Generalized submersiveness of second-order ordinary differential equations

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Abstract. We generalize the notion of submersive second-order differential equations by relaxing the condition that the decoupling stems from the tangent lift of a basic distribution. It is shown that this leads to adapted coordinates in which a number of first-order equations decouple from the remaining second-order ones.

1 Introduction and preliminaries

Second-order ordinary differential equations (SODEs for short) were said to be submersive by Kossowski and Thompson [6], if they partially decouple into an independent second-order system of lower dimension, plus a second part which depends on all variables. SODEs (which we will assume to be autonomous here, for simplicity) are geometrically represented by vector fields on the tangent bundle TM of a manifold M, and a characterization of a property such as submersiveness only makes sense if it can be described by intrinsic test criteria, i.e. conditions which can be verified prior to the identification of coordinates in which the partial decoupling takes place. The conditions described in [6] come from the identification of suitable distributions on TM which, not surprisingly, fully exploit the special features of tangent bundle geometry and are constructed, more specifically, from complete and vertical lifts of vector fields on M. Such specific tangent bundle characteristics very often have a more compact formulation in terms of the calculus of so-called vector fields along the tangent bundle projection $\tau : TM \to M$ (see [10], [11]), in the sense that a single set of conditions on vector fields along τ can capture the requirements that have to be met by both complete and vertical lifts, or alternatively, horizontal and vertical lifts. In fact, a successful first application of the calculus along τ was the full characterization of complete decoupling of SODEs in [12].

There are many ways in which (partial) decoupling of equations can play a role in applications. Apart from the obvious direct interest which any form of explicit decoupling of the given dynamics will have in the integration process, there may also be more indirect aspects of separability around. Decoupling can be inherent, for example, in the reduction of various kinds of mechanical systems through symmetries. A well known and much investigated feature of separability, which may or may not be related to decoupling of the dynamical equations, is that of separability of the Hamilton-Jacobi equation. Even further afield is the so-called separable case, which Douglas distinguished in his celebrated study of the inverse problem of the calculus of variations for mechanical systems with two degrees of freedom [5]. There separable refers to decoupling of the integrability conditions to be satisfied for having a solution of the Helmholtz conditions in the inverse problem. Yet, as was shown in a generalization of this 'separable case' to n degrees of freedom in [3], there is a perhaps unexpected link with a form of decoupling of the Lagrangian equations in this situation: these equations are not submersive in the sense described above, but decouple into a set of n systems of two first-order equations.

In the study of SODEs it is quite natural, both from an analytical and a geometrical point of view, to focus in the first place on techniques which preserve the second-order character of the system or, in other words, preserve the tangent bundle structure of the underlying manifold. Coordinate transformations then are restricted to be point transformations, as is the case in the concept of submersiveness described above. But as the example of the inverse problem suggests, there may be situations where it is less appropriate to insist on the preservation of the second-order character. In the present paper, starting from the characterization of submersive SODEs in its most economical form, that of the existence of a distribution along τ which has appropriate invariance properties (section 2), we shall see that there is a natural way of relaxing those invariance requirements and show that it leads to a generalized form of submersiveness in which the quotient system does not preserve its second-order character. A number of related issues will be discussed which are reflected in the titles of subsequent sections.

In the hope of keeping the paper more or less self-contained, we end this section by recalling the basics of the geometry of SODEs and the main ingredients of the calculus along τ relevant for the study of SODEs.

A SODE field on TM, say

$$\Gamma = u^\alpha \frac{\partial}{\partial q^\alpha} + F^\alpha(q,u)\frac{\partial}{\partial u^\alpha},$$

comes with a canonically defined connection on $\tau : TM \to M$, determined by a horizontal lift construction which, in coordinates (q^α, u^α) on TM, is given by

$$X = X^\alpha(q)\frac{\partial}{\partial q^\alpha} \;\mapsto\; X^H = X^\alpha H_\alpha, \quad \text{with} \quad H_\alpha = \frac{\partial}{\partial q^\alpha} - \Gamma^\beta_\alpha \frac{\partial}{\partial u^\beta}, \quad \Gamma^\alpha_\beta = -\frac{1}{2}\frac{\partial F^\alpha}{\partial u^\beta}.$$

The domain of the horizontal lift operator naturally extends to the $C^\infty(TM)$-module $\mathfrak{X}(\tau)$ of vector fields along τ, whose elements have a coordinate expression like the X above, but with components X^α which are functions on TM. In fact, with the aid of the projection operators P_H and P_V of the above non-linear connection on TM, one can construct a linear connection on the pullback bundle $\tau^*\tau : \tau^*TM \to TM$ (see e.g. [9]). Vector fields along τ are sections of $\tau^*\tau$. The linear connection $\mathrm{D} : \mathfrak{X}(TM) \times \mathfrak{X}(\tau) \to \mathfrak{X}(\tau)$, said to be of Berwald type, then essentially defines *vertical* and *horizontal covariant derivative operators* D^V_X and D^H_X on $\mathfrak{X}(\tau)$. In coordinates these are determined by the following action on functions $F \in C^\infty(TM)$ and basic vector fields.

$$\mathrm{D}^V_X F = X^\alpha V_\alpha(F), \qquad \mathrm{D}^V_X \frac{\partial}{\partial q^\alpha} = 0 \qquad (V_\alpha := \frac{\partial}{\partial u^\alpha})$$

$$\mathrm{D}^H_X F = X^\alpha H_\alpha(F), \qquad \mathrm{D}^H_X \frac{\partial}{\partial q^\alpha} = X^\beta V_\alpha(\Gamma^\gamma_\beta)\frac{\partial}{\partial q^\gamma}\,.$$

The action of these covariant derivatives extends to 1-forms along τ by duality and then further to arbitrary tensor fields along τ as degree zero derivations. For the general theory of derivations of forms along τ see [10, 11].

