{\mathbb{R}^m} \underline{x}^j \; \psi_{t,k_2}(\underline{x}) \; dV(\underline{x}) = 0 \quad , \quad j = 0, 1, \ldots, t-1 \quad .$$

16.2.3 The generalized anisotropic Clifford-Hermite CWT

Similar to the orthogonal Clifford setting, the *generalized anisotropic Clifford-Hermite CWT* (generalized g-CHCWT) of a function $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ is defined by

$$\begin{aligned} T_{t,k_2}[f](a,\underline{b},s) &= F_{t,k_2}(a,\underline{b},s) = < \psi^{a,\underline{b},s}_{t,k_2}, f > \\ &= \int_{\mathbb{R}^m} \left(\psi^{a,\underline{b},s}_{t,k_2}(\underline{x})\right)^\dagger f(\underline{x})\, dV(\underline{x}) \end{aligned}$$

where, still for $(t,k_2) \neq (0,0)$, the continuous family of wavelets $\psi^{a,\underline{b},s}_{t,k_2}(\underline{x})$ is given by

$$\psi^{a,\underline{b},s}_{t,k_2}(\underline{x}) = \frac{1}{a^{m/2}}\, s\, \psi_{t,k_2}\left(\frac{s^\dagger(\underline{x}-\underline{b})s}{a}\right) s^\dagger \quad ,$$

with $a \in \mathbb{R}_+$, $\underline{b} \in \mathbb{R}^m$, $s \in \mathrm{Spin}(m)$, originating from the mother wavelet ψ_{t,k_2} by dilation, translation and spinor-rotation. Recall that $\mathrm{Spin}(m)$ denotes the anisotropic Spin-group introduced in Section 14.3.
Their anisotropic Fourier transform is given by

$$\begin{aligned} &\mathcal{F}_g[\psi^{a,\underline{b},s}_{t,k_2}](\underline{\xi}) \\ = \ & \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{x}, \underline{\xi} >\right) \frac{1}{a^{m/2}}\, s\, \psi_{t,k_2}\left(\frac{s^\dagger(\underline{x}-\underline{b})s}{a}\right) s^\dagger\, dV(\underline{x}) \\ = \ & \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) \int_{\mathbb{R}^m} \exp\left(-i < \underline{t}, \underline{\xi} >\right) \frac{1}{a^{m/2}}\, s\, \psi_{t,k_2}\left(\frac{s^\dagger \underline{t} s}{a}\right) \\ & s^\dagger\, dV(\underline{t}) \\ = \ & \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) \int_{\mathbb{R}^m} \exp\left(-i < \underline{z}, s^\dagger \underline{\xi} s >\right) \frac{1}{a^{m/2}}\, s\, \psi_{t,k_2}\left(\frac{\underline{z}}{a}\right) \\ & s^\dagger\, dV(\underline{z}) \\ = \ & \frac{1}{(2\pi)^{m/2}} \exp\left(-i < \underline{b}, \underline{\xi} >\right) a^{m/2} \int_{\mathbb{R}^m} \exp\left(-i < \underline{u}, a s^\dagger \underline{\xi} s >\right) s\, \psi_{t,k_2}(\underline{u}) \\ & s^\dagger\, dV(\underline{u}) \\ = \ & a^{m/2} \exp\left(-i < \underline{b}, \underline{\xi} >\right) s\, \mathcal{F}_g[\psi_{t,k_2}](a s^\dagger \underline{\xi} s)\, s^\dagger \quad , \end{aligned}$$

where we have applied the successive substitutions $\underline{t} = \underline{x} - \underline{b}$, $\underline{z} = s^\dagger \underline{t} s$ and $\underline{u} = \dfrac{\underline{z}}{a}$ for which respectively $dV(\underline{x}) = dV(\underline{t})$, $dV(\underline{z}) = dV(\underline{t})$ and $dV(\underline{u}) = a^{-m} dV(\underline{z})$.

In the above calculation we have also used the fact that

$$\begin{aligned}
< s^\dagger \underline{x} s, s^\dagger \underline{\xi} s > &= -(s^\dagger \underline{x} s) \bullet (s^\dagger \underline{\xi} s) \;=\; -\frac{1}{2}\{s^\dagger \underline{x} s, s^\dagger \underline{\xi} s\} \\
&= -\frac{1}{2}(s^\dagger \underline{x} s \; s^\dagger \underline{\xi} s + s^\dagger \underline{\xi} s \; s^\dagger \underline{x} s) \;=\; -\frac{1}{2}(s^\dagger \underline{x}\underline{\xi} s + s^\dagger \underline{\xi}\underline{x} s) \\
&= s^\dagger \left(-\frac{1}{2}\right)(\underline{x}\underline{\xi} + \underline{\xi}\underline{x}) s \;=\; s^\dagger < \underline{x}, \underline{\xi} > s \;= < \underline{x}, \underline{\xi} > s^\dagger s \\
&= < \underline{x}, \underline{\xi} > \quad .
\end{aligned}$$

By means of the Parseval formula (14.10), the definition of the generalized g-CHCWT may be rewritten in the frequency domain as

$$\begin{aligned}
F_{t,k_2}(a, \underline{b}, s) &= \lambda_1 \ldots \lambda_m < \mathcal{F}_g[\psi^{a,\underline{b},s}_{t,k_2}], \mathcal{F}_g[f] > \\
&= \lambda_1 \ldots \lambda_m \; (2\pi)^{m/2} \; a^{m/2} \; s \\
&\quad \mathcal{F}_g \left[\left(\mathcal{F}_g[\psi_{t,k_2}](a s^\dagger \underline{\xi} s) \right)^\dagger \; s^\dagger \; \mathcal{F}_g[f](\underline{\xi}) \right] (-\underline{b}) \quad . \qquad (16.14)
\end{aligned}$$

It is clear that the generalized g-CHCWT will map $L_2(\mathbb{R}^m, dV(\underline{x}))$ into a weighted L_2-space on $\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m)$. The weight function, depending on the so-called admissibility constant, has to be chosen in such a way that the CWT is an isometry, or in other words that the Parseval formula should hold. Introducing the inner product

$$\begin{aligned}
&[F_{t,k_2}, Q_{t,k_2}] \\
&= \frac{1}{C_{t,k_2}} \int_{\mathrm{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} (F_{t,k_2}(a, \underline{b}, s))^\dagger \;\; Q_{t,k_2}(a, \underline{b}, s) \; \frac{da}{a^{m+1}} \; dV(\underline{b}) \; ds \quad ,
\end{aligned}$$

where ds stands for the Haar measure on $\mathrm{Spin}(m)$, we hence search for the constant C_{t,k_2} in order to have the Parseval formula

$$[F_{t,k_2}, Q_{t,k_2}] = < f, q >$$

fulfilled. In order to derive this *admissibility constant* C_{t,k_2}, we rewrite the proof of Theorem 4.1 stating the isometry property of the orthogonal CCWT, to the present metric dependent context invoking the results obtained in the preceding two chapters.

