Relative equilibria of invariant Lagrangian systems on a Lie group

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Abstract

The reduction of a left-invariant Lagrangian system on a Lie group is usually approached from the viewpoint of the calculus of variations (see e.g. the standard reference [7]). In this paper, we will explain the reduction process in terms of a suitable choice of coordinates. We show in detail how in the presence of symmetry the Euler-Lagrange equations of a left-invariant Lagrangian reduce to the so-called Euler-Poincaré equations and give some illustrative examples. Next, we consider the case of a bi-invariant Lagrangian and we discuss briefly its relation to the inverse problem of the canonical connection on a Lie group [8]. In section 6, we give different characterizations of the notion of a geodesic vector (relative equilibrium) and extend a criterion about their stability (in Arnold's 'Riemannian' paper [1]) to arbitrary invariant Lagrangians. Along the way, we rediscover results that can also be found in the papers [6,11,10]. At the end of the paper we use one of the examples to test the stability criterium.

1 Basic machinery for Lie groups

Throughout the paper, G will be a connected Lie group and $\mathfrak{g}$ its Lie algebra. λ_g and ρ_g will denote left and right multiplication, respectively. Both maps can be extended to actions $T\lambda_g$ and $T\rho_g$ of G on TG. By left translating a basis $\{E_i\}$ of the Lie algebra, we obtain a left invariant basis $\{\hat{E}_i\}$ of $\mathcal{X}(G)$. Similarly, $\{\tilde{E}_i\}$ will denote the right-invariant basis of $\mathcal{X}(G)$ obtained via right translation. We will use the following convention: if C_{ij}^k are the structure

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constants of the Lie algebra, then $[\hat{E}_i, \hat{E}_j] = C_{ij}^k \hat{E}_k$ and $[\tilde{E}_i, \tilde{E}_j] = -C_{ij}^k \tilde{E}_k$ (This is e.g. the convention in [7]).

In the following, a vector v_g in T_gG will have coordinates (w^i) w.r.t. $\{\hat{E}_i\}$, i.e. $v_g = w^i \hat{E}_i(g)$. Each vector v_g can be left translated to an element $T\lambda_{g^{-1}} v_g$ in the Lie algebra and (w^i) are the coordinates of this element w.r.t. the basis $\{E_i\}$.

The coefficients $(A_j^i) \in C^\infty(G)$ will denote the relation between the two bases on $\mathcal{X}(G)$,

$$\hat{E}_i(g) = A_i^j(g) \tilde{E}_j(g), \tag{1}$$

and we will suppose that the two bases coincide at the identity e of the group, $\hat{E}_i(e) = \tilde{E}_i(e) = E_i \in T_eG \simeq \mathfrak{g}$, and thus $A_i^j(e) = \delta_i^j$. A more geometric interpretation of the matrix (A_j^i) is the following: let's right translate both sides of expression (1) by means of $T\rho_{g^{-1}}$. Then, taking the left-invariance of the vector field $\hat{E}_i$ into account, the left-hand side is in fact $T\rho_{g^{-1}} T\lambda_g E_i = Ad_g(E_i)$. On the other hand, due to the right invariance of the vector fields $\tilde{E}_j$, the right hand side is $A_i^j(g) E_j$. To conclude, $(A_i^j(g))$ are the components of the adjoint map Ad_g. We will use $(\bar{A}_i^j)$ for the inverse of (A_i^j).

The following property is true for any action of a connected Lie group on a manifold: a tensor field is invariant under an action if and only if its Lie derivative under any fundamental vector field vanishes. Here, in the case where the manifold is the Lie group, we will always assume that the action of interest is given by left multiplication. In that case, the fundamental vector fields are exactly the right-invariant vector fields, for which $\{\tilde{E}_i\}$ is a basis. A function f on G is left-invariant if and only if all $\tilde{E}_i(f) = 0$ and a vector field X on G is left-invariant if and only if all $[\tilde{E}_i, X] = 0$. In particular, for the left-invariant $\hat{E}_j$, $[\tilde{E}_i, \hat{E}_k] = 0$. This has some immediate consequences for the coefficients A_i^j, in view of the bracket relations in both bases:

$$\tilde{E}_j(A_i^k) + A_i^l C_{lj}^k = 0 \qquad \text{and} \qquad A_j^i A_l^k C_{ik}^m = A_n^m C_{jl}^n. \tag{2}$$

A (Riemannian) metric k on G is left-invariant if $\mathcal{L}_{\tilde{E}_i} k = 0$. So, if $k_{jl} = k(\hat{E}_j, \hat{E}_l)$ are its coefficients w.r.t. the left-invariant basis,

$$\begin{aligned} 0 &= (\mathcal{L}_{\tilde{E}_i} k)(\hat{E}_j, \hat{E}_l) = \tilde{E}_i(k(\hat{E}_j, \hat{E}_l)) - k([\tilde{E}_i, \hat{E}_j], \hat{E}_l) - k(\hat{E}_j, [\tilde{E}_i, \hat{E}_l]) \\ &= \tilde{E}_i(k_{jl}). \end{aligned}$$

From this, k_{jl} are constants and, as a consequence, left-invariant metrics can be interpreted as symmetric bilinear products on the Lie algebra.

