On polynomial solutions of generalized Moisil-Théodoresco systems and Hodge-de Rham systems

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Abstract. The aim of the paper is to study relations between polynomial solutions of generalized Moisil-Théodoresco (GMT) systems and polynomial solutions of Hodge-de Rham systems and, using these relations, to describe polynomial solutions of GMT systems. We decompose the space of homogeneous solutions of GMT system of a given homogeneity into irreducible pieces under the action of the group $O(m)$ and we characterize individual pieces by their highest weights and we compute their dimensions.

1. Introduction

In its classical form Clifford analysis is the study of properties of solutions of the Dirac operator D acting on functions defined on $\mathbf{R}^m$ with values in the corresponding Clifford algebra $\mathbf{R}_{0,m}$. It is, however, often important (and interesting) to consider special types of solutions obtained by considering functions taking values in suitable subspaces of $\mathbf{R}_{0,m}$. To describe some of such important cases, we shall use the language of differential forms.

Let G be an open subset of the Euclidean space $\mathbf{R}^m$ and let

$$\mathcal{E}(G) = \bigoplus_{s=0}^{m} \mathcal{E}^s(G)$$

be the space of (smooth) differential forms on G. It is well known that the Clifford algebra $\mathbf{R}_{0,m}$ is isomorphic (as a vector space) with the Grassmann algebra $\Lambda^*(\mathbf{R}^m)$. Hence the space of (smooth) Clifford algebra valued functions on G can

be identified with the space $\mathcal{E}(G)$ of (smooth) differential forms on G. As was explained in detail in [3], the Dirac operator D corresponds under this identification to the operator $d + d^*$ acting on the space of differential forms, where d and d^* are the standard de Rham differential and its adjoint, respectively.

In recent years, there was a growing interest in the study and better understanding of properties of solutions of generalized Moisil-Théodoresco systems (see [13, 1, 9, 10, 6, 2]), which are defined as follows (see [1]).

Definition 1. Assume that r, p and q are non-negative integers such that $p \leq q$ and $r + 2q \leq m$. Furthermore, denote by $\mathcal{E}^{(r,p,q)}(G)$ the subspace of $\mathcal{E}(G)$ determined by

$$\mathcal{E}^{(r,p,q)}(G) = \bigoplus_{j=p}^{q} \mathcal{E}^{r+2j}(G).$$

A generalized Moisil-Théodoresco system of type (r, p, q) (GMT system for short) is then defined as the homogeneous system obtained by restricting the operator $d + d^*$ to the space $\mathcal{E}^{(r,p,q)}(G)$, i.e.

$$(d + d^*)\,\omega = 0,\ \omega \in \mathcal{E}^{(r,p,q)}(G).$$

Notice that if $\omega \in \mathcal{E}^{(r,p,q)}(G)$ is written out as

$$\omega = \sum_{j=p}^{q} \omega^{r+2j}$$

with $\omega^{r+2j} \in \mathcal{E}^{r+2j}(G)$, then the equation $(d + d^*)\,\omega = 0$ means that

$$
\begin{aligned}
d^* \omega^{r+2p} &= 0, \\
d\omega^{r+2j} + d^* \omega^{r+2j+2} &= 0,\ j = p, \ldots, q-1, \\
d\omega^{r+2q} &= 0.
\end{aligned}
\tag{1}
$$

Some special cases of GMT systems are well known and well understood. It is possible to choose (r, p, q) in such a way that $\mathcal{E}^{(r,p,q)}(G)$ is equal to the space $\mathcal{E}^+(G)$ (resp. $\mathcal{E}^-(G)$) of all even (resp. odd) forms. In these cases, the corresponding GMT system is (equivalent to) the classical Dirac equation for functions with values in the even (resp. odd) part of the Clifford algebra. Properties of these functions were carefully studied in classical Clifford analysis.

Another very important special case is the GMT system of type $(r, 0, 0)$. In this case, the space $\mathcal{E}^{(r,p,q)}(G)$ reduces to the space $\mathcal{E}^r(G)$ of forms of pure degree r and the corresponding GMT system coincides with the Hodge-de Rham system

$$d\omega = 0,\ d^*\omega = 0,\quad \omega \in \mathcal{E}^r(G).$$

Solutions of the Hodge-de Rham system will play a key role in our study of solutions of GMT systems. As discussed in [1], it is important to understand the structure of homogeneous polynomial solutions of these systems.