By means of (16.14), we find

$$[F_{t,k_2}, Q_{t,k_2}] = \frac{(\lambda_1 \ldots \lambda_m)^2 \ (2\pi)^m}{C_{t,k_2}} \int_{\text{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}_g \left[\left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger s^\dagger \ \mathcal{F}_g[f](\underline{\xi}) \right] (-\underline{b}) \right)^\dagger s^\dagger \ s \ \mathcal{F}_g \left[\left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger \ s^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \right] (-\underline{b}) \ \frac{da}{a} \ dV(\underline{b}) \ ds \ .$$

Moreover, taking into account that $s^\dagger s = 1$ for all $s \in \text{Spin}(m)$ and applying the Parseval formula for the anisotropic Fourier transform, yields consecutively

$$\begin{aligned} &[F_{t,k_2}, Q_{t,k_2}] \\ &= \frac{(\lambda_1 \ldots \lambda_m)^2 \ (2\pi)^m}{C_{t,k_2}} \int_{\text{Spin}(m)} \int_0^{+\infty} \left\langle \mathcal{F}_g \left[\left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger \ s^\dagger \ \mathcal{F}_g[f](\underline{\xi}) \right] , \right. \\ &\quad \left. \mathcal{F}_g \left[\left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger \ s^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \right] \right\rangle \frac{da}{a} \ ds \\ &= \frac{(\lambda_1 \ldots \lambda_m) \ (2\pi)^m}{C_{t,k_2}} \int_{\text{Spin}(m)} \int_{\mathbb{R}^m} \int_0^{+\infty} \left(\mathcal{F}_g[f](\underline{\xi}) \right)^\dagger \ s \ \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \\ &\quad \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger \ s^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \ \frac{da}{a} \ dV(\underline{\xi}) \ ds \\ &= \frac{(\lambda_1 \ldots \lambda_m) \ (2\pi)^m}{C_{t,k_2}} \int_{\mathbb{R}^m} \left(\mathcal{F}_g[f](\underline{\xi}) \right)^\dagger \left(\int_{\text{Spin}(m)} \int_0^{+\infty} s \ \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right. \\ &\quad \left. \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s) \right)^\dagger s^\dagger \ \frac{da}{a} \ ds \right) \mathcal{F}_g[q](\underline{\xi}) \ dV(\underline{\xi}) \ . \end{aligned} \tag{16.15}$$

From Lemma 16.3 we obtain

$$\begin{aligned} &\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi}) \ \left(\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi}) \right)^\dagger \\ &\qquad = \frac{1}{\lambda_1 \ldots \lambda_m} \ \exp\left(- \|\underline{\xi}\|^2\right) \ \underline{\xi}^t \ P_{k_2}(\underline{\xi}) \ \left(P_{k_2}(\underline{\xi}) \right)^\dagger \ (-\underline{\xi})^t \ . \end{aligned} \tag{16.16}$$

Let us now assume that $P_{k_2}(\underline{\xi}) \left(P_{k_2}(\underline{\xi}) \right)^\dagger$ is scalar-valued. Then equation (16.16) becomes

$$\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi}) \ \left(\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi}) \right)^\dagger = \frac{1}{\lambda_1 \ldots \lambda_m} \ \exp\left(- \|\underline{\xi}\|^2\right) \ \|\underline{\xi}\|^{2t} \ P_{k_2}(\underline{\xi}) \ \left(P_{k_2}(\underline{\xi}) \right)^\dagger \ .$$

As the above expression is scalar-valued, the integral between brackets in (16.15) takes the form

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\ \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\right)^\dagger\ s^\dagger\ \frac{da}{a}\ ds$$
$$= \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\ \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\right)^\dagger\ \frac{da}{a}\ ds\ .$$

By means of the substitution

$$\underline{\xi} = \frac{t}{a}\ \underline{\eta}\ \ ;\ \ \underline{\eta} \in S^{m-1} \quad \text{for which} \quad \frac{da}{a} = \frac{dt}{t}\ ,$$

we find

$$\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\ \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\right)^\dagger\ s^\dagger\ \frac{da}{a}\ ds$$
$$= \int_{\mathrm{Spin}(m)} \int_0^{+\infty} \mathcal{F}_g[\psi_{t,k_2}](ts^\dagger \underline{\eta} s)\ \left(\mathcal{F}_g[\psi_{t,k_2}](ts^\dagger \underline{\eta} s)\right)^\dagger\ \frac{dt}{t}\ ds\ .$$

Taking into account that $s^\dagger \underline{\eta} s = \underline{\nu} \in S^{m-1}$ for all $\underline{\eta} \in S^{m-1}$, this integral may be further simplified to

$$\begin{aligned}
&\int_{\mathrm{Spin}(m)} \int_0^{+\infty} s\ \mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\ \left(\mathcal{F}_g[\psi_{t,k_2}](as^\dagger \underline{\xi} s)\right)^\dagger\ s^\dagger\ \frac{da}{a}\ ds \\
&= \int_0^{+\infty} \int_{S^{m-1}} \mathcal{F}_g[\psi_{t,k_2}](t\underline{\nu})\ \left(\mathcal{F}_g[\psi_{t,k_2}](t\underline{\nu})\right)^\dagger\ \frac{dt}{t}\ L(\underline{\nu}, d\underline{\nu}) \\
&= \int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_{t,k_2}](\underline{u})\ \left(\mathcal{F}_g[\psi_{t,k_2}](\underline{u})\right)^\dagger}{\|\underline{u}\|^m}\ dV(\underline{u})\ .
\end{aligned}$$