In this paper, most of the objects of interest live in fact on the tangent manifold TG and therefore we will often use the infinitesimal generators of the induced action $T\lambda_g$ of G on TG. They are exactly the complete lifts $\{\tilde{E}_i^{\rm C}\}$ of the infinitesimal generators of the action λ_g of G on G. So, a function $F \in C^\infty(TG)$ is left-invariant if $\tilde{E}_i^{\rm C}(F) = 0$. Left-invariant functions are in 1 to 1 correspondence with functions on the Lie algebra: we can always look at the restriction of such a function to vectors in the fibre $T_eG \simeq \mathfrak{g}$. In the other direction, any function on the Lie algebra can be extended to a left-invariant function on the whole tangent manifold by claiming it to be constant along an orbit.

A vector field $Z = Z^j \hat{E}_j^{\rm C} + F^j \hat{E}_j^{\rm V} \in \mathcal{X}(TG)$ is left-invariant if and only if $[\tilde{E}_i^{\rm C}, Z] = 0$. That is, if $\tilde{E}_i^{\rm C}(Z^j) = 0$ and $\tilde{E}_i^{\rm C}(F^j) = 0$, so if all Z^j and F^j are invariant functions. In particular, for an invariant second order field $\Gamma = w^i \hat{E}_j^{\rm C} + F^j \hat{E}_j^{\rm V} \in \mathcal{X}(TG)$, we find that, next to $\tilde{E}_i^{\rm C}(w^j) = 0$, also $\tilde{E}_i^{\rm C}(F^j) = 0$. So, all coefficients F^j can be interpreted as functions on the Lie algebra and therefore Γ reduces to a vector field $F = F^i \dfrac{\partial}{\partial w^i}$ on the Lie algebra $\mathfrak{g}$. If $w(t)$ is an integral curve of F, then we can recover an base integral curve of Γ by integrating $g^{-1}(t)\dot{g}(t) = w(t)$.

2 The Euler-Poincaré equations

Let $L \in C^\infty(TG)$ be a left-invariant regular Lagrangian with Lagrangian second order vector field Γ. This vector field can be characterized by the equation

$$\mathcal{L}_\Gamma \theta_L - dL = 0, \qquad \text{where} \quad \theta_L = S(dL). \tag{3}$$

We will show that if L is left-invariant, then so is also Γ, and we will compute its reduced vector field F on $\mathfrak{g}$.

First, recall the following general statement concerning the Lagrangian field $\Gamma \in \mathcal{X}(TM)$ of a Lagrangian L on a manifold M. If Z is a vector field on TM which is such that $Z(L) = 0$, $[\Delta, Z] = 0$ and $\mathcal{L}_Z S = 0$, then Z must be a symmetry, that is $[Z, \Gamma] = 0$ (Δ is the Liouville vector field, S is the vertical endomorphism). In our case where $M = G$, left-invariance of the Lagrangian means that $\tilde{E}_i^{\rm C} L = 0$. It is not difficult to see that the vector fields $Z = \tilde{E}_i^{\rm C}$ also satisfy all the other conditions in the statement. For example, $\mathcal{L}_{\tilde{E}_i^{\rm C}} S(\hat{E}_j^{\rm C}) = [\tilde{E}_i^{\rm C}, S(\hat{E}_j^{\rm C})] - S([\tilde{E}_i^{\rm C}, \hat{E}_j^{\rm C}]) = [\tilde{E}_i, \hat{E}_j]^{\rm V} - S([\tilde{E}_i, \hat{E}_j]^{\rm C}) = 0$ and $\mathcal{L}_{\tilde{E}_i^{\rm C}} S(\hat{E}_j^{\rm V}) = -S([\tilde{E}_i^{\rm C}, \hat{E}_j^{\rm V}]) = -S([\tilde{E}_i, \hat{E}_j]^{\rm V}) = 0$. We can therefore conclude that also the Lagrangian field Γ is left-invariant, $[\tilde{E}_i^{\rm C}, \Gamma] = 0$.

We will apply (3) to all $\tilde{E}_i^{\mathrm{C}}$. Alternatively, we could equally use $\hat{E}_i^{\mathrm{C}}$. Indeed, with the help of (2), the difference between the complete and vertical lifts of elements in both bases is given by

$$\hat{E}_i^{\mathrm{C}} = A_i^j \tilde{E}_j^{\mathrm{C}} + w^k C_{ki}^j \hat{E}_j^{\mathrm{V}} \qquad \text{and} \qquad \hat{E}_i^{\mathrm{V}} = A_i^j \tilde{E}_j^{\mathrm{V}}. \tag{4}$$

Given that the left-hand side of (3) is a semi-basic form, it is clear from (4) that the application of (3) to elements of the basis $\{\hat{E}_i^{\mathrm{C}}\}$ will only add a factor A_i^j to the result of doing the same for the basis $\{\tilde{E}_j^{\mathrm{C}}\}$.

