# Triangular norms which are meet-morphisms in interval-valued fuzzy set theory

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**Abstract**

In this paper we study t-norms on the lattice of closed subintervals of the unit interval. Unlike for t-norms on a product lattice for which there exists a straightforward characterization of t-norms which are join-morphisms, respectively meet-morphisms, the situation is more complicated for t-norms in interval-valued fuzzy set theory. In previous papers several characterizations were given of t-norms in interval-valued fuzzy set theory which are join-morphisms and which satisfy additional properties, but little attention has been paid to meet-morphisms. Therefore, in this paper, we focus on t-norms which are meet-morphisms. We consider a general class of t-norms and investigate under which conditions t-norms belonging to this class are meet-morphisms. We also characterize the t-norms which are both a join- and a meet-morphism and which satisfy an additional border condition.

*Keywords:* interval-valued fuzzy set, t-norm, meet-morphism

## 1 Introduction

Interval-valued fuzzy set theory [11, 15] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [8] it is shown that the underlying lattice of intuitionistic fuzzy set theory is isomorphic to the underlying lattice $\mathcal{L}^I$ of interval-valued fuzzy set theory.

In [6, 7, 5, 18] several characterizations of t-norms on $\mathcal{L}^I$ in terms of t-norms on the unit interval are given. In [13, 19, 20] t-norms on related and more general lattices are investigated. However all the characterizations in these papers only deal with t-norms which are join-morphisms. Unlike for t-norms on a product lattice for which there exists a straightforward characterization of t-norms which are join-morphisms [3], respectively meet-morphisms, the situation is more complicated for t-norms in interval-valued fuzzy set theory. Therefore, in this paper, we focus on t-norms which are meet-morphisms. We consider a general class of t-norms (given in [7]) and investigate under which conditions t-norms belonging to this class are meet-morphisms.

## 2 The lattice $\mathcal{L}^I$

**Definition 2.1** *We define* $\mathcal{L}^I = (L^I, \leq_{L^I})$*, where*

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \text{ for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.$$

Similarly as Lemma 2.1 in [8] it can be shown that $\mathcal{L}^I$ is a complete lattice.

**Definition 2.2** [11, 15] *An interval-valued fuzzy set on* $U$ *is a mapping* $A : U \to L^I$*.*

**Definition 2.3** [1] *An intuitionistic fuzzy set on* $U$ *is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

*where* $\mu_A(u) \in [0,1]$ *denotes the membership degree and* $\nu_A(u) \in [0,1]$ *the non-membership degree of* $u$ *in* $A$ *and where for all* $u \in U$*,* $\mu_A(u) + \nu_A(u) \leq 1$*.*

An intuitionistic fuzzy set $A$ on $U$ can be represented by the $\mathcal{L}^I$-fuzzy set $A$ given by

$$\begin{aligned} A \;:\; U &\to L^I \;: \\ u &\mapsto [\mu_A(u), 1 - \nu_A(u)], \end{aligned}$$

In Figure 1 the set $L^I$ is shown. Note that to each element $x = [x_1, x_2]$ of $L^I$ corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

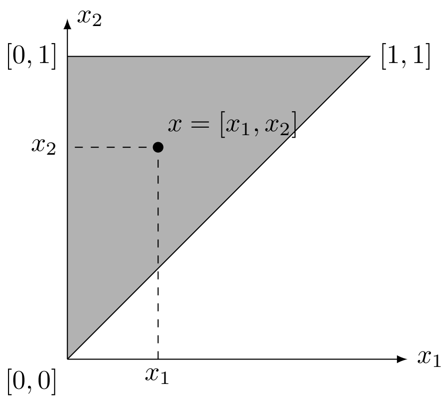


Figure 1: The grey area is $L^I$.

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1$ and $x_2$, i.e. $x = [x_1, x_2]$. The length $x_2 - x_1$ of the interval $x \in L^I$ is called the degree of uncertainty and is denoted by $x_\pi$. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. Note that, for $x, y$ in $L^I$, $x <_{L^I} y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We define for further usage the set $D = \{[x_1, x_1] \mid x_1 \in [0,1]\}$.