Consequently, if we put

$$\begin{aligned}
&C_{t,k_2} \\
&= (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_{t,k_2}](\underline{u})\ \left(\mathcal{F}_g[\psi_{t,k_2}](\underline{u})\right)^\dagger}{\|\underline{u}\|^m}\ dV(\underline{u}) \\
&= \frac{(2\pi)^m}{\lambda_1 \ldots \lambda_m} \int_0^{+\infty} \exp\left(-t^2\right)\ t^{2t+2k_2-1}\ dt \int_{S^{m-1}} P_{k_2}(\underline{\nu})\left(P_{k_2}(\underline{\nu})\right)^\dagger\ L(\underline{\nu}, d\underline{\nu}) \\
&= \frac{(2\pi)^m}{\lambda_1 \ldots \lambda_m} \frac{(t+k_2-1)!}{2} \int_{S^{m-1}} |P_{k_2}(\underline{\nu})|^2\ L(\underline{\nu}, d\underline{\nu})\ ,
\end{aligned}$$

expression (16.15) becomes

$$[F_{t,k_2}, Q_{t,k_2}] = \lambda_1 \ldots \lambda_m \int_{\mathbb{R}^m} \left(\mathcal{F}_g[f](\underline{\xi})\right)^\dagger \ \mathcal{F}_g[q](\underline{\xi}) \ dV(\underline{\xi}) = < f \ , \ q > \quad .$$

We have thus proved that, provided that $P_{k_2}(\underline{\xi}) \left(P_{k_2}(\underline{\xi})\right)^\dagger$ is scalar-valued, the generalized anisotropic Clifford-Hermite wavelets satisfy the *admissibility condition*:

$$C_{t,k_2} = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi}) \ \left(\mathcal{F}_g[\psi_{t,k_2}](\underline{\xi})\right)^\dagger}{\left\|\underline{\xi}\right\|^m} \ dV(\underline{\xi}) < \infty \quad .$$

Summarizing, on condition that $P_{k_2}(\underline{\xi}) \left(P_{k_2}(\underline{\xi})\right)^\dagger$ is scalar-valued, the generalized g-CHCWT maps $L_2(\mathbb{R}^m, dV(\underline{x}))$ isometrically into $L_2(\mathbb{R}_+ \times \mathbb{R}^m \times \mathrm{Spin}(m),$ $C_{t,k_2}^{-1} \ a^{-(m+1)} \ da \ dV(\underline{b}) \ ds)$.

Remark 16.1 Although we have restricted ourselves to the construction of the anisotropic Clifford-Hermite wavelets, it should be clear that the other types of basic Clifford-wavelets discussed in Part I of this work, can also be developed in the metric dependent setting of Clifford analysis. Furthermore, the same remark holds for the general isotropic Clifford wavelet theory presented in Chapter 4.

Nederlandse samenvatting

Hoofdstuk 1: Inleiding

Deze doctoraatsscriptie bestaat uit drie delen: (i) continue clifford-wavelettransformatie, (ii) clifford-fouriertransformatie en (iii) metriekafhankelijke clifford-analyse, en bestrijkt drie wiskundige analysedomeinen: (i) continue wavelet-transformatie, (ii) fouriertransformatie en (iii) cliffordanalyse, die in elk van deze drie delen interfereren.

Wavelettransformatie is de laatste decennia uitgegroeid tot een veelgebruikt instrument in theoretisch en toepassingsgericht onderzoek en heeft een stevige reputatie verworven voor praktische toepassingen (zie bvb. [45, 51, 76]). Kernidee in wavelettheorie is schaalafhankelijke signaalanalyse. Wavelets zijn oscillerende functies in een begrensd gebied van ruimte of tijd die daarbuiten (bijna) nul zijn, m.a.w. sterk in ruimte of tijd gelokaliseerde golfverschijnselen. Een specifieke wavelet wordt gedilateerd om het in een bepaalde schaal te persen of uit te smeren en vervolgens getranslateerd langs en gecorreleerd met het te analyseren signaal. Op grote schaal worden aldus de grove kenmerken van een signaal gedetecteerd, op kleine schaal de fijnere structuren.
Bij het aftasten van een signaal met een wavelet kan men schaal en positie laten variëren in discrete stappen of op continue wijze. Dit laatste geval resulteert in de zgn. continue wavelettransformatie (CWT), terwijl in het eerste geval de zgn. discrete wavelettransformatie (DWT) ontstaat. Beide vertonen min of meer tegengestelde eigenschappen en hebben elk hun specifiek toepassingsveld. De CWT is een succesvol instrumentarium voor signaalanalyse, terwijl de DWT een krachtig hulpmiddel is bij bvb. datacompressie en signaalreconstructie. De

DWT-methodiek is veruit de meest gebruikte van beide, maar de CWT en, algemener, redundante signaalrepresentaties, bieden in sommige gevallen welbepaalde voordelen.

Belichten we even de hoogtepunten uit de wavelethistoriek. Als een eerste aanzet kan het werk van Jean-Baptiste Joseph Fourier (1768-1830) worden beschouwd. Zijn frequentieanalysetechniek (1807) zou zeer belangrijk blijken te zijn en heeft een zeer sterke impact gehad. De eerste vermelding van wat thans een "wavelet" wordt genoemd, dateert van 1909, en is te vinden in het doctoraatswerk over orthogonale functiesystemen van Alfred Haar (1885-1933). Hij introduceerde de waveletfamilie die thans zijn naam draagt; het zijn de eenvoudigst denkbare wavelets met compacte drager. Een volgende stap werd gezet door de natuurkundige Paul Levy (1886-1971) die bij zijn onderzoek op brownse beweging tot de bevinding kwam dat de haarwavelets met variërende schaal een efficiëntere basis vormden dan de circulaire basisfuncties van Fourier. Het theoretisch waveletconcept in zijn huidige vorm is te danken aan ingenieur Jean Morlet en het onderzoeksteam van het centrum voor theoretische fysica te Marseille o.l.v. de natuurkundige Alex Grossmann. Morlet (1980) gebruikte als eerste de term wavelet ("ondelette"), meer bepaald "wavelets met constante helling". Morlet en Grossmann legden ook de fundamenten voor datacompressie. Het multiresolutieconcept, dat leidde tot de DWT, is toe te schrijven aan Yves Meyer en Stéphane Mallat. De tweedimensionale CWT duikt op in het doctoraatswerk [94] van Romain Murenzi die samenwerkte met Jean-Pierre Antoine, Alex Grossmann en Ingrid Daubechies. Deze laatste gebruikte het multiresolutieconcept om een waveletfamilie te construeren die de belangrijke eigenschappen: compacte drager, orthonormaliteit en regulariteit vertonen.

