On hyperovals of polar Grassmannians

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be

Abstract

A hyperoval of a point-line geometry is a nonempty set of points meeting each line in either 0 or 2 points. In this paper, we study hyperovals in line Grassmannians of finite polar spaces of rank 3, hereby often imposing some extra regularity conditions. We determine an upper bound and two lower bounds for the size of such a hyperoval. If equality occurs in one of these bounds, then there is an associated interesting point set of the polar space, like a tight set, an m-ovoid or a set of points having two possible intersection sizes with generators. With the aid of a computer, we have determined all hyperovals of the line Grassmannians of $Q^+(5,2)$, $Q(6,2)$ and $Q^-(7,2)$. Several of the bounds we have found are actually tight for these geometries.

Keywords: hyperoval, line Grassmannian, polar space, tight set, m-ovoid
MSC2010: 51A50, 51E20, 05B25

1 Introduction

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial linear space with (nonempty) point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. A *hyperoval* of $\mathcal{S}$ is a nonempty set X of points having the property that every line of $\mathcal{S}$ has either 0 or 2 of its points in X. The notion of a hyperoval of a partial linear space generalizes the well-known notion of a hyperoval of a projective plane. A hyperoval of a finite projective plane can only exist if the order of that plane is even. Hyperovals of finite polar spaces have already been studied, see [2, 3, 4, 10, 11, 13, 14, 15] for generalized quadrangles (i.e. polar spaces of rank 2) and [4, 5, 6, 12] for polar spaces of rank 3.

In the present paper, we study hyperovals of line Grassmannians of polar spaces of rank 3, and point out some connections between these objects and other interesting combinatorial objects like hyperovals of GQ's and certain nice point sets of polar spaces. These nice point sets comprise tight sets, m-ovoids and sets of points having two possible intersection sizes with generators.

Suppose Π is a polar space of rank 3. For every incident point-plane pair (x, π) of Π, we denote by $L(x, \pi)$ the set of all lines of π through x. Then the line Grassmannian $\mathcal{G}_\Pi$

of Π is the point-line geometry whose points are the lines of Π and whose lines are all the sets $L(x, \pi)$, where π is some plane of Π and $x \in \pi$ (incidence is containment).

Suppose Π is a finite polar space of rank 3 defined by a quadric or symplectic/Hermitian polarity of a finite projective space. If $\mathcal{G}_\Pi$ admits a hyperoval H and π is a plane of Π through an element of H, then every point $x \in \pi$ is contained in either 0 or 2 lines of π that belong to H, showing that the lines of π belonging to H form a hyperoval in the dual plane of π. This is only possible when the underlying field of Π has characteristic 2.

Suppose therefore that q is an even prime power and that Π is one of the following polar spaces of rank 3:

$$Q^+(5, q),\ Q(6, q),\ Q^-(7, q),\ H(5, q) \text{ for } q \text{ square},\ H(6, q) \text{ for } q \text{ square}.$$

For each of these polar spaces, we associate a parameter ϵ, as indicated in the following table:

Polar space	$Q^+(5,q)$	$Q(6,q)$	$Q^-(7,q)$	$H(5,q)$	$H(6,q)$
ϵ	0	1	2	$\frac{1}{2}$	$\frac{3}{2}$

The polar space Π then has $(q^{\epsilon+2}+1)(q^2+q+1)$ points, $(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)(q^2+q+1)$ lines and $(q^\epsilon+1)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$ planes. For every point x of Π, let $\mathrm{Res}(x)$ denote the point-line geometry whose points and lines are the lines and planes of Π through x, with containment as incidence relation. Then $\mathrm{Res}(x)$ is a generalized quadrangle of order (q, q^ϵ).

Suppose H is a hyperoval of the line Grassmannian $\mathcal{G}_\Pi$ of Π. So, H is a set of lines of Π. We call a line of Π *black* if it belongs to H, otherwise we call it *white*. We call a point of Π *black* if it belongs to some black line, otherwise we call it *white*. We call a plane of Π *black* if it contains a black line, otherwise we call it *white*.

If $\alpha \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, then we call the hyperoval H *α-regular* if every black point is incident with precisely α black lines. The hyperoval H is called *regular* if it is α-regular for some $\alpha \in \mathbb{N}^*$. In this case, α is called the *degree* of H.

If π is a black plane then the black lines contained in π form a hyperoval in the dual plane of π. So, π contains precisely $q+2$ black lines and $\frac{(q+2)(q+1)}{2}$ black points incident with at least one of these $q+2$ black lines. We call the black plane π *thin* if it contains no further black points, that means, if all black points of π are incident with some black line of π.

If x is a black point, then we denote by $\mathcal{L}_x$ the set of black lines through x. The following lemma, which is a direct consequence of the definition of hyperoval, describes a close relationship between hyperovals of polar line Grassmannians and hyperovals of GQ's.

Lemma 1.1 *For every black point x, the set $\mathcal{L}_x$ is a hyperoval of the generalized quadrangle* $\mathrm{Res}(x)$.

Among all the polar line Grassmannians that we consider here, the three examples with the smallest number of points are the line Grassmannian $\mathcal{G}^+(2)$ of $Q^+(5, 2)$, the line

Grassmannian $\mathcal{G}(2)$ of $Q(6,2)$ and the line Grassmannian $\mathcal{G}^-(2)$ of $Q^-(7,2)$. (These are the only three examples having fewer than 1500 points.) We have used a computer to determine all hyperovals of these geometries. These hyperovals have been listed in tables that can be found in the appendix. We extract the following information from these tables, hereby following the convention to call a hyperoval H_1 "more symmetric" than another hyperoval H_2 if the order of the setwise stabilizer of H_1 is bigger than the order of the setwise stabilizer of H_2 (in the full group of automorphisms), or equivalently, if the isomorphism class of H_1 is smaller than the isomorphism class of H_2.

- $\mathcal{G}^+(2)$ has up to isomorphism 19 hyperovals. Three of them are regular, namely the three most symmetric hyperovals, and five of them have the property that every black plane is thin, namely the five most symmetric hyperovals.
- $\mathcal{G}(2)$ has up to isomorphism 39 hyperovals. Five of them have the property that every black plane is thin, namely the five most symmetric hyperovals. There are also six regular hyperovals (all of which belong to the 13 most symmetric hyperovals).
- $\mathcal{G}^-(2)$ has up to isomorphism 54 hyperovals. Two of them are regular, namely the two most symmetric hyperovals, and four of them have the property that every black plane is thin, namely the four most symmetric hyperovals.

If the number of symmetries is a measure of how "nice" a hyperoval is, then the above indicates that it might be worthwhile to study hyperovals that are regular and/or have the property that every black plane is thin.

In this paper, we look at line Grassmannians of finite polar spaces of rank 3 that are regular and have the property that every black plane is thin. We derive lower and upper bounds for the size of such a hyperoval. In case one of these bounds is achieved, the set of black points will be a tight set, an m-ovoid or a set of points having two possible intersection sizes with generators.