Note that for any non-empty subset $A$ of $L^I$ it holds that

$$\sup A = [\sup\{x_1 \mid [x_1, x_2] \in A\}, \sup\{x_2 \mid [x_1, x_2] \in A\}],$$
$$\inf A = [\inf\{x_1 \mid [x_1, x_2] \in A\}, \inf\{x_2 \mid [x_1, x_2] \in A\}].$$

**Theorem 2.1 (Characterization of supremum in $\mathcal{L}^I$)** [6] *Let $A$ be an arbitrary non-empty subset of $L^I$ and $a \in L^I$. Then $a = \sup A$ if and only if*

$$\begin{aligned}&(\forall x \in A)(x \leq_{L^I} a)\\ &\textit{and } (\forall \varepsilon_1 > 0)(\exists z \in A)(z_1 > a_1 - \varepsilon_1)\\ &\textit{and } (\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 > a_2 - \varepsilon_2).\end{aligned}$$

**Definition 2.4** *A t-norm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{T} : (L^I)^2 \to L^I$ which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.*

**Example 2.1** [7, 9] We give some special classes of t-norms on $\mathcal{L}^I$. Let $T$, $T_1$ and $T_2$ be t-norms on $([0,1], \leq)$ such that $T_1(x_1, y_1) \leq T_2(x_1, y_1)$ for all $x_1, y_1$ in $[0,1]$, and let $t \in [0,1]$. Then we have the following classes:

- t-representable t-norms:

  $$\mathcal{T}_{T_1,T_2}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)],$$

  for all $x, y$ in $L^I$;

- pseudo-t-representable t-norms:

  $$\mathcal{T}_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))],$$

  for all $x, y$ in $L^I$;

- $\mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), \max(T(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))]$, for all $x, y$ in $L^I$;

- $\mathcal{T}'_T(x, y) = [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)]$, for all $x, y$ in $L^I$;

- $\mathcal{T}_{T_1,T_2,t}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))]$, for all $x, y$ in $L^I$, where $T_1$ and $T_2$ additionally satisfy, for all $x_1, y_1$ in $[0,1]$,

  $$T_2(x_1, y_1) > T_2(t, T_2(x_1, y_1)) \implies T_1(x_1, y_1) = T_2(x_1, y_1). \tag{1}$$

  In Theorem 5 of [7] it is shown that $\mathcal{T}_{T_1,T_2,t}$ is indeed a t-norm on $\mathcal{L}^I$ if $T_1$ and $T_2$ satisfy (1).[1]

**Definition 2.5** *We say that a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is*

- *a join-morphism if for all $x$, $y$, $z$ in $L^I$,*

  $$\mathcal{T}(x, \sup(y, z)) = \sup(\mathcal{T}(x, y), \mathcal{T}(x, z));$$

- *a meet-morphism if for all $x$, $y$, $z$ in $L^I$,*

  $$\mathcal{T}(x, \inf(y, z)) = \inf(\mathcal{T}(x, y), \mathcal{T}(x, z));$$

[1] Note that the condition in Theorem 5 of [7] that $T_1$ and $T_2$ are left-continuous is not used to prove that $\mathcal{T}_{T_1,T_2,t}$ is a t-norm.

- *a sup-morphism if for all $x \in L^I$ and $\emptyset \neq Z \subseteq L^I$,*

$$\mathcal{T}(x, \sup Z) = \sup\{\mathcal{T}(x, z) \mid z \in Z\};$$

- *an inf-morphism if for all $x \in L^I$ and $\emptyset \neq Z \subseteq L^I$,*

$$\mathcal{T}(x, \inf Z) = \inf\{\mathcal{T}(x, z) \mid z \in Z\}.$$

**Definition 2.6** *Let $n \in \mathbb{N} \setminus \{0\}$. If for an $n$-ary mapping $f$ on $[0,1]$ and an $n$-ary mapping $F$ on $L^I$ it holds that*

$$F([a_1, a_1], \ldots, [a_n, a_n]) = [f(a_1, \ldots, a_n), f(a_1, \ldots, a_n)],$$