The aim of the paper is to study relations between polynomial solutions of GMT systems and polynomial solutions of Hodge-de Rham systems. Using these

relations and results obtained by Y. Homma in [16] on solutions of Hodge-de Rham systems, it is then possible to describe fully polynomial solutions of GMT systems. To describe them in more details, we shall introduce a suitable notation.

Denote by $\mathcal{P}$ the space of real-valued polynomials in $\mathbf{R}^m$ and by $\mathcal{P}_k$ the space of polynomials of $\mathcal{P}$ which are homogeneous of degree k. Moreover, let $\Lambda^*(\mathbf{R}^m)$ and $\Lambda^s(\mathbf{R}^m)$ stand for the exterior algebra and the space of s-vectors over $\mathbf{R}^m$, respectively. Of course,

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_k \ \text{ and } \ \Lambda^*(\mathbf{R}^m) = \bigoplus_{s=0}^{m} \Lambda^s(\mathbf{R}^m).$$

Let us now introduce the following spaces of differential forms with polynomial coefficients: $\mathcal{P}_k^* = \mathcal{P}_k \bigotimes_{\mathbf{R}} \Lambda^*(\mathbf{R}^m)$, $\mathcal{P}_k^s = \mathcal{P}_k \bigotimes_{\mathbf{R}} \Lambda^s(\mathbf{R}^m)$ and, finally,

$$\mathcal{P}_k^{(r,p,q)} = \bigoplus_{j=p}^{q} \mathcal{P}_k^{r+2j}.$$

In what follows, we shall mainly study the space

$$MT_k^{(r,p,q)} = \{P \in \mathcal{P}_k^{(r,p,q)} : \ (d + d^*)P = 0\} \tag{2}$$

of polynomial solutions of the generalized Moisil-Théodoresco system of homogeneity k. We shall relate them to the spaces

$$H_k^s = \{P \in \mathcal{P}_k^s : \ dP = 0, \ d^*P = 0\} \tag{3}$$

of polynomial solutions of the Hodge-de Rham system of homogeneity k and use the results obtained by Y. Homma in [16] for H_k^s.

In Section 2, we decompose $MT_k^{(r,p,q)}$ into a sum of pieces isomorphic to spaces H_k^s of polynomial solutions of various Hodge-de Rham systems (see Theorem 1). In such a way, we shall be able to characterize irreducible components of these spaces under the action of the group $O(m)$ and count the dimension of the space of polynomial solutions of given homogeneity. In Section 3, we shall review results from [16] on the decomposition of the space $\operatorname{Ker}_k^s \Delta$ of harmonic polynomials of homogeneity k with values in $\Lambda^s(\mathbf{R}^m)$ into four different pieces and the description of the spaces H_k^s. In the last section, we shall add an alternative description of the four pieces in the decomposition of $\operatorname{Ker}_k^s \Delta$.

2. The decomposition of $MT_k^{(r,p,q)}$

In this part, we shall describe a decomposition of the spaces $MT_k^{(r,p,q)}$ into a direct sum of spaces isomorphic to spaces of solutions of various Hodge-de Rham systems. It will give us (as we shall see later) a decomposition of the spaces $MT_k^{(r,p,q)}$ into a sum of irreducible components under the $O(m)$-action. We shall need the following refined version of the Poincaré Lemmas (see also [3] and [10]).

Lemma 1. *The following properties hold:*
(i) For $P_k^s \in \mathcal{P}_k^s$ with $s > 0$, $dP_k^s = 0$ if and only if there exists $P_{k+1}^{s-1} \in \mathcal{P}_{k+1}^{s-1}$ such that $d^* P_{k+1}^{s-1} = 0$ and $dP_{k+1}^{s-1} = P_k^s$.
(ii) For $P_k^s \in \mathcal{P}_k^s$ with $s < m$, $d^* P_k^s = 0$ if and only if there exists $P_{k+1}^{s+1} \in \mathcal{P}_{k+1}^{s+1}$ such that $dP_{k+1}^{s+1} = 0$ and $d^* P_{k+1}^{s+1} = P_k^s$.