Suppose X is a set of points of Π. Then X is called an *m-ovoid* (for some $m \in \mathbb{N}$) if it intersects every plane of Π in precisely m points. In general, the number of ordered pairs of distinct collinear points of X is bounded above by $(q+1) \cdot |X| \cdot \left(\frac{|X|}{q^2+q+1} + (q-1) \right)$. If equality occurs, then the set X is called *tight*. Then $i := \frac{|X|}{q^2+q+1} \in \mathbb{N}$ and X is also called *i-tight*. In case of equality, we also have that every point $x \in X$ is collinear with precisely $(i+q-1)(q+1)$ points of $X \setminus \{x\}$ and that every point $y \notin X$ is collinear with precisely $i(q+1)$ points of X. Tight sets were introduced by Payne [16] for generalized quadrangles and by Drudge [8] for arbitrary polar spaces. We refer to these references for proofs of the above-mentioned facts.

We will prove the following results in this paper.

Theorem 1.2 (Section 3) *Suppose H is an α-regular hyperoval of $\mathcal{G}_\Pi$ having the property that every black plane is thin. Then the numbers $\frac{|H|\cdot(q+1)}{\alpha}$ and $\frac{|H|\cdot(q^\epsilon+1)}{q+2}$ are integral. Moreover,*

$$\frac{1}{2}(q+2)(q^{\epsilon+1}+1)\Big(\alpha(q+1) - 2q(q^\epsilon+1)\Big) \le |H| \le \frac{1}{2}\alpha(q+2)(q^{\epsilon+2}+1),$$

with equality in (at least) one of these bounds if and only if every white plane contains a constant number of black points. If the lower bound occurs, then every white plane contains precisely $\frac{(q+1)(q+2)}{2} - \frac{q(q+2)(q^\epsilon+1)}{\alpha}$ black points. If the upper bound occurs, then the set of black points is an m-ovoid with $m = \frac{(q+1)(q+2)}{2}$.

Theorem 1.3 (Section 3) *Suppose H is an α-regular hyperoval of $\mathcal{G}_\Pi$ having the property that every black plane is thin. Then we have*

$$|H| \geq \frac{1}{2}\alpha(q^2+q+1)(q^{\epsilon+1}-q+2),$$

with equality if and only if the set of black points is a tight set of points.

Inspection of all hyperovals of $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$ tells us that the bounds of Theorems 1.2 and 1.3 are actually tight for several of the possible values of α. In fact, for regular hyperovals for which all black planes are thin, the divisibility conditions and bounds of Theorems 1.2, 1.3 in combination with an additional argument allows to almost completely determine the spectrum of possible hyperoval sizes for the geometries $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$, see the appendix.

The lower bound in Theorem 1.2 can obviously only be achieved if $\frac{q(q+2)(q^\epsilon+1)}{\alpha} \in \mathbb{N}$. By refining some of our arguments, we will derive an inequality in Section 4 which is stronger than this lower bound if $\frac{q(q+2)(q^\epsilon+1)}{\alpha} \notin \mathbb{N}$.

From the fact that $\mathcal{L}_x$ is a hyperoval of $\mathrm{Res}(x)$ for every black point x, it can be deduced that $\max(2q^\epsilon+2, (q^\epsilon-q+2)(q+1)) \leq \alpha \leq 2(q^{\epsilon+1}+1)$, see Lemma 2.2. The upper and lower bound in Theorem 1.2 coincide if and only if $\alpha = 2(q^{\epsilon+1}+1)$. So, if $\alpha = 2(q^{\epsilon+1}+1)$, then we should have $|H| = (q+2)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$. In fact, in this case the requirement that every black plane is thin is not necessary. We will see later (Proposition 2.9) that in the case of a regular hyperoval of degree $2(q^{\epsilon+1}+1)$, every black plane needs to be thin.

Put $\alpha^* := \frac{2(q+2)(q^\epsilon+1)(q^{\epsilon+1}+1)}{2q^{\epsilon+1}+q^\epsilon+q^2+2}$. Both Theorems 1.2 and 1.3 give lower bounds for $|H|$. Depending on whether $\alpha < \alpha^*$ or $\alpha > \alpha^*$, the lower bound given in Theorem 1.3 will be stronger or weaker than the one given in Theorem 1.2. It can easily be shown that $\max(2q^\epsilon+2, (q^\epsilon-q+2)(q+1)) < \alpha^* < 2(q^{\epsilon+1}+1)$. So, in some cases Theorem 1.2 will provide the best lower bound. In other cases, Theorem 1.3 will do that.

2 Basic properties of hyperovals

A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *generalized quadrangle* (GQ) of order (s,t) if every line is incident with precisely $s+1$ points, every point is incident with precisely $t+1$ lines and for every non-incident point-line pair (x, L), there exists a unique point on L collinear with x. The following lemma is a special case of [1, Lemmas 3.9 and 3.11], see also Theorems 2.1 and 2.2 of [7] for the lower bounds.

Lemma 2.1 *Suppose X is a hyperoval of a generalized quadrangle $\mathcal{Q}$ of order (s,t). Then $|X|$ is even and* $\max(2(t+1),(t-s+2)(s+1)) \leq |X| \leq 2(st+1)$.

Suppose now that q is an even prime power and that Π is one of the following polar spaces of rank 3:

$$Q^+(5,q),\ Q(6,q),\ Q^-(7,q),\ H(5,q) \text{ for } q \text{ square},\ H(6,q) \text{ for } q \text{ square}.$$

Let $\epsilon \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ be the parameter associated to Π as defined in Section 1.

Suppose that H is a hyperoval of the line Grassmannian $\mathcal{G}_\Pi$ of Π. As we saw in Section 1, we can then talk about black and white points, black and white lines, black and white planes. For every black point x, we put $\alpha_x := |\mathcal{L}_x|$, where $\mathcal{L}_x$ denotes the set of black lines through x. The following is an immediate consequence of Lemmas 1.1 and 2.1.

Lemma 2.2 *For every black point x, the number α_x is even and* $\max(2q^\epsilon+2,(q^\epsilon-q+2)(q+1)) \leq \alpha_x \leq 2(q^{\epsilon+1}+1)$.

Lemma 2.3 (1) *Every black plane π contains precisely $q+2$ black lines and $\frac{(q+2)(q+1)}{2}$ black points that are incident with at least one of these $q+2$ black lines. The $q+2$ black lines of π form a hyperoval in the dual plane of π.*

(2) *The total number of black planes is equal to $\frac{q^\epsilon+1}{q+2} \cdot |H|$. If H is α-regular, then the total number of black points is equal to $\frac{q+1}{\alpha} \cdot |H|$.*

Proof. The first claim was already explained in Section 1. As for the second claim, this follows from straightforward double counting. ∎

Lemma 2.4 *Suppose π is a black plane and L is a white line of π. Then L contains precisely $\frac{q+2}{2}$ black points that are incident with some black line of π. As consequence, if π is a thin black plane, then every white line of π is incident with precisely $\frac{q+2}{2}$ black points.*

Proof. Each of the $q+2$ black lines of π intersects L in a black point, and every black point of L that arises in this way is contained in precisely two black lines of π. Hence, L contains precisely $\frac{q+2}{2}$ black points that are incident with some black line of π. ∎

