Operator identities in q-deformed Clifford analysis

K. Coulembier and F. Sommen

Abstract. In this paper, we define a q-deformation of the Dirac operator as a generalization of the one dimensional q-derivative. This is done in the abstract setting of radial algebra. This leads to a q-Dirac operator in Clifford analysis. The q-integration on $\mathbb{R}^m$, for which the q-Dirac operator satisfies Stokes' formula, is defined. The orthogonal q-Clifford-Hermite polynomials for this integration are briefly studied.

MSC 2000 : 30G35, 15A66,
Keywords : q-derivative, Dirac operator, radial algebra

1. Introduction

In [13] Jackson originally introduced the q-analogues of differentiation, integration and special functions in the context of q-hypergeometric series (also known as basic hypergeometric series). An overview of the theory of q-calculus can be found in [9, 10]. In this paper we aim to define a q-deformation of the Dirac operator, which implies a q-deformation of partial derivatives in higher dimensions. Generalizations of the q-derivative to higher dimensions have also been developed in the theory of quantum spaces, see e.g. [2].

In [4] another approach to develop a q-Dirac operator was taken. In the present paper the behavior of the q-Dirac operator with respect to vectors plays a central role. It is therefore logical to define the q-Dirac operator in setting of radial algebra, see [15]. The q-Dirac operator on radial algebra is defined by a list of axioms based on q-calculus and Clifford analysis. It is then proven that this list of axioms uniquely defines a q-Dirac operator.

The q-Dirac operator on radial algebra leads to a q-Dirac operator in specific realizations of radial algebra, such as Clifford analysis or super Clifford analysis, see e.g. [6]. The expression for the q-Dirac operator in Clifford analysis shows that the deformation is in fact purely radial. An alternative theory of radial deformations of the Dirac operator is presented in [5]. For completeness we also derive the list of axioms that uniquely define the q-Dirac operator in Clifford analysis. Although the resulting q-Dirac operator differs from the one obtained in [4] the integration as defined in [4] also leads to Stokes' formula for the q-Dirac operator in the present paper.

Since the integration corresponds to the one in [4], the theory of orthogonal polynomials also coincides. Although the q-Dirac operator in the current paper is introduced more naturally, it seems that the theory of q-orthogonal polynomials in Clifford analysis is more elegantly described using the q-Dirac operator developed in [4].

This paper is organized as follows. First we recall the basic notions of q-calculus, Clifford analysis and radial algebra. The q-Dirac operator on radial algebra is defined by a list of axioms. Explicit calculations show that the axioms uniquely define the action of the q-Dirac operator on elements of the radial algebra of low degree. The applied techniques are then used to prove the unicity of the q-Dirac operator. Then it is proven that the list of axioms is not inconsistent and therefore uniquely defines a q-Dirac operator. This implies the definition of the q-Dirac operator in Clifford analysis. The action on terms of the Fischer decomposition is calculated and the unicity of the q-Dirac operator in Clifford analysis is proven. It turns out that the notion of q-monogenic polynomials and monogenic polynomials coincide, which implies that the deformation is purely radial. Then the q-integration on $\mathbb{R}^m$ is defined as q-deformed radial integration combined with undeformed spherical integration. It is proven that the q-Dirac operator satisfies Stokes' formula. Then it is argued that the q-Dirac operator as developed in [4] is better suited to study orthogonal polynomials. Finally in the conclusion the results are reviewed and a comparison is made between the q-Dirac operator in the current paper and the one in [4].

2. Preliminaries

First we give a short introduction to q-calculus, see e.g. [9, 10, 13]. We define for u a number or operator, and the deformation parameter q,

$$[u]_q = \frac{q^u - 1}{q - 1}.$$

It's clear that $\lim_{q\to 1}[u]_q = u$. In this paper we always assume $0 < q < 1$.

The q-derivative of a function $f(t)$ is given by the expression

$$D_t^q(f(t)) \quad = \quad \frac{f(qt) - f(t)}{(q-1)t},$$

which implies

$$D_t^q(t^k) = \frac{q^k - 1}{q - 1} t^{k-1} = [k]_q t^{k-1} \tag{1}$$

and

$$D_t^q t = qtD_t^q + 1. \tag{2}$$

The q-derivative satisfies the Leibniz rule

$$D_t^q(f_1(t)f_2(t)) \quad = \quad D_t^q(f_1(t))f_2(t) + f_1(qt)D_t^q(f_2(t)). \tag{3}$$

The q-integration or Jackson integration on an interval $[0,a]$ with $a \in \mathbb{R}$ is given by

$$\int_0^a f(t)\, d_q t \quad = \quad (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k. \tag{4}$$

The infinite q-integration is given by

$$\int_0^{a\cdot\infty} f(t)\, d_q t \quad = \quad (1-q)a\sum_{k=-\infty}^{\infty} f(aq^k)q^k, \tag{5}$$

for $a \in \mathbb{R}$, see [11]. The infinite integration is therefore a function of a. However it is a q-constant, $D_a^q \int_0^{a\cdot\infty} = 0$. More general intervals are defined by $\int_a^b = \int_0^b - \int_0^a$, and satisfy the important property

$$\int_a^b D_t^q f(t)\, d_q t \quad = \quad f(b) - f(a). \tag{6}$$

There also exists an extensive theory of q-polynomials and -special functions, see e.g. [9, 12]. For a general $\alpha > -1$ we define the q-Laguerre polynomials, see e.g. [12, 14] as

$$\mathcal{L}_t^\alpha(u|q^2) \quad = \quad \sum_{i=0}^{t} q^{(t-i)(t-i+1)} \frac{(-u)^i}{[t-i]_{q^2}![i]_{q^2}!} \frac{\Gamma_{q^2}(t+\alpha+1)}{\Gamma_{q^2}(i+\alpha+1)},$$

with the q-factorial given by $[k]_q! = [k]_q[k-1]_q \cdots [1]_q$ and the q-Gamma function Γ_q which satisfies $\Gamma_q(u+1) = u\Gamma_q(u)$. We also introduce a q-exponential by

$$e_{q^2}(u) \quad = \quad \sum_{j=0}^{\infty} \frac{q^{j(j-1)}u^j}{[j]_{q^2}!}.$$

Now we briefly recall the basic notions of Clifford analysis. For more details we refer the reader to [1, 8]. The complex Clifford algebra $\mathbb{C}_m$ is generated by an orthonormal basis $(e_1, \cdots, e_m)$ for $\mathbb{R}^m$ with multiplication rules

$$e_i e_j + e_j e_i \quad = \quad -2\delta_{ij} \qquad \text{for} \quad 1 \leq i, j \leq m. \tag{7}$$

The algebra generated by these Clifford numbers and the m variables x_j, which commute with the e_i, is the algebra of Clifford-valued polynomials $\mathcal{P} = \mathbb{R}[x_1, \cdots, x_m] \otimes \mathbb{C}_m$. The vector variable $\underline{x}$ on $\mathbb{R}^m$ can be identified with the first order Clifford polynomial of the form $\underline{x} = \sum_{j=1}^m e_j x_j$. The multiplication rules (7) imply that the square of this vector variable is scalar valued,

$$\underline{x}^2 = -\sum_{j=1}^m x_j^2 = -r^2.$$

The corresponding Dirac operator or vector derivative in the vector variable $\underline{x}$ is the operator

$$\partial_{\underline{x}} = -\sum_{j=1}^{m} e_j \partial_{x_j}.$$

The square of the Dirac operator is again scalar and is the Laplace operator. Clifford analysis deals with the function theory of solutions of the equation $\partial_{\underline{x}} f(\underline{x}) = 0$, called monogenic functions. In particular we study monogenic polynomials of degree k. Denote by $\mathbb{E} = \sum_{j=1}^{m} x_j \partial_{x_j}$ the Euler operator.