Because of the invariance of L and Γ,

$$0 = (\mathcal{L}_\Gamma \theta_L - dL)(\tilde{E}_i^{\mathrm{C}}) = \Gamma(\tilde{E}_i^{\mathrm{V}}(L)), \tag{5}$$

so, as a consequence of the symmetry of the problem, the 'momenta' $\tilde{E}_i^{\mathrm{V}} L$ are constants of motion. By means of expression (4), Γ can be rewritten as

$$\begin{aligned}\Gamma &= w^k \hat{E}_k^{\mathrm{C}} + f^k \hat{E}_k^{\mathrm{V}} = w^k A_k^j \tilde{E}_j^{\mathrm{C}} + w^k w^m A_l^j C_{jm}^l \hat{E}_j^{\mathrm{V}} + f^k \hat{E}_k^{\mathrm{V}} \\ &= w^k A_k^j \tilde{E}_j^{\mathrm{C}} + f^k \hat{E}_k^{\mathrm{V}}.\end{aligned}$$

Both terms of Γ give a term in (5). By taking again the invariance of L into account, we get for the first term

$$\begin{aligned}w^k A_k^j \tilde{E}_j^{\mathrm{C}} \tilde{E}_i^{\mathrm{V}}(L) &= w^k A_k^j \tilde{E}_i^{\mathrm{V}} \tilde{E}_j^{\mathrm{C}}(L) + w^k A_k^j [\tilde{E}_j^{\mathrm{C}}, \tilde{E}_i^{\mathrm{V}}](L) = w^k A_k^j C_{ij}^l \tilde{E}_l^{\mathrm{V}}(L) \\ &= w^k A_k^j C_{ij}^l \bar{A}_l^m \hat{E}_m^{\mathrm{V}}(L).\end{aligned}$$

The last term is

$$f^k \hat{E}_k^{\mathrm{V}} \tilde{E}_i^{\mathrm{V}}(L) = f^k \tilde{E}_i^{\mathrm{V}} \hat{E}_k^{\mathrm{V}}(L) = f^k \bar{A}_i^l \hat{E}_l^{\mathrm{V}} \hat{E}_k^{\mathrm{V}}(L) = f^k \bar{A}_i^l \hat{E}_k^{\mathrm{V}} \hat{E}_l^{\mathrm{V}}(L).$$

Let $l \in C^\infty(\mathfrak{g})$ be the restriction of the left-invariant Lagrangian $L \in C^\infty(TG)$ to the Lie algebra. Then, in the current coordinate system, $\hat{E}_k^{\mathrm{V}}(L)$ is $\dfrac{\partial l}{\partial w^k}$, when we restrict it to a function on $\mathfrak{g}$. The defining relation for the reduced vector field $F = F^i \dfrac{\partial}{\partial w^i} \in \mathcal{X}(\mathfrak{g})$ of Γ in (5) is (deleting the factor $\bar{A}_a^b$)

$$F\Big(\frac{\partial l}{\partial w^l}\Big) = C_{ml}^j w^m \frac{\partial l}{\partial w^j}. \tag{6}$$

These are the so-called Euler-Poincaré equations [7] and they should be interpreted as differential equations with solution $w(t)$ in the Lie algebra. As was mentioned before, we can find the corresponding solution $g(t) \in G$ of the Euler-Lagrange equations by integrating also $g^{-1}\dot{g}(t) = w(t)$.

3 Examples

Euler's equations for the free rigid body. Consider a rigid body that can rotate freely around a fixed point. If $\mathbf{L}$ is its angular momentum, then the equation of motion is $\dot{\mathbf{L}} = 0$. If $(e) = (\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1)$ is a basis that moves along with the body and if ω is the angular velocity of that basis, then this equation can be rewritten as

$$\dot{\mathbf{L}}^{(e)} = \omega \times \mathbf{L}, \tag{7}$$

where $\dot{\mathbf{L}}^{(e)}$ stands for relative derivative of $\mathbf{L}$ w.r.t. the moving basis. Without loss of generality, we can suppose that the 3 unit vectors of the basis (e) are axes of inertia of the body. Then $\mathbf{L} = I_1\omega_1\mathbf{e}_1 + I_2\omega_2\mathbf{e}_2 + I_3\omega_3\mathbf{e}_3$ and equation (7) should determine the components of ω along the moving basis:

$$\begin{cases} I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \\ I_2\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3, \\ I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2. \end{cases}$$

The above equations are sometimes referred to as the 'dynamical equations of Euler'. They are of Euler-Poincaré type in the following way. Recall the identification

$$\omega \simeq \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3).$$

The basis $(e) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ can then be identified with the basis

$$\left\{ e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

satisfying $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$ and $[e_2, e_3] = e_1$. It is easy to see that the above equations are Euler-Poincaré equations on the Lie algebra $so(3)$ for the Lagrangian $l(\omega) = \frac{1}{2}\langle\omega, I\omega\rangle = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \in C^\infty(\mathfrak{g})$.

The corresponding Lagrangian $L \in C^\infty(TG)$ at the level of the Lie group $SO(3)$ can be found by left-translating the Lagrangian l. Having solved the dynamical equations, one still needs to integrate the relation $\omega = R^{-1}\dot{R}$, to find the corresponding solution $R(t) \in SO(3)$ of the Euler-Lagrange equations. This last relation is known as the 'kinematical equations of Euler'.