Lemma 2.5 *Suppose H is an α-regular hyperoval having the property that every black plane is thin. Then every black point is contained in $\frac{\alpha(q^\epsilon+1)}{2}$ black planes and $\frac{(q^\epsilon+1)(2q^{\epsilon+1}+2-\alpha)}{2}$ white planes.*

Proof. The total number of planes through x is equal to $(q^\epsilon+1)(q^{\epsilon+1}+1)$. We count the number of black planes through x containing at least one and hence precisely two black lines through x. Each of the α black lines through x is contained in $q^\epsilon+1$ (necessarily black) planes. So, we get a list of $\alpha(q^\epsilon+1)$ (not necessarily distinct) black planes, and every black plane occurs either 0 or 2 times in this list. So, the total number of planes through x containing at least one black line through x is equal to $\frac{1}{2}\alpha(q^\epsilon+1)$. Let π be one of the $(q^\epsilon+1)(q^{\epsilon+1}+1)-\frac{1}{2}\alpha(q^\epsilon+1)=\frac{1}{2}(q^\epsilon+1)(2q^{\epsilon+1}+2-\alpha)$ planes through x not containing any black line through x. Then π cannot be a black plane, since otherwise the plane would contain a black point (namely x) that is not contained in any of the $q+2$ black lines of π, in contradiction with the fact that π would be thin. ∎

Lemma 2.6 (1) *If π is a black plane, then there are $(q+2)q^\epsilon$ black planes meeting π in a black line, $\frac{1}{q+2}\Big(\sum_{x\in\pi}\alpha_x\Big)-(q+1)(q^\epsilon+1)$ black planes meeting π in a white line and $(q+1)(q^{\epsilon+1}+1)-\frac{1}{q+2}\Big(\sum_{x\in\pi}\alpha_x\Big)$ white planes meeting π in a (necessarily white) line.*

(2) *If π is a white plane, then there are precisely $\frac{1}{q+2}\cdot\sum_{x\in\pi}\alpha_x$ black planes meeting π in a (necessarily white) line and $(q^2+q+1)q^\epsilon-\frac{1}{q+2}\sum_{x\in\pi}\alpha_x$ white planes meeting π in a (necessarily white) line.*

Proof. Suppose π is a black plane. There are $q+2$ black lines contained in π and each of these black lines is incident with precisely q^ϵ black planes distinct from π. So, there are $(q+2)q^\epsilon$ black planes that meet π in a black line. Now, call N the number of black planes that meet π in a white line. Counting in two different ways the number of pairs (L,π'), where L is a black line meeting π in a singleton and π' is the (necessarily unique) black plane through L meeting π in a line, we find

$$\Big(\sum_{x\in\pi}\alpha_x\Big)-(q+1)(q+2)=N\cdot(q+2)+(q+2)q^\epsilon\cdot(q+1). \tag{1}$$

Indeed, since there are $(q+2)(q+1)$ possibilities for an incident point-line pair in π where both objects are black, the number of possibilities for L is equal to $\Big(\sum_{x\in\pi}\alpha_x\Big)-(q+1)(q+2)$. On the other hand, for each of the N black planes meeting π in a white line, there are $q+2$ possibilities for L, and for each of the $(q+2)q^\epsilon$ planes meeting π in a black line, there are $q+1$ possibilities for L. From equation (1), we deduce that $N=\frac{1}{q+2}\Big(\sum_{x\in\pi}\alpha_x\Big)-(q+1)(q^\epsilon+1)$. Now, since there are $(q^2+q+1)q^\epsilon$ planes meeting π in a line, the total number of white planes meeting π in a line is equal to $(q^2+q+1)q^\epsilon-(q+2)q^\epsilon-N=(q+1)(q^{\epsilon+1}+1)-\frac{1}{q+2}\cdot\sum_{x\in\pi}\alpha_x$.

The case where π is a white plane is similar. In this case, we obtain the equality $\sum_{x\in\pi}\alpha_x=N\cdot(q+2)$, where N is the total number of black planes meeting π in a (necessarily white) line. ∎

Proposition 2.7 (1) *If π is a black plane, then $(q+1)(q+2)(q^\epsilon+1)\leq\sum_{x\in\pi}\alpha_x\leq(q+1)(q+2)(q^{\epsilon+1}+1)$. The lower bound is achieved if and only if there are no black planes meeting π in a white line. The upper bound is achieved if and only if every plane meeting π in a line is black. If the lower bound is achieved, then π is thin and $\alpha_x=2(q^\epsilon+1)$ for every black point x of π.*

(2) *If π is a white plane, then $\sum_{x\in\pi}\alpha_x \leq (q^2+q+1)(q+2)q^\epsilon$, with equality if and only if every plane meeting π in a line is black.*

Proof. The numbers occurring in Lemma 2.6 need to be nonnegative implying the lower and upper bounds, and most other claims of the proposition. If π is a black plane, then π contains at least $\frac{(q+2)(q+1)}{2}$ black points and $\alpha_x \geq 2(q^\epsilon+1)$ for each of these black points (see Lemma 2.2). So, we also have $\sum_{x\in\pi}\alpha_x \geq (q^\epsilon+1)(q+2)(q+1)$, with equality if and only if π is thin and $\alpha_x = 2(q^\epsilon+1)$ for every black point x of π. ∎

Proposition 2.8 *We have $|H| \leq (q+2)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$, with equality if and only if every plane is black.*

Proof. By Lemma 2.3, the total number of black planes is equal to $\frac{q^\epsilon+1}{q+2}\cdot|H|$. The proposition immediately follows from the observation that this number is at most the total number $(q^\epsilon+1)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$ of planes of Π. ∎

By Lemma 2.2, the degree of a regular hyperoval is bounded above by $2(q^{\epsilon+1}+1)$. We now show that if the degree attains this maximal value, then the size of the hyperoval must attain the upper bound stated in Proposition 2.8.

Proposition 2.9 *Suppose H is α-regular with $\alpha = 2(q^{\epsilon+1}+1)$. Then $|H| = (q+2)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$ and every black plane is thin.*

Proof. To show that $|H| = (q+2)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)$, it suffices by Proposition 2.8 to show that every plane is black. Now, the graph on the planes of Π, where two planes are adjacent whenever they intersect in a line, is connected. So, in order to prove the proposition, it suffices to prove the following for every black plane π of Π: (1) every plane meeting π in a line is black; (2) π is thin.

Denote by N the total number of black points contained in π. Then $\frac{(q+2)(q+1)}{2} \leq N$ with equality if and only if π is thin. By Proposition 2.7, $\frac{(q+1)(q+2)}{2}\cdot 2(q^{\epsilon+1}+1) \leq N\cdot 2(q^{\epsilon+1}+1) = \sum_{x\in\pi}\alpha_x \leq (q+1)(q+2)(q^{\epsilon+1}+1)$. So, we have that $N = \frac{(q+1)(q+2)}{2}$, implying that π is thin, and $\sum_{x\in\pi}\alpha_x = (q+1)(q+2)(q^{\epsilon+1}+1)$ implying that every plane meeting π in a line is black. ∎

3 Proofs of Theorems 1.2 and 1.3

Suppose H is an α-regular hyperoval of the line Grassmannian $\mathcal{G}_\Pi$ of Π having the property that every black plane is thin. We denote by B the set of black points and by W the set of white planes. For every black plane π, let W_π denote the set of all white planes meeting π in a line. For every white plane π, let N_π denote the number of black points inside π.