*for all $(a_1, \ldots, a_n) \in [0,1]^n$, then we say that $F$ is a natural extension of $f$ to $L^I$.*

Clearly, for any mapping $F$ on $L^I$, $F(D, \ldots, D) \subseteq D$ if and only if there exists a mapping $f$ on $[0,1]$ such that $F$ is a natural extension of $f$ to $L^I$. E.g. $\mathcal{T}_{T,T}$, $\mathcal{T}_T$, $\mathcal{T}_{T,t} = \mathcal{T}_{T,T,t}$ and $\mathcal{T}'_T$ are all natural extensions of $T$ to $L^I$, $\mathcal{N}_s$ is a natural extension of $N_s$.

**Example 2.2** Let, for all $x, y$ in $[0,1]$,

$$\begin{aligned} T_W(x,y) &= \max(0, x+y-1), \\ T_P(x,y) &= xy, \\ T_D(x,y) &= \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Then $T_W$, $T_P$ and $T_D$ are t-norms on $([0,1], \leq)$. Let now, for all $x, y$ in $L^I$,

$$\begin{aligned} \mathcal{T}_W(x,y) &= [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)], \\ \mathcal{T}_P(x,y) &= [x_1 y_1, \max(x_1 y_2, x_2 y_1)]. \end{aligned}$$

Then $\mathcal{T}_W$ and $\mathcal{T}_P$ are t-norms on $\mathcal{L}^I$. Furthermore, $\mathcal{T}_W$ and $\mathcal{T}_P$ are natural extensions of $T_W$ and $T_P$ respectively.

We will also need the following result and definition (see [2, 12, 14, 16, 17]).

**Theorem 2.2** *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined by, for all $x, y$ in $[0,1]$,*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right), & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y), & \text{otherwise,} \end{cases} \tag{2}$$

*is a t-norm on $([0,1], \leq)$.*

**Definition 2.7** *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm $T$ defined by (2) is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we will write*

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}.$$

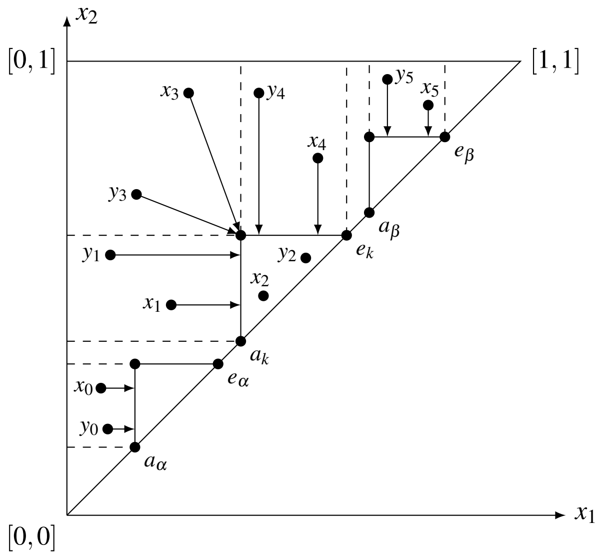


Figure 2: The different positions of $x, y \in L^I$, where $\mathcal{T}_\alpha([0,1],[0,1]) = [0,1]$, $\mathcal{T}_k([0,1],[0,1]) = [0,t]$ and $\mathcal{T}_\beta([0,1],[0,1]) = [0,0]$. The value of $(\mathcal{T}(x,y))_2$ is calculated at the ending points of the arrows.