Cliffordalgebra en cliffordanalyse ontwikkelden zich in min of meer dezelfde periode. In 1882 verscheen een postuum artikel [46] van William Kingdon Clifford (1845-1879) waarin hij de algebra's invoerde die later zijn naam zouden dragen. Hij beoogde ermee zowel de uitwendige algebra van Grassmann als de quaternionen van Hamilton in één enkele structuur te veralgemenen. Cliffordalgebra's worden ook *meetkundige algebra's* genoemd omdat ze zowel de meetkundige als de algebraïsche eigenschappen van de euclidische ruimte in zich verenigen.
Cliffordalgebra's werden op verschillende momenten "herontdekt", i.h.b. door natuurkundigen. Toen bvb. in 1928 Paul A.M. Dirac in zijn beroemd artikel [58] over het elektron de zgn. γ-matrices invoerde om de klein-gordon-differentiaalvergelijking te lineariseren, construeerde hij in feite de basisvectoren van de cliffordalgebra $\mathbb{R}_{1,3}$.

De complexe getallenalgebra is de universele cliffordalgebra over $\mathbb{R}$. Het product van de cauchy-riemannoperator met zijn complex toegevoegde is de tweedimensionale laplaciaan. M.a.w. holomorfe functies in het vlak zijn nuloplossingen van een eerste orde rotatie-invariante differentiaaloperator die de laplaciaan factoriseert.

Het is in deze zin dat cliffordanalyse als de "natuurlijke" veralgemening tot hogere dimensie van de theorie van de holomorfe functies in het complexe vlak moet worden beschouwd. De veralgemeende holomorfe functies, meestal *monogene* functies genoemd, zijn de nuloplossingen van een eerste orde rotatie-invariante differentiaaloperator, de diracoperator, die de laplaciaan factoriseert. Het is precies deze factorizering van de laplaciaan die het speciaal verband oplevert tussen monogene functies enerzijds en harmonische functies in de ruimte anderzijds: de eigenschappen van monogene functies houden steeds een verfijning in van deze van harmonische functies.
De eerste cliffordanalyseresultaten werden, onafhankelijk van elkaar, bekomen in de jaren 1930 door R. Fueter, G. Moisil en N. Théodorescu. Een grondige cliffordanalysestudie waarbij de parallellen tussen de complexe en de clifford-functietheorie worden getrokken, is te vinden in het boek [15] van F. Brackx, R. Delanghe en F. Sommen uit 1982. Ondertussen heeft cliffordanalyse aan interesse gewonnen en is zij uitgegroeid tot een zelfstandige analysediscipline.

Eén van de belangrijkste kenmerken van cliffordanalyse is het simultaan, gelijkberechtend en globaal gebruik van alle dimensies, in tegenstelling tot een tensoriële benadering met producten van ééndimensionale fenomenen. Het is precies deze intrinsiek meerdimensionale natuur die de mogelijkheid schept om meerdimensionale wavelets te construeren en de hieraan geassocieerde CWT-theorie te ontwikkelen. De constructieprocedure voor wavelets wordt in het kader van de standaard cliffordanalyse uiteengezet in het eerste deel van dit werk. In deel III wordt dit veralgemeend naar metriekafhankelijke cliffordanalyse.
Deel II is gewijd aan de clifford-fouriertransformatie, waarbij de invoering van een cliffordalgebrawaardige fourierkern leidt tot een factorizatie van de klassieke fouriertransformatie in de geest van de reeds vermelde verfijning van harmonische functies door monogene.

We geven nu een algemeen overzicht van de inhoud van deze scriptie.

Twee inleidende hoofdstukken moeten de tekst op zichzelf staand maken. In hoofdstuk 2 worden de basisresultaten van cliffordalgebra en orthogonale cliffordanalyse vermeld. Orthogonaal moet hier worden begrepen in de zin dat de

fundamentele groep waarvan de actie de diracoperator invariant laat, de speciale orthogonale groep $\mathrm{SO}(m)$ is. Ook enkele veel gebruikte resultaten van de fouriertransformatie en van sferische harmonieken worden aangehaald.
In hoofdstuk 3 wordt, nogal gedetailleerd, de ééndimensionale CWT beschreven; hierbij hebben we ons hoofdzakelijk geïnspireerd op [5].

Deel I: De continue clifford-wavelettransformatie

In deel I komen de beide hierboven vermelde disciplines samen in de studie van de meerdimensionale continue clifford-wavelettransformatie. Na de studie van de clifford-hermite CWT van Brackx en Sommen en de constructie van een reeks specifieke clifford-wavelets, realiseerden we ons dat een algemene theorie kon worden opgesteld waarin al deze clifford-wavelets pasten. We spreken hier van de isotrope clifford-wavelettheorie omdat wordt gewerkt in het traditionele kader met de standaard euclidische metriek.
De constructiemethodiek van clifford-wavelets verloopt als volgt. In een eerste fase worden orthogonale veeltermen in de ruimte geconstrueerd als veralgemening van klassieke orthogonale veeltermen op de reële rechte. Een fundamentele rol is hierbij weggelegd voor een typische cliffordanalysetechniek, de zgn. cauchy-kowalewskaia-extensie van reëel-analytische functies in $\mathbb{R}^m$ tot monogene functies in $\mathbb{R}^{m+1}$. Recursieve betrekkingen en een rodriguesformule voor deze nieuwe clifford-veeltermen leiden, samen met de clifford-stokesstelling, tot de onontbeerlijke orthogonaliteitsrelaties in de ruimte. Deze stellen ons namelijk in staat kandidaat-moederwavelets te selecteren die, onder bepaalde voorwaarden, als kernfuncties kunnen worden gebruikt in een meerdimensionale CWT.
Deze methodiek wordt driemaal toegepast, in de hoofdstukken 5, 6 en 7, voor de constructie van telkens de orthogonale veeltermen, de wavelets en de geassocieerde CWT van het type "bi-axiaal clifford-hermite", "clifford-gegenbauer" en "clifford-laguerre".

In het geval van de jacobiveeltermen echter, leidt voornoemde constructie tot functies die niet langer aan de moederwaveletcondities voldoen. Toch zijn we erin geslaagd clifford-jacobiveeltermen te gebruiken als kernfunctie in een clifford-CWT (hoofdstuk 9), maar dan niet voor $L_2(\mathbb{R}^m)$-functies, maar voor functies in de hardyruimte $H^2(\mathbb{R})$. Meer zelfs, het bleek mogelijk ook hier een algemene theorie op te stellen voor deze zgn. "half-clifford CWT" (hoofdstuk 8).