Other important operators are: the *dynamical covariant derivative* ∇, a self-dual derivation of degree zero on tensor fields along τ, and a $(1,1)$ tensor Φ along τ called the *Jacobi endomorphism*. These can implicitly be defined by the following formula for the decomposition of the vector field $\mathcal{L}_\Gamma X^H$ on TM into its horizontal and vertical part:

$$\mathcal{L}_\Gamma X^H = (\nabla X)^H + \Phi(X)^V.$$

For practical purposes it suffices to know that:

$$\nabla F = \Gamma(F) \qquad \nabla\frac{\partial}{\partial q^\alpha} = \Gamma^\beta_\alpha \frac{\partial}{\partial q^\beta} \qquad \nabla dq^\alpha = -\Gamma^\alpha_\beta dq^\beta\,,$$

$$\Phi^\alpha_\beta = -\frac{\partial F^\alpha}{\partial q^\beta} - \Gamma^\alpha_\gamma\Gamma^\gamma_\beta - \Gamma(\Gamma^\alpha_\beta)\,.$$

2 Submersive systems and a natural generalization

Submersiveness of a SODE, as defined in [6], is characterized by the following result (see [8] or [13]).

Theorem 1. *A SODE Γ on TM is (locally) submersive if and only if there exists a distribution K along $\tau : TM \to M$, such that*

$$\Phi(K) \subset K, \quad \nabla K \subset K, \quad \mathrm{D}^V_Z K \subset K \;\; \forall Z \in \mathfrak{X}(\tau).$$

The third condition expresses that K is a basic distribution, i.e. is generated by a distribution on M. It is also D^H-invariant and therefore generated by a Frobenius integrable distribution. If coordinates on integral submanifolds of K are denoted by x^a, and y^i denote transversal coordinates (so we write $(q^\alpha) = (x^a, y^i)$ for a complete set of adapted coordinates on M and shall denote corresponding fibre coordinates on TM by (v^a, w^i)), ∇- and Φ-invariance of K imply that the forces F^i do not depend on (x^a, v^a). Hence, the equations for the y^i decouple from the rest; they will be referred to as the *driving system*, while the equations for the x^a then constitute the *driven system* (adapting this way to the present situation a suggestive terminology, introduced in [7], and also used in [13]).

A natural question to ask is: "what happens if K is not basic?". Is there a chance that the distribution $\mathcal{K}$ on TM, spanned by K^H and K^V remains integrable?

Recall [11] that we have the following general bracket relations:

$$\begin{aligned}
[\Gamma, X^V] &= -X^H + (\nabla X)^V, \\
[\Gamma, X^H] &= (\nabla X)^H + (\Phi X)^V, \\
[X^V, Y^V] &= (\mathrm{D}^V_X Y - \mathrm{D}^V_Y X)^V, \\
[X^H, Y^V] &= (\mathrm{D}^H_X Y)^V - (\mathrm{D}^V_Y X)^H, \\
[X^H, Y^H] &= (\mathrm{D}^H_X Y - \mathrm{D}^H_Y X)^H + R(X,Y)^V,
\end{aligned} \tag{1}$$

where R is the curvature tensor of the non-linear connection. Assume now that K is a distribution along τ, which is ∇- and Φ-invariant, but not necessarily D^V-invariant. It is clear then that

$$[\Gamma, \mathcal{K}] \subset \mathcal{K}.$$

Keeping the commutator property

$$[\nabla, \mathrm{D}^V_X] = \mathrm{D}^V_{\nabla X} - \mathrm{D}^H_X$$

in mind, if we want $\mathcal{K}$ to be integrable on TM, it is clear from the bracket of horizontal and vertical lifts that we must assume

$$\mathrm{D}^V_X Y \in K, \qquad \forall X, Y \in K,$$

which is a weaker assumption though than D^V-invariance, whereas it then further follows that also

$$\mathrm{D}^H_X Y \in K, \qquad \forall X, Y \in K.$$

Moreover, since

$$\begin{aligned}
3\,R(X,Y) &= \mathrm{D}^V_X \Phi(Y) - \mathrm{D}^V_Y \Phi(X) \\
&= \mathrm{D}^V_X(\Phi Y) - \mathrm{D}^V_Y(\Phi X) - \Phi(\mathrm{D}^V_X Y) + \Phi(\mathrm{D}^V_Y X),
\end{aligned}$$

we automatically have that the bracket of horizontal lifts also preserves $\mathcal{K}$.

Hence, we shall study distributions K along τ, satisfying the conditions

$$\nabla K \subset K, \quad \Phi(K) \subset K, \quad \mathrm{D}^V_Z K \subset K \ \ \forall Z \in K, \tag{2}$$

and investigate to what extent this represents a generalized notion of submersiveness of SODEs.

3 Construction of adapted coordinates for the distribution $\mathcal{K}$

Since K need not be basic and adapted coordinates accordingly are no longer going to be obtained by a point transformation, it is useful to understand in detail how coordinates can be selected which preserve as much as possible the tangent bundle structure on TM. The following account is inspired by the proof of the Frobenius theorem in [2] and is specifically aimed at preserving a structure of horizontal and vertical lifts in setting up a local basis for the co-distribution $\mathcal{K}^0$ of constraint forms of $\mathcal{K}$.

Let $\{X_a\}_{a=1,\ldots,\mathsf{k}}$ be a basis for K, so that $\mathcal{K} = \mathrm{sp}\,\{X_a{}^H, X_a{}^V\}$. If θ is a constraint form for $\mathcal{K}$, then so is its image $S(\theta)$ under the vertical endomorphism S on TM. Indeed, we have $\langle X_a{}^V, S(\theta)\rangle = 0$ trivially, and $\langle X_a{}^V, \theta\rangle = 0$ implies that $\langle X_a{}^H, S(\theta)\rangle = \langle S(X_a{}^H), \theta\rangle = 0$. Hence the constraint forms also come in pairs, say $\{\theta^i, S(\theta^i)\},\ i = 1, \ldots, \mathsf{l}$ ($\mathsf{k} + \mathsf{l} = \mathsf{n} = \dim M$).