THE 'BLOCH-ISERLES'-EQUATIONS [3]. The space of interest is $Sym(n)$, the linear space of symmetric $n \times n$ matrices. The equation is

$$\dot{X} = [X^2, N], \qquad X \in Sym(n), \tag{8}$$

where $N \in so(n)$ is a skew-symmetric $n \times n$ matrix. Any such N gives $Sym(n)$ the structure of a Lie algebra with bracket

$$[X, Y]_N = XNY - YNX, \qquad X, Y \in Sym(n).$$

Can we find a Lagrangian $l \in C^\infty(Sym(n))$ for which these equations are of Euler-Poincaré type w.r.t. the above Lie algebra? The answer is given in [4]: the corresponding Lagrangian is

$$l(X) = \frac{1}{2} trace(X^2). \tag{9}$$

To make things more accessible, we will only consider the easiest case of $Sym(2)$. Then, a basis is given by the matrices

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad e_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and, without any problem we can take N to be $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The Lie algebra brackets are $[e_1, e_2]_N = 2e_1$, $[e_1, e_3]_N = e_2$ and $[e_2, e_3]_N = 2e_3$. An arbitrary element of the Lie algebra is of the form $X = xe_1 + ye_2 + ze_3 = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and the equations take the form

$$\begin{bmatrix} \dot{x} & \dot{y} \\ \dot{y} & \dot{z} \end{bmatrix} = \begin{bmatrix} -2y(x+z) & x^2 - z^2 \\ x^2 - z^2 & 2y(x+z) \end{bmatrix}. \tag{10}$$

It can easily be verified that the Euler-Poincaré equations (6) for the Lagrangian $l(x, y, z) = \frac{1}{2}(x^2 + 2y^2 + z^2)$ on the above Lie algebra are exactly equations (10).

The Lie algebra $(Sym(n), [\cdot, \cdot]_N)$ is not so unfamiliar as it may seem at first. In [4] it shown to be isomorphic to $sp(n, \mathbf{R})$, by means of (for $n = 2$)

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \mapsto NX = \begin{bmatrix} y & z \\ -x & -y \end{bmatrix}.$$

However, the Euler-Poincaré-structure of the equations and the Lagrangian are most clear in the Lie algebra $(Sym(n), [\cdot, \cdot]_N)$.

4 Bi-invariant Lagrangians

In this section, we will suppose that the Lagrangian L is both left- and right-invariant. If $Ad : G \times \mathfrak{g} \to \mathfrak{g} : (g, w) \mapsto T\rho_{g^{-1}} T\lambda_g w$ is the adjoint action, then L's restriction l to $\mathfrak{g}$ is obviously Ad-invariant. Also the converse is true: By means of left translation, we can obtain a bi-invariant Lagrangian from an Ad-invariant function on the Lie algebra.

An equivalent condition for bi-invariance is that both $\tilde{E}_i^{\mathrm{C}} L = 0$ and $\hat{E}_j^{\mathrm{C}} L = 0$. Relation (4) then shows that bi-invariance of a left-invariant Lagrangian is equivalent with

$$w^k C_{ki}^j \hat{E}_j^{\mathrm{V}} L = 0 \quad \text{or} \quad w^k C_{ki}^j \frac{\partial l}{\partial w^j} = 0. \tag{11}$$

The condition (11) appears also in the context of the inverse problem for invariant Lagrangians on a Lie group [8]. It is the necessary and sufficient condition for the existence of a left-invariant Lagrangian whose Lagrangian field coincides with that of the spray associated to the canonical connection. This is easy to see as follows. The canonical connection is given by

$$\nabla_X Y = \frac{1}{2}[X, Y], \qquad X, Y \text{ left-invariant vector fields on } G.$$

In the basis $\{\hat{E}_i\}$, the connection coefficients are $\Gamma^k_{ij} = \frac{1}{2}C^k_{ij}$ and therefore is the associated second order field simply $\Gamma = w^i\hat{E}^{\rm C}_i$ and the reduced vector field on $\mathfrak{g}$, $F = 0$. On the other hand, from (6) it is clear that the reduced vector field vanishes in the Euler-Poincaré equations if and only if (11) is satisfied.

The integral curves of the vector field $F = 0$ on the Lie algebra are of the form $w(t) = w_0$. Recall that the relation between solutions $w(t)$ of the Euler-Poincaré equations (at the Lie algebra level) and the corresponding solutions of the Euler-Lagrange equations (at the group level) was $w = g^{-1}\dot{g}$. So, the solutions through e at $t = 0$ are of the form $g(t) = \exp(tw_0)e$. Therefore, all solutions of the Euler-Lagrange equations are exponentials if and only if L is bi-invariant. This property is well-known for Riemann metrics on a Lie group: The two notions of the exponential map coincide if and only if the metric is bi-invariant.

Let's come back to the condition (11). After a first derivation by w^l, we find

$$C^j_{li}\frac{\partial l}{\partial w^j} + w^k C^j_{ki}k_{jl} = 0, \qquad k_{ij} = \frac{\partial^2 l}{\partial w^i \partial w^j},$$

and by taking the linear combination with w^l, the above becomes (taking into account (11))

$$w^k w^l C^j_{ki}k_{jl} = 0. \tag{12}$$

After two derivations of (11) by w^l and w^m, we find

$$C^k_{il}k_{km} + C^k_{im}k_{lk} + C^k_{ij}w^j\frac{\partial k_{lk}}{\partial w^m} = 0, \tag{13}$$

($\dfrac{\partial k_{lk}}{\partial w^m}$ is the (totally symmetric) 'Cartan torsion' of l on $\mathfrak{g}$). In the case that L is a Finsler function, l is a so-called 'Minkowski function' on the vector space $\mathfrak{g}$ (in the terminology of [2]) and $w^k\dfrac{\partial l}{\partial w^k} = 2l$ and thus $w^k\dfrac{\partial^2 l}{\partial w^k\partial w^m} = \dfrac{\partial l}{\partial w^m}$. The conditions (12) and (13) are then both equivalent with (11): you just have to take a suitable linear combination with w's. This observation can also be found in [6], where a k satisfying (13) is called a 'Minkowski Lie algebra' on $\mathfrak{g}$.