Definition 1. *An element $F \in \mathcal{P}$ is a spherical monogenic of degree k if it satisfies $\partial_{\underline{x}} F = 0$ and $\mathbb{E}F = kF$, i.e. $F \in \mathcal{P}_k$. Moreover the space of all spherical monogenics of degree k is denoted by $\mathcal{M}_k$.*

The space of Clifford algebra-valued polynomials can be decomposed as follows.

Lemma 1 (Fischer decomposition). *The space $\mathcal{P}_k$ decomposes as*

$$\mathcal{P}_k = \bigoplus_{i=0}^{k} \underline{x}^i \mathcal{M}_{k-i}.$$

This decompositions is unique, so $\sum_i \underline{x}^i M_{k-i} = 0$ (with $M_{k-i} \in \mathcal{M}_{k-i}$) implies $M_{k-i} = 0$ for every i.

The role of the special orthogonal group $SO(m)$ in harmonic analysis is taken over by the spin group $Spin(m)$ in Clifford analysis,

$$Spin(m) = \{s \in \mathbb{C}_m | \quad \exists k \in \mathbb{N}, s = \underline{\omega}_1 \cdots \underline{\omega}_{2k}, \quad \underline{\omega}_i \in \mathbb{S}^{m-1}, i = 1, \cdots, 2k\}.$$

A real vector $\underline{\omega}$ in the Clifford algebra belongs to unit sphere $\mathbb{S}^{m-1}$ if $\underline{\omega}^2 = -1$. The spin group is a double cover of the special orthogonal group. The L-representation of the spin group on Clifford algebra-valued functions is given by

$$L(s)[f(\underline{x})] \quad = \quad s f(s^{-1} \underline{x} s).$$

The Dirac operator is is spin-invariant,

$$\left[\partial_{\underline{x}}, L(s)\right] \quad = \quad 0. \tag{8}$$

We will also need the main anti-involution $\bar{\cdot}$ on the Clifford algebra $\mathbb{C}_m$, defined by

- $\bar{\cdot}$ is equal to the complex conjugation on scalars,
- $\overline{e_i} = -e_i$,
- $\overline{ab} = \overline{b}\,\overline{a}$ for all $a, b \in \mathbb{C}_m$.

Using the Clifford algebra multiplication rules (7) yields

$$\{\underline{x}, \partial_{\underline{x}}\} = \partial_{\underline{x}} \underline{x} + \underline{x} \partial_{\underline{x}} = 2\mathbb{E} + m. \tag{9}$$

In particular the relations $\partial_{\underline{x}}(\underline{x}) = m$ and

$$\partial_{\underline{x}} \underline{x}^2 = \underline{x}^2 \partial_{\underline{x}} + 2\underline{x} \tag{10}$$

hold.

The relation $\{\underline{x}, e_j\} = -2x_j$ implies that $\{\underline{x}, \underline{z}\}$ is scalar valued for a general vector $\underline{z} \in \mathbb{R}^m$ identified with $\underline{z} = \sum_{j=1}^m e_j z_j \in \mathbb{C}_m$.

This leads us to radial algebra, see [15]. The starting object in the definition of radial algebra is a set S of 'abstract vector variables'. In this paper we will always assume an infinite set S. The radial algebra $R(S)$ is the universal algebra generated by S and subject to the constraints

$$[\{x, y\}, z] \quad = \quad 0 \qquad \text{for any} \quad x, y, z \in S. \tag{11}$$

The subset of $R(S)$ which commutes with all the elements of $R(S)$ is called the set of scalars, and denoted by $R_0(S)$. The formal inner product of two elements of the radial algebra is given by

$$\langle u, v\rangle \quad = \quad \frac{1}{2}\{u, v\}$$

and is an element of $R_0(S)$, by (11). In [15] it was proven that $R_0(S)$ is generated by the formal inner products. The space $R_1(S)$ is defined as the space of $R_0(S)$-linear combinations of elements of S. Clifford analysis is obtained again when we take the set $S = \{\underline{x}, e_1, \cdots, e_m\}$, therefore only viewing the 'prefered vector' $\underline{x}$ as a variable. Since this set S is finite, not all results from radial algebra will be immediately applicable to Clifford analysis.

We can construct a vector derivative (Dirac operator) with respect to each element of S. Mostly we will choose x. This means that all the other operators, such as the Euler operator, are defined with respect to x. So we will always use the notation $\mathbb{E}$ in stead of $\mathbb{E}_x$. The subspace of the radial algebra $R(S)$ which is of degree k with respect to x is denoted by $[R(S)]_k$.

In order to define the vector derivative $\partial_x \in \mathrm{End}(R(S))$, there has to be a unique constant scalar m for which

$$(A0) \qquad\qquad \partial_x(x) = m, \;\; \forall x \in S.$$

On the level of radial algebra this m is a parameter wich can take any value in $\mathbb{R}$. The Dirac operator is defined uniquely by this axiom and by

$$\begin{array}{ll}
(A1) & \partial_x(fF) = \partial_x(f)F + f\partial_x(F), \;\; f \in R_0(S)\,, \quad F \in R(S) \\
(A2) & \partial_x(FG) = \partial_x(F)G, \quad F \in R(S)\,, \quad G \in R(S\backslash\{x\}) \\
(A3) & \partial_x(x^2) = 2x \\
(A4) & \partial_x(\langle x, z\rangle) = z\,, \;\; z \neq x.
\end{array}$$

An important result of radial algebra is that it allows to develop a theory of super Clifford analysis, see e.g. [6]. Using p commuting variables and $2n$ anti-commuting variables we obtain a model for radial algebra with (super-)dimension $M = p - 2n$. So here the dimension of the radial algebra is an element of $\mathbb{Z}$. In [7] it was proven that there is a Fischer decomposition (lemma 1) in superspace if $M \notin -2\mathbb{N}$.

3. The q-Dirac operator on radial algebra

3.1. Definition

Our aim is to define a q-deformed vector derivative on the level of radial algebra. Property (2) implies that the following relation holds for the one dimensional q-derivative:

$$D_t^q t^2 = q^2 t^2 D_t^q + (q+1)t.$$

Comparing this with property (10) we find that the q-Dirac operator should satisfy $\partial_x^q x^2 = q^2 x^2 \partial_x^q + (q+1)x$. Formula (1) implies $D_t^q(t) = [1]_q$. Therefore we also impose $\partial_x^q(x) = [m]_q$ for some parameter $m \in \mathbb{R}$ and $\partial_x^q(\langle z, x\rangle) = \frac{[m]_q}{m} z$. We also assume ∂_x^q to be vector valued, i.e. ∂_x^q acting on $R_0(S)$ should be inside $R_1(S)$ and $\partial_v \langle v, \partial_x^q\rangle = \partial_x^q$ on $R(S\backslash\{v\})$ with $2\langle v, \partial_x^q\rangle = \{v, \partial_x^q\}$. In undeformed radial algebra the anticommutator of x and ∂_x commute with other vectors, see equation (9). It is logical to extend this property to the q-anticommutator of x and ∂_x^q. Thus we are led to the following axioms for $\partial_x^q \in \mathrm{End}(R(S))$,

$$
\begin{array}{ll}
(B0) & \partial_x^q(x) = [m]_q \\
(B1) & \partial_x^q(f) \in R_1(S), \quad f \in R_0(S) \quad \text{and } \partial_v\langle v, \partial_x^q\rangle = \partial_x^q \text{ on } R(S\backslash\{v\}) \\
(B2) & \partial_x^q(FG) = \partial_x^q(F)G \qquad F \in R(S)\,,\ G \in R(S\backslash\{x\}) \\
(B3) & \partial_x^q x^2 = q^2 x^2 \partial_x^q + (q+1)x \\
(B4) & \partial_x^q(\langle x, z\rangle) = \dfrac{[m]_q}{m} z \quad \forall\, z \neq x \\
(B5) & [(\partial_x^q x + q x \partial_x^q), z] = 0 \quad \forall\, z \neq x \\
(B6) & \lim\limits_{z \to u} \partial_x^q F(z) = \partial_x^q F(u)\,, \quad z \neq x \neq u \\
(B7) & \mathbb{E}\partial_x^q = \partial_x^q(\mathbb{E} - 1) \quad \text{and} \quad \mathbb{E}_u \partial_x^q = \partial_x^q \mathbb{E}_u, \quad u \neq x.
\end{array}
$$