The aim of this section is to prove Theorems 1.2 and 1.3. The proofs of these results will make use of the precise values of $\sum_{\pi\in W}1$, $\sum_{\pi\in W}N_\pi$ and $\sum_{\pi\in W}N_\pi^2$ that we will determine. The computations of $\sum_{\pi\in W}1$ and $\sum_{\pi\in W}N_\pi$ will be rather straightforward. This seems however not to be the case for the sum $\sum_{\pi\in W}N_\pi^2$ whose value will be computed via a detour, hereby making use of some of the lemmas of Section 2.

Proposition 3.1 *We have*

$$\sum_{\pi \in W} 1 = (q^\epsilon + 1)(q^{\epsilon+1} + 1)(q^{\epsilon+2} + 1) - \frac{|B| \cdot \alpha(q^\epsilon + 1)}{(q+1)(q+2)}, \qquad (2)$$

$$\sum_{\pi \in W} N_\pi = |B| \cdot \frac{1}{2}(q^\epsilon + 1)(2q^{\epsilon+1} + 2 - \alpha). \qquad (3)$$

Proof. The first equality follows from the fact that Π has $(q^\epsilon + 1)(q^{\epsilon+1} + 1)(q^{\epsilon+2} + 1)$ planes, $\frac{|B| \cdot \alpha(q^\epsilon+1)}{(q+2)(q+1)}$ of which are black (see Lemma 2.3). The second equality can be obtained from counting the number of pairs (x, π), where x is a black point and π is a white plane incident with x, hereby taking into account Lemma 2.5. ■

Lemma 3.2 *We have $W = \emptyset$ if and only if $\alpha = 2(q^{\epsilon+1} + 1)$.*

Proof. If $W = \emptyset$, then equation (3) implies that $\alpha = 2q^{\epsilon+1} + 2$. Conversely, if $\alpha = 2(q^{\epsilon+1} + 1)$, then Propositions 2.8 and 2.9 imply that $W = \emptyset$. ■

Lemma 3.3 *For every black plane π, we have*

$$|B| = \frac{(q+1)^2(q+2)}{4} \cdot \alpha - \frac{q(q+1)(q+2)}{2} \cdot (q^\epsilon + 1) + \sum_{\pi' \in W_\pi} N_{\pi'}.$$

Proof. For every plane π' intersecting π in a line, let $M_{\pi'}$ denote the number of black points in $\pi' \setminus \pi$. Since every point outside π is incident with a unique plane meeting π in a line, we have $|B| = M_1 + \sum M_{\pi'}$, where the summation ranges over all planes π' intersecting π in a line and $M_1 = \frac{(q+2)(q+1)}{2}$ denotes the total number of black points inside π. Now, by Lemma 2.6 there are

- $(q+2)q^\epsilon$ black planes meeting π in a black line, each contributing $\frac{(q+2)(q+1)}{2} - (q+1) = \frac{q(q+1)}{2}$ extra black points;
- $\frac{1}{2}(q+1)(\alpha - 2q^\epsilon - 2)$ black planes meeting π in a white line, each contributing $\frac{(q+2)(q+1)}{2} - \frac{q+2}{2} = \frac{q(q+2)}{2}$ extra black points.
- $\frac{1}{2}(q+1)(2q^{\epsilon+1} + 2 - \alpha)$ white planes meeting π in a (necessarily white) line. Each such white plane π' contributes $N_{\pi'} - \frac{q+2}{2}$ extra black points.

Hence, the total number of black points is equal to

$$\begin{aligned} &\frac{1}{2}(q+1)(q+2) + (q+2)q^\epsilon \cdot \frac{q(q+1)}{2} + \frac{1}{2}(q+1)(\alpha - 2q^\epsilon - 2) \cdot \frac{q(q+2)}{2} \\ &+ \sum_{\pi' \in W_\pi} N_{\pi'} - \frac{1}{2}(q+2) \cdot \frac{1}{2}(q+1)(2q^{\epsilon+1} + 2 - \alpha) \\ &= \frac{(q+1)^2(q+2)}{4}\alpha - \frac{q(q+1)(q+2)}{2}(q^\epsilon + 1) + \sum_{\pi' \in W_\pi} N_{\pi'}. \end{aligned}$$

■

Proposition 3.4 *We have*

$$\sum_{\pi \in W} N_\pi^2 = \frac{q^\epsilon + 1}{q+1} \cdot |B|^2 - \Big(\frac{(q+1)(q+2)(q^\epsilon+1)}{4}\alpha - \frac{q(q+2)}{2}(q^\epsilon+1)^2 \Big) \cdot |B|. \qquad (4)$$

Proof. For every black plane π, we have

$$|B| = \frac{(q+1)^2(q+2)}{4} \cdot \alpha - \frac{q(q+1)(q+2)}{2} \cdot (q^\epsilon + 1) + \sum_{\pi' \in W_\pi} N_{\pi'}.$$

If we sum the last equation over all $\frac{|B| \cdot \alpha(q^\epsilon+1)}{(q+1)(q+2)}$ black planes, then taking into account that for every white plane π', there are $\frac{1}{q+2} \cdot \alpha \cdot N_{\pi'}$ black planes meeting π in a line, we see that

$$\frac{|B|^2 \cdot \alpha(q^\epsilon+1)}{(q+1)(q+2)} = \Big(\frac{(q+1)^2(q+2)}{4}\alpha - \frac{q(q+1)(q+2)}{2}(q^\epsilon+1) \Big) \cdot \frac{|B| \cdot \alpha(q^\epsilon+1)}{(q+1)(q+2)} + \frac{\alpha}{q+2} \sum_{\pi \in W} N_\pi^2,$$

from which the stated equality follows. ∎

We have now collected enough information to prove Theorems 1.2 and 1.3. The Cauchy-Schwartz inequality $\Big(\sum_{\pi \in W} 1\Big) \cdot \Big(\sum_{\pi \in W} N_\pi^2\Big) \geq \Big(\sum_{\pi \in W} N_\pi\Big)^2$ implies that

$$\Big((q^{\epsilon+1}+1)(q^{\epsilon+2}+1) - \frac{|B| \cdot \alpha}{(q+1)(q+2)} \Big) \cdot \Big(\frac{|B|}{q+1} - \Big(\frac{(q+1)(q+2)}{4}\alpha - \frac{q(q+2)}{2}(q^\epsilon+1) \Big) \Big)$$
$$\geq \frac{|B|}{4} \cdot (2q^{\epsilon+1} + 2 - \alpha)^2,$$

i.e.

$$\frac{\alpha \cdot |B|^2}{(q+2)(q+1)^2} - \Big(\frac{\alpha \cdot (2q^{\epsilon+2} + q^{\epsilon+1} + q + 2)}{2(q+1)} - \frac{q(q^\epsilon+1)(q^{\epsilon+1}+1)}{q+1} \Big) \cdot |B|$$
$$+ \frac{(q+2)(q^{\epsilon+1}+1)(q^{\epsilon+2}+1)}{4} \Big((q+1)\alpha - 2q(q^\epsilon+1) \Big) \leq 0.$$