5 Geodesic vectors, relative equilibria and their stability

For a bi-invariant Lagrangian, all solutions of the Euler-Lagrange equations are exponentials. But when do we have an exponential solution for a left-invariant Lagrangian? The papers [10,11] by J. Szenthe and the paper [1] by V. Arnold deal with this question. Such solutions are usually called 'relative equilibria' in a 'mechanics' context [1,9] and 'geodesic vectors' in a more 'geometric' context [10,11]. A solution $\gamma(t)$ is called stationary if there is a $w_0 \in \mathfrak{g}$ such that $\gamma(t) = \exp(tw_0)\gamma(0)$. $w_0 \in \mathfrak{g}$ is a geodesic vector or relative equilibrium if the solution through $w_0 \in T_eG$ at $t = 0$ (and thus with $e = \gamma(0) \in G$) is stationary. The group element where the solution starts does not really play a role: if γ is stationary for some w_0 and $\gamma(0) = g \in G$ (i.e. if $\gamma(t) = \exp(tw_0)g$), then $Ad_{g^{-1}}w_0$ is a geodesic vector with corresponding solution $\bar{\gamma}(t) = \exp(Ad(g^{-1})w_0)e$, through e at $t = 0$.

If w_0 is a geodesic vector for L, then its corresponding solution $\gamma(t) = \exp(tw_0)$ satisfies $\gamma^{-1}(t) = \exp(-tw_0)$ and $\dot{\gamma}(t) = T\lambda_{\exp(tw_0)}w_0$ (λ denotes left-translation as before). So $\gamma^{-1}(t)\dot{\gamma}(t) = T\lambda_{\exp(-tw_0)} \circ T\lambda_{\exp(tw_0)}w_0 = w_0$ is constant. So, the search for geodesic vectors is equivalent with the search for constant solutions of the Euler-Poincaré equations (6) and that explains the name relative 'equilibria'. From the Euler-Poincaré equations, we see that $w_0 = (w_0^k)$ is such a solution if and only if

$$w_0^k C_{ki}^j \frac{\partial l}{\partial w^j}(w_0) = 0. \tag{14}$$

In the case that L is Finslerian or Riemannian, we find equivalent characterizations by taking derivatives (as in the previous section). For example in the case that the Lagrangian comes from a Riemannian metric, condition (14) for $l = \frac{1}{2}k_{ij}w^iw^j$ is equivalent with

$$w_0^k w_0^l C_{ki}^j k_{jl} = 0 \qquad \text{or} \qquad k(w_0, [w_0, w]) = 0, \forall w \in \mathfrak{g}.$$

If we identify a geodesic vector w_0 as an element in $T_eG \subset TG$, then - next to the supposed left-invariance of the Lagrangian $\tilde{E}_i^{\mathrm{C}}L(v) = 0$, for all $v \in TG$ - also

$$\hat{E}_i^{\mathrm{C}}L(w_0) = 0. \tag{15}$$

The two characterizations (14) and (15) can also be found in [11], in a somehow disguised form. The characterization (15) is, in Szenthe's particular coordinate system, the condition $\dfrac{\partial L}{\partial x^i}\,|_{w_0} = 0$ in his proposition 2.2. However, the proof of this proposition in [11] is wrong, and the assumption that the Lagrangian is a first integral of its Lagrangian field is unnecessary, as indeed are the assumptions that the group is compact and that the basis of T_eG is orthonormal.

The curve $\dot\gamma(t) = TL_{\exp(tw_0)}w_0$ is also an integral curve through $w_0 \in T_eG$ of the vector field $\tilde w_0^{\rm C}$ ($\tilde w_0$ is the fundamental vector field associated to left translations and is thus right-invariant). So, if γ is a base integral curve of the Lagrangian field Γ then $\ddot\gamma(t) = \Gamma(\dot\gamma(t)) = \tilde w_0^{\rm C}(\dot\gamma)$. Due to the invariance, this property is equivalent with $\Gamma(w_0) = \tilde w_0^{\rm C}(w_0)$. In general, $\Gamma = y^i\tilde E_i^{\rm C} + f^i\hat E_i^{\rm V}$. So, w_0 is a geodesic vector if the functions f^i vanish at w_0 and geodesic vectors (relative equilibria) are equilibria of the reduced vector field $F = f^i\dfrac{\partial}{\partial w^i}$ on $\mathfrak{g}$.

Finally, condition (14) can be interpreted as the condition for the critical points of the restriction of the Lagrangian (quadratic function in the Riemannian case) to some orbit. There are even two interpretations: the orbit of w_0 under Ad as in [11], or a dynamical orbit as in [1].