Remark 1. *It would seem logical to demand the relation $\partial_x^q\langle x, z\rangle = q\langle x, z\rangle\partial_x^q + z$ as an axiom. However, this leads to a contradiction with axiom $(B6)$ when we evaluate $\partial_x^q(\langle x, z\rangle\langle x, v\rangle)$.*

3.2. Unicity

Now we prove some properties of an operator which satisfies axioms $(B0) - (B7)$. This will lead to a proof of the unicity of the q-Dirac operator. The q-deformation of the Euler operator is given by

$$E = [\mathbb{E}]_q = \frac{q^{\mathbb{E}} - 1}{q - 1}$$

and satisfies

$$Ex - qxE = x \quad \text{and} \quad Ez = zE. \tag{12}$$

Lemma 2. *An operator $\partial_x^q \in End(R(S))$ satisfying axioms $(B0)$, $(B3)$ and $(B5)$ also satisfies the following q-deformed version of formula (9):*

$$\partial_x^q x + q x \partial_x^q = [m]_q + (q^m + q)E = [m + \mathbb{E}]_q + q[\mathbb{E}]_q. \tag{13}$$

Proof. The combination of axioms $(B2)$ and $(B7)$ imply $\partial_x^q(1) = 0$. We define the operator A on $R(S)$ as

$$A \quad = \quad \partial_x^q x + qx\partial_x^q - [m]_q - (q^m + q)E.$$

Axiom $(B0)$ and the relation $\partial_x^q(1) = 0$ imply $A(1) = 0$. Using the definition of A and equation (12) we calculate

$$\begin{aligned}
\partial_x^q x^2 &= (A - qx\partial_x^q + [m]_q + (q^m + q)E)\, x \\
&= (A + [m]_q + (q^m + q)E)\, x - qx\,(A - qx\partial_x^q + [m]_q + (q^m + q)E) \\
&= Ax - qxA + q^2x^2\partial_x^q + [m]_q(1-q)x + (q^m + q)(Ex - qxE) \\
&= Ax - qxA + q^2x^2\partial_x^q + (q+1)x.
\end{aligned}$$

Comparing this with axiom $(B3)$ yields $Ax = qxA$. Using axiom $(B5)$ and equation (12) we also find $Az = zA$ $\forall z \neq x$, so $A = 0$ as an operator on $R(S)$. □

We will also need the following calculation.

Lemma 3. *An operator $\partial_x^q \in End(R(S))$ satisfying axiom $(B5)$ also satisfies*

$$\partial_x^q\langle x, z\rangle - q\langle x, z\rangle\partial_x^q \quad = \quad \langle z, \partial_x^q\rangle x - qx\langle z, \partial_x^q\rangle.$$

Proof. We calculate, using $(B5)$

$$\begin{aligned}
2\partial_x^q\langle x, z\rangle + 2qx\langle z, \partial_x^q\rangle &= \partial_x^q xz + qx\partial_x^q z + \partial_x^q zx + qxz\partial_x^q \\
&= z\partial_x^q x + zqx\partial_x^q + \partial_x^q zx + qxz\partial_x^q \\
&= 2\langle z, \partial_x^q\rangle x + 2q\langle x, z\rangle\partial_x^q,
\end{aligned}$$

which gives the desired result. □

Now we calculate some explicit evaluations of the Dirac operator. Because of lemma 3 and axiom $(B4)$ we find

$$\langle z, \partial_x^q\rangle(x) = \frac{[m]_q}{m} z \quad \text{and} \quad \langle z, \partial_x^q\rangle(\langle x, u\rangle) = \frac{[m]_q}{m}\langle z, u\rangle.$$

Therefore all the first order evaluations are completely determined from axioms $(B0) - (B7)$. Now we consider second order evaluations. Axiom $(B3)$ implies

$$\partial_x^q(x^2) \quad = \quad (q+1)x \tag{14}$$

and $\langle z, \partial_x^q\rangle(x^2) = (q+1)\langle x, z\rangle$. Equation (13) combined with axiom $(B4)$ yields

$$\partial_x^q(x\langle x, z\rangle) \quad = \quad -qx\frac{[m]_q}{m} z + ([m+1]_q + q)\langle x, z\rangle. \tag{15}$$

In order to find the other second order evaluations, we use a technique that will be generalized in the proof of theorem 1.

Lemma 4. *Consider $z, u \in S$, different from x. For an operator $\partial_x^q \in End(R(S))$ satisfying axioms $(B0) - (B7)$, the following relations hold:*

$$\partial_x^q(\langle x,z\rangle\langle x,u\rangle) = \frac{[m+1]_q+q}{m+2}(u\langle x,z\rangle + z\langle x,u\rangle) + \frac{1+q-2q[m]_q/m}{m+2}x\langle u,z\rangle$$

and

$$\langle z,\partial_x^q\rangle(x\langle x,u\rangle) = \frac{[m+1]_q+q}{m+2}(x\langle u,z\rangle + z\langle x,u\rangle) + \frac{1+q-2q[m]_q/m}{m+2}u\langle x,z\rangle.$$

Proof. We start by using lemma 3 on $\partial_x^q(\langle x,u\rangle\langle x,z\rangle)$,

$$\partial_x^q(\langle x,z\rangle\langle x,u\rangle) \quad = \quad q\frac{[m]_q}{m}\langle x,z\rangle u + \langle z,\partial_x^q\rangle(x\langle x,u\rangle) - q\frac{[m]_q}{m}x\langle z,u\rangle. \tag{16}$$

So it suffices to calculate $\partial_x^q(\langle x,z\rangle\langle x,u\rangle)$. Axioms $(B7)$ and $(B1)$ imply that there must be coefficients α, γ and β such that

$$\partial_x^q(\langle x,z\rangle\langle x,u\rangle) \quad = \quad \alpha z\langle x,u\rangle + \gamma u\langle x,z\rangle + \beta x\langle z,u\rangle$$

holds. The symmetry between z and u (axiom $(B6)$) implies $\alpha = \gamma$. Consider $v \in S$ and different from x, u, z, the calculation above implies

$$\begin{aligned}\langle v,\partial_x^q\rangle(\langle x,z\rangle\langle x,u\rangle) &= \alpha(\langle v,z\rangle\langle x,u\rangle + \langle v,u\rangle\langle x,z\rangle) + \beta\langle v,x\rangle\langle z,u\rangle \quad \text{and}\\ \langle v,\partial_x^q\rangle(x\langle x,u\rangle) &= \alpha(v\langle x,u\rangle + \langle v,u\rangle x) + \beta\langle v,x\rangle u.\end{aligned}$$