Uit het voorgaande moge ten overvloede blijken dat de constructie van meerdimensionale orthogonale veeltermen cruciaal is geweest voor het vinden van onze specifieke clifford-wavelets. Wat daarbij opvalt, is dat het steeds mogelijk blijkt gebruik te maken van een reële gewichtsfunctie. Wat uiteraard de vraag deed rijzen wat de impact zou kunnen zijn van cliffordalgebrawaardige gewichtsfuncties in de orthogonaliteitscondities. Het was onmiddellijk duidelijk dat een alternatief voor de cauchy-kowalewskaia-extensie-techniek moest worden gezocht. Cliffordanalyse zelf snelde ons hierbij ter hulp, in de gedaante van de clifford-heavisidefuncties; dit zijn twee functies P^+ en P^- die de eenheid ontbinden: $P^+ + P^- = 1$, elkaar annihileren: $P^+P^- = P^-P^+ = 0$, en idempotent zijn: $(P^+)^2 = P^+$, $(P^-)^2 = P^-$. Zoals de naam aangeeft, kunnen ze worden beschouwd als meerdimensionale heavisidefuncties. Met behulp van deze clifford-heavisidefuncties konden de gewenste orthogonaliteitsbetrekkingen in de ruimte herleid worden tot orthogonaliteitsbetrekkingen op de reële rechte, waardoor een veralgemening kon worden opgesteld, zowel in de hele ruimte als in de eenheidsbal, van bepaalde traditionele ééndimensionale orthogonale veeltermen. Dit is het onderwerp van hoofdstuk 10.

Deel II: De clifford-fouriertransformatie

Het centrale thema van Deel II is de clifford-fouriertransformatie. Meestal wordt in cliffordanalyse gebruik gemaakt van de standaard fouriertransformatie die in feite tensorieel is. Het cliffordkarakter van het fourierspectrum wordt daarbij enkel bepaald door het cliffordkarakter van het origineel signaal, vermits de fourierkern complex is. Veralgemeningen van de fouriertransformatie met een cliffordalgebrawaardige kernfunctie zijn talrijk, vertonen uiteenlopende eigenschappen en worden met succes aangewend bij twee- en driedimensionale signaalanalyse.

De clifford-fouriertransformatie die we in dit werk opvoeren, heeft zijn oorsprong in de poging om een meerdimensionale fractionele fouriertransformatie te ontwerpen. De ééndimensionale fractionele fouriertransformatie wordt in integraalgedaante gebracht via de zgn. mehlerformule voor de klassieke hermiteveeltermen. In hoofdstuk 11 maken we een omtrekkende beweging: we definiëren een meerdimensionale fractionele clifford-fouriertransformatie, tonen aan dat deze, verrassend, samenvalt met de klassieke tensoriële fractionele fouriertransformatie en bekomen aldus de clifford-mehlerformule voor de clifford-hermiteveeltermen.

In hoofdstuk 12 bestuderen we de clifford-fouriertransformatie. Bij de studie van de fractionele fouriertransformatie bleek de voorstelling van de fourierintegraal als een exponentiële operator met scalaire differentiaaloperatorkern $\mathcal{H}$ van wezenlijk belang. Om een waarlijk cliffordkarakter aan een fouriertransformatie te koppelen zochten we naar een vervanger van deze kern $\mathcal{H}$, in de vorm van een cliffordalgebrawaardige operator. Daartoe wordt die operator $\mathcal{H}$ gesplitst in twee differentiaaloperatoren van de tweede orde, wat leidt tot een paar fouriertransformaties $(\mathcal{F}_{\mathcal{H}^+}, \mathcal{F}_{\mathcal{H}^-})$, waarvan het harmonisch gemiddelde precies de standaard fouriertransformatie is, en die kunnen worden beschouwd als een verfijning van de traditionele fouriertransformatie, analoog aan de verfijning van het begrip "harmoniciteit" door "monogeniteit". Het volgende doel was dit paar clifford-fouriertransformaties in integraalgedaante te brengen met een gesloten vorm voor de clifford-fourierkern. Daarin zijn we, voor algemene dimensie, niet geslaagd. Wel voor dimensie twee, zodat in dit speciale geval verdere eigenschappen van de clifford-fouriertransformaties konden worden aangetoond.

Het laatste onderwerp dat aan bod komt in het tweede deel, is de gabor- en hermitefilters. Beide worden gebruikt bij de modellering van de receptorprofielen in het menselijk visueel systeem; daarom worden deze filters vaak "eerste-zichtfilters" genoemd. De meerdimensionale filters die we kunnen construeren met gebruik van cliffordanalyse, doen beroep op enerzijds de clifford-hermiteveeltermen van Sommen en anderzijds op de tweedimensionale clifford-fourierintegraalkern van het vorig hoofdstuk.

Deel III: Metriekafhankelijke cliffordanalyse en anisotrope clifford-hermitewavelets

Terwijl het eerste en het tweede deel van deze doctoraatsscriptie is ontwikkeld binnen het kader van de orthogonale cliffordanalyse, is het derde deel gewijd aan een nieuwsoortige cliffordanalyse, waarbij de basis van de onderliggende kwadratische ruimte niet langer orthogonaal is. De hele theorie wordt daarmee "metriekafhankelijk" of "anisotroop". De doelstelling van de ontwikkeling van deze metriekafhankelijke cliffordanalyse is het aanpassingsvermogen van het coördinatensysteem aan bepaalde preferentiële, niet noodzakelijk orthogonale, richtingen in de signalen of weefselstructuren die met wavelets worden geanalyseerd. In deze nieuwe context hebben we daartoe anisotrope clifford-hermitewavelets geconstrueerd.

Hoofdstuk 2: Cliffordalgebra en orthogonale cliffordanalyse: basisbegrippen

Cliffordanalyse is uitgegroeid tot een zelfstandige discipline binnen de klassieke analyse. Zij vertoont raakvlakken met harmonische analyse, maar is ook complementair eraan, en biedt een alternatief voor de theorie van functies van meer complexe variabelen. Aanhangers ervan bestempelen cliffordanalyse graag als

de meest directe, elegante en krachtige veralgemening van de complexe analyse in het vlak.
Is $(e_1, \ldots, e_m)$ een orthonormale basis voor de euclidische ruimte $\mathbb{R}^m$ dan wordt de zgn. *diracoperator* gegeven door $\sum_{j=1}^{m} e_j \partial_{x_j}$. Deze operator staat centraal in deze meerdimensionale functietheorie. De nuloplossingen ervan worden *monogene* (sommigen gebruiken zelfs het woord *holomorfe*) functies genoemd. De diracoperator is invariant onder de groep SO(m); daarom spreken we van *orthogonale* cliffordanalyse. Een aanzienlijk aantal boeken, artikels en congresbijdragen - vermelden we [15, 55, 70, 71, 72, 93, 99, 103] - hebben deze orthogonale cliffordanalyse gestalte gegeven en de toepassingsmogelijkheden in de verf gezet. Dit inleidend hoofdstuk 2 is in deze scriptie opgenomen om de niet-gespecialiseerde lezer te laten kennismaken met de basisbegrippen uit cliffordalgebra en cliffordanalyse. Ook enkele vaak gebruikte resultaten i.v.m. de fouriertransformatie en sferische harmonieken worden aangehaald.