Notice that $H_0^s = \mathcal{P}_0^s$ and $MT_0^{(r,p,q)} = \mathcal{P}_0^{(r,p,q)}$.
    Now we can state the main result of this section.

Theorem 1. *Let $k \in \mathbf{N}$. Denote by Φ the restriction of the operator d to the space $MT_k^{(r,p,q)}$. Then*

$$\operatorname{Ker} \Phi = \bigoplus_{j=p}^{q} H_k^{r+2j} \quad and \quad \operatorname{Im} \Phi = \bigoplus_{j=p}^{q-1} H_{k-1}^{r+2j+1}.$$

Moreover, we have that

$$MT_k^{(r,p,q)} \simeq \operatorname{Ker} \Phi \oplus \operatorname{Im} \Phi.$$

Proof. Let $P_k \in MT_k^{(r,p,q)}$, that is, $P_k = P_k^{r+2p} + P_k^{r+2p+2} + \cdots + P_k^{r+2q}$ with $P_k^{r+2j} \in \mathcal{P}_k^{r+2j}$. By (1) we have that

$$\begin{aligned} d^* P_k^{r+2p} &= 0, \\ dP_k^{r+2j} &= -d^* P_k^{r+2j+2}, \ j = p, \ldots, q-1, \\ dP_k^{r+2q} &= 0. \end{aligned} \quad (4)$$

For $j = p, \ldots, q-1$, put $P_{k-1}^{r+2j+1} = dP_k^{r+2j}$. Obviously, $P_{k-1}^{r+2j+1} = -d^* P_k^{r+2j+2}$ and $\Phi(P_k) = P_{k-1}^{r+2p+1} + P_{k-1}^{r+2p+3} + \cdots + P_{k-1}^{r+2q-1}$. Moreover, by virtue of (4), we have that

$$\operatorname{Ker} \Phi = \bigoplus_{j=p}^{q} H_k^{r+2j}.$$

Using the fact that $d^2 = 0$ and $(d^*)^2 = 0$, we get that

$$\operatorname{Im} \Phi \subset \bigoplus_{j=p}^{q-1} H_{k-1}^{r+2j+1}.$$

To show the opposite inclusion, consider an arbitrary form

$$P_{k-1} = P_{k-1}^{r+2p+1} + P_{k-1}^{r+2p+3} + \cdots + P_{k-1}^{r+2q-1}$$

with $P_{k-1}^{r+2j+1} \in H_{k-1}^{r+2j+1}$. Then it is sufficient to find a form $P_k \in MT_k^{(r,p,q)}$ such that $\Phi(P_k) = P_{k-1}$. By using Lemma 1, we shall construct such a P_k as follows.
(i) For $dP_k^{r+2p+1} = 0$ we can find $P_k^{r+2p} \in \mathcal{P}_k^{r+2p}$ such that

$$dP_k^{r+2p} = P_{k-1}^{r+2p+1} \quad and \quad d^* P_k^{r+2p} = 0.$$

For $d^* P_k^{r+2p+1} = 0$ we can take $\tilde{P}_k^{r+2p+2} \in \mathcal{P}_k^{r+2p+2}$ such that

$$d^* \tilde{P}_k^{r+2p+2} = -P_{k-1}^{r+2p+1} \quad \text{and} \quad d\tilde{P}_k^{r+2p+2} = 0.$$

(ii) For $dP_k^{r+2p+3} = 0$ we can find $\bar{P}_k^{r+2p+2} \in \mathcal{P}_k^{r+2p+2}$ such that

$$d\bar{P}_k^{r+2p+2} = P_{k-1}^{r+2p+3} \quad \text{and} \quad d^* \bar{P}_k^{r+2p+2} = 0.$$

For $d^* P_k^{r+2p+3} = 0$ we can take $\tilde{P}_k^{r+2p+4} \in \mathcal{P}_k^{r+2p+4}$ such that

$$d^* \tilde{P}_k^{r+2p+4} = -P_{k-1}^{r+2p+3} \quad \text{and} \quad d\tilde{P}_k^{r+2p+4} = 0.$$