The latter equation is equivalent with $\frac{\alpha}{(q+1)^2(q+2)}(|B| - r_1) \cdot (|B| - r_2) \leq 0$, where

$$r_1 = \frac{(q+1)(q+2)(q^{\epsilon+1}+1)}{2\alpha} \Big(\alpha(q+1) - 2q(q^\epsilon+1) \Big),$$
$$r_2 = \frac{(q+1)(q+2)(q^{\epsilon+2}+1)}{2}.$$

Since $\alpha \leq 2(q^{\epsilon+1}+1)$, we have $r_2 - r_1 = \frac{q(q+1)(q+2)(q^\epsilon+1)}{2\alpha}(2q^{\epsilon+1} + 2 - \alpha) \geq 0$. Hence,

$$r_1 \leq |B| \leq r_2 \qquad \text{and} \qquad \frac{r_1 \alpha}{q+1} \leq |H| \leq \frac{r_2 \alpha}{q+1}.$$

We have equality in the above-mentioned Cauchy-Schwartz inequality if and only if N_π is constant. In the case where $W \neq \emptyset$, this happens precisely when every white plane contains precisely $\overline{N}$ black points, where $\overline{N} := \frac{\sum_{\pi\in W} N_\pi}{\sum_{\pi \in W} 1}$ is the average number of black points inside a white plane.

If $|B| = r_1$, then one computes that every white plane is incident with precisely $\frac{(q+1)(q+2)}{2} - \frac{q(q+2)(q^\epsilon+1)}{\alpha}$ black points. If $|B| = r_2$, then one computes that every white plane is incident with precisely $\frac{(q+2)(q+1)}{2}$ black points. So, in the latter case, the set B is an m-ovoid with $m = \frac{(q+2)(q+1)}{2}$.

In the general case, the inequality $r_1 \leq |B| \leq r_2$ implies that

$$\frac{(q+1)(q+2)}{2} - \frac{q(q+2)(q^\epsilon+1)}{\alpha} \leq \overline{N} \leq \frac{(q+2)(q+1)}{2}$$

if $W \neq \emptyset$. We have now finished the proof of Theorem 1.2. Recall that the divisibility conditions of that theorem have already been proved in Lemma 2.3. ■

We are now also able to prove Theorem 1.3. The total number of black planes is equal to $\frac{|B|\cdot\alpha(q^\epsilon+1)}{(q+1)(q+2)}$. Since every line of Π is incident with precisely $q^\epsilon + 1$ planes of Π, the total number of ordered pairs of distinct collinear points of B is equal to

$$\frac{1}{q^\epsilon+1}\Big(\frac{|B|\cdot\alpha(q^\epsilon+1)}{(q+1)(q+2)} \cdot \frac{(q+2)(q+1)}{2} \cdot (\frac{(q+2)(q+1)}{2} - 1) + \sum_{\pi\in W} N_\pi(N_\pi - 1)\Big).$$

Making use of Propositions 3.1 and 3.4, we see that this number is equal to

$$\frac{|B|^2}{q+1} + \frac{q(q+2)}{2}(q^\epsilon+1)\cdot|B| - (q^{\epsilon+1}+1)\cdot|B|.$$

This number is at most $(q+1)\cdot|B|\cdot\Big(\frac{|B|}{q^2+q+1} + (q-1)\Big)$, with equality if and only if the set of points is tight. The obtained inequality is equivalent to $|B| \geq \frac{(q+1)(q^2+q+1)}{2}(q^{\epsilon+1} - q + 2)$. Hence, $|H| \geq \frac{1}{2}\alpha(q^2+q+1)(q^{\epsilon+1} - q + 2)$. ■

Remark. Suppose $W \neq \emptyset$. Then we can consider the adjacency matrix A of the black and white planes, that is the 0-1 matrix whose rows are indexed by the black planes and whose columns are indexed by the white planes, with an entry being equal to 1 if and only if the corresponding black plane intersects the corresponding white plane in a line. Let W' denote the set of white planes containing a black point, and let A' denote the submatrix of A obtained by considering only those columns that correspond to the elements of W'. The equations in Lemma 3.3 determine a linear system with $|W'|$ unknowns whose corresponding matrix is A'. The rank of A' is at most $|W'|$. If the rank equals $|W'|$, then the linear system implies that all values N_π, $\pi \in W'$, are constant, say equal to $\widetilde{N}$. So, in this case every white plane has either 0 or $\widetilde{N}$ black points. Inspection of all hyperovals of $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$ tells us that it is quite common for the rank of A' to be equal to $|W'|$. However, it seems to be less common for the most symmetric hyperovals, in particular for the regular hyperovals that only admit black planes that are thin.

4 An improved lower bound for $|H|$

Let H be an α-regular hyperoval of the line Grassmannian $\mathcal{G}_\Pi$ of Π having the property that every black plane is thin. We continue with the notation introduced in Section 3. Suppose $W \neq \emptyset$. Since $N_\pi \in \mathbb{N}$ for every $\pi \in W$, we have

$$\sum_{\pi \in W} \Big(N_\pi - n \Big) \Big(N_\pi - (n+1) \Big) \geq 0, \tag{5}$$

for every $n \in \mathbb{Z}$, with equality if and only if every white plane contains either n or $n+1$ black points. The inequality (5) is equivalent to

$$\Big(\sum_{\pi \in W} 1 \Big) \cdot \Big(\sum_{\pi \in W} N_\pi^2 \Big) - (2n+1) \cdot \Big(\sum_{\pi \in W} 1 \Big) \cdot \Big(\sum_{\pi \in W} N_\pi \Big) + (n^2+n) \cdot \Big(\sum_{\pi \in W} 1 \Big)^2 \geq 0. \tag{6}$$

If we put $x := y - n$, where $y := \frac{\sum_{\pi \in W} N_\pi}{\sum_{\pi \in W} 1}$, then the inequality (6) becomes

$$\Big(\sum_{\pi \in W} 1 \Big) \cdot \Big(\sum_{\pi \in W} N_\pi^2 \Big) - \Big(\sum_{\pi \in W} N_\pi \Big)^2 + (x^2 - x) \cdot \Big(\sum_{\pi \in W} 1 \Big)^2 \geq 0. \tag{7}$$

The value for $x^2 - x$ is always positive, except when $x \in [0,1]$. In view of the fact that $y - x = n$ needs to be integral, we thus see that the strongest bound in (7) is obtained if we put $x = y - \lfloor y \rfloor \in [0,1[$. Notice also that if $y \in \mathbb{N}$, then the strongest bound in (7) is obtained for two values of x, namely the values $x = y - \lfloor y \rfloor = 0$ and $x = 1$. In this case, the inequality (7) reduces to the Cauchy-Schwartz inequality $\Big(\sum_{\pi \in W} 1 \Big) \cdot \Big(\sum_{\pi \in W} N_\pi^2 \Big) - \Big(\sum_{\pi \in W} N_\pi \Big)^2 \geq 0$, which we already employed earlier. Observe also that if we have equality in (7), then every white plane contains precisely n or $n+1$ black points, where $n = y - x = \lfloor y \rfloor$. Making use of the computations done in Section 3, we can thus conclude:

Proposition 4.1 *Suppose H is an α-regular hyperoval of $\mathcal{G}_\Pi$ having the property that every black plane is thin. Let B denote the set of black points of H. Suppose $y_1 := (q^\epsilon + 1)(q^{\epsilon+1} + 1)(q^{\epsilon+2} + 1) - \frac{|B| \cdot \alpha (q^\epsilon + 1)}{(q+1)(q+2)} > 0$. Put $y_2 := |B| \cdot \frac{1}{2}(q^\epsilon + 1)(2q^{\epsilon+1} + 2 - \alpha)$, $x := \frac{y_2}{y_1} - \lfloor \frac{y_2}{y_1} \rfloor \in [0,1[$, $r_1 := \frac{(q+1)(q+2)(q^{\epsilon+1}+1)}{2\alpha} \Big(\alpha(q+1) - 2q(q^\epsilon + 1) \Big)$ and $r_2 := \frac{1}{2}(q+1)(q+2)(q^{\epsilon+2}+1)$. Then*

$$|B| \cdot (q^\epsilon + 1)^2 \cdot (|B| - r_1)(r_2 - |B|) \geq y_1^2 (x - x^2), \tag{8}$$

with equality if and only if there exist two consecutive integers n and $n+1$ such that every white plane contains either n or $n+1$ black points. In fact, if equality holds, then every white plane contains either n_1 or n_2 black points, where $n_1 = \lfloor \frac{y_2}{y_1} \rfloor$ and $n_2 = \lceil \frac{y_2}{y_1} \rceil$.

Remarks. (1) Since $x \in [0,1[$, we have $y_1^2(x-x^2) \geq 0$ and so the inequality (8) is stronger then the previous obtained inequality, which was equivalent to $(|B|-r_1)(r_2-|B|) \geq 0$.

(2) The numbers n_1 and n_2 are consecutive integers if $\frac{y_2}{y_1} \notin \mathbb{N}$. If $\frac{y_2}{y_1} \in \mathbb{N}$, then $n_1 = n_2 = \frac{y_2}{y_1}$ and we find again that if $|B| \in \{r_1, r_2\}$, then every white plane contains precisely $\frac{y_2}{y_1}$ black points.

A The hyperovals of $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$

We have used the computer algebra system GAP [9] to determine all hyperovals of the geometries $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$. The aim of this appendix is to list all those hyperovals together with some of their properties. The computer code and a library of the hyperplanes can be found online (see `http://cage.ugent.be/geometry`).

We denote by Π the underlying polar space of $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ or $\mathcal{G}^-(2)$. Since all lines of $\mathcal{G}_\Pi$ are incident with precisely three points, we observe that a nonempty set H of points of $\mathcal{G}_\Pi$ is a hyperoval if and only if its characteristic vector is orthogonal with the characteristic vectors of all lines of $\mathcal{G}_\Pi$. Here, all characteristic vectors are regarded over the smallest field $\mathbb{F}_2 = \mathrm{GF}(2)$. Based on this observation, we have determined all hyperovals of $\mathcal{G}_\Pi$. The hyperovals we found in this way are listed in Tables 1 till 4. It turns out that $\mathcal{G}^+(2)$ has 32767 hyperovals which fall into 19 isomorphism classes, $\mathcal{G}(2)$ has 2097151 hyperovals which fall into 39 isomorphism classes and $\mathcal{G}^-(2)$ has 268435455 hyperovals which fall into 54 isomorphism classes. We have also written GAP code to determine some of the elementary properties of these hyperplanes. This information can also be found in the tables. For each hyperoval H, we have listed the following information:

- the number N of hyperovals in its isomorphism class;
- its size $|H|$;
- the numbers $|\mathcal{D}_i|$, $i \in I$, where I denotes the set of possible degrees for the points and $\mathcal{D}_i$ denotes the set of points of degree i (by convention, a white point has degree 0);
- the number $|\mathcal{B}_1|$, where $\mathcal{B}_1$ denotes the set of thin black planes;
- the number $|\mathcal{B}_2|$, where $\mathcal{B}_2$ denotes the set of non-thin black planes;
- in case there are white planes, WP denotes the array of the form $n_1^{e_1} n_2^{e_2} \ldots n_k^{e_k}$, where $k \in \{1,\ldots,8\}$, $n_1, n_2, \ldots, n_k \in \{0,\ldots,7\}$ with $n_1 < n_2 < \cdots < n_k$ and $e_1, e_2, \ldots, e_k \in \mathbb{N}^*$ such that for every $i \in \{1,2,\ldots,k\}$, there are precisely e_i white planes containing precisely n_i black points (note that $e_1 + e_2 + \ldots + e_k$ is the total number of white planes).

For some of the hyperovals occurring in Tables 1 through 4, it is possible to give an easy explicit description. The hyperoval H_3 of $\mathcal{G}^+(2)$ consists of all lines that are disjoint with two given mutually disjoint planes. We have also verified that the remaining

H	N	$\vert H\vert$	$\vert\mathcal{D}_0\vert$	$\vert\mathcal{D}_4\vert$	$\vert\mathcal{D}_6\vert$	$\vert\mathcal{B}_1\vert$	$\vert\mathcal{B}_2\vert$	WP
H_1	56	60	5	0	30	30	0	–
H_2	105	32	11	24	0	16	0	$0^2 4^{12}$
H_3	120	28	14	21	0	14	0	$0^2 3^{14}$
H_4	210	48	7	12	16	24	0	4^6
H_5	280	36	11	18	6	18	0	3^{12}
H_6	672	60	0	15	20	0	30	–
H_7	840	60	1	12	22	6	24	–
H_8	1120	48	2	27	6	6	18	6^6
H_9	1260	48	3	24	8	8	16	$4^2 6^4$
H_{10}	1344	60	0	15	20	0	30	–
H_{11}	1680	44	5	24	6	10	12	$3^2 5^6$
H_{12}	2520	56	1	18	16	4	24	6^2
H_{13}	2520	52	3	18	14	10	16	5^4
H_{14}	2520	44	6	21	8	14	8	$3^2 4^2 5^4$
H_{15}	2880	56	0	21	14	0	28	7^2
H_{16}	2880	56	0	21	14	0	28	7^2
H_{17}	3360	56	2	15	18	10	18	6^2
H_{18}	3360	52	2	21	12	8	18	6^4
H_{19}	5040	52	2	21	12	6	20	$5^2 6^2$

Table 1: The hyperovals of the line Grassmannian $\mathcal{G}^+(2)$ of $Q^+(5,2)$