Axiom $(B1)$ then implies

$$\begin{aligned}\partial_x^q x\langle x,u\rangle &= \partial_v\langle v,\partial_x^q\rangle x\langle x,u\rangle\\ &= \alpha(m\langle x,u\rangle + ux) + \beta xu.\end{aligned}$$

Comparing this with formula (15), yields $\alpha = \frac{[m+1]_q+q}{m+2}$ and $\beta = \frac{1+q-2q[m]_q/m}{m+2}$. □

This lemma and the calculation before imply that the operator ∂_x^q is uniquely determined by axioms $(B0)-(B7)$ on the space $[R(S)]_0 \oplus [R(S)]_1 \oplus [R(S)]_2$. By generalizing the used techniques we can prove the unicity on $R(S)$. First the symmetry is used in the following lemma to limit the possible expressions for ∂_x^q acting on elements of $R(S)$. We use the convention that the notation $\langle x,u_i\rangle\langle x,u_{i+1}\rangle\cdots\langle x,u_k\rangle$ is considered to be 1 if $k < i$ and introduce some notations. For u_i elements of S and S_{j-1} the permutation group on $\{1,2,\cdots,j-1\}$, define the element of $R_0(S)$ given by

$$\begin{aligned}&\Lambda_i^j(u_1,\cdots,u_{j-1})\\ =& \sum_{\sigma\in S_{j-1}}\langle u_{\sigma(1)},u_{\sigma(2)}\rangle\cdots\langle u_{\sigma(2i-1)},u_{\sigma(2i)}\rangle\langle x,u_{\sigma(2i+1)}\rangle\cdots\langle x,u_{\sigma(j-1)}\rangle\end{aligned}$$

and the element of $R_1(S)$ given by

$$\begin{aligned}&\Pi_i^j(u_1,\cdots,u_{j-1})\\ =& \sum_{\sigma\in S_{j-1}}u_{\sigma(1)}\langle u_{\sigma(2)},u_{\sigma(3)}\rangle\cdots\langle u_{\sigma(2i)},u_{\sigma(2i+1)}\rangle\langle x,u_{\sigma(2i+2)}\rangle\cdots\langle x,u_{\sigma(j-1)}\rangle.\end{aligned}$$

The following properties of Λ and Π are straightforward to obtain

$$\Pi_i^j\, x \quad = \quad 2\Lambda_i^j - x\,\Pi_i^j \tag{17}$$

and

$$\Lambda_i^j(u_1,\cdots,u_{j-1}) = \sum_{\sigma\in S_{j-1}} u_{\sigma(1)}u_{\sigma(2)}\langle u_{\sigma(3)},u_{\sigma(4)}\rangle\cdots \tag{18}$$
$$\cdots\langle u_{\sigma(2i-1)},u_{\sigma(2i)}\rangle\langle x,u_{\sigma(2i+1)}\rangle\cdots\langle x,u_{\sigma(j-1)}\rangle.$$

Lemma 5. *With $x, v, u_1, u_2, \cdots, u_j$ all different from each other and ∂_x^q an operator satisfying axioms $(B0)-(B7)$, the expression*

$$\langle v,\partial_x^q\rangle\left(\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle\right)$$

has to be of the following form

$$\sum_{i=0}^{\lfloor\frac{j-1}{2}\rfloor}\alpha_i x^{2i}\langle v,\Pi_i^{j+1}(u_1,\cdots,u_j)\rangle + \sum_{i=0}^{\lfloor\frac{j-2}{2}\rfloor}\beta_i x^{2i}\langle v,x\rangle\Lambda_{i+1}^{j+1}(u_1,\cdots,u_j)$$

for some constants $\{\alpha_i\}$ and $\{\beta_i\}$.

Proof. The fact that every term is accompanied by permutations of the u_l is immediately clear from the fact that $u_1,\cdots u_j$ all have the same indistinguishable role, as implied by axiom $(B6)$. Axioms $(B1)$ and $(B7)$ imply that

$$\langle v,\partial_x^q\rangle\left(\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle\right)$$

is an element of $R_0(S)$ of degree $j-1$ in x and of degree 1 in v and u_l for $1\leq l\leq j$.

First we consider all the possible terms without an x^2, these correspond to $i=0$ in the summation above. So we need to make inner products of x with $j-1$ of the $v, u_1,\cdots,u_j$. Besides these inner products with x, there either has to be an inner product $\langle v,u_\lambda\rangle$ or an inner product of the form $\langle u_\lambda,u_\rho\rangle$. For the first option to be possible, $j\geq 1$ has to hold. For the second option to be possible, $j\geq 2$ has to hold. These are the two possibilities that correspond to α_0 and β_0. A completely similar reasoning leads to the terms with $i\geq 1$. □

Now lemma 4 can be generalized to arbitrary degrees.

Theorem 1. *For an operator $\partial_x^q\in End(R(S))$ satisfying axioms $(B0)-(B7)$, the expression*

$$\partial_x^q\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle$$

with all the u_i different from each other and from x, is completely determined for $j\in\mathbb{N}$.

Proof. This expression is determined for $j=1$, by axiom $(B4)$. Now, define

$$F_j(u_1,u_2,\cdots,u_j) \quad = \quad \partial_x^q\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle.$$

Lemma 5 shows which form $\langle v, F_j\rangle$ takes. This implies that

$$\begin{aligned} & F_j(u_1, u_2, \cdots, u_j) \\ = & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i} \Pi_i^{j+1}(u_1, \cdots, u_j) + \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \beta_i^j x^{2i+1} \Lambda_{i+1}^{j+1}(u_1, \cdots, u_j) \end{aligned}$$

for some coefficients α and β. Now we will prove that the coefficients $\{\alpha_i^j, i \le \lfloor \frac{j-1}{2} \rfloor\}$ and $\{\beta_i^j, i \le \lfloor \frac{j-2}{2} \rfloor\}$ are determined in terms of the coefficients $\{\alpha_i^{j-1}, i \le \lfloor \frac{j-2}{2} \rfloor\}$ and $\{\beta_i^{j-1}, i \le \lfloor \frac{j-3}{2} \rfloor\}$. This proves by induction that the coefficients are determined, since they are known for $j = 1$.

Define $G_j(u_1, u_2, \cdots, u_{j-1})$ by $\langle G_j, u_j\rangle = \langle v, F_j\rangle$, which is equivalent with

$$G_j = \langle v, \partial_x^q\rangle \langle x, u_1\rangle \cdots \langle x, u_{j-1}\rangle x. \tag{19}$$

From the expression of $\langle v, F_j\rangle$ we can calculate G_j,

$$\begin{aligned} G_j(u_1, u_2, \cdots, u_{j-1}) = & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i} v \Lambda_i^j(u_1, \cdots, u_{j-1}) \\ + & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i} 2i [\sum_{\sigma \in S_{j-1}} \langle v, u_{\sigma(1)}\rangle u_{\sigma(2)} \langle u_{\sigma(3)}, u_{\sigma(4)}\rangle \cdots \\ & \cdots \langle u_{\sigma(2i-1)}, u_{\sigma(2i)}\rangle \langle x, u_{\sigma(2i+1)}\rangle \cdots \langle x, u_{\sigma(j-1)}\rangle] \\ + & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i+1} (j - 2i - 1) \langle v, \Pi_i^j(u_1, \cdots, u_{j-1})\rangle \\ + & \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \beta_i^j x^{2i} (2i + 2) \langle v, x\rangle \Pi_i^j(u_1, \cdots, u_{j-1}) \\ + & \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \beta_i^j x^{2i+1} (j - 2i - 2) \langle v, x\rangle \Lambda_{i+1}^j(u_1, \cdots, u_{j-1}). \end{aligned}$$