Define $P_k^{r+2p+2} = \tilde{P}_k^{r+2p+2} + \bar{P}_k^{r+2p+2}$. By (i) and (ii), obviously,

$$d^* P_k^{r+2p+2} = -P_{k-1}^{r+2p+1} \quad \text{and} \quad dP_k^{r+2p+2} = P_{k-1}^{r+2p+3}.$$

By induction, we can thus construct, for each $j = p, \ldots, q$, a form $P_k^{r+2j} \in \mathcal{P}_k^{r+2j}$ such that

$$d^* P_k^{r+2j} = -P_{k-1}^{r+2j-1} \quad \text{and} \quad dP_k^{r+2j} = P_{k-1}^{r+2j+1}$$

where $P_{k-1}^{r+2p-1} = 0$ and $P_{k-1}^{r+2q+1} = 0$. Then the form

$$P_k = P_k^{r+2p} + P_k^{r+2p+2} + \cdots + P_k^{r+2q}$$

has the required properties. □

Not too much is known in general about the spaces $MT_k^{(r,p,q)}$. In the case $\mathbf{R}^3$, a basis for the space $MT_k^{(1,0,0)} = H_k^1$ is given in [15] and orthonormal bases for the spaces $MT_k^{(1,0,0)} = H_k^1$ and $MT_k^{(0,0,1)}$ are constructed in [5]. In the case $\mathbf{R}^4$, a procedure has been worked out in [10] for constructing bases for the spaces $MT_k^{(r,p,q)}$. In the general case $\mathbf{R}^m$, bases for the space $MT_k^{(1,0,0)} = H_k^1$ have been given in [9] and [17]. Furthermore, denote by $\mathcal{M}_k^+$ the real vector space of left monogenic polynomials in $\mathbf{R}^m$ which are homogeneous of degree k and which take values in the even part $\mathbf{R}_{0,m}^+$ of the Clifford algebra $\mathbf{R}_{0,m}$ and put $n = [\frac{m}{2}]$. Then the following result is well-known (see [1] and [11]).

Lemma 2. $MT_k^{(0,0,n)} \simeq \mathcal{M}_k^+$ and $\dim MT_k^{(0,0,n)} = \dim \mathcal{M}_k^+ = c(k,m)$ where

$$c(k,m) = 2^{m-1} \binom{k+m-2}{m-2}. \tag{5}$$

3. The $O(m)$-modules H_k^s

In this section, we shall use some known facts from representation theory of the group $O(m)$. We refer to [14, 5.2.2] for more details. The classification of irreducible $O(m)$-modules is closely related to the classification of irreducible $SO(m)$-modules. Let us recall that the latter classification is standardly given in terms of the highest weight of the module. In the even dimensional case $m = 2n$, a highest weight of an

irreducible $SO(m)$-module is a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers satisfying the relation $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq |\lambda_n|$. On the other hand, in the odd dimensional case $m = 2n + 1$, the vector λ satisfies the condition $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$.

The classification of $O(m)$-modules also differs in even and odd dimensions. In the case when the dimension m is odd, each irreducible representation V of the group $O(m)$ remains irreducible even as a representation over the special orthogonal group $SO(m)$. Moreover, the irreducible $O(m)$-representation $V = V_{(\lambda,\epsilon)}$ is uniquely determined by the highest weight λ of V for $SO(m)$ and by a number $\epsilon \in \{\pm 1\}$.

In the even dimensional case, the situation is a bit more complicated. In what follows, denote by V_λ an irreducible $SO(m)$-module with the highest weight λ. Assume now that $m = 2n$ and V is an irreducible $O(m)$-module. Then there are two possibilities. The first one is that the module V remains irreducible even as $SO(m)$-module and V is isomorphic to V_λ for some $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, 0)$. In this case, the irreducible $O(m)$-representation $V = V_{(\lambda,\epsilon)}$ is again uniquely determined by the highest weight λ and a number $\epsilon \in \{\pm 1\}$. On the other hand, there is yet another possibility. It may happen that, as $SO(m)$-module, V is reducible and $V \simeq V_\lambda \oplus V_{\bar{\lambda}}$ for some $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_n > 0$ and $\bar{\lambda} = (\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n)$. In that case, we denote V by $V_{(\lambda,0)}$. See [14, 5.2.2] for details.