Without loss of generality (using an appropriate matrix multiplication if necessary) we can arrange that each constraint form starts with a corresponding coordinate 1-form. Specifically, this can be done pairwise again, i.e. we can take θ^i of the form

$$\theta^i = du^i + \Gamma^i_\alpha dq^\alpha + \theta^i_a(du^a + \Gamma^a_\beta dq^\beta),$$

summation over repeated indices always being understood (from 1 to n for Greek indices, 1 to k for indices such as $a, b, \ldots$, and from 1 to l for indices such as $i, j, \ldots$). We then have

$$S(\theta^i) = dq^i + \theta^i_a dq^a,$$

and, if X^α_a are the components of the basis vector X_a of K, we have the *constraint relations*

$$X^i_a + \theta^i_b X^b_a = 0, \qquad i = 1, \ldots, \mathsf{l},\ a = 1, \ldots, \mathsf{k}.$$

It is easy to verify that these are the relations which guarantee that the forms θ^i and $S(\theta^i)$ vanish on $X_a{}^V$ and $X_a{}^H$, spanning $\mathcal{K}$.

Now Frobenius guarantees that there exist combinations of the constraint forms which are exact, meaning that there exists a non-singular $2\mathsf{l} \times 2\mathsf{l}$ matrix such that, say

$$\begin{pmatrix} A^i_j & B^i_j \\ C^i_j & D^i_j \end{pmatrix} \begin{pmatrix} S(\theta^j) \\ \theta^j \end{pmatrix} = \begin{pmatrix} df^i \\ dg^i \end{pmatrix}, \quad i = 1, \ldots, \mathsf{l},$$

for some functions f^i, g^i. In turn, this implies that

$$\begin{aligned} \frac{\partial f^i}{\partial u^j} &= B^i_j, & \frac{\partial f^i}{\partial q^j} &= A^i_j + B^i_k(\Gamma^k_j + \theta^k_a \Gamma^a_j), \\ \frac{\partial g^i}{\partial u^j} &= D^i_j, & \frac{\partial g^i}{\partial q^j} &= C^i_j + D^i_k(\Gamma^k_j + \theta^k_a \Gamma^a_j), \end{aligned}$$

from which it follows that the Jacobian $\partial(f^i, g^i)/\partial(q^j, u^j)$ is non-singular. The level sets of the functions $(f^i, g^i),\ i = 1, \ldots, \mathsf{l}$, define the integral submanifolds of $\mathcal{K}$ and the non-singularity of the Jacobian just discussed further implies that the $(q^a, u^a),\ a = 1, \ldots, \mathsf{k}$ can be used as local coordinates on the leaves. As before we now write coordinates, adapted to the distribution $\mathcal{K}$, as (x^a, y^i, v^a, w^i). It is clear that transformation formulas from the original variables (q^α, u^α) to those new coordinates can be taken to be of the form

$$\begin{array}{lll} x^a = q^a, & v^a = u^a, & a = 1, \ldots, \mathsf{k} \\ y^i = f^i(q,u), & w^i = g^i(q,u), & i = 1, \ldots, \mathsf{l}, \end{array}$$

and $(\partial/\partial x^a, \partial/\partial v^a)$ should span $\mathcal{K}$. As a matter of fact, using the constraint relations mentioned above, one easily obtains that in the new coordinates:

$$X_a{}^V = X_a^b \frac{\partial}{\partial v^b}, \qquad X_a{}^H = X_a^b \left(\frac{\partial}{\partial x^b} - \Gamma_b^c \frac{\partial}{\partial v^c} \right) - X_a^i \Gamma_i^b \frac{\partial}{\partial v^b}.$$

The given SODE Γ transforms in the following way:

$$\Gamma = u^\alpha \frac{\partial}{\partial q^\alpha} + F^\alpha \frac{\partial}{\partial u^\alpha} = v^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial v^a} + \tilde{F}^i \frac{\partial}{\partial y^i} + \tilde{G}^i \frac{\partial}{\partial w^i}, \tag{3}$$

where we keep the original notation F^a for force functions which are merely expressed in the new variables, while $\tilde{F}^i = \Gamma(f^i)$, $\tilde{G}^i = \Gamma(g^i)$. We know that Γ preserves the distribution $\mathcal{K}$, hence

$$\left[\Gamma, \frac{\partial}{\partial x^a}\right], \left[\Gamma, \frac{\partial}{\partial v^a}\right] \in \mathrm{sp}\left\{ \frac{\partial}{\partial x^a}, \frac{\partial}{\partial v^a} \right\}.$$

It follows that the functions $\tilde{F}^i$, $\tilde{G}^i$ do not depend on the variables (x^a, v^a). The result is a submersive system, where the driving part is a first-order system, while the driven part, representing the dynamics on the leaves, preserves its second-order character. Explicitly,

$$\begin{array}{lll} \dot{y}^i = \tilde{F}^i(y^j, w^j), & \dot{w}^i = \tilde{G}^i(y^j, w^j), & i = 1, \ldots, \mathsf{l} \\ \dot{x}^a = v^a, & \dot{v}^a = F^a(x^b, v^b, y^i, w^i), & a = 1, \ldots, \mathsf{k}. \end{array}$$

We summarize the results in the following theorem.