This allows to calculate $\partial_v G_j$ using equations (17) and (18),

$$\begin{aligned} \partial_v G_j(u_1, u_2, \cdots, u_{j-1}) = & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i} (m + 2i) \Lambda_i^j(u_1, \cdots, u_{j-1}) \\ + & \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \alpha_i^j x^{2i} (j - 2i - 1) \left[2\Lambda_i^j(u_1, \cdots, u_{j-1}) - x\Pi_i^j(u_1, \cdots, u_{j-1})\right] \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \beta_i^j x^{2i+1}(2i+2)\Pi_i^j(u_1,\cdots,u_{j-1}) \\
&+ \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} \beta_i^j x^{2i+2}(j-2i-2)\Lambda_{i+1}^j(u_1,\cdots,u_{j-1}).
\end{aligned}
$$

By combining terms and taking into account zero terms we obtain

$$
\begin{aligned}
& \partial_v G_j(u_1,u_2,\cdots,u_{j-1}) \\
= \;& (m+2j-2)\alpha_0^j(j-1)!\langle x,u_1\rangle\cdots\langle x,u_{j-1}\rangle \\
+ \;& \sum_{i=1}^{\lfloor \frac{j-1}{2} \rfloor} [(m+2j-2i-2)\alpha_i^j + (j-2i)\beta_{i-1}^j]x^{2i}\Lambda_i^j(u_1,\cdots,u_{j-1}) \\
+ \;& \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} [(2i+2)\beta_i^j - (j-2i-1)\alpha_i^j]x^{2i+1}\Pi_i^j(u_1,\cdots,u_{j-1}).
\end{aligned}
$$

The expression $\partial_v G_j$ can also be calculated starting from equation (19), using axiom $(B1)$, the expression of F_{j-1} in terms of Λ and Π and lemma 2,

$$
\begin{aligned}
& \partial_v G_j \\
= \;& \partial_x^q x\langle x,u_1\rangle\cdots\langle x,u_{j-1}\rangle \\
= \;& ([m+j-1]_q + q[j-1]_q)\,\langle x,u_1\rangle\cdots\langle x,u_{j-1}\rangle \\
- \;& \sum_{i=0}^{\lfloor \frac{j-2}{2} \rfloor} q\alpha_i^{j-1}x^{2i+1}\Pi_i^j(u_1,\cdots,u_{j-1}) - \sum_{i=0}^{\lfloor \frac{j-3}{2} \rfloor} q\beta_i^{j-1}x^{2i+2}\Lambda_{i+1}^j(u_1,\cdots,u_{j-1})
\end{aligned}
$$

Comparing coefficients yields

$$
\begin{aligned}
\alpha_0^j &= \frac{[m+j-1]_q + q[j-1]_q}{(m+2j-2)(j-1)!}, \\
\left[(m+2j-2i-2)\alpha_i^j + (j-2i)\beta_{i-1}^j\right] &= -q\beta_{i-1}^{j-1} \quad \text{and} \\
\left[(2i+2)\beta_i^j - (j-2i-1)\alpha_i^j\right] &= -q\alpha_i^{j-1}.
\end{aligned}
$$

Assuming that the coefficients for $j-1$ are known, this can clearly be solved by using the third equation to calculate β_0^j, then the second to calculate α_1^j and so forth. The solution will always be unique. However it will only exist if $m \notin -2\mathbb{N}$. □

Now we can prove the unicity of the q-Dirac operator if the parameter m is not even and negative. If $m \in -2\mathbb{N}$ the axioms are not consistent as can be seen from the proof of theorem 1.

Theorem 2. *There can only be one q-Dirac operator $\partial_x^q \in End(R(S))$ satisfying axioms $(B0)$ – $(B7)$.*

Proof. We need to prove that ∂_x^q acting on all elements of $R(S)$ is determined from axioms $(B0)-(B7)$. The radial algebra $R(S)$ is additively generated by elements of the form

$$x^k\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle y_1\cdots y_l,$$

with $k\in\mathbb{N}$, $u_i,y_i\in S$ and different from x. Because of axiom $(B6)$ we can consider all the u_i to be different from each other. Lemma 2 and axiom $(B2)$ imply that the q-Dirac operator acting on such elements of $R(S)$ is determined if the q-Dirac operator on $\langle x,u_1\rangle\langle x,u_2\rangle\cdots\langle x,u_j\rangle$ is determined. Theorem 1 therefore proves this theorem. □

It should still be proven that the axioms $(B0)-(B7)$ are in general not inconsistent, this will be done in section 3.3 by explicit construcion of ∂_x^q if $m\not\in-2\mathbb{N}$.

Remark 2. *The calculations in theorem* 1 *imply that there cannot be a q-deformed Dirac operator in models of radial algebra for super Clifford analysis if the superdimension $M=p-2n$ (with p bosonic and n fermionic variables) is even and negative. This corresponds to the case without Fischer decomposition, see* [7].

We call an element P of the radial algebra q-monogenic if it satisfies $\partial_x^qP=0$. Here we will find that a certain type of second degree polynomial is q-monogenic if it is monogenic. In section 3.3 we will find that every monogenic polynomial is q-monogenic.

Lemma 6. *If the element of the radial algebra,*

$$\langle x,z\rangle^2+a\langle x,z\rangle xz+bx^2z^2$$

with $a,b\in\mathbb{R}$ is monogenic, it is also q-monogenic if $m\not\in-2\mathbb{N}$.

Proof. Using the calculations in equations (14) and (15) and in lemma 4 we find

$$\begin{aligned}
&\partial_x^q(\langle x,z\rangle^2+a\langle x,z\rangle xz+bx^2z^2)\\
=\;&(\frac{[m+1]_q+q}{m+2})2z\langle x,z\rangle+\frac{1+q-2q[m]_q/m}{m+2}xz^2\\
+\;&a([m+1]_q+q)\langle x,z\rangle z-aq\frac{[m]_q}{m}xz^2+b(q+1)xz^2.
\end{aligned}$$

Therefore it is q-monogenic if $a=\frac{-2}{m+2}$ and $b=\frac{-1}{m+2}$. This is independent from q and therefore the polynomial is q-monogenic if it is monogenic. □

In case the parameter m satisfies $[m+1]_q+q=0$ (which requires m to be negative) the element of $R(S)$ above is q-monogenic for all (a,b) such that $1+a+mb=0$ holds. So for specific values of q and m the space of q-monogenic elements of $R(S)$ can include elements which are not monogenic.

3.3. Existence

We give an explicit expression for ∂_x^q in case $m \notin -2\mathbb{N}$ in terms of the classical vector derivative ∂_x defined by axioms $(A0)-(A4)$. In this way we prove that the set of axioms $(B0)-(B7)$ is consistent, and complete the unicity and existence of an operator defined by $(B0)-(B7)$. This explicit expression for ∂_x^q in terms of ∂_x also leads to a definition for the q-deformed Dirac operator in (super) Clifford analysis.