Let $A$ be an arbitrary countable index-set and $\mathcal{T}_\alpha$ a t-norm on $\mathcal{L}^I$, for all $\alpha \in A$. Define, for all $\alpha \in A$ and for all $a_\alpha$, $e_\alpha$ in $D$ with $a_\alpha \leq_{L^I} e_\alpha$, the following sets and mappings:[2]

$$
\begin{aligned}
&J_\alpha = \{x \mid x \in L^I \text{ and } a_\alpha \leq_{L^I} x \leq_{L^I} e_\alpha\};\\
&J_\alpha^* = \{x \mid x \in L^I \text{ and } x_1 > (a_\alpha)_1 \text{ and } x_2 \leq (e_\alpha)_2\};\\
&\Phi_\alpha : J_\alpha \to L^I :\\
&\qquad x \mapsto \left[\frac{x_1 - (a_\alpha)_1}{(e_\alpha)_1 - (a_\alpha)_1}, \frac{x_2 - (a_\alpha)_2}{(e_\alpha)_2 - (a_\alpha)_2}\right], \forall x \in J_\alpha;\\
&\Phi_\alpha^{-1} : L^I \to J_\alpha :\\
&\qquad x \mapsto [(a_\alpha)_1 + x_1((e_\alpha)_1 - (a_\alpha)_1), (a_\alpha)_2 + x_2((e_\alpha)_2 - (a_\alpha)_2)], \forall x \in L^I;\\
&\mathcal{T}'_\alpha = \Phi_\alpha^{-1} \circ \mathcal{T}_\alpha \circ (\Phi_\alpha \times \Phi_\alpha).
\end{aligned}
$$

In Figure 2 the three smaller triangles are $J_\alpha$, $J_k$ and $J_\beta$. Assume that $J_\alpha^* \cap J_\beta^* = \varnothing$, for any $\alpha, \beta \in A$. Our aim is to construct a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ such that $\mathcal{T}\big|_{J_\alpha^* \times J_\alpha^*} = \mathcal{T}'_\alpha$, for all $\alpha \in A$.

Let arbitrarily $k \in A$ and define the sets $A_< = \{\alpha \mid \alpha \in A \text{ and } a_\alpha <_{L^I} a_k\}$ and $A_> = \{\alpha \mid \alpha \in A \text{ and } a_\alpha >_{L^I} a_k\}$. Assume furthermore that $\mathcal{T}_\alpha([0,1],[0,1]) = [0,1]$, for all $\alpha \in A_<$, and $\mathcal{T}_\alpha([0,1],[0,1]) = [0,0]$, for all $\alpha \in A_>$. For $\mathcal{T}_k$ we do not impose any restriction, so $\mathcal{T}_k([0,1],[0,1]) = [0,t]$ with $t \in [0,1]$. In [4, Theorem 4.2] it is shown that if $\mathcal{T}_\alpha$ is continuous for all $\alpha \in A$ and if we want to construct a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ which satisfies the residuation principle and for which $\mathcal{T}\big|_{J_\alpha^* \times J_\alpha^*} = \mathcal{T}'_\alpha$ for all $\alpha \in A$, then there must exist a $k \in A$ such that the previously mentioned assumptions for $\mathcal{T}_\alpha([0,1],[0,1])$, for all $\alpha \in A$, hold.

[2] In [4] it is shown that if $a_\alpha \notin D$ or $e_\alpha \notin D$, then there does not exist an increasing bijection $\Phi$ from $J_\alpha$ to $L^I$ such that $\Phi^{-1}$ is increasing. In this case the ordinal sum construction cannot be extended to $L^I$.

**Theorem 2.3** [4] *Let, for all $\alpha \in A$, $T_\alpha : [0,1]^2 \to [0,1]$ be the mapping defined by*

$$T_\alpha(x_1, y_1) = (\mathcal{T}_\alpha([x_1, x_1], [y_1, y_1]))_1, \forall (x_1, y_1) \in [0,1]^2,$$