Theorem 2. *Let Γ be a given second-order vector field on TM with its associated horizontal distribution. Denote by ∇ and Φ the corresponding dynamical covariant derivative and Jacobi endomorphism, respectively, and assume K is a distribution along $\tau : TM \to M$, satisfying the conditions*

$$\nabla K \subset K, \quad \Phi(K) \subset K, \quad \mathrm{D}^V_Z K \subset K \ \ \forall Z \in K.$$

Then the distribution $\mathcal{K}$ on TM, spanned by K^H, K^V, is integrable and invariant under Γ. It follows that there exist adapted coordinates in which Γ partially decouples into a driving system of first-order equations and a driven second-order system.

4 The almost tangent structure on each leaf of $\mathcal{K}$

Consider the restriction of S to a leaf $\mathcal{L}$ of $\mathcal{K}$. The tangent space to $\mathcal{L}$ at any point $p \in \mathcal{L}$ has the basis $\{{X_a}^H(p), {X_a}^V(p)\}$. Now

$$S_p({X_a}^H(p)) = {X_a}^V(p),$$
$$S_p({X_a}^V(p)) = 0.$$

Thus, firstly, S_p maps $T_p\mathcal{L}$ into itself, and so S defines by restriction a type $(1,1)$ tensor field on $\mathcal{L}$, which we denote by $\overline{S}$. Secondly, $\ker \overline{S}_p = \operatorname{im} \overline{S}_p$, so $\overline{S}$ defines an almost tangent structure on $\mathcal{L}$. Thirdly, since the bracket of vector fields tangent to $\mathcal{L}$ is also tangent to $\mathcal{L}$, it easily follows that the Nijenhuis torsion $N_{\overline{S}}$, which is also the restriction of N_S to $\mathcal{L}$, vanishes; thus the almost tangent structure on $\mathcal{L}$ is integrable. That is to say, the vertical endomorphism S (which is of course an integrable almost tangent structure on TM) defines by restriction an integrable almost tangent structure $\overline{S}$ on each leaf of $\mathcal{K}$.

It is well-known (see e.g. [4]) that, on any manifold with an integrable almost tangent structure, local coordinates (x^a, v^a) may be found with respect to which the tensor defining the structure takes the form

$$\frac{\partial}{\partial v^a} \otimes dx^a.$$

We now take an alternative (dual) look at the construction of adapted coordinates with this in mind.

We could first normalize the given basis of K to vector fields along τ of the form

$$X_a = \frac{\partial}{\partial q^a} + X_a^i \frac{\partial}{\partial q^i}.$$

Then,

$${X_a}^V = \frac{\partial}{\partial u^a} + X_a^i \frac{\partial}{\partial u^i},$$

and in order to complete the basis for the lifted distribution $\mathcal{K}$, one may choose to replace the horizontal vector fields ${X_a}^H$ by

$$\widehat{{X_a}^H} = {X_a}^H + (\Gamma_a^b + X_a^i \Gamma_i^b){X_b}^V.$$

The effect of this shift to a new basis of $\mathcal{K}$ (which no longer consists of horizontal and vertical lifts of a basis of K) is that integrability of $\mathcal{K}$ will now imply that $\widehat{{X_a}^H}$ and ${X_a}^V$ commute. Of course we still have that $S({X_a}^V) = 0$ and $S(\widehat{{X_a}^H}) = {X_a}^V$. On the one hand, integrability of $\mathcal{K}$ still means that there exist functions (f^i, g^i) which are killed by $(\widehat{{X_a}^H}, {X_a}^V)$; on the other, $\widehat{{X_a}^H}$ and ${X_a}^V$, since they commute, can be straightened out simultaneously to vector fields of the form $(\partial/\partial x^a, \partial/\partial v^a)$ say. In fact, this is precisely what the same type of coordinate transformation

$$x^a = q^a, \qquad v^a = u^a,$$
$$y^i = f^i(q,u), \qquad w^i = g^i(q,u),$$

will achieve in this case, since $\langle \widehat{X_a{}^H}, dx^b \rangle = \langle \widehat{X_a{}^H}, dq^b \rangle = \delta^b_a$ and likewise $\langle X_a{}^V, dv^b \rangle = \langle X_a{}^V, du^b \rangle = \delta^b_a$. It then follows that, as desired, $\overline{S}$ will take the form

$$\overline{S} = \frac{\partial}{\partial v^a} \otimes dx^a$$

in these coordinates.

A remark about the coordinates of the previous section: we could have modified the basis for $\mathcal{K}^0$ in the previous section in a similar way, by putting simply

$$\widehat{\theta^i} = \theta^i - (\Gamma^i_j + \theta^i_a \Gamma^a_j)\, S(\theta^j).$$

Which of these alternative constructions of adapted coordinates is the better one, will probably be dictated by the kind of application one has in mind. As will be briefly discussed in the final section, one can think of applications where, for example, the distribution K comes from eigenspaces of Φ and it will then be more transparent to keep the structure of horizontal and vertical lifts in choosing a local basis for $\mathcal{K}$.

5 Geometrical meaning of generalized submersiveness

Section 2 showed us the way to a natural generalization of the necessary and sufficient conditions for local submersiveness of a SODE. In sections 2 and 3, we have understood the details of the effect of relaxing the conditions, so that we can come now to a proper definition of what 'generalized submersiveness' geometrically means, in comparison to the original concept, as defined in [6] and [8].

Definition 1. *A* SODE Γ *on* TM *is said to have the* generalized submersiveness property*, if there exists a surjective submersion* ψ *of* TM *onto some manifold* N*, such that:*

(i) *There exists a vector field* X *on* N *which is* ψ*-related to* Γ*.*

(ii) *The vertical endomorphism* S *on* TM *defines by restriction an almost tangent structure* $\overline{S}$ *on each fibre of* ψ*.*

Note that it follows from the assumption (ii) that the restricted almost tangent structure $\overline{S}$ will be integrable also (as argued in the previous section). It is further worth mentioning in relation to the first assumption that the following general property holds: if $\psi : N' \to N$ is a surjective submersion (fibration), and $\mathcal{K} = \ker T\psi$ denotes the distribution tangent to the fibres of ψ, then for any vector field Z on N' which is projectable, we have that $\mathcal{L}_Z \mathcal{K} \subset \mathcal{K}$.