Theorem 3. *If $m \notin -2\mathbb{N}$ there is exactly one operator ∂_x^q in End$(R(S))$ which satisfies axioms $(B0)-(B7)$. The operator is given by*

$$\partial_x^q = \sum_{j=0}^{\infty} x^j f_j(\mathbb{E})\partial_x^{j+1},$$

with, for $j \in \mathbb{N}$, the functions $f_j : \mathbb{N} \to \mathbb{R}$ defined by $f_0(u) = \frac{[m+u]_q + q[u]_q}{m+2u}$ and

$$\begin{aligned} f_{2j+1}(u) &= \frac{1}{2j+2}\left[f_{2j}(u+1) - qf_{2j}(u)\right] \ \textit{and} \\ f_{2j+2}(u) &= -\frac{1}{2u+2j+m+2}\left[f_{2j+1}(u+1) + qf_{2j+1}(u)\right]. \end{aligned}$$

Proof. The operator $f_j(\mathbb{E})$ is defined as the diagonal operator on $R(S)$ which has the same eigenvectors as $\mathbb{E}$ but with eigenvalues $k \in \mathbb{N}$ replaced by $f_j(k)$. Since each element of $R(S)$ is of finite maximal degree in x, the expression above for ∂_x^q is an element of End$(R(S))$.

Because of the unicity of an operator which satisfies axioms $(B0)-(B7)$, we only have to check that the proposed form of ∂_x^q satisfies the axioms. Axioms $(B0)$ and $(B4)$ hold since $f_0(0) = [m]_q/m$. Because x and ∂_x are vector valued operators axiom $(B1)$ holds. Axioms $(B2)$, $(B6)$ and $(B7)$ trivially hold. Now we will prove that equation (13) holds, from which $(B3)$ and $(B5)$ follow. Substituting the proposed expression for ∂_x^q and using the formulas $\{\partial_x^{2j+1}, x\} = \partial_x^{2j}(2\mathbb{E}+m)$ and $[\partial_x^{2j}, x] = 2j\partial_x^{2j-1}$ yields

$$\begin{aligned} \partial_x^q x + qx\partial_x^q &= -\sum_{j=0}^{\infty} x^{2j+1} f_{2j}(\mathbb{E}+1)\partial_x^{2j+1} + \sum_{j=0}^{\infty} x^{2j} f_{2j}(\mathbb{E})(2\mathbb{E}+m+2j)\partial_x^{2j} \\ &+ \sum_{j=0}^{\infty} x^{2j+2} f_{2j+1}(\mathbb{E}+1)\partial_x^{2j+2} + \sum_{j=0}^{\infty} x^{2j+1}(2j+2) f_{2j+1}(\mathbb{E})\partial_x^{2j+1} \\ &+ \sum_{j=0}^{\infty} x^{2j+1} q f_{2j}(\mathbb{E})\partial_x^{2j+1} + \sum_{j=0}^{\infty} x^{2j+2} q f_{2j+1}(\mathbb{E})\partial_x^{2j+2} \\ &= f_0(\mathbb{E})(2\mathbb{E}+m) = [m+\mathbb{E}]_q + q[\mathbb{E}]_q, \end{aligned}$$

which proves the theorem. □

Remark 3. *The explicit expression in theorem 3 implies that lemma 6 can be generalized immediately. Every monogenic polynomial is q-monogenic. The other direction is not true, a polynomial can be q-monogenic for specific values of m and q without being monogenic.*

Example 1. *As an easy example we can calculate the expression for ∂_x^q acting on elements of $R(S)$ of degree 2 in x,*

$$\partial_x^q|_{[R(S)]_2} = \frac{[m+1]_q+q}{m+2}\partial_x + \frac{1}{2}\frac{1+q-2q[m]_q/m}{m+2}x\partial_x^2.$$

From this we immediately re-obtain the expressions in lemma 4.

4. The q-Dirac operator in Clifford analysis

The expression for the q-Dirac operator in terms of ∂_x in theorem 3 allows to construct it for specific models of radial algebra. For Clifford analysis on $\mathbb{R}^m$, the q-Dirac operator on $\mathcal{P} = \mathbb{R}[x_1, \cdots, x_m] \otimes \mathbb{C}_m$ is therefore defined as

$$\partial_{\underline{x}}^q = \sum_{j=0}^{\infty} \underline{x}^j f_j(\mathbb{E}) \partial_{\underline{x}}^{j+1}, \tag{20}$$

with f_j the functions in theorem 3. Although this expression is not very transparent, the action on terms of the Fischer decomposition can be calculated easily. This operator can also be determined uniquely by a list of axioms, closely related to the B-axioms. The only essential change is axiom $(B6)$, which does not have the same implications in Clifford analysis, since the set S is finite for Clifford analysis. Therefore it is replaced by the condition of $Spin(m)$-invariance.

4.1. The q-Dirac operator and the Fischer decompostion

It is straightforward to check that the Dirac operator in Clifford analysis (20) still satisfies the relation

$$\partial_{\underline{x}}^q \underline{x} + q\underline{x}\partial_{\underline{x}}^q = [m+\mathbb{E}]_q + q[\mathbb{E}]_q, \tag{21}$$

which also implies $\partial_{\underline{x}}^q \underline{x}^2 = q^2\underline{x}^2\partial_{\underline{x}}^q + (q+1)\underline{x}$. The q-deformed Euler operator also satisfies $[\mathbb{E}]_q = rD_r^q$. In Clifford analysis (with dimension $m > 0$) we can prove that a polynomial is monogenic if and only if it is q-monogenic (with q an arbitrary fixed constant satisfying $0 < q < 1$), see the discussion in remark 3.

Theorem 4. *Given $M_k \in \mathcal{M}_k$ a spherical monogenic of degree k, then the following relations holds:*

$$\begin{aligned}
\partial_{\underline{x}}^q \underline{x}^{2l} M_k &= [2l]_q \underline{x}^{2l-1} M_k \qquad l \in \mathbb{N}, \\
\partial_{\underline{x}}^q \underline{x}^{2l+1} M_k &= \left([2l+k+m]_q + q^{2l+1}[k]_q\right) \underline{x}^{2l} M_k \qquad l \in \mathbb{N}.
\end{aligned}$$

Given $P_k \in \mathcal{P}_k$ a polynomial of degree k, $\partial_{\underline{x}}^q P_k = 0$ holds if and only if $P_k \in \mathcal{M}_k$.

Proof. The expression (20) immediately implies that $\partial_{\underline{x}}^q M_k = 0$ if $M_k \in \mathcal{M}_k$. Applying the commutation relation (21) yields

$$\begin{aligned}\partial_{\underline{x}}^q \underline{x}^{2l} M_k &= (q^{2l} x^{2l} \partial_{\underline{x}}^q + \frac{q^{2l}-1}{q-1} \underline{x}^{2l-1}) M_k \\ &= [2l]_q \underline{x}^{2l-1} M_k\end{aligned}$$

and

$$\begin{aligned}\partial_{\underline{x}}^q \underline{x}^{2l+1} M_k &= \left[q^{2l} \underline{x}^{2l} \partial_{\underline{x}}^q \underline{x} + \frac{q^{2l}-1}{q-1} \underline{x}^{2l} \right] M_k \\ &= \left[q^{2l} \underline{x}^{2l} ([m+k]_q + q[k]_q) + [2l]_q \right] \underline{x}^{2l} M_k \\ &= ([2l+m+k]_q + q^{2l+1}[k]_q) \underline{x}^{2l} M_k,\end{aligned}$$

which completes the proof of the first part of the theorem.