Hoofdstuk 3: Klassieke continue wavelettransformatie

Ook dit hoofdstuk 3 is inleidend. De ééndimensionale continue wavelettransformatie wordt, nogal gedetailleerd, erin uiteengezet, zodat onze resultaten over meerdimensionale continue wavelettransformatietheorie eraan kunnen worden getoetst.

Hoofdstuk 4: Isotrope continue clifford-wavelettransformatie: algemene theorie

Het is pas na de studie van de clifford-hermite CWT ingevoerd door Brackx en Sommen in [32, 33] en na een reeks van nieuwsoortige clifford-CWTs te hebben geconstrueerd, dat we ons hebben gerealiseerd dat er een algemeen theoretisch kader binnen de orthogonale cliffordanalyse kon worden opgesteld, waarin deze specifieke CCWTs konden worden ingepast. Deze algemene theorie is het onderwerp van dit hoofdstuk. Daarnaast presenteren we ook een methodologie voor de constructie van clifford-waveletfuncties. Zowel deze algemene theorie als de blauwdruk voor waveletconstructie, genieten ten volle van de intrinsieke eigenschappen van en de mogelijkheden geboden door cliffordanalyse. Eén van die belangrijke karakteristieken is het waarlijk meerdimensionaal karakter waarbij alle dimensies samen in acht worden genomen en dit in tegenstelling tot een vaak voorkomende tensoriële benadering met tensorproducten van ééndimensionale fenomenen.

Het belangrijkste begrip in dit hoofdstuk is dat van *moederwavelet*; in deze definitie schuilt namelijk de zgn. *admissibiliteitsvoorwaarde*, die een voldoende voorwaarde is waaraan kandidaat-moederwavelets moeten voldoen.

Definitie 4.1

Een cliffordalgebrawaardige functie $\psi \in L_1(\mathbb{R}^m) \cap L_2(\mathbb{R}^m)$ is een moederwavelet in $\mathbb{R}^m$ als zij in de frequentieruimte voldoet aan de volgende twee voorwaarden:

1. $\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger$ *is reëelwaardig;*

2.

$$C_\psi = (2\pi)^m \int_{\mathbb{R}^m} \frac{\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger}{|\underline{\xi}|^m}\ dV(\underline{\xi}) < \infty \quad .$$

Hierbij is $\mathcal{F}$ de notatie voor de fouriertransformatie en stelt $\lambda^\dagger$ de hermitisch toegevoegde voor van het cliffordgetal λ. De constante C_ψ heet de *admissibiliteitsconstante*.

In het speciaal geval dat de moederwavelet ψ invariant is onder de actie van de groep Spin(m), die een dubbele bedekking van SO(m) vormt, dit is

$$s\ \psi(\overline{s}\underline{x}s)\ \overline{s} = \psi(\underline{x}) \quad , \quad s \in \text{Spin}(m) \quad ,$$

wordt de bovenstaande conditie 1 vervangen door

1bis. $\mathcal{F}[\psi](\underline{\xi})\ \left(\mathcal{F}[\psi](\underline{\xi})\right)^\dagger$ *is radiaal-symmetrisch, d.w.z. enkel afhankelijk van* $|\underline{\xi}|$.

In dit hoofdstuk ontwerpen we ook de methodologie voor de constructie van clifford-wavelets, die dan in de volgende hoofdstukken zal worden toegepast.

Hoofdstuk 5: Clifford-hermiteveeltermen en de clifford-hermite CWT

In dit hoofdstuk construeren we de bi-axiale clifford-hermitepolynomen en de eraan geassocieerde CCWT. Het is een veralgemening van de clifford-hermite CWT van Brackx en Sommen; voor de volledigheid wordt eerst kort deze theorie uiteengezet.

Hoofdstuk 6: Clifford-gegenbauerveeltermen en de clifford-gegenbauer CWT

In dit hoofdstuk passen we de algemene theorie, ontwikkeld in hoofdstuk 4, toe vertrekkende van een cliffordveralgemening van de traditionele gegenbauer-gewichtsfunctie.

Hoofdstuk 7: Clifford-laguerreveeltermen en de clifford-laguerre CWT

Weer binnen het algemeen kader worden in dit hoofdstuk de zgn. clifford-laguerreveeltermen geconstrueerd die als basis dienen voor de zgn. clifford-laguerre CWT.

Hoofdstuk 8: De half CCWT: algemene theorie

De voorwaarden opgelegd aan de moederwavelet in het orthogonaal cliffordanalysekader (zie hoofdstuk 4) zijn voldoende voorwaarden. Het is mogelijk een CCWT op te stellen als de kandidaat-moederwavelets aan meer algemene voorwaarden voldoen. Daartegenover staat dat moet worden beroep gedaan op de orthogonale decompositie van de ruimte $L_2(\mathbb{R}^m)$ in termen van de hardyruimte $H^2(\mathbb{R}^m)$ en haar orthogonaal complement. Op deze wijze wordt een mooi verband gelegd tussen de CCWT-theorie en de theorie van clifford-hardyruimten. Deze laatste kunnen als volgt worden geïntroduceerd. Neem $f \in L_2(\mathbb{R}^m)$ en beschouw haar cauchyintegraal

$$\mathcal{C}[f](x_0, \underline{x}) = E(x_0, \ . \ ) * f(\ . \ )(\underline{x}) = \int_{\mathbb{R}^m} E(x_0, \underline{x} - \underline{y}) \ f(\underline{y}) \ dV(\underline{y})$$

in de halfruimten

$$\mathbb{R}^{m+1}_{\pm} = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1} \ : \ x_0 \gtrless 0\} \ .$$

Hierbij stelt

$$E(x_0, \underline{x}) = \frac{1}{A_{m+1}} \frac{x_0 - \underline{x}}{|x_0 + \underline{x}|^{m+1}}$$

de fundamentele oplossing van de cauchy-riemannoperator in $\mathbb{R}^{m+1}$ voor. De notatie A_{m+1} staat voor de oppervlakte van de eenheidssfeer S^m in $\mathbb{R}^{m+1}$.