Obviously, as explored in detail in the previous sections, the content of theorem 2 is that the conditions (2) imply that a SODE locally matches the criteria of the above definition. We now show that these conditions are actually also necessary for having local generalized submersiveness. In other words we have the following generalization of theorem 1.

Theorem 3. *A SODE Γ locally has the generalized submersiveness property if and only if there exists a distribution K along $\tau : TM \to M$ having the invariance properties (2).*

Proof. As indicated above, it follows from assumption (i) in the definition that $\mathcal{L}_\Gamma \mathcal{K} \subset \mathcal{K}$, where $\mathcal{K} = \ker T\psi$. Furthermore, from

$$\mathcal{L}_\Gamma S(\mathcal{K}) = \mathcal{L}_\Gamma(S(\mathcal{K})) - S(\mathcal{L}_\Gamma \mathcal{K}) \subset \mathcal{K},$$

we conclude that also $\mathcal{L}_\Gamma S(\mathcal{K}) \subset \mathcal{K}$. Since the horizontal projector of the Ehresmann connection defined by Γ is given by $P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$, this means that the horizontal and vertical parts of vector fields in $\mathcal{K}$ both belong to $\mathcal{K}$. Take $p \in TM$, and consider $\mathcal{K}_p \subset T_pTM$. Let V_p and H_p be the vertical and horizontal subspaces of T_pTM, and set $V(\mathcal{K}_p) = V_p \cap \mathcal{K}_p$, $H(\mathcal{K}_p) = H_p \cap \mathcal{K}_p$. We know that each of these subspaces contains nonzero vectors, by the remark above. Furthermore, $\overline{S}_p$ maps $H(\mathcal{K}_p)$ into $V(\mathcal{K}_p)$; thus $\overline{S}_p$ defines by restriction a linear map $H(\mathcal{K}_p) \to V(\mathcal{K}_p)$, say $\mathcal{S}$; and $\mathcal{S}$ is injective. Now $\overline{S}$ is an almost tangent structure, so $V(\mathcal{K}_p) = \ker \overline{S}_p = \operatorname{im} \overline{S}_p$; that is to say, for any $\xi \in V(\mathcal{K}_p)$ there is some $\eta \in \mathcal{K}_p$ such that $\xi = \overline{S}_p(\eta)$. Let ζ be the horizontal component of η: then $\zeta \in H(\mathcal{K}_p)$, and $\mathcal{S}(\zeta) = \overline{S}_p(\zeta) = \overline{S}_p(\eta) = \xi$. Thus $\mathcal{S} : H(\mathcal{K}_p) \to V(\mathcal{K}_p)$ is surjective as well as injective, and is therefore an isomorphism. We have $\mathcal{K}_p = H(\mathcal{K}_p) \oplus V(\mathcal{K}_p)$, where the two summands are isomorphic via $\mathcal{S}$.

Now define K by $H(\mathcal{K}_p) = K_p^{H}$. Then $V(\mathcal{K}_p) = \mathcal{S}(H(\mathcal{K}_p)) = K_p^{V}$. The Γ-invariance of $\mathcal{K}$ then implies (from the first of the bracket relations (1)) that $\nabla K \subset K$. Subsequently, the second of the bracket relations (1) shows that $\Phi(K) \subset K$. Finally, since the distribution $\mathcal{K}$ is integrable, if $X^H \in \mathcal{K}$ and $Y^V \in \mathcal{K}$ then $[X^H, Y^V] \in \mathcal{K}$. It then follows from the fourth bracket relation in (1) that $\mathrm{D}^V_Z K \subset K$ for all $Z \in K$, which completes the proof. $\square$

6 Generalized submersiveness by stages

Let K_1 be a k_1-dimensional distribution along τ satisfying the conditions (2). Assume that a second distribution K_2, which may or may not have a non-empty intersection with K_1, has the properties

$$\nabla K_2 \subset K_1 + K_2, \quad \Phi(K_2) \subset K_1 + K_2, \quad \mathrm{D}^V_Z(K_1 + K_2) \subset K_1 + K_2 \;\; \forall Z \in K_1 + K_2,$$

and let k_2 denote the dimension of $K_1 + K_2$. It follows from the commutator $[\nabla, \mathrm{D}^V_X]$ that also $\mathrm{D}^H_Z(K_1 + K_2) \subset K_1 + K_2 \;\; \forall Z \in K_1 + K_2$. In turn, the bracket relations of section 2 imply that both $\mathcal{K}_1 = \mathrm{sp}\{{K_1}^H, {K_1}^V\}$ and $\mathcal{K}_1 + \mathcal{K}_2 = \mathrm{sp}\{(K_1 + K_2)^H, (K_1 + K_2)^V\}$ are integrable distributions on TM, invariant under Γ. Consider the annihilating co-distributions

$$(\mathcal{K}_1 + \mathcal{K}_2)^0 \subset {\mathcal{K}_1}^0.$$

Let $\{df^{i_2}, dg^{i_2}\}$, $i_2 = 1, \ldots, \mathsf{l}_2$ (with $\mathsf{l}_2 + \mathsf{k}_2 = n$) be a set of exact forms spanning $(\mathcal{K}_1 + \mathcal{K}_2)^0$ and extend this to a basis for the module ${\mathcal{K}_1}^0$. Since ${\mathcal{K}_1}^0$ is also integrable, we know that there exists a further number of exact 1-forms, $\{df^{i_1}, dg^{i_1}\}$, $i_1 = 1, \ldots, \mathsf{l}_1$ say, linear combinations of the basis elements just constructed, which together with the $\{df^{i_2}, dg^{i_2}\}$