Now let n be a positive integer such that either $m = 2n$ or $m = 2n + 1$. For $1 \leq s \leq n$ and $k \in \mathbf{N}_0$, let λ_k^s be the vector in $\mathbf{R}^n$ having s non-zero components and given by

$$\lambda_k^s = (k+1, 1, \ldots, 1, 0, \ldots, 0).$$

Moreover, put $\lambda_0^0 = (0, \ldots, 0) \in \mathbf{R}^n$.

The following theorem tells us that all (non-trivial) $O(m)$-modules H_k^s are irreducible and mutually inequivalent. Moreover, it gives a characterization of the corresponding irreducible $O(m)$-modules using the classification mentioned above. Using the Weyl dimensional formula, we can then compute the dimension of all these spaces. As a corollary, a formula for the dimension of the space $MT_k^{(r,p,q)}$ is obtained.

Theorem 2. *The following properties hold:*

(a1) $H_0^0 \simeq V_{(\lambda_0^0, 1)}$ *and* $H_0^m \simeq V_{(\lambda_0^0, -1)}$.
     Moreover, for $k \geq 1$, $H_k^0 \simeq \{0\}$ *and* $H_k^m \simeq \{0\}$.
(a2) *Let* $m = 2n + 1$, $1 \leq s \leq n$ *and let* $k \in \mathbf{N}_0$.
     Then $H_k^s \simeq V_{(\lambda_k^s, 1)}$ *and* $H_k^{m-s} \simeq V_{(\lambda_k^s, -1)}$.
(a3) *Let* $m = 2n$, $1 \leq s \leq n - 1$ *and let* $k \in \mathbf{N}_0$.
     Then $H_k^s \simeq V_{(\lambda_k^s, 1)}$, $H_k^{m-s} \simeq V_{(\lambda_k^s, -1)}$ *and* $H_k^n \simeq V_{(\lambda_k^n, 0)}$.
(a4) *Let* $1 \leq s \leq m - 1$ *and let* $k \in \mathbf{N}_0$. *Putting*

$$d(k, m, s) = \binom{m-2}{s-1} \binom{k+m-2}{m-2} \frac{(2k+m)(k+m-1)}{(k+s)(k+m-s)}, \qquad (6)$$

we have that $\dim H_k^s = d(k, m, s)$.

Corollary 1. *Let r, p and q be non-negative integers such that $p \leq q$ and $r + 2q \leq m$. Then*

$$\dim MT_k^{(r,p,q)} = \sum_{j=p}^{q} d(k, m, r + 2j) + \sum_{j=p}^{q-1} d(k - 1, m, r + 2j + 1)$$

where for $1 \leq s \leq m - 1$, $d(k, m, s)$ is given by (6). Furthermore,

$$d(k, m, s) = \begin{cases} 1 & \text{for } k = 0 \text{ and } s = 0, m; \\ 0 & \text{for } k \geq 1 \text{ and } s = 0, m; \\ 0 & \text{for } k = -1. \end{cases}$$

There is a useful additional information describing a decomposition of harmonic forms.

Lemma 3. *Let $0 \leq s \leq m$ and let $k \in \mathbf{N}_0$. Then we have that*

$$\operatorname{Ker} {}_k^s \Delta = H_k^s \oplus U_k^s \oplus V_k^s \oplus W_k^s$$

where H_k^s denotes the space of solutions of the Hodge-de Rham equations and U_k^s, V_k^s and W_k^s are irreducible $O(m)$-modules with the following properties:

(a1) $H_0^0 \simeq V_{(\lambda_0^0, 1)}$ *and* $H_0^m \simeq V_{(\lambda_0^0, -1)}$.
  Moreover, for $k \geq 1$, $H_k^0 \simeq \{0\}$ and $H_k^m \simeq \{0\}$.

