Improved Polynomial Chaos Discretization Schemes to Integrate Interconnects into Design Environments

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***Abstract*—Recently, an efficient stochastic modeling method for interconnects with inherent variability in their physical parameters was proposed, based on applying the so-called polynomial chaos (PC) approach in conjunction with a Stochastic Galerkin Method (SGM) onto telegrapher's equations. Although this approach was already very successful from a numerical point of view, the novel technique could not be conveniently integrated into SPICE-like solvers, limiting the applicability of the method. In this letter, the PC-SGM scheme for telegrapher's equations is revisited, pinpointing the origin of this inconvenience and immediately allowing to mitigate the issue. By adapting the traditional discretization of the stochastic telegrapher's equations approach, an augmented, yet deterministic, set of ordinary differential equations is obtained that turns out to be of the same type as the telegrapher's equations, and hence, the physical property of reciprocity is preserved. Consequently, it can be directly and more efficiently handled using SPICE-like solvers, which usually assume matrix symmetries. As an application example, the variability analysis of a state-of-the-art on-chip line for millimeter-wave applications is performed in a SPICE solver.**

***Index Terms*—Discretization scheme, polynomial chaos, reciprocity, stochastic analysis, transmission lines, uncertainty.**

I. Introduction

In recent years, great attention has been drawn to the availability of models for the efficient inclusion of inherent parameter variability in the early-stage simulation of microwave and millimeter-wave electronic circuits and interconnections. As the technology is pushing towards further miniaturization, the impact of manufacturing process tolerances on high-frequency designs is becoming increasingly critical. However, the traditional Monte Carlo analysis is often computationally prohibitive due to the complex nature of the structures under investigation. Therefore, a novel methodology to efficiently include variations of on-chip interconnects into the standard framework of transmission-line theory was developed [1]. The approach is based on the so-called polynomial chaos (PC) expansion in conjunction with the Stochastic Galerkin Method (SGM) [2]. The *stochastic* problem was discretized by first projecting it onto a finite set of suitable multivariate orthogonal polynomials and next testing the result with the same set. This Galerkin discretization scheme yielded a new set of *deterministic* ordinary differential equations (ODE), albeit of a larger size. However, it was demonstrated to provide a considerable speed-up with respect to Monte Carlo, e.g., for design purposes.

Although the aforementioned method turned out to be very successful from a numerical point of view, its applicability was limited. In fact, an ad hoc MATLAB implementation needed to be developed for every different circuit topology to be analyzed. For reasons explained below, the application of the standard PC-SGM hindered the convenient integration of the technique into standard design environments, except for lossless and dispersion-free lines [3]. Hence, so far, a SPICE-compatible PC-based variability analysis of on-chip lines, exhibiting dispersion and slow-wave effects, has never been presented.

In this letter, the PC-SGM framework is revisited from an altogether different point of view. By focusing on the discretization scheme, the nature of the aforementioned inconvenience is discovered and mitigated. For the first time in literature, the practical implications of the traditional PC-SGM-discretization are exposed, and it is shown that better choices are available, allowing a more efficient implementation in commercial circuit-analysis tools as well as the inclusion of losses and dispersion. As an application example, the stochastic analysis of an on-chip differential line for millimeter-wave applications is performed with HSPICE.

II. Discretization of Stochastic Telegrapher's Equations

Variations in the interconnect properties, due for instance to uncertainties in the geometry and/or material parameters, reflect into a variability of the p.u.l. parameters. Therefore, the frequency-domain telegrapher's equations [4] of a multiconductor transmission line (MTL) consisting of N signal conductors and a reference conductor can be written as

$$\frac{d}{dz}\mathbf{V}(z,s,\boldsymbol{\xi}) = -\overline{\mathbf{Z}}(s,\boldsymbol{\xi})\cdot\mathbf{I}(z,s,\boldsymbol{\xi}), \tag{1a}$$

$$\frac{d}{dz}\mathbf{I}(z,s,\boldsymbol{\xi}) = -\overline{\mathbf{Y}}(s,\boldsymbol{\xi})\cdot\mathbf{V}(z,s,\boldsymbol{\xi}), \tag{1b}$$

where the N-vectors $\mathbf{V}$ and $\mathbf{I}$ contain the voltages and currents along the line and with $\overline{\mathbf{Z}}$ and $\overline{\mathbf{Y}}$ the $(N\times N)$-p.u.l. impedance and admittance matrices, respectively. The vector $\boldsymbol{\xi}$ encompasses d random variables (RVs), describing the uncertainties, and s is the Laplace variable.