will span ${\mathcal{K}_1}^0$ (here $\mathsf{l}_1 + \mathsf{l}_2 + \mathsf{k}_1 = n$). Extending the constructions which were described in detail in section 3, we define now a coordinate transformation of the following form:

$$\begin{array}{lll} x^a = q^a, & v^a = u^a, & a = 1, \dots, \mathsf{k}_1, \\ y^{i_2} = f^{i_2}(q,u), & w^{i_2} = g^{i_2}(q,u), & i_2 = 1, \dots, \mathsf{l}_2, \\ y^{i_1} = f^{i_1}(q,u), & w^{i_1} = g^{i_1}(q,u), & i_1 = 1, \dots, \mathsf{l}_1. \end{array}$$

We then have that

$$\mathcal{K}_1 = \mathrm{sp}\left\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial v^a}\right\}, \qquad \mathcal{K}_1 + \mathcal{K}_2 = \mathrm{sp}\left\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^{i_1}}, \frac{\partial}{\partial v^a}, \frac{\partial}{\partial w^{i_1}}\right\},$$

and Γ takes the form

$$\Gamma = v^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial v^a} + \tilde{F}^{i_1} \frac{\partial}{\partial y^{i_1}} + \tilde{G}^{i_1} \frac{\partial}{\partial w^{i_1}} + \tilde{F}^{i_2} \frac{\partial}{\partial y^{i_2}} + \tilde{G}^{i_2} \frac{\partial}{\partial w^{i_2}},$$

with

$$\tilde{F}^{i_k} = \Gamma(f^{i_k}),\ \tilde{G}^{i_k} = \Gamma(g^{i_k}), \qquad k = 1, 2.$$

Now, since Γ preserves $\mathcal{K}_1$, it follows that

$$\frac{\partial \tilde{F}^{i_k}}{\partial x^a} = \frac{\partial \tilde{F}^{i_k}}{\partial v^a} = \frac{\partial \tilde{G}^{i_k}}{\partial x^a} = \frac{\partial \tilde{G}^{i_k}}{\partial v^a} = 0, \qquad k = 1, 2,$$

and the invariance of $\mathcal{K}_1 + \mathcal{K}_2$ further implies that also

$$\frac{\partial \tilde{F}^{i_2}}{\partial y^{i_1}} = \frac{\partial \tilde{F}^{i_2}}{\partial w^{i_1}} = \frac{\partial \tilde{G}^{i_2}}{\partial y^{i_1}} = \frac{\partial \tilde{G}^{i_2}}{\partial w^{i_1}} = 0.$$

Hence, the transformed differential equations partially decouple in stages, in the following way:

$$\begin{array}{lll} \dot{y}^{i_2} = \tilde{F}^{i_2}(y^{j_2}, w^{j_2}), & \dot{w}^{i_2} = \tilde{G}^{i_2}(y^{j_2}, w^{j_2}), & i_2 = 1, \dots, \mathsf{l}_2 \\ \dot{y}^{i_1} = \tilde{F}^{i_1}(y^{j_2}, w^{j_2}, y^{j_1}, w^{j_1}), & \dot{w}^{i_1} = \tilde{G}^{i_1}(y^{j_2}, w^{j_2}, y^{j_1}, w^{j_1}), & i_1 = 1, \dots, \mathsf{l}_1 \\ \dot{x}^a = v^a, & \dot{v}^a = F^a(x^b, v^b, y^i, w^i), & a = 1, \dots, \mathsf{k}_1. \end{array}$$

Proceeding in the same way to more than two distributions, we reach the following conclusion.

Theorem 4. *Let Γ be a given second-order vector field on TM with its associated horizontal distribution. Denote by ∇ and Φ the corresponding dynamical covariant derivative and Jacobi endomorphism, respectively. Assume that $K_1, \dots, K_s$ are distributions along $\tau : TM \to M$, satisfying the conditions*

$$\nabla K_r \subset \sum_{i=1}^r K_i, \quad \Phi(K_r) \subset \sum_{i=1}^r K_i, \quad \mathrm{D}^{\mathrm{V}}_Z(\sum_{i=1}^r K_i) \subset \sum_{i=1}^r K_i \ \forall Z \in \sum_{i=1}^r K_i,$$

for $r = 1, \dots s$. Then all distributions $\sum_{i=1}^r \mathcal{K}_i$ on TM are integrable and invariant under Γ. It follows that there exist adapted coordinates in which Γ decouples in stages into a hierarchy of first-order systems, each driving the next one, and a second-order system driven by all the first-order ones.

7 Examples and other forms of submersiveness

We shall illustrate first that, in the construction of the previous section, $\mathcal{K}_1$ and $\mathcal{K}_2$ indeed may or may not have a non-empty intersection.

Suppose we are in dimension three and start from a local basis (X_1, X_2, X_3) for $\mathfrak{X}(\tau)$, with dual basis (ϕ_1, ϕ_2, ϕ_3). Assume that distributions, satisfying the conditions of the previous section are:

$$\mathcal{K}_1 = \mathrm{sp}\{X_3{}^H, X_3{}^V\} \qquad \mathcal{K}_2 = \mathrm{sp}\{X_2{}^H, X_2{}^V\},$$

so that

$$(\mathcal{K}_1 + \mathcal{K}_2)^0 = \mathrm{sp}\{\phi_1{}^H, \phi_1{}^V\}, \qquad \mathcal{K}_1{}^0 = \mathrm{sp}\{\phi_1{}^H, \phi_1{}^V, \phi_2{}^H, \phi_2{}^V\}.$$