In the first case, let l_{w_0} be the restriction of the Lagrangian l to only elements in the Ad-orbit of w_0. A point is critical if $v(l_{w_0}) = 0$ for all tangent vectors v to the orbit of w_0. Any such tangent vector can be constructed in the following way. For any $w \in \mathfrak{g}$, the curve

$$c_w : t \mapsto Ad_{\exp(tw)}w_0$$

lies in the orbit of w_0 and its tangent vector at $t = 0$ is in fact

$$\dot c_w(0) = [w, w_0], \tag{16}$$

(regarding a tangent vector in $T_{w_0}\mathfrak{g}$ as an element in $\mathfrak{g}$). Since any tangent vector to the orbit of w_0 can be written in this way, a point is critical if for all $w = (w^i)$,

$$0 = \dot c_w(0)(l_{w_0}) = [w, w_0]l = w^iC_{ij}^k w_0^j\frac{\partial l}{\partial w^k}(w_0),$$

which is exactly condition (14).

Szenthe uses the above characterization, in the case that G is compact, to prove that there exists at least one stationary geodesic. Let $ad : G\times G \to G, (g,h) \mapsto g^{-1}hg$ and $Ad : G \times \mathfrak{g} \to \mathfrak{g}, (g,x) \mapsto Tad_g(x)$ be the two notions of adjoint actions. Let $G(x) \subset \mathfrak{g}$ be the orbit of $x \in \mathfrak{g}$ under Ad and $G_x \subset G$ the (closed) isotropy group of x. Then, there is a bijection $\rho : G/G_x \to G(x), [h]_{G_x} \mapsto Ad_hx$. In the case that G is compact, so is also G/G_x and thus has the function $L \circ \rho$ at least one critical point.

The characterization in [1] is a bit different from the above. Although Arnold only deals with the case when L is Riemannian, we can extend it to the more general case.

Recall that we could write the Euler-Poincaré equations in the form $\Gamma(\tilde{E}_i^{\mathrm{V}} L) = 0$. So, due to the symmetry of the equations we have conservation of the momenta, and all $\tilde{E}_i^{\mathrm{V}} L \in C^\infty(TG)$ are first integrals of our system. These functions are *not* left-invariant: $\tilde{E}_j^{\mathrm{C}}(\tilde{E}_i^{\mathrm{V}}(L)) = C_{ji}^k \tilde{E}_k^{\mathrm{V}} L \neq 0$ (they are also not right-invariant), although their restriction to the Lie algebra $\mathfrak{g}$ is $\dfrac{\partial l}{\partial w^i}$. The left-invariant function on TG that corresponds to $\dfrac{\partial l}{\partial w^i}$ on the Lie algebra is in fact $\hat{E}_i^{\mathrm{V}} L = A_i^j \tilde{E}_j^{\mathrm{V}} L$ (recall that A_j^i was δ_j^i at e).

Let $g(t)$ be a solution of the Euler-Lagrange equations, going through $g(0) = e$ and $\dot{g}(0) = w_0 \in T_e G = \mathfrak{g}$. Then, $w(t) = g^{-1}(t)\dot{g}(t)$ is a solution of the Euler-Poincaré equations (with $w(0) = w_0$). The conservation law is then

$$\tilde{E}_i^{\mathrm{V}} L(\dot{g}(t)) = \tilde{E}_i^{\mathrm{V}} L(w_0).$$

The right-hand side is $\dfrac{\partial l}{\partial w^i}(w_0)$. The left-hand side is $\bar{A}_i^j(g(t))\hat{E}_j^v L(g(t)w(t)) = \bar{A}_i^j(g(t))\hat{E}_j^v L(w(t)) = \bar{A}_i^j(g(t))\dfrac{\partial l}{\partial w^j}(w(t))$. Since the matrix $(A_i^j(g))$ corresponds to the operator Ad_g (see section 1), $\bar{A}_i^j(g)$ corresponds with $Ad_{g^{-1}}$. To conclude, the conservation of momentum can be written as

$$((Ad_{g^{-1}(t)}x)l)(w(t)) = (x(l))(w_0), \tag{17}$$

for all $x \in \mathfrak{g}$, here interpreted as tangent vectors in $T_w\mathfrak{g}$.

The rest of the reasoning will be more clear if we take first a step backwards and suppose that the Lagrangian is Riemannian. Then $l(w) = \frac{1}{2}k(w, w)$, for some symmetric bilinear form k on $\mathfrak{g}$ and (17) is

$$k(Ad_{g^{-1}(t)}x, w(t)) = k(x, w_0), \qquad \forall x \in \mathfrak{g}. \tag{18}$$

We will use the first integrals to establish an equivalence relation on $\mathfrak{g}$. Two elements w and $\bar{w}$ of the Lie algebra are said to be equivalent if there exists a $g \in G$ such that

$$k(w, Ad_g x) = k(\bar{w}, x), \qquad \forall x \in \mathfrak{g}. \tag{19}$$

In general $k(w, Ad_g x) \neq k(Ad_g w, x)$. There is only equality when k is Ad-invariant, and this is only the case when $L \in C^\infty(TG)$ is bi-invariant (and thus when all $w \in \mathfrak{g}$ are relative equilibria). From this, an orbit under the above equivalence relation is different form an Ad-orbit, where two vectors w and $\bar{w}$ are equivalent if there is a $g \in G$ such that $Ad_g w = \bar{w}$ and thus

$$k(Ad_g w, x) = k(\bar{w}, x), \qquad \forall x \in \mathfrak{g}.$$

The advantage of the above equivalence relation is that it is 'invariant' (and we will use this for checking stability later), that is to say: each solution $w(t)$ of the Euler-Poincaré equations stays in one and the same equivalence class.