The original PC-SGM approach [2], conceived to deal with randomness in differential equations such as (1), is now first

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briefly revisited from a computational point of view, displaying the nature of the above mentioned limitations. Thereto, we start by constructing a finite-dimensional Hilbert space $\mathcal{V}$ over the real numbers $\mathbb{R}$ having domain $\Omega \subset \mathbb{R}^d$, containing the d-vectors of RVs $\boldsymbol{\xi}$. We equip $\mathcal{V}$ with an inner product:

$$<f(\boldsymbol{\xi}), g(\boldsymbol{\xi})> = \int_{\Omega} f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) W(\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{2}$$

where the weighting function $W(\boldsymbol{\xi})$ coincides with the multivariate probability density function of $\boldsymbol{\xi}$ in the domain Ω. Consider further the set of linearly independent basis functions $\{\phi_k(\boldsymbol{\xi})\}_{k=0}^{P}$ that span $\mathcal{V}$, where $P+1 = \dim(\mathcal{V})$. All stochastic quantities of (1) are now projected onto this basis:

$$\mathcal{X}(\boldsymbol{\xi}) \approx \sum_{k=0}^{P} \mathcal{X}_k \phi_k(\boldsymbol{\xi}), \tag{3}$$

where $\mathcal{X}(\boldsymbol{\xi})$ stands for $\mathbf{V}(z,s,\boldsymbol{\xi})$, $\mathbf{I}(z,s,\boldsymbol{\xi})$, $\overline{\mathbf{Z}}(s,\boldsymbol{\xi})$, and $\overline{\mathbf{Y}}(s,\boldsymbol{\xi})$, and with corresponding expansion coefficients $\mathcal{X}_k$, $k = 0,\ldots,P$. Discretization of the stochastic telegrapher's equations (1), i.e. inserting (3) into (1), and testing the result using the inner product (2) and a set of $P+1$ testing functions $\{\psi_k(\boldsymbol{\xi})\}_{k=0}^{P}$, yields:

$$\overline{\mathbf{D}} \cdot \frac{d}{dz}\tilde{\mathbf{V}}(z,s) = -\tilde{\overline{\mathbf{Z}}}(s) \cdot \tilde{\mathbf{I}}(z,s), \tag{4a}$$

$$\overline{\mathbf{D}} \cdot \frac{d}{dz}\tilde{\mathbf{I}}(z,s) = -\tilde{\overline{\mathbf{Y}}}(s) \cdot \tilde{\mathbf{V}}(z,s). \tag{4b}$$

In (4), the new $N(P+1)$-vectors $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{I}}$ contain the sought-for coefficients for the voltage and current variables, and the $N(P+1) \times N(P+1)$-matrices $\tilde{\overline{\mathbf{Z}}}$ and $\tilde{\overline{\mathbf{Y}}}$ consist of $(P+1) \times (P+1)$ block matrices of size $N \times N$, each block given by: $\tilde{\overline{\mathbf{Z}}}_{ml} = \sum_{k=0}^{P} \overline{\mathbf{Z}}_k < \phi_k(\boldsymbol{\xi})\phi_l(\boldsymbol{\xi}), \psi_m(\boldsymbol{\xi}) >$ $(m,l = 0,\ldots,P)$, and similarly for $\tilde{\overline{\mathbf{Y}}}$. The matrix $\overline{\mathbf{D}}$ consists of $(P+1) \times (P+1)$ blocks of size $N \times N$, where each block is diagonal with N entries $<\phi_k(\boldsymbol{\xi}), \psi_m(\boldsymbol{\xi})>$. The matrix $\overline{\mathbf{D}}$ is of course symmetrical. In short, (4) constitutes a new pair of coupled ordinary differential equations (ODE), which is $P+1$ times larger than the original one (1), but the dependence on $\boldsymbol{\xi}$ has vanished.

In the standard PC-SGM-approach, as always adopted in (electrical) engineering applications (see e.g. [5]), the Wiener-Askey scheme is used, meaning that the basis functions are chosen to be polynomials that are orthogonal w.r.t. the probability density function $W(\boldsymbol{\xi})$, i.e. $<\phi_k(\boldsymbol{\xi}), \phi_l(\boldsymbol{\xi})> = <\phi_k(\boldsymbol{\xi}), \phi_k(\boldsymbol{\xi})>\, \delta_{kl}$, where the Kronecker delta was used. For many engineering problems, this guarantees rapid convergence of the series (3). As already anticipated by its name, the SGM leverages a set of testing functions that are identical to the basis functions, i.e. $\psi_k(\boldsymbol{\xi}) \equiv \phi_k(\boldsymbol{\xi})$. At this point it is important to mention that the matrices $\tilde{\overline{\mathbf{Z}}}$ and $\tilde{\overline{\mathbf{Y}}}$ so obtained are also *symmetrical* matrices, as is usually the case with Galerkin testing schemes in computational problems. Additionally, the inner product matrix $\overline{\mathbf{D}}$ becomes a diagonal *Gramian* matrix with only $N(P+1)$ non-zero entries. The augmented set of ODEs can be solved with great accuracy using standard techniques in, e.g., MATLAB, as explained in [1]. However, the question now arises whether:

1) the set of ODEs (4) can be interpreted again as a set of *augmented telegrapher's equations*;
2) and consequently, implemented as such by integration into standard SPICE-like circuit solvers.