The obtained relations can be summarized as $\partial_{\underline{x}}^q \underline{x}^l M_k = c_{l,k} \underline{x}^{l-1} M_k$ for $l > 0$. The coefficients $c_{k,l}$ are strictly positive since $m > 0$, $l > 0$ and $k \geq 0$. Consider a fixed polynomial $P_k \in \mathcal{P}_k$ with Fischer decomposition (lemma 1)

$$P_k = \sum_{j=0}^{k} \underline{x}^j M_{k-j} \quad \Rightarrow \quad \partial_{\underline{x}}^q P_k = \sum_{j=1}^{k} c_{j,k-j} \underline{x}^{j-1} M_{k-j}.$$

The unicity of the Fischer decomposition then implies that if P_k satisfies $\partial_{\underline{x}}^q P_k = 0$, then $M_j = 0$ must hold for $j < k$ and therefore $P_k = M_k$ holds. This completes the proof. □

The fact that the space of spherical q-monogenics is independent of q, shows that the only real q-deformation is radial. This is closely related to the fact that the q-deformed Dirac operator is invariant under the spin group as will be proven in the following section, and not under a q-deformed version of the spin group.

The action of the q-Dirac operator can be extended to general radial functions, not necessarily in $\mathcal{P}$. When f is a scalar function, axiom $(B3)$ can be extended to

$$\partial_{\underline{x}}^q f(\underline{x}^2) = -\frac{\underline{x}}{r} \left(D_r^q f(\underline{x}^2) \right) + f(q^2 \underline{x}^2) \partial_{\underline{x}}^q. \tag{22}$$

The space of all scalar radial functions for which the expression above exists is denoted by $\mathcal{J}$.

4.2. Unicity of the q-Dirac operator on Clifford analysis

Theorem 5. *The Dirac operator $\partial_{\underline{x}}^q$ on $\mathcal{P} = \mathbb{R}[x_1, \cdots, m] \otimes \mathbb{C}_m$ in formula (20) is uniquely determined by the following list of properties:*

$(C0)$	$\partial_{\underline{x}}^q(\underline{x}) = [m]_q$
$(C3)$	$\partial_{\underline{x}}^q \underline{x}^2 = q^2 \underline{x}^2 \partial_x^q + (q+1)\underline{x}$
$(C5)$	$(\partial_{\underline{x}}^q x + qx\partial_{\underline{x}}^q)$ *is a scalar operator*
$(C6)$	$\partial_{\underline{x}}^q$ *is Spin(m)-invariant*
$(C7)$	$\mathbb{E}\partial_{\underline{x}}^q = \partial_{\underline{x}}^q(\mathbb{E} - 1)$.

Proof. First, it can be checked that the q-Dirac operator in equation (20) satisfies these properties. Properties $(C0)$ and $(C7)$ are trivial. Properties $(C3)$ and $(C5)$ follow from equation (21). Property $(C6)$ holds since $\underline{x}$, $\partial_{\underline{x}}$ and $\mathbb{E}$ are $Spin(m)$-invariant.

In the exact same way as in the proof of lemma 2 properties $(C0)$, $(C3)$ and $(C5)$ imply that equation (21) must hold. The action on a polynomial of the form $\underline{x}^l M_k$ with $M_k \in \mathcal{M}_k$ is therefore uniquely determined when $\partial_{\underline{x}}^q M_k$ is determined.

Axiom $(C7)$ implies that the image of $\partial_{\underline{x}}^q$ acting on the space $\mathcal{M}_k$ is a subspace of $\mathcal{P}_{k-1}$. As a $Spin(m)$-representation $\mathcal{M}_k$ decomposes into irreducible pieces as (for simplicity we consider the case $m = 2n+1$)

$$\mathcal{M}_k = \bigoplus_{i=0}^{2^n} \mathcal{M}_k^i.$$

Each $\mathcal{M}_k^i$ is an irreducible highest weight representation for $Spin(m)$, with highest weight $(k+\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, where the term $\frac{1}{2}$ is repeated $n-1$ times. This essentially follows from the decomposition of $\mathbb{C}_m$ into its spinor spaces. The decomposition of the space $\mathcal{P}_{k-1}$ in lemma 1,

$$\mathcal{P}_{k-1} = \bigoplus_{j=0}^{k-1} \underline{x}^j \mathcal{M}_{k-1-j}$$

implies that there does not appear a $Spin(m)$-representation of highest weight $(k+\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ in $\mathcal{P}_{k-1}$. Since the q-Dirac operator is $Spin(m)$-invariant and the image of $\partial_{\underline{x}}^q$ acting on the space $\mathcal{M}_k$ is inside $\mathcal{P}_{k-1}$, we find that $\partial_{\underline{x}}^q$ must be zero on $\mathcal{M}_k$. The case $m = 2n$ is completely equivalent.

Summarizing, the axioms imply that $\partial_{\underline{x}}^q \underline{x}^l M_k$ is given by the expressions in theorem 4 which implies that $\partial_{\underline{x}}^q$ is uniquely determined and corresponds to the expression in equation (20). $\square$

5. q-integration

Since the deformation is purely radial, it is logical to define integration as a combination of q-deformed (Jackson) radial integration with undeformed spherical integration, $\int_{\mathbb{S}^{m-1}} d\sigma$, as was also done in [4]. The spherical integration satisfies $\int_{\mathbb{S}^{m-1}} d\sigma \underline{x} \mathcal{M}_k = 0$ for $k \in \mathbb{N}$ and $\int_{\mathbb{S}^{m-1}} d\sigma \mathcal{M}_k = 0$ for $k > 0$. We define integration for functions of the form $\mathcal{P} \otimes \mathcal{J}$ with $\mathcal{J}$ the scalar radial functions such that the q-Dirac operator is defined on them.

Definition 2. *The q-integration on $\mathcal{P} \otimes \mathcal{J}$ is defined as*

$$\int_{\mathbb{B}^m(\lambda)} f(\underline{x}) d_q V(\underline{x}) = \int_0^\lambda d_q r r^{m-1} \int_{\mathbb{S}^{m-1}} d\sigma f(\underline{x}),$$

with the one dimensional q-integration as defined in equation (4) *for $\lambda \in \mathbb{R}^+$ or equation* (5) *for $\lambda = a \cdot \infty$ with $a \in \mathbb{R}^+$.*

The vector $\underline{\xi}$ will be used for the normalized vector variable, $\underline{\xi} = \underline{x}/r$. The integration over the boundary of $\mathbb{B}^m(\lambda)$, the ball with radius λ in $\mathbb{R}^m$, is denoted by $\int_{\partial\mathbb{B}^m(\lambda)} d\sigma$ and satisfies

$$\int_{\partial\mathbb{B}^m(\lambda)} d\sigma f(\underline{x}) = \lambda^{m-1}\int_{\mathbb{S}^{m-1}} d\sigma f(\lambda\underline{\xi}).$$

Now we prove that Stokes' theorem holds for this q-integration and the q-Dirac operator.