Then, the result is that, in suitable coordinates, the equations will decouple by stages into two first-order and one second-order system:

$$\begin{array}{ll}
\dot{y}^1 = \tilde{F}^1(y^1, w^1), & \dot{w}^1 = \tilde{G}^1(y^1, w^1), \\
\dot{y}^2 = \tilde{F}^2(y^1, y^2, w^1, w^2), & \dot{w}^2 = \tilde{G}^2(y^1, y^2, w^1, w^2), \\
\dot{x} = v, & \dot{v} = F(x, y, v, w).
\end{array}$$

In dimension four, for example, take

$$\mathcal{K}_1 = \mathrm{sp}\{X_3{}^H, X_4{}^H, X_3{}^V, X_4{}^V\} \qquad \mathcal{K}_2 = \mathrm{sp}\{X_2{}^H, X_4{}^H, X_2{}^V, X_4{}^V\},$$

with again

$$(\mathcal{K}_1 + \mathcal{K}_2)^0 = \mathrm{sp}\{\phi_1{}^H, \phi_1{}^V\}, \qquad \mathcal{K}_1{}^0 = \mathrm{sp}\{\phi_1{}^H, \phi_1{}^V, \phi_2{}^H, \phi_2{}^V\}.$$

This time, there will be two first-order systems and a two-degree-of-freedom second-order system of the following form:

$$\begin{array}{ll}
\dot{y}^1 = \tilde{F}^1(y^1, w^1), & \dot{w}^1 = \tilde{G}^1(y^1, w^1), \\
\dot{y}^2 = \tilde{F}^2(y^1, y^2, w^1, w^2), & \dot{w}^2 = \tilde{G}^2(y^1, y^2, w^1, w^2), \\
\dot{x}^1 = v^1, & \dot{v}^1 = F^1(x, y, v, w), \\
\dot{x}^2 = v^2, & \dot{v}^2 = F^2(x, y, v, w).
\end{array}$$

It is pretty clear that we can model also other types of partial decoupling of the system, in which, for example, a number of first-order systems decouple individually from all the rest and together serve as the driving system for the surviving second-order part. Take for example the case $n = 3$ again, but assume this time that both

$$K_1 = \mathrm{sp}\{X_2, X_3\} \quad \text{and} \quad K_2 = \mathrm{sp}\{X_1, X_3\}$$

satisfy the conditions (2). It follows that $\mathrm{sp}\{\phi_1{}^H, \phi_1{}^V\}$ and $\mathrm{sp}\{\phi_2{}^H, \phi_2{}^V\}$ are integrable co-distributions, which therefore can be spanned by exact 1-forms, say (df^1, dg^1) and (df^2, dg^2) respectively. In new coordinates $(y^i = f^i(q,u), w^i = g^i(q,u), x = q^3, v = u^3)$, Γ will take the form

$$\Gamma = v\frac{\partial}{\partial x} + F\frac{\partial}{\partial v} + \tilde{F}^i\frac{\partial}{\partial y^i} + \tilde{G}^i\frac{\partial}{\partial w^i},$$

and since Γ must preserve the two distributions

$$\mathcal{K}_1 = \mathrm{sp}\left\{\frac{\partial}{\partial y^2}, \frac{\partial}{\partial w^2}, \frac{\partial}{\partial y^3}, \frac{\partial}{\partial w^3}\right\}, \quad \mathcal{K}_2 = \mathrm{sp}\left\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial w^1}, \frac{\partial}{\partial y^3}, \frac{\partial}{\partial w^3}\right\},$$

it follows that the functions $\tilde{F}^1, \tilde{G}^1$ will depend on (y^1, w^1) only, and similarly $\tilde{F}^2, \tilde{G}^2$ will depend on (y^2, w^2) only. Obviously, this can be generalized to arbitrary dimension and any number of integrable co-distributions $\mathrm{sp}\{{\phi_i}^H, {\phi_i}^V\}$.

We end this section with an explicit example, which exhibits both the features of submersiveness and generalized submersiveness. Consider the following system of second-order equations (we use lower indices for coordinates here, to avoid confusion with powers):

$$\begin{aligned} \ddot{q}_1 &= 0, \\ \ddot{q}_2 &= 2\dot{q}_1\dot{q}_2 + \mathrm{e}^{q_1}\dot{q}_1^2. \end{aligned}$$

A set of eigenvectors of the Jacobi endomorphism Φ is given by

$$X_1 = u_1\frac{\partial}{\partial q_1} + u_2\frac{\partial}{\partial q_2}, \qquad X_2 = \frac{\partial}{\partial q_2}.$$

Obviously, $K_1 = \mathrm{sp}\{X_1\}$ and $K_2 = \mathrm{sp}\{X_2\}$ are invariant under Φ. Moreover we have

$$\nabla X_1 = 0, \qquad \text{and} \qquad \nabla X_2 = -u_1\, X_2,$$

so that both distributions along τ are invariant under ∇ as well. Finally,

$$\mathrm{D}^V_{X_1} X_1 = X_1 \qquad \text{and} \qquad \mathrm{D}^V_{X_2} X_2 = 0,$$

so that K_1 and K_2 satisfy all conditions (2).

K_2 is spanned by a basic vector field and therefore actually matches the requirements of standard submersiveness. The integrable distribution $\mathcal{K}_2$ on TM is spanned by $\{\partial/\partial q_2, \partial/\partial u_2\}$ and the SODE Γ projects on the quotient to $\ddot{q}_1 = 0$. In other words, the given system is already written in coordinates adapted to this foliation and displays the corresponding partial decoupling.