In the Riemannian case we can define B such that

$$k([w, x], y) = k(B(y, w), x), \qquad \forall x, y, w \in \mathfrak{g},$$

with coefficients $B^m_{jk} = k^{lm} k_{ij} C^i_{kl}$. The Euler-Poincaré equations for $l = \frac{1}{2} k_{ij} w^i w^j$ can then be rewritten in the form

$$\dot{w}^m = k^{lm} k_{ij} C^i_{kl} w^j w^m \qquad \text{or} \qquad \dot{w} = B(w, w).$$

From this, w_0 is a relative equilibrium if and only if $B(w_0, w_0) = 0$.

A tangent vector to the equivalence class (19) of w_0 can be constructed as follows. For any given w, define a curve $t \mapsto c_w(t)$ by means of

$$k(c(t), x) = k(w_0, Ad_{\exp(tw)} x), \qquad \forall x \in \mathfrak{g}.$$

Its tangent vector at $t = 0$ is

$$\dot{c}_w(0) = B(w_0, w) \tag{20}$$

(any fibre of $T\mathfrak{g}$ is interpreted as $\mathfrak{g}$), because all coefficients k_{ij} are constants in the Riemanian case and therefore $k(\dot{c}_w(0), x) = k(w_0, \frac{d}{dt} |_0 (Ad_{\exp(wt)} x)) = k(w_0, [w, x])$. Remark again that the two approaches (16) and (20) do not coincide: $B(w_0, x) = [x, w_0]$ if and only if the metric (Lagrangian) is bi-invariant.

The function l_{w_0} is now the restriction of l to the equivalence class (19) of $w_0 \in \mathfrak{g}$. A point w_0 is critical if for all w, $\dot{c}_w(0)(l_{w_0}) = B(w_0, w) l = 0$. In the current case where $l(\xi) = \frac{1}{2} k(\xi, \xi)$ is Riemannian, this relation is simply

$$0 = k(w_0, B(w_0, w)) = -k(B(w_0, w_0), w) \qquad \Leftrightarrow \qquad B(w_0, w_0) = 0,$$

which is the condition for w_0 to be a relative equilibrium.

Still continuing the Riemannian case, Arnold [1] uses the following argument to say something about stability. Let l be a first integral of a first-order dynamical system $\dot{w} = f(w)$; let the system have an invariant equivalence relation and let w_0 be an extremum of l on an equivalence class, then, if the second derivatives of l on the equivalence class of w_0 form a positive-definite or a negative-definite quadratic form, the point w_0 is stable.

In the Riemannian case, the Lagrangian l is of course a first integral and the second derivatives of the restricted Lagrangian l_{w_0} give us the quadratic form Q of interest. Here, for $w_0 = (w_0^i)$ and $w = (w^i)$, the tangent vector $\dot{c}_w(0)$ is locally $v_w = B_{ij}^k w_0^i w^j \frac{\partial}{\partial w^k} |_{w_0}$. An easy calculation shows that the quadratic form that should be positive- or negative-definite is

$$\begin{aligned} Q(w) &= v_w(v_w l_{w_0}) = B(w_0, w)(B(w_0, w)l) \\ &= k(B(w_0, w), B(w_0, w)) + k([w, w_0], B(w_0, w)). \end{aligned} \tag{21}$$

This observation is Théorème 4 of [1].

Let's see now how we can extend all this to the case of a non-Riemannian Lagrangian. Then l is not a first integral anymore, but (the Hamiltonian) $E_l = \frac{\partial l}{\partial w^i} w^i - l \in C^\infty(\mathfrak{g})$ is! So, we will need to look at the restriction of E_l to some equivalence class. As before, this equivalence class comes from the conservation of momentum (17): two vectors w and $\bar{w}$ are in the same class if there exist a $g \in G$ such that

$$((Ad_g x)l)(w) = (x(l))(\bar{w}), \qquad \forall x \in \mathfrak{g}.$$

It is easy to see that this does indeed define an equivalence relation. By (17), the curve $w(t) = g^{-1}(t)\dot{g}(t)$ of a solution $g(t)$ of the Euler-Lagrange equations stays in the same class for all t. As before, for every given $\bar{w}$ in $\mathfrak{g}$, we can define a curve $c_{\bar{w}}(t)$ by means of

$$((Ad_{\exp(\bar{w}t)} x)l)(w_0) = (x(l))(c_{\bar{w}}(t)), \qquad \forall x \in \mathfrak{g}.$$

This curve lies in the equivalence class of w_0 (and is well-defined if we suppose that l is regular). Its tangent vector at $t = 0$ satisfies

$$C_{ij}^k \bar{w}^i x^j \frac{\partial l}{\partial w^k} = x^j \frac{\partial^2 l}{\partial w^j \partial w^m} (\dot{c}_w)^m(0)$$

(all functions are evaluated at $c_{\bar{w}}(t = 0) = w_0$), or, a tangent vector to the equivalence class at w_0 is given by

$$\dot{c}_{\bar{w}}(0) = k^{jm} C_{ij}^k \bar{w}^i \frac{\partial l}{\partial w^k} \frac{\partial}{\partial w^m} |_{w_0} .$$