De cauchyintegraal is een isomorfisme tussen $L_2(\mathbb{R}^m)$ en de hardyruimten $H^2(\mathbb{R}^{m+1}_\pm)$, gedefinieerd door

$$H^2(\mathbb{R}^{m+1}_\pm) = \Big\{ F(x_0,\underline{x}) \text{ monogeen in } \mathbb{R}^{m+1}_\pm \text{ zodanig dat } \sup_{x_0 \gtrless 0} \int_{\mathbb{R}^m} |F(x_0,\underline{x})|^2 \, dV(\underline{x}) < +\infty \Big\} \quad .$$

De niet-tangentiële randwaarden voor $x_0 \to 0\pm$ van de cauchyintegraal worden gegeven door

$$\mathcal{C}^\pm[f](\underline{x}) := \lim_{x_0 \to 0\pm} \mathcal{C}[f](x_0,\underline{x}) = \pm\frac{1}{2} f(\underline{x}) + \frac{1}{2} H[f](\underline{x}) \quad .$$

Hierbij stelt $H[f]$ de clifford-hilbertgetransformeerde van f voor, gegeven door

$$H[f](\underline{x}) = -\frac{2}{A_{m+1}} \, \mathrm{Pv}\frac{\underline{x}}{r^{m+1}} * f \quad .$$

De hardyruimte $H^2(\mathbb{R}^m)$ wordt dan gedefinieerd als de afsluiting in $L_2(\mathbb{R}^m)$ van de ruimte van de niet-tangentiële randwaarden voor $x_0 \to 0+$ van de functies in $H^2(\mathbb{R}^{m+1}_+)$. Deze hardyruimte $H^2(\mathbb{R}^m)$ is een gesloten deelruimte van $L_2(\mathbb{R}^m)$, wat aanleiding geeft tot de orthogonale directe som ontbinding

$$L_2(\mathbb{R}^m) = H^2(\mathbb{R}^m) \oplus H^2(\mathbb{R}^m)^\perp \quad .$$

De projecties van $L_2(\mathbb{R}^m)$ op deze orthogonale deelruimten heten de szegöprojecties.

Hoofdstuk 9: Clifford-jacobiveeltermen en de half clifford-jacobi CWT

De theorie van de half CCWT, uiteengezet in hoofdstuk 8, wordt hier toegepast om clifford-waveletfuncties te construeren op basis van de cliffordveralgemening van de jacobiveeltermen.

Hoofdstuk 10: Cliffordalgebrawaardige orthogonale polynomen

In de voorgaande hoofdstukken werd systematisch gebruik gemaakt van het cauchy-kowalewskaia-extensieprincipe om meerdimensionale orthogonale veeltermen te construeren. Dit hoofdstuk is gewijd aan een andere methode om hetzij in de hele euclidische ruimte, hetzij in de open eenheidsbal orthogonale veeltermen te construeren. Terzelfdertijd wordt de beschouwde klasse van gewichtsfuncties, die in de vorige hoofdstukken reëelwaardig waren, uitgebreid tot specifieke cliffordalgebrawaardige gewichtsfuncties. De methode bestaat erin de gewenste meerdimensionale orthogonaliteitsrelatie om te vormen tot een orthogonaliteitsbetrekking op de reële as. Dat dit mogelijk is, is te danken aan het bestaan van de zgn. clifford-heavisidefuncties. Deze wederzijds-annihilerende idempotente functies zijn een zeer typisch fenomeen uit de cliffordanalyse en kunnen, zoals de naam aangeeft, beschouwd worden als meerdimensionale veralgemeningen van de heavisidefunctie op de reële as.
We zijn erin geslaagd op deze wijze drie soorten orthogonale veeltermen te construeren in de open eenheidsbal: de clifford-jacobiveeltermen, de verschoven clifford-jacobiveeltermen en de clifford-gegenbauerveeltermen. De eerste twee soorten zijn nieuw. De derde soort valt, op een constante na, samen met de clifford-gegenbauerveeltermen die op een heel andere wijze zijn ingevoerd in [47].
Onze methode heeft ook geleid tot drie soorten orthogonale veeltermen in de hele ruimte: de clifford-laguerreveeltermen van het tweede type, de half-clifford-hermiteveeltermen en de clifford-hermiteveeltermen van het tweede type. De eerste twee soorten zijn nieuw. De derde soort valt, op een constante na, samen met de radiale clifford-hermiteveeltermen van Sommen.

Hoofdstuk 11: Fractionele fouriertransformatie in de clifforanalysecontext

De fractionele fouriertransformatie op de reële as kan als een integraaltransformatie worden geschreven dankzij de zgn. mehlerformule, een identiteit waaraan de klassieke hermiteveeltermen voldoen. We zijn, nog niet, erin geslaagd een rechtstreeks bewijs te leveren voor een meerdimensionaal analogon waarin de veralgemeende clifford-hermiteveeltermen optreden. Dit hoofdstuk levert, op eerder verrassende wijze, wel een dergelijke clifford-mehlerformule op, maar dan onrechtstreeks. We voeren de meerdimensionale fractionele fouriertrans-

formatie in binnen het cliffordanalysekader. Het blijkt dat ze samenvalt met de klassieke tensoriële meerdimensionale fractionele fouriertransformatie, waaruit dan de clifford-mehlerformule volgt. Deze finale identiteit ziet eruit als volgt:
Stelling 11.1 *De veralgemeende clifford-hermiteveeltermen voldoen aan de mehleridentiteit:*

$$\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} \frac{\exp(-i(s+k)\alpha)}{\gamma_{s,k}}\; H_{s,k}(\sqrt{2}\underline{\xi}) \left(\sum_{j=1}^{\dim(M_\ell^+(k))} P_k^{(j)}(\sqrt{2}\underline{\xi}) \big(P_k^{(j)}(\sqrt{2}\underline{x})\big)^\dagger \right)$$

$$\begin{aligned} \big(H_{s,k}(\sqrt{2}\underline{x})\big)^\dagger \quad = \quad & \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-\exp(-2i\alpha)}} \right)^m \\ & \exp\left(\frac{2 < \underline{x}, \underline{\xi} > \exp(-i\alpha) - (|\underline{x}|^2 + |\underline{\xi}|^2)\exp(-2i\alpha)}{1-\exp(-2i\alpha)} \right) \quad . \end{aligned}$$

Hierin stellen $H_{s,k}(\sqrt{2}\underline{x})$ deze veralgemeende clifford-hermiteveeltermen voor; $P_k^{(j)}$ is een monogene homogene veelterm van de k^{de} graad. De verzameling

$$\left\{ H_{s,k}(\sqrt{2}\underline{x})\; P_k^{(j)}(\sqrt{2}\underline{x})\; \exp\left(-\frac{|\underline{x}|^2}{2} \right) \right\}$$

vormt een basis voor de hilbertruimte $L_2(\mathbb{R}^m)$. De parameter α is afkomstig uit de fractionele fouriertransformatie. Voor $\alpha = \frac{\pi}{2}$ herleidt deze zich tot de standaard fouriertransformatie.