Since $\mathrm{D}^V_{X_2} X_1 = X_2$ and so is not in K_1, this distribution is not basic and determines a true case of generalized submersiveness. According to the general theory, it must be possible to find adapted coordinates (y, w) on the quotient manifold N of the foliation determined by $\mathcal{K}_1$, such that a decoupled system of first-order equations for (y, w) becomes the driving system, and $\ddot{q}_1 = 0$ plays the role of the driven part, albeit of a very special nature. It turns out that also the driving part will be of a particularly simple form here. This is due to the fact that Γ itself belongs to the integrable distribution $\mathcal{K}_1$ in this case: we have ${X_1}^V = \Delta$, the dilation vector field on TM, and ${X_1}^H = \Gamma$ (because Γ is a quadratic spray). As a result, the new coordinates (y, w) are bound to be first integrals of Γ, homogeneous of degree zero in the u-coordinates, and the functions $(\tilde{F}, \tilde{G})$ determining the right-hand sides of the first-order equations for (y, w) will be zero as well. The computer algebra packages DIMSYM [15] and EXCALC [14] have been very efficient tools to actually compute two exact forms which span the co-distribution of $\mathcal{K}_1$. The corresponding first integrals are found to be

$$
\begin{aligned}
y &= \frac{u_2}{u_1}\mathrm{e}^{-2q_1} + \mathrm{e}^{-q_1},\\
w &= \mathrm{e}^{q_1} + 2q_2 - \frac{u_2}{u_1}.
\end{aligned}
$$

8 Discussion and areas of application

The theory outlined in the previous sections offers a kind of general scheme, which can then be applied to various situations where the integrable distributions actually come from additional geometrical data.

One such situation occurs in the study of degenerate Lagrangian systems (see e.g. [1]). For a Lagrangian L denote by ω_L its Cartan 2-form. If L is not regular then the characteristic distribution of ω_L, $\mathrm{char}\,\omega_L$, is nonzero; it is integrable, and $S(\mathrm{char}\,\omega_L) \subset \mathrm{char}\,\omega_L$. Any vector field Z such that $\mathcal{L}_Z\omega_L = 0$ has the property that $\mathcal{L}_Z(\mathrm{char}\,\omega_L) \subset \mathrm{char}\,\omega_L$. Let $V(\mathrm{char}\,\omega_L)$ be the vertical part of $\mathrm{char}\,\omega_L$. Degenerate Lagrangians are said to be of type II if $\dim(\mathrm{char}\,\omega_L) = 2\dim(V(\mathrm{char}\,\omega_L))$, and it is shown in [1] that this is equivalent to $S(\mathrm{char}\,\omega_L) = V(\mathrm{char}\,\omega_L)$. Under the general assumptions that L admits a global dynamics, i.e. that the equation $i_Z\omega_L = -dE_L$ (where E_L is the energy function associated to L) has solutions, and that $\mathrm{char}\,\omega_L$ defines a fibration, one further shows that there is a SODE Γ among the dynamical fields Z, and each such Z is $\Gamma \mod \mathrm{char}\,\omega_\mathrm{L}$. Then S defines an almost tangent structure on the leaves of the fibration defined by $\mathrm{char}\,\omega_L$, and $\mathcal{L}_\Gamma(\mathrm{char}\,\omega_L) \subset \mathrm{char}\,\omega_L$. Most of the analysis in [1] in fact is concerned with the case in which $\mathrm{char}\,\omega_L$ is the tangent distribution of a distribution on the base, and thus with the usual case of submersiveness in the sense of [6]. But if one does not impose this extra assumption, one clearly would be looking at a case of generalized submersiveness, as discussed in the present paper.

Another potential area of application is the inverse problem of the calculus of variation, where the additional geometrical element is the availability of (or search for) a suitable metric g along τ. In fact, part of the inspiration for the general set-up we have explained comes from the inverse problem, more specifically from the so-called *separable* case, referred to already in the introduction (see [3]), and the explicit example of the preceding section actually belongs to this category of systems. Work is in progress to use the tools and insights which have been developed in the present paper to identify more of such classes of SODEs of arbitrary dimension, for which a Lagrangian exists. We content ourselves here to some general considerations about the way in which distributions along τ can make their appearance in the inverse problem, and about the interaction of the Helmholtz conditions with the assumptions of generalized submersiveness. The classification of different cases in the study of the inverse problem is most of the time carried out in terms of properties of eigenspaces of the Jacobi endomorphism Φ. So let K_A be an eigenspace of Φ, corresponding to the eigenvalue λ_A. Assume K_A satisfies the conditions of generalized submersiveness, meaning here that we impose two more restrictions. One easily verifies that these restrictions require that

$$[\nabla\Phi, \Phi]|_{K_A} = 0 \quad \text{and} \quad R(K_A, K_A) \subset K_A.$$

These are assumptions on Φ and the curvature tensor R, which are natural generalizations of assumptions made in previous case studies in the inverse problem. A different source of interesting distributions might arise as follows. Let g be a (non-degenerate) symmetric type $(0, 2)$ tensor field along τ and assume K is a distribution satisfying the conditions of generalized submersiveness. In fact, let's assume that g is positive definite for simplicity, so that K has a disjunct orthogonal complement $K^\perp$ and $K \oplus K^\perp$ spans the whole tangent space. The question of interest then becomes how the Helmholtz conditions for g can help to transfer properties from K to $K^\perp$. Obviously, $\nabla g = 0$ will imply that also $\nabla K^\perp \subset K^\perp$, whereas the symmetry of Φ with respect to g will bring about that also $\Phi(K^\perp) \subset K^\perp$. The remaining Helmholtz condition, however, namely $\mathrm{D}^{\mathrm{v}}_X g\,(Y, Z) = \mathrm{D}^{\mathrm{v}}_Y g\,(X, Z)$, does not immediately imply that D^{v}-invariance is inherited by $K^\perp$ as well.

Note that in a different context, the interplay between a distribution K and a Riemannian metric g is the key to understanding so-called "driven cofactor systems", introduced in [7], from a geometrical perspective (see [13]).

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