The restriction of $E_l = \dfrac{\partial l}{\partial w^s} w^s - l$ to the equivalence class of w_0 has a critical point at w_0 if for all $\bar{w}$,

$$\begin{aligned} 0 = \dot{c}_{\bar{w}}(0)(E_l)(w_0) &= k^{jm} C_{ij}^k \bar{w}^i \frac{\partial l}{\partial w^k} \left(\frac{\partial^2 l}{\partial w^m \partial w^s} w^s + \frac{\partial l}{\partial w^m} - \frac{\partial l}{\partial w^m} \right) (w_0) \\ = w_0^j C_{ij}^k \bar{w}^i \frac{\partial l}{\partial w^k} (w_0). \end{aligned}$$

Not surprisingly, this means that w_0 is a relative equilibrium. We can also apply the stability theorem. By means of the second derivatives, we find that the quadratic form

$$\begin{aligned} Q(\bar{w}) &= v_{\bar{w}}(v_{\bar{w}} E_l)(w_0) \\ &= \Big(k^{ln} C_{hl}^m C_{ij}^k \frac{\partial l}{\partial w^m}(w_0) \frac{\partial l}{\partial w^k}(w_0) + C_{hk}^m C_{ij}^k w_0^j \frac{\partial l}{\partial w^m}(w_0) \Big) \bar{w}^h \bar{w}^i \end{aligned}$$

should be either positive or negative definite in order for w_0 to be stable.

6 Coming back to the examples

Let's consider the example of the Bloch-Iserles equations (10) again. The relative equilibria lie either on the line $(\lambda, 0, \lambda)$ or on the plane $(\lambda, \rho, -\lambda)$. In this example, the Lagrangian $l(x, y, z) = \frac{1}{2}(x^2 + 2y^2 + z^2)$ is clearly Riemannian. The only non-zero coefficients of B are

$$\begin{aligned} B_{12}^1 &= -2, & B_{11}^2 &= 1, & B_{21}^3 &= 2, \\ B_{23}^1 &= 2, & B_{33}^2 &= -1, & B_{32}^2 &= 2, \end{aligned}$$

and the coefficient matrix of the quadratic form (21) is, for $w_0 = (x, y, z)$

$$Q = \begin{bmatrix} 2x^2 + 4y^2 - 2xz & 2xy + 2yz & x^2 + 4y^2 + z^2 - 2xz \\ 2xy + 2yz & 0 & 2xy + 2yz \\ x^2 + 4y^2 + z^2 - 2xz & 2xy + 2yz & 4y^2 + z^2 - 2zx \end{bmatrix}$$

In the points $(\lambda, \rho, -\lambda)$, the quadratic form $Q(w)$ in $w = (w_x, w_y, w_z)$ is

$$Q(w) = (4\lambda^2 + 4\rho^2)(w_x^2 + 2w_y^2 + w_z^2),$$

which is clearly positive definite. We can therefore conclude that these geodesic vectors must be stable. For $(\lambda, 0, \lambda)$, however, we get

$$Q(w) = 0,$$

so we can not draw a conclusion from the analysis of the previous section. For this particular example, however, the behaviour of all solutions can easily be analyzed analytically (see figure 1). Treating the configuration space locally as

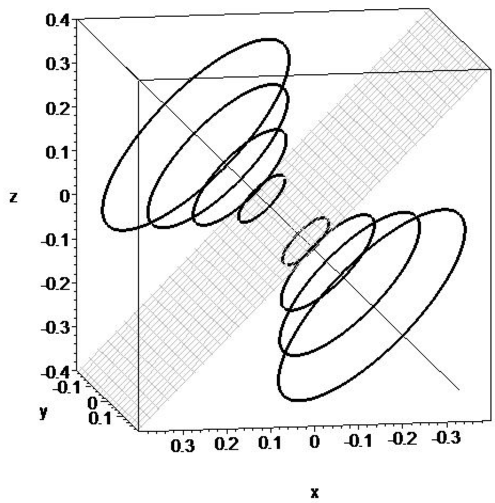

Fig. 1. Some solutions with initial conditions $(0, 0, i/10)$ for $i = -4 \ldots 4$.

$\mathbf{R}^3$, it is easy to see that all non-equilibrium solutions lie in a plane $x+z = C$, for some constant C. In each such plane, these solutions are closed (they are in fact circles) and centered around the equilibrium $(\frac{C}{2}, 0, \frac{C}{2})$. Therefore all relative equilibria are stable.

7 Outlook

It should be possible to extend the above results and methods to Lagrangian systems on a manifold M, invariant under an action $G \times M \to M$ of a Lie group G on M. Some immediate questions are the following ones: If L is an invariant Lagrangian (Finsler function/Riemannian metric) on M, when does it have a stationary geodesic (relative equilibria), i.e. a solution of the form $\gamma(t) = \exp(t\xi)m$, for some $m \in M$ and $\xi \in \mathfrak{g}$? In particular, is there a similar characterization for these solutions as critical points of some function? It is well-known that the equations of motion for such invariant Lagrangian systems reduce to the so-called Lagrange-Poincaré equations (see [5]). These equations can be decomposed into a vertical and horizontal component. In this more general set-up, the vertical equation represents in fact conservation of momentum $\Gamma(\tilde{E}_i^{\rm v} L) = 0$. A standard reference for the Hamiltonian approach to relative equilibria is [9].

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