H	N	$\lvert H\rvert$	$\lvert\mathcal{D}_0\rvert$	$\lvert\mathcal{D}_6\rvert$	$\lvert\mathcal{D}_8\rvert$	$\lvert\mathcal{D}_{10}\rvert$	$\lvert\mathcal{B}_1\rvert$	$\lvert\mathcal{B}_2\rvert$	WP
H_1	315	128	15	0	48	0	96	0	$0^3 4^{36}$
H_2	336	120	18	0	45	0	90	0	3^{45}
H_3	378	160	11	0	20	32	120	0	4^{15}
H_4	630	96	19	32	12	0	72	0	$0^6 4^{57}$
H_5	1008	120	16	20	15	12	90	0	$3^{15} 4^{30}$
H_6	1120	144	0	36	27	0	0	108	7^{27}
H_7	2880	168	0	0	63	0	0	126	7^9
H_8	3780	160	3	0	60	0	24	96	$4^3 6^{12}$
H_9	3780	128	7	32	24	0	32	64	$0^1 4^{14} 6^{24}$
H_{10}	4032	160	3	0	60	0	30	90	6^{15}
H_{11}	4320	112	14	28	21	0	56	28	$0^2 3^{14} 4^{14} 5^{21}$
H_{12}	5040	168	4	8	27	24	42	84	5^9
H_{13}	5040	152	6	0	57	0	42	72	$3^3 5^{18}$
H_{14}	10080	176	0	12	27	24	0	132	7^3
H_{15}	11340	160	3	16	28	16	24	96	$4^3 6^{12}$
H_{16}	12096	160	1	20	30	12	10	110	$6^5 7^{10}$
H_{17}	15120	176	1	8	30	24	12	120	6^3
H_{18}	15120	168	2	12	29	20	18	108	$5^3 6^6$
H_{19}	15120	152	4	20	27	12	26	88	$3^1 4^2 5^6 6^{12}$
H_{20}	15120	136	8	24	23	8	42	60	$3^6 5^{27}$
H_{21}	40320	168	2	10	33	18	24	102	$6^6 7^3$
H_{22}	40320	136	6	30	21	6	36	66	$4^6 5^9 6^{18}$
H_{23}	45360	144	5	24	26	8	36	72	$4^6 6^{21}$
H_{24}	45360	136	6	28	25	4	34	68	$3^2 4^4 5^9 6^{18}$
H_{25}	48384	160	0	20	35	8	0	120	7^{15}
H_{26}	60480	160	0	24	27	12	0	120	7^{15}
H_{27}	60480	160	3	16	28	16	30	90	6^{15}
H_{28}	60480	152	4	20	27	12	30	84	$5^9 6^{12}$
H_{29}	80640	168	0	18	27	18	0	126	7^9
H_{30}	90720	144	6	20	29	8	40	68	$3^2 4^4 5^9 6^{12}$
H_{31}	103680	168	0	14	35	14	0	126	7^9
H_{32}	120960	152	2	22	33	6	16	98	$6^{14} 7^7$
H_{33}	120960	152	2	26	25	10	12	102	$5^3 6^{12} 7^6$
H_{34}	145152	160	2	20	25	16	20	100	$6^{10} 7^5$
H_{35}	181440	168	0	16	31	16	0	126	7^9
H_{36}	181440	160	2	16	33	12	16	104	$5^3 6^8 7^4$
H_{37}	181440	160	1	20	30	12	8	112	$6^7 7^8$
H_{38}	181440	160	1	20	30	12	10	110	$6^5 7^{10}$
H_{39}	181440	152	2	24	29	8	14	100	$5^3 6^{10} 7^8$

Table 2: The hyperovals of the line Grassmannian $\mathcal{G}(2)$ of $Q(6,2)$

H	N	$\|H\|$	$\|\mathcal{D}_0\|$	$\|\mathcal{D}_{12}\|$	$\|\mathcal{D}_{16}\|$	$\|\mathcal{B}_1\|$	$\|\mathcal{B}_2\|$	WP
H_1	1071	512	23	0	96	640	0	$0^5 4^{120}$
H_2	1632	336	35	84	0	420	0	$0^{30} 4^{315}$
H_3	3808	432	29	36	54	540	0	$3^{90} 4^{135}$
H_4	4284	384	31	64	24	480	0	$0^{15} 4^{270}$
H_5	24192	544	0	68	51	0	680	7^{85}
H_6	45696	480	14	60	45	240	360	$3^{30} 5^{135}$
H_7	64260	512	7	64	48	128	512	$0^1 4^{28} 6^{96}$
H_8	128520	576	3	32	84	80	640	$4^5 6^{40}$
H_9	128520	448	11	96	12	176	384	$0^2 4^{51} 6^{152}$
H_{10}	171360	560	5	36	78	124	576	$3^2 4^3 5^{24} 6^{36}$
H_{11}	171360	464	11	84	24	196	384	$0^2 4^{51} 6^{132}$
H_{12}	274176	512	8	60	51	160	480	$3^5 5^{60} 6^{60}$
H_{13}	304640	576	2	36	81	72	648	$6^{18} 7^{27}$
H_{14}	342720	528	7	52	60	180	480	$4^{15} 6^{90}$
H_{15}	391680	448	14	84	21	224	336	$0^2 3^{14} 4^{42} 5^{63} 6^{84}$
H_{16}	411264	544	6	44	69	160	520	$5^{25} 6^{60}$
H_{17}	548352	576	0	44	75	0	720	7^{45}
H_{18}	548352	544	5	48	66	140	540	6^{85}
H_{19}	548352	480	9	80	30	180	420	$4^{30} 6^{135}$
H_{20}	685440	480	10	76	33	192	408	$3^6 4^{24} 5^{27} 6^{108}$
H_{21}	1028160	512	7	64	48	160	480	$4^{15} 6^{110}$
H_{22}	1028160	496	9	68	42	172	448	$3^6 4^9 5^{52} 6^{78}$
H_{23}	2056320	528	3	68	48	68	592	$4^5 6^{52} 7^{48}$
H_{24}	2056320	496	5	84	30	92	528	$3^2 4^3 5^{24} 6^{68} 7^{48}$
H_{25}	2193408	528	0	80	39	0	660	7^{105}
H_{26}	2741760	512	2	84	33	40	600	$6^{50} 7^{75}$
H_{27}	3084480	512	3	80	36	64	576	$4^5 6^{56} 7^{64}$
H_{28}	3290112	512	3	80	36	60	580	$4^5 6^{60} 7^{60}$
H_{29}	4112640	560	1	52	66	28	672	$6^{17} 7^{48}$
H_{30}	4112640	512	4	76	39	80	560	$3^1 4^4 5^{12} 6^{60} 7^{48}$
H_{31}	4700160	560	0	56	63	0	700	7^{65}
H_{32}	4700160	560	0	56	63	0	700	7^{65}
H_{33}	4935168	544	1	64	54	20	660	$6^{25} 7^{60}$
H_{34}	4935168	512	4	76	39	80	560	$5^{20} 6^{60} 7^{45}$
H_{35}	5483520	544	2	60	57	56	624	$6^{34} 7^{51}$
H_{36}	5806080	544	0	68	51	0	680	7^{85}
H_{37}	6168960	544	2	60	57	48	632	$5^5 6^{32} 7^{48}$
H_{38}	6168960	544	0	68	51	0	680	7^{85}

Table 3: The hyperovals of the line Grassmannian $\mathcal{G}^-(2)$ of $Q^-(7,2)$, I