Hoofdstuk 12: De clifford-fouriertransformatie

Meestal wordt in cliffordanalyse gebruik gemaakt van de standaard fouriertransformatie met een complexe fourierkern. Recentelijk is vnl. met het oog op toepassingen in beeldverwerking, een aantal veralgemeningen van de fouriertransformatie ingevoerd waarbij gebruik wordt gemaakt van een cliffordalgebrawaardige fourierkern. In dit hoofdstuk geven we een overzicht van al deze integraaltransformaties in de cliffordalgebracontext.
Onze eigen clifford-fouriertransformatie wordt uitvoerig behandeld. Speciale aandacht gaat daarbij naar het tweedimensionale geval. Enkel in dit geval zijn we, tot nu toe, erin geslaagd de integraalkern in een gesloten gedaante te brengen. Deze ziet er als volgt uit:

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{\xi}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\big(\pm(\underline{\xi} \wedge \underline{x})\big)\; f(\underline{x})\; dV(\underline{x}) \quad .$$

De clifford-fourierkern $\exp\big(\pm(\underline{\xi} \wedge \underline{x})\big)$ is functie van de bivector $\underline{\xi} \wedge \underline{x}$ die ontstaat als het uitwendig product in de cliffordalgebra van de vectoren $\underline{x}$ en $\underline{\xi}$. Op te merken valt dat de clifford-fouriertransformatie van Ebling en Scheuermann in bovenstaande gedaante kan worden gebracht; in [62] wordt getoond hoe deze transformatie kan worden toegepast voor de analyse van vectoriële patronen in het frequentiedomein.

Hoofdstuk 13: Clifford-filters voor "het eerste zicht"

Het gabormodel wordt alom gebruikt om de receptorprofielen in het menselijk visueel systeem te modelleren. Een ander, minder gebruikt, model daarvoor is het hermitemodel, dat gebaseerd is op de analysefilters van de hermitetransformatie; het biedt enkele voordelen zoals het orthogonaal karakter van de basisfuncties en klaarblijkelijk is er een betere overeenkomst met de experimentele data.
In dit hoofdstuk ontwikkelen we de filterfuncties van de klassieke hermitetransformatie in termen van de veralgemeende clifford-hermiteveeltermen. We construeren ook een nieuwe meerdimensionale hermitetransformatie binnen het cliffordanalysekader, die we vergelijken met de clifford-hermite CWT. M.b.v. de clifford-fouriertransformatie uit het voorgaand hoofdstuk, voeren we tweedimensionale clifford-gaborfilters in, onderzoeken we in welke mate ze gelokaliseerd zijn in het ruimte- en het frequentiedomein en leggen we hun verband met andere types gaborfilters bloot.

Hoofdstuk 14: Metriekafhankelijke cliffordanalyse

Metriekafhankelijke cliffordanalyse, ook wel anisotrope of metrodynamische cliffordanalyse genoemd, is ontwikkeld met het doel het traditioneel cartesiaans assenstelsel te richten volgens bepaalde, niet noodzakelijk loodrechte, voorkeursrichtingen.
Daartoe construeren we de cliffordalgebra over $\mathbb{R}^m$ op basis van een symmetrische bilineaire vorm waarvoor het scalair product van een vector $\underline{x} = \sum_{j=1}^{m} e_j x^j$ met zichzelf, de vorm krijgt:

$$< \underline{x}, \underline{x} > = \sum_{j=1}^{m} \sum_{k=1}^{m} g_{jk} x^j x^k \quad .$$

De tensor g_{jk} noemen we de metrische tensor van deze cliffordalgebra; we onderstellen dat deze metrische tensor reëel, symmetrisch en positief definiet is. Het idee zelf is niet nieuw; lokale metrische tensoren werden reeds beschouwd in [49], [70] en [83] in het kader van cliffordanalyse op variëteiten. Een metrische tensor in drie dimensies waarbij de derde dimensie onaangeroerd blijft, werd reeds gebruikt in [67] voor tweedimensionale beeld- en weefselstructuuranalyse. Dit hoofdstuk biedt echter de volledige uitwerking van cliffordanalyse in een metriekafhankelijk kader. Nieuwe mogelijkheden voor toepassing worden aldus gecreëerd, zoals de metriekafhankelijke meerdimensionale hilberttransformatie [14] en de anisotrope clifford-hermitewavelets van het hoofdstuk 16.

Hoofdstuk 15: Metriekinvariante integralen

Met het oog op integratie over hyperoppervlakken, worden in dit hoofdstuk specifieke clifford-differentiaalvormen in het metriekafhankelijke cliffordanalysekader bestudeerd. Aanbevolen lectuur voor clifford-differentiaalvormen in het algemeen is bvb. [63]. Uitwendig differentiëren en contractie van differentiaalvormen komt aan bod en, uiteraard, stellen we de ook hier zeer fundamentele stokesstelling op. Speciale aandacht gaat uit naar de leray- en sigma-differentiaalvorm vanwege hun cruciale rol bij het opstellen van de orthogonaliteitsrelaties van sferisch monogenen op de eenheidssfeer.

Hoofdstuk 16: Anisotrope clifford-hermitewavelets

Zoals reeds aangekondigd, worden in dit hoofdstuk clifford-hermitewavelets geconstrueerd in het kader van metriekafhankelijke cliffordanalyse. Deze zgn. anisotrope clifford-hermitewavelets bieden het voordeel te kunnen worden aangepast aan voorkeursrichtingen, niet noodzakelijk onderling loodrecht, in de te analyseren signalen of weefselstructuren. I.h.b. zijn ze geschikt voor analyse van meerdimensionale signalen of texturen die een min of meer constant kenmerk vertonen in bepaalde richtingen.

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