(a2) *Let $m = 2n + 1$, $1 \leq s \leq n$ and let $k \in \mathbf{N}_0$.*
  Then $H_k^s \simeq V_{(\lambda_k^s, 1)}$ and $H_k^{m-s} \simeq V_{(\lambda_k^s, -1)}$.

(a3) *Let $m = 2n$, $1 \leq s \leq n - 1$ and let $k \in \mathbf{N}_0$.*
  Then $H_k^s \simeq V_{(\lambda_k^s, 1)}$, $H_k^{m-s} \simeq V_{(\lambda_k^s, -1)}$ and $H_k^n \simeq V_{(\lambda_k^n, 0)}$.

(a4) *For $0 \leq s \leq m$ and $k \geq 0$, we have that $\dim H_k^s = d(k, m, s)$.*

(b) *For $1 \leq s \leq m$ and $k \geq 1$, we have that $U_k^s \simeq H_{k-1}^{s-1}$ and $U_k^0 \simeq \{0\}$.*

(c) *For $0 \leq s \leq m - 1$ and $k \geq 1$, we have that $V_k^s \simeq H_{k-1}^{s+1}$ and $V_k^m \simeq \{0\}$.*

(d) *For $1 \leq s \leq m - 1$ and $k \geq 2$, we have that $W_k^s \simeq H_{k-2}^s$, $W_1^s \simeq \{0\}$, $W_k^0 \simeq \{0\}$ and $W_k^m \simeq \{0\}$.*

Proofs of Theorem 2 and Lemma 3 can be easily deduced from results proved in [16].

4. The decomposition of the kernel of the Hodge Laplacian

In Lemma 3, the kernel of the Hodge Laplacian has been decomposed into irreducible $O(m)$-modules

$$\operatorname{Ker} {}_k^s \Delta = H_k^s \oplus U_k^s \oplus V_k^s \oplus W_k^s.$$

Now we would like to give an analytic description of the spaces U_k^s, V_k^s and W_k^s.

Theorem 3. *Let $0 \leq s \leq m$ and let $k \in \mathbf{N}_0$. Then the following properties hold:*

(a) $U_k^s \simeq \operatorname{Ker} {}_k^s dd^* / \operatorname{Ker} {}_k^s d^*$ *and* $V_k^s \simeq \operatorname{Ker} {}_k^s d^*d / \operatorname{Ker} {}_k^s d$,

(b) $W_k^s \simeq \operatorname{Ker} {}_k^s \Delta / (\operatorname{Ker} {}_k^s dd^* \cap \operatorname{Ker} {}_k^s d^*d)$,

(c) $\operatorname{Ker}^s_k dd^* \cap \operatorname{Ker}^s_k d = H^s_k \oplus U^s_k$ and $\operatorname{Ker}^s_k d^*d \cap \operatorname{Ker}^s_k d^* = H^s_k \oplus V^s_k$,
(d) $\operatorname{Ker}^s_k dd^* \cap \operatorname{Ker}^s_k d^*d = H^s_k \oplus U^s_k \oplus V^s_k$.

The Fisher decomposition [13, p. 167] tells us that

$$\mathcal{P}_k = \bigoplus_{j=0}^{[k/2]} r^{2j}\mathcal{H}_{k-2j} \quad\text{and thus}\quad \mathcal{P}^s_k = \bigoplus_{j=0}^{[k/2]} r^{2j}\operatorname{Ker}^s_{k-2j}\Delta.$$

From Lemma 3, we obtain the decomposition

$$\mathcal{P}^s_k = H^s_k \oplus \bigoplus_{j=0}^{[k/2]} r^{2j}U^s_{k-2j} \oplus \bigoplus_{j=0}^{[k/2]} r^{2j}V^s_{k-2j} \oplus \bigoplus_{j=0}^{[k/2]} r^{2j}Z^s_{k-2j} \tag{7}$$

where $Z^s_k = r^2 H^s_{k-2} \oplus W^s_k$.