H	N	$\|H\|$	$\|\mathcal{D}_0\|$	$\|\mathcal{D}_{12}\|$	$\|\mathcal{D}_{16}\|$	$\|\mathcal{B}_1\|$	$\|\mathcal{B}_2\|$	WP
H_{39}	6580224	560	0	56	63	0	700	7^{65}
H_{40}	6580224	496	6	80	33	120	500	$4^{10}5^{15}6^{90}7^{30}$
H_{41}	8225280	544	1	64	54	28	652	$6^{17}7^{68}$
H_{42}	8225280	512	6	68	45	128	512	$3^{2}4^{8}5^{19}6^{72}7^{24}$
H_{43}	9400320	560	0	56	63	0	700	7^{65}
H_{44}	10967040	528	2	72	45	48	612	$6^{42}7^{63}$
H_{45}	10967040	528	2	72	45	48	612	$6^{42}7^{63}$
H_{46}	11612160	544	0	68	51	0	680	7^{85}
H_{47}	11612160	544	0	68	51	0	680	7^{85}
H_{48}	11612160	544	0	68	51	0	680	7^{85}
H_{49}	12337920	544	1	64	54	24	656	$6^{21}7^{64}$
H_{50}	12337920	528	3	68	48	68	592	$5^{10}6^{47}7^{48}$
H_{51}	16450560	544	2	60	57	56	624	$6^{34}7^{51}$
H_{52}	16450560	544	2	60	57	48	632	$5^{5}6^{32}7^{48}$
H_{53}	19740672	528	2	72	45	40	620	$5^{5}6^{40}7^{60}$
H_{54}	21934080	528	2	72	45	48	612	$6^{42}7^{63}$

Table 4: The hyperovals of the line Grassmannian $\mathcal{G}^-(2)$ of $Q^-(7,2)$, II

hyperovals H of $\mathcal{G}^+(2)$, $\mathcal{G}(2)$ and $\mathcal{G}^-(2)$ for which all black planes are thin are constructed in a uniform way, namely by taking a subspace U of co-dimension 2 of the ambient space of Π, and putting H equal to the set of lines of Π disjoint from U. Depending on what type of quadric $U \cap \Pi$ is, we then obtain one of the possibilities for H.

By Theorem 1.3, we know that the set $\mathcal{D}_4$ for the hyperoval H_3 of $\mathcal{G}^+(2)$ should be a tight set. This tight set is the complement of the union of two disjoint planes. By Theorem 1.3, we also know that the set $\mathcal{D}_{12}$ for the hyperoval H_2 of $\mathcal{G}^-(2)$ is a tight set. This tight set is the complement of a hyperbolic quadric $Q^+(5,2) \subset Q^-(7,2)$.

Suppose H is an α-regular hyperplane of $\mathcal{G}_\Pi$ for which every black plane is thin. Suppose also that π is a white plane containing a black point, and let $\mathcal{L}$ denote the set of lines π contained in a black plane. Since every line of π is contained in at most 2^ϵ black planes, and every black point of π is contained in $\frac{\alpha}{2} \geq 2^\epsilon + 1$ black planes meeting π in a line, we see that every black point of π is incident with at least two lines of $\mathcal{L}$. Together with the fact that every line of $\mathcal{L}$ contains precisely two black points (recall Lemma 2.4), we now see that there are at most four black points inside π. Lemma 3.3 then implies that

$$|H| = \frac{|B| \cdot \alpha}{3} \leq 3\alpha^2 - 4(2^\epsilon + 1)\alpha + 2\alpha(2^{\epsilon+2} + 2 - \alpha). \tag{9}$$

Suppose $\Pi = Q^+(5,2)$. Since every hyperoval of $Q^+(3,2)$ has size 4 or 6, every regular hyperoval of $\mathcal{G}_\Pi$ has degree $\alpha \in \{4,6\}$. By Theorems 1.2, 1.3 and equation (9), we have that $(\alpha, |H|)$ must be equal to either $(4,28)$, $(4,32)$ or $(6,60)$. By Table 1, these are indeed the possible values of $(\alpha, |H|)$.

If $\Pi = Q(6,2)$, then a similar treatment as in the previous paragraph would show that $\alpha \in \{6,8,10\}$ and that $(\alpha, |H|)$ must be equal to either $(6,84)$, $(8,120)$, $(8,128)$ or $(10,180)$. Table 2 tells us that the possibilities $(6,84)$ and $(10,180)$ cannot occur.

If $\Pi = Q^-(7,2)$, then a similar treatment as above tells us that $\alpha \in \{12,16\}$ and that $(\alpha, |H|)$ must be equal to either $(12,336)$ or $(16,512)$. By Tables 3 and 4, these are indeed the possible values for $(\alpha, |H|)$.

References

[1] P. J. Cameron, D. R. Hughes and A. Pasini. Extended generalized quadrangles. *Geom. Dedicata* 35 (1990), 193–228.

[2] A. Cossidente. Hyperovals on $H(3,q^2)$. *J. Combin. Theory Ser. A* 118 (2011), 1190–1195.

[3] A. Cossidente, O. H. King and G. Marino. Hyperovals of $H(3,q^2)$ when q is even. *J. Combin. Theory Ser. A* 120 (2013), 1131–1140.

[4] A. Cossidente and G. Marino. Hyperovals of Hermitian polar spaces. *Des. Codes Cryptogr.* 64 (2012), 309–314.

[5] A. Cossidente, G. Marino and T. Penttila. The action of the group $G_2(q) < PSU_6(q^2)$, q even, and related combinatorial structures. *J. Combin. Des.* 21 (2013), 81–88.

[6] B. De Bruyn. On hyperovals of polar spaces. *Des. Codes Cryptogr.* 56 (2010), 183–195.

[7] A. Del Fra, D. Ghinelli and S. E. Payne. $(0,n)$-sets in a generalized quadrangle. Combinatorics '90 (Gaeta, 1990). *Ann. Discrete Math.* 52 (1992), 139–157.

[8] K. Drudge. Extremal sets in projective and polar spaces. Ph.D. thesis. The University of Western Ontario, 1998.

[9] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)

[10] A. A. Makhnev. Extensions of GQ(4,2), the description of hyperovals. *Discrete Math. Appl.* 7 (1997), 419–435.

[11] A. A. Makhnev. Locally GQ(3,5)-graphs and geometries with short lines. *Discrete Math. Appl.* 8 (1998), 275–290.

[12] D. V. Pasechnik. Extending polar spaces of rank at least 3. *J. Combin. Theory Ser. A* 72 (1995), 232–242.

[13] D. V. Pasechnik. The triangular extensions of a generalized quadrangle of order (3,3). *Bull. Belg. Math. Soc. Simon Stevin* 2 (1995), 509–518.

[14] D. V. Pasechnik. The extensions of the generalized quadrangle of order (3,9). *European J. Combin.* 17 (1996), 751–755.

[15] F. Pavese. Hyperovals on $H(3, q^2)$ left invariant by a group of order $6(q+1)^3$. *Discrete Math.* 313 (2013), 1543–1546.

[16] S. E. Payne. Tight pointsets in finite generalized quadrangles. *Congr. Numer.* 60 (1987), 243–260.