*and let $T$ be the ordinal sum of $\langle (a_\alpha)_1, (e_\alpha)_1, T_\alpha \rangle$, $\alpha \in A$. Define the mapping $\mathcal{T} : (L^I)^2 \to L^I$ by, for all $x, y \in L^I$,*

$$\begin{aligned}
&(\mathcal{T}(x,y))_1 = T(x_1, y_1),\\
&(\mathcal{T}(x,y))_2\\
&= \begin{cases}
(\mathcal{T}'_\alpha([\max(x_1,(a_\alpha)_1), \min(x_2,(e_\alpha)_2)], [\max(y_1,(a_\alpha)_1), \min(y_2,(e_\alpha)_2)]))_2, \\
\qquad \textit{if } (x_2 \in \,](a_\alpha)_2, (e_\alpha)_2] \textit{ and } y_2 > (a_\alpha)_2 \textit{ and } y_1 \le (e_\alpha)_1 \textit{ and } \alpha \in A_<) \\
\qquad \textit{or } (y_2 \in \,](a_\alpha)_2, (e_\alpha)_2] \textit{ and } x_2 > (a_\alpha)_2 \textit{ and } x_1 \le (e_\alpha)_1 \textit{ and } \alpha \in A_<) \\
\qquad \textit{or } (x_1 \in \,](a_\alpha)_1, (e_\alpha)_1] \textit{ and } y_2 > (a_\alpha)_2 \textit{ and } y_1 \le (e_\alpha)_1 \textit{ and } \alpha \in A_>) \\
\qquad \textit{or } (y_1 \in \,](a_\alpha)_1, (e_\alpha)_1] \textit{ and } x_2 > (a_\alpha)_2 \textit{ and } x_1 \le (e_\alpha)_1 \textit{ and } \alpha \in A_>) \\
\qquad \textit{or } (x_2 > (a_\alpha)_2 \textit{ and } x_1 \le (e_\alpha)_1 \textit{ and } y_2 > (a_\alpha)_2 \textit{ and } y_1 \le (e_\alpha)_1 \textit{ and } \alpha = k), \\
\min(x_2, y_2), \ \textit{if the previous conditions do not hold} \\
\qquad \textit{and } (x_2 \le (a_k)_2 \textit{ or } y_2 \le (a_k)_2), \\
\min(x_2, y_1), \ \textit{if the previous conditions do not hold and } x_1 \le y_1, \\
\min(y_2, x_1), \ \textit{else.}
\end{cases}
\end{aligned}$$

*Then $\mathcal{T}$ is a t-norm on $\mathcal{L}^I$ called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle$, $\alpha \in A$, and we write*

$$\mathcal{T} = ((\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A_<} \ / \ \langle a_k, e_k, \mathcal{T}_k \rangle \ / \ (\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A_>}).$$

In Figure 2 the construction of $(\mathcal{T}(x_i, y_i))_2$ is shown for $(x_i, y_i) \in (L^I)^2$ where $i \in \{0, \ldots, 5\}$. The value of $(\mathcal{T}(x_i, y_i))_2$ is calculated at the ending points of the arrows for each $i \in \{0, \ldots, 5\}$. In the figure, $k$ is defined as in the paragraph before Theorem 2.3, $\alpha \in A_<$ and $\beta \in A_>$.

In the following example we show that there exist *different* t-norms $T_1$ and $T_2$ on $([0,1], \le)$ such that the mapping $\mathcal{T}_{T_1,T_2,t}$ defined in Example 2.1 is a t-norm on $\mathcal{L}^I$.

**Example 2.3** Let $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ be t-norms on $([0,1], \le)$ such that $\hat{T}_1 \le \hat{T}_2$. Let furthermore $t \in [0,1]$. Define the t-norms $T_1$ and $T_2$ by

$$\begin{aligned}
T_1 &= (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \hat{T}_3 \rangle),\\
T_2 &= (\langle 0, t, \hat{T}_2 \rangle, \langle t, 1, \hat{T}_3 \rangle).
\end{aligned}$$

Then

$$\begin{aligned}
&T_2(x_1, y_1) > T_2(t, T_2(x_1, y_1)) \quad (= \min(t, T_2(x_1, y_1)))\\
&\iff T_2(x_1, y_1) > t\\
&\implies \min(x_1, y_1) > t,
\end{aligned}$$

for all $x_1, y_1$ in $[0,1]$. It can be easily verified that $T_1 \le T_2$ and $T_1(x_1, y_1) = T_2(x_1, y_1)$, for all $x_1, y_1$ in $]t, 1]^2$. Clearly, if $\hat{T}_1 \ne \hat{T}_2$, then $T_1 \ne T_2$.