Unfortunately, using the traditional PC-SGM discretization scheme and despite the symmetry of the matrices $\tilde{\overline{\mathbf{Z}}}$ and $\tilde{\overline{\mathbf{Y}}}$, this turns out not to be entirely the case, due to the presence of the (diagonal) matrix $\overline{\mathbf{D}}$. This can be understood as follows. When rewriting (4) using $\tilde{\overline{\mathbf{Z}}}_a = \overline{\mathbf{D}}^{-1} \cdot \tilde{\overline{\mathbf{Z}}}$ and $\tilde{\overline{\mathbf{Y}}}_a = \overline{\mathbf{D}}^{-1} \cdot \tilde{\overline{\mathbf{Y}}}$, then indeed a pair of telegrapher's equations is obtained, but the new *augmented* p.u.l. impedance and admittance matrices $\tilde{\overline{\mathbf{Z}}}_a$ and $\tilde{\overline{\mathbf{Y}}}_a$ are no longer symmetrical, leading to a non-reciprocal system [6]. Whereas the lack of reciprocity has never been a problem in the numerical solution of (4) before, it does impede the integration into standard SPICE-like solvers, which usually do not support non-reciprocal MTLs.

In order to mitigate this issue, a better discretization scheme has to be chosen, rendering symmetrical matrices $\tilde{\overline{\mathbf{Z}}}_a$ and $\tilde{\overline{\mathbf{Y}}}_a$. An, in hindsight, straightforward choice is to retain the basis functions $\phi_k(\boldsymbol{\xi})$, $k = 0,\ldots,P$, and changing the testing functions as follows: $\psi_k(\boldsymbol{\xi}) = \phi_k(\boldsymbol{\xi}) / < \phi_k(\boldsymbol{\xi}), \phi_k(\boldsymbol{\xi}) >$ $(k = 0,\ldots,P)$, yielding an inner product matrix $\overline{\mathbf{D}}$ equal to the identity matrix. It is worth noting that this scheme is equivalent to choosing a set of identical basis and testing functions $\phi'_k(\boldsymbol{\xi}) = \psi'_k(\boldsymbol{\xi}) = \phi_k(\boldsymbol{\xi}) / \sqrt{<\phi_k(\boldsymbol{\xi}), \phi_k(\boldsymbol{\xi})>}$, i.e. using only *orthonormal* polynomials. Other choices of basis and testing functions can maybe be found that would also yield symmetrical augmented p.u.l. matrices. However, the proposed choice of orthonormal polynomials comes from the fact that, according to the Wiener-Askey scheme, a good convergence of the series (3) can still be expected.

We like to stress here again that the original PC-SGM discretization, adopted in all engineering problems so far, is flawless and sound, leading to very accurate and efficient numerical computation schemes. However, the discretization scheme needs to be altered to conveniently allow the PC-SGM technique to be integrated into commercial, SPICE-like software. Thus, the above presented finding has important consequences in the circuit design, and by extension maybe also in other engineering domains where PC-modeling is used.

III. Practical Implications: Application Example

As a practical advantage of the modified formulation introduced in this letter, the new p.u.l. matrices can be directly supplied as an input to MTL elements available in commercial solvers, such as the W-element in HSPICE or Agilent's Advanced Design System (ADS). In fact, this inherently assumes symmetric matrices, as it is for physical reciprocal lines. Since the W-element can manage lossy and dispersive lines, it is now possible to conveniently model the effect of stochastic frequency-dependent line parameters within the design environment.

As a validation of the proposed important improvement of the PC-SGM-formulation, we analyze the response of the differential transmission line depicted in Fig. 1. This is a differential on-chip line exhibiting strong dispersive behavior

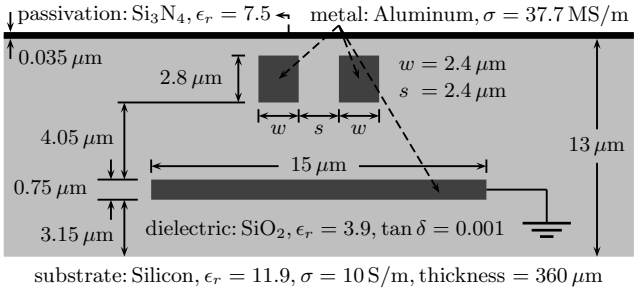

Fig. 1. Cross-section of the differential line considered for the application.

and also slow-wave effects, due to the presence of the semiconductor. The geometric dimensions and material properties are shown in the figure and a length of 1 mm is assumed. This interconnect provides connectivity among components in integrated millimeter-wave applications [7]. The line is excited at the near ends by low-impedance voltage sources, having an internal impedance of 1 Ω, and terminated by 1 pF capacitive loads. In order to model the variability arising from the fabrication process (e.g., etching and photolithography), the gap s between the lines and their widths w are considered as independent Gaussian variables with relative standard deviations of 8% and 10%, respectively.