Theorem 6. *For $f \in \mathcal{P}\otimes\mathcal{J}$ and the integration in defintion 2, the following relations hold,*

$$\begin{aligned}\int_{\mathbb{B}^m(\lambda)} (\partial_{\underline{x}}^q f) d_q V(\underline{x}) &= -\int_{\partial\mathbb{B}^m(\lambda)} d\sigma\, \underline{\xi} f(\underline{x}) \qquad \text{if } \lambda \in \mathbb{R}^+ \\ &= 0 \quad \text{if } \lambda = a\cdot\infty,\ a \in \mathbb{R}^+ \\ &\qquad \text{and} \quad \lim_{k\to\infty}\int_{\partial\mathbb{B}^m(q^{-k}a)} d\sigma\underline{\xi} f = 0.\end{aligned}$$

Proof. The space $\mathcal{P}\otimes\mathcal{J}$ is generated by functions of the form $M_k g(\underline{x}^2)$ and $\underline{x}M_k g(\underline{x}^2)$ with $M_k \in \mathcal{M}_k$ and $g(\underline{x}^2)$ a scalar valued radial function. It is immediately clear that both the left-hand and right-hand side of the proposed equations are zero unless $k = 0$. The first possibility we must check is therefore $f(\underline{x}) = g(\underline{x}^2)$. Equation (22) shows that the left-hand side is equal to

$$\int_{\mathbb{B}(\lambda)} (\partial_{\underline{x}}^q f) d_q V(\underline{x}) = -\int_{\mathbb{B}(\lambda)} \frac{\underline{x}}{r}\left(D_r^q g(\underline{x}^2)\right) dV(\underline{x}).$$

Therefore both left-hand and right-hand side are zero. The only case left is $f(\underline{x}) = \underline{x}g(\underline{x}^2)$. The left-hand side can be calculated using equations (21) and (22) as

$$\begin{aligned}&\int_{\mathbb{B}(\lambda)} (\partial_{\underline{x}}^q f) d_q V(\underline{x}) \\ =\ & [m]_q \int_{\mathbb{B}(\lambda)} g(\underline{x}^2) d_q V(\underline{x}) + (q^m + q)\int_{\mathbb{B}(\lambda)} r D_r^q g(\underline{x}^2) d_q V(\underline{x}) \\ +\ & q\int_{\mathbb{B}(\lambda)} \frac{\underline{x}^2}{r} D_r^q g(\underline{x}^2) d_q V(\underline{x}) \\ =\ & [m]_q \int_{\mathbb{B}(\lambda)} g(\underline{x}^2) d_q V(\underline{x}) + q^m \int_{\mathbb{B}(\lambda)} r D_r^q g(\underline{x}^2) d_q V(\underline{x}).\end{aligned}$$

In case $\lambda \in \mathbb{R}^+$, the following relation can be calculated using equations (3) and (6)

$$\begin{aligned}q^m \int_0^\lambda d_r r^m D_r^q h(r) &= \int_0^\lambda d_q r D_r^q (r^m h(r)) - [m]_q \int_0^\infty d_q r r^{m-1} h(r) \\ &= \lambda^m h(\lambda) - [m]_q \int_0^\infty d_q r r^{m-1} h(r).\end{aligned}$$

For $\lambda \in \mathbb{R}^+$ we therefore obtain

$$\begin{aligned}\int_{\mathbb{B}(\lambda)} (\partial_{\underline{x}}^q f) d_q V(\underline{x}) &= \lambda^m g(-\lambda^2) \int_{\mathbb{S}^{m-1}} d\sigma 1 \\ &= -\int_{\partial \mathbb{B}^m(\lambda)} d\sigma \underline{\xi} \underline{x} g(\underline{x}^2) = -\int_{\partial \mathbb{B}^m(\lambda)} d\sigma\, \underline{\xi} f(\underline{x}).\end{aligned}$$

The infinite integration can be calculated in the exact same way. It can also be obtained from the first property by observing the relation $\int_0^{a\cdot\infty} d_q r = \lim_{k\to\infty} \int_0^{q^{-k}a} d_q r$. □

6. Orthogonal polynomials

Now that the integration corresponding to the q-Dirac operator is defined, one can consider orthogonal polynomials. Therefore we consider a fixed basis $\{M_k^{(l)}\}$ for the space of spherical monogenics $\mathcal{M}_k$ which satisfies

$$\int_{\mathbb{S}^{m-1}} d\sigma \left[\overline{M_k^{(l)}} M_k^{(t)}\right]_0 = \delta_{lt},$$

with $[\cdot]_0 : \mathbb{C}_m \to \mathbb{C}$ the projection of the Clifford algebra onto its scalar part.

In theorem 10 in [4] it was proven that for the inner product on $\mathcal{P}$ defined as

$$\langle f|g\rangle = \int_{\mathbb{B}^m\left(\frac{1}{\sqrt{1-q^2}}\right)} \left[\overline{f(\underline{x})} g(\underline{x})\right]_0 e_{q^2}(\underline{x}^2) d_q V(\underline{x}),$$

the q-Clifford-Hermite polynomials given by

$$\phi_{j,k,l} = \begin{cases} \mathcal{L}_i^{\frac{m}{2}+k-1}(-\underline{x}^2|q^2) M_k^{(l)} & \text{if } j = 2i \\ \underline{x}\mathcal{L}_i^{\frac{m}{2}+k}(-\underline{x}^2|q^2) M_k^{(l)} & \text{if } j = 2i+1, \end{cases}$$

form an orthogonal basis for $\mathcal{P}$. Although these polynomials are orthogonal for the integration connected to the q-Dirac operator, they do not behave well with respect to the q-Dirac operator developed in the present paper. For instance it can be calculated that the relation

$$\partial_{\underline{x}}^q \phi_{j,k,l} = C_{j,k,l} \phi_{j-1,k,l}$$

will not hold for any constants $C_{j,k,l}$. Such an equation does hold for the q-Dirac operator developed in [4], which therefore is better suited to generalize the theory of q-orthogonal polynomials (see e.g. [3, 12]) to Clifford analysis. Such an equation also holds for the q-Dirac operator developed in the present paper and the polynomials $\psi_{j,k,l}$ defined as

$$\psi_{j,k,l} = \begin{cases} \mathcal{L}_i^{\alpha_{k,m}-1}(-\underline{x}^2|q^2) M_k^{(l)} & \text{if } j = 2i \\ \underline{x}\mathcal{L}_i^{\alpha_{k,m}}(-\underline{x}^2|q^2) M_k^{(l)} & \text{if } j = 2i+1 \end{cases}$$

with $\alpha_{k,m}$ satisfying $[2\alpha_{k,m}]_q = [k+m]_q + q[k]_q$. However, these polynomials are not orthogonal with respect to the inner product introduced above.

7. Conclusion

The q-Dirac operator defined by the list of axioms $(B0) - (B7)$ is more naturally defined compared to the Dirac operator in [4]. In the case of Clifford analysis the q-Dirac operator in the current paper is uniquely defined from the axioms $(C0)$, $(C3)$, $(C5)$, $(C6)$ and $(C7)$. The real difference between this list and the list of axioms $(A1) - (A4)$ on p7 of [4] is that in the present paper the q-anticommutator of $\partial^q_{\underline{x}}$ and $\underline{x}$ has to be scalar while in [4] the square of the q-Dirac operator has to be scalar. The first condition is logical in the framework of Clifford analysis, while the second condition is necessary in order to obtain a q-deformed Laplace operator.

Both q-Dirac operators satisfy Stokes' formula for the same type of q-integration on $\mathbb{R}^m_q$. However, as the theory of q-Clifford-Hermite polynomials, which are orthogonal with respect to this integration, is more closely linked to the q-Dirac operator in [4], that operator is better suited to study q-polynomials.

There are still other possible lists of axioms which define q-Dirac operators. Each definition seems to have its own specific advantage with respect to integration, the Fischer decomposition, axial functions or q-polynomials. An important direction for further research is a systematic study of the link between changes in the axioms and properties of the resulting q-Dirac operator.

Acknowledgment

K. Coulembier is supported by a Ph.D. Fellowship of the the Research Foundation - Flanders (FWO).

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K. Coulembier
Krijgslaan 281
9000 Gent
Belgium
e-mail: `coulembier@cage.ugent.be`

F. Sommen
Krijgslaan 281
9000 Gent
Belgium
e-mail: `fs@cage.ugent.be`