Putting

$$X^s_{k,j} = r^{2j}Z^s_{k-2j} \cap \operatorname{Ker}^s_k d \quad\text{and}\quad Y^s_{k,j} = r^{2j}Z^s_{k-2j} \cap \operatorname{Ker}^s_k d^*,$$

we first prove

Lemma 4. *The following properties hold:*
(a) $r^{2j}Z^s_{k-2j} = X^s_{k,j} \oplus Y^s_{k,j}$ and $X^s_{k,j} \simeq Y^s_{k,j} \simeq H^s_{k-2-2j}$;
(b) $\operatorname{Ker}^s_k d = H^s_k \oplus \bigoplus_{j=0}^{[k/2]} r^{2j}U^s_{k-2j} \oplus \bigoplus_{j=0}^{[k/2]} X^s_{k,j}$;
(c) $\operatorname{Ker}^s_k d^* = H^s_k \oplus \bigoplus_{j=0}^{[k/2]} r^{2j}V^s_{k-2j} \oplus \bigoplus_{j=0}^{[k/2]} Y^s_{k,j}$.

Proof. The proof will be given by induction on the degree s. For $s = 0$, the statements are obvious. Assume that the statements (a), (b) and (c) of Lemma 4 are true for some $s = 0,\dots,m-1$. Then we need to verify them for $s+1$.

(α) By Lemma 1 and by using the assumption (c), we have that

$$\operatorname{Ker}^{s+1}_{k-1} d = d(\operatorname{Ker}^s_k d^*) = d(V^s_k) \oplus \bigoplus_{j=1}^{[k/2]} d(r^{2j}V^s_{k-2j}) \oplus \bigoplus_{j=0}^{[k/2]} d(Y^s_{k,j}).$$

Using the invariance of the differential d and Lemma 3, we get that $H^{s+1}_{k-1} = d(V^s_k)$, while for $j = 0,\dots,[(k-1)/2]$

$$X^{s+1}_{k-1,j} = d(r^{2(j+1)}V^s_{k-2-2j}) \quad\text{and}\quad r^{2j}U^{s+1}_{k-1-2j} = d(Y^s_{k,j}).$$

Of course, $X^{s+1}_{k-1,j} \simeq H^{s+1}_{k-3-2j}$.

(β) By Lemma 1 and by (α), we have that

$$d^*(\operatorname{Ker}^{s+1}_{k+1} d) = \bigoplus_{j=0}^{[(k+1)/2]} d^*(r^{2j}U^{s+1}_{k+1-2j}) \oplus \bigoplus_{j=0}^{[(k+1)/2]} d^*(X^{s+1}_{k+1,j}).$$

Using the invariance of the codifferential d^* and Lemma 3, we get that for $j = 0,\dots,[k/2]$

$$r^{2j}V^s_{k-2j} = d^*(X^{s+1}_{k+1,j}), \; H^s_k = d^*(U^{s+1}_{k+1}) \quad\text{and}\quad Y^s_{k,j} = d^*(r^{2(j+1)}U^{s+1}_{k-1-2j}).$$

Hence, as $d^*(\mathrm{Ker}\,_{k+1}^{s+1}\,d) = d^*(\mathcal{P}_{k+1}^{s+1})$, by virtue of (7), we may conclude that

$$d^*(r^{2j}Z_{k+1-2j}^{s+1}) = d^*(X_{k+1,j}^{s+1}) \quad \text{and thus} \quad r^{2j}Z_{k+1-2j}^{s+1} \simeq d^*(X_{k+1,j}^{s+1}) \oplus Y_{k+1,j}^{s+1}.$$

This shows that $r^{2j}Z_{k+1-2j}^{s+1} = X_{k+1,j}^{s+1} \oplus Y_{k+1,j}^{s+1}$, which completes the proof. $\qquad\square$

Proof of Theorem 3. The arguments used in the steps (α) and (β) of the proof of Lemma 4 show that

$$\mathrm{Ker}\,_k^s\,dd^* = \mathrm{Ker}\,_k^s\,d^* \oplus U_k^s \quad \text{and} \quad \mathrm{Ker}\,_k^s\,d^*d = \mathrm{Ker}\,_k^s\,d \oplus V_k^s.$$

This easily implies all statements of Theorem 3. $\qquad\square$

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