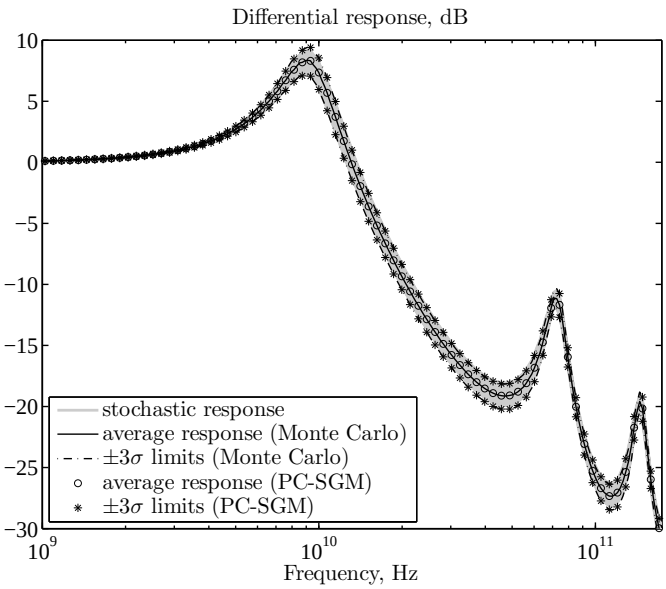

Fig. 2. Bode plot of the differential response.

Fig. 2 shows the Bode plot of the differential response. The black lines represent the average response and the $\pm 3\sigma$ bounds, estimated after 10000 Monte Carlo simulations. The markers indicate the same statistical information obtained with the PC-SGM technique. The good agreement between the two approaches can be appreciated. Finally, a subset of 100 random responses is plotted (gray area) to provide a qualitative idea of their fluctuation.

Stochastic functions such as probability density functions (PDFs) can be obtained via the PC-SGM as well. Fig. 3 compares the PDF of the response at the 9-GHz resonance, obtained with both Monte Carlo and the PC-SGM. The accuracy of the proposed implementation is established also for

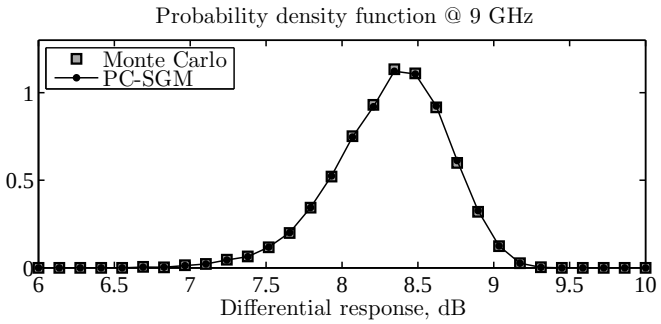

Fig. 3. Probability density functions of the differential response at 9 GHz.

this case. For this analysis up to 170 GHz, the dispersive p.u.l. parameters were supplied to HSPICE as frequency-dependent tabulated data for a W-element. The Monte Carlo circuit simulation required about 4 hours, while the simulation of the PC-SGM-augmented line took 21.3 s. An impressive speed-up of 700$\times$ is thus achieved. We stress here again that it is not possible to implement this example in SPICE with previously presented techniques.

IV. Conclusions

This letter presents a considerable improvement of the PC-SGM discretization scheme, which allows for the first time the stochastic simulation of state-of-the-art on-chip interconnects for microwave and millimeter-wave applications into commercial SPICE-like solvers. Thanks to a formal study of the discretization scheme, it was discovered that by normalization of basis and testing functions, augmented, but deterministic, reciprocal telegrapher's equations are obtained. This allows the augmented p.u.l. matrices to be supplied as standard input parameters to MTL models available in commercial software. The augmented problem can then be solved as a classical lossy and dispersive MTL. This in an important finding for the community active in the development of PC-based high-frequency design techniques. The advocated PC-SGM technique shows good agreement and a large speed-up with respect to the conventional Monte Carlo approach in the statistical assessment of a millimeter-wave on-chip differential line with uncertain gap and line width.

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