Define the mapping $\mathcal{T}_{T_1,T_2,t}$ by $\mathcal{T}_{T_1,T_2,t}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))]$, for all $x, y$ in $L^I$. Then $\mathcal{T}_{T_1,T_2,t}$ is a t-norm on $\mathcal{L}^I$ (see Example 2.1).

Finally we need a metric on $L^I$. Well-known metrics include the Euclidean distance and the Hamming distance. In the two-dimensional space $\mathbb{R}^2$ they are defined as follows:

- the Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^E(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\ ,$$
- the Hamming distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^H(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

If we restrict these distances to $L^I$ then we obtain the metric spaces $(L^I, d^E)$ and $(L^I, d^H)$. In these metric spaces, denote by $B(a; \varepsilon)$ the open ball with center $a$ and radius $\varepsilon$ defined as $B(a; \varepsilon) = \{x \mid x \in L^I \text{ and } d(x, a) < \varepsilon\}$. In the sequel, when we speak about continuity on $\mathcal{L}^I$, we mean continuity w.r.t. one of the above mentioned metric spaces.

## 3 Characterization of t-norms which are meet-morphisms

Since $([0, 1], \leq)$ is a chain, any t-norm on the unit interval is a join- and a meet-morphism. Furthermore, it is well-known that continuous t-norms on $([0, 1], \leq)$ are sup- and inf-morphisms. For t-norms on product lattices, the following result holds.

**Theorem 3.1** [3] *Consider two bounded lattices $\mathcal{L}_1 = (L_1, \leq_{L_1})$ and $\mathcal{L}_2 = (L_2, \leq_{L^2})$ and a t-norm $\mathcal{T}$ on the product lattice $\mathcal{L}_1 \times \mathcal{L}_2 = (L_1 \times L_2, \leq)$, where $(x_1, x_2) \leq (y_1, y_2) \iff (x_1 \leq_{L_1} y_1$ and $x_2 \leq_{L_2} y_2)$, for all $(x_1, x_2)$, $(y_1, y_2)$ in $L_1 \times L_2$. The t-norm $\mathcal{T}$ is a join-morphism (resp. meet-morphism) if and only if there exist t-norms $T_1$ on $\mathcal{L}_1$ and $T_2$ on $\mathcal{L}_2$ which are join-morphisms (resp. meet-morphisms), such that for all $(x_1, x_2)$, $(y_1, y_2)$ in $L_1 \times L_2$,*
$$\mathcal{T}((x_1, x_2), (y_1, y_2)) = [T_1(x_1, y_1), T_2(x_2, y_2)].$$

On $\mathcal{L}^I$, the situation is more complicated. Not all t-norms on $\mathcal{L}^I$ are join- and meet-morphisms. Consider the t-norm $\mathcal{T}'_{T_P}$ given by $\mathcal{T}'_{T_P}(x, y) = [\min(x_1 y_2, x_2 y_1), x_2 y_2]$, for all $x, y$ in $L^I$. Then we have $\mathcal{T}'_{T_P}([0.2, 0.5], \sup([0.5, 0.5], [0, 1])) = \mathcal{T}'_{T_P}([0.2, 0.5], [0.5, 1]) = [0.2, 0.5] \neq [0.1, 0.5] = \sup([0.1, 0.25], [0, 0.5]) = \sup(\mathcal{T}'_{T_P}([0.2, 0.5], [0.5, 0.5]), \mathcal{T}'_{T_P}([0.2, 0.5], [0, 1]))$. So $\mathcal{T}'_{T_P}$ is not a join-morphism. Similarly the t-norm $\mathcal{T}_{T_P}$ is not a meet-morphism.

Gehrke *et al.* [10] used the following definition for a t-norm on $\mathcal{L}^I$: a commutative, associative binary operation $\mathcal{T}$ on $\mathcal{L}^I$ is a t-norm if for all $x, y, z$ in $L^I$,

(G.1) $\mathcal{T}(D, D) \subseteq D$,

(G.2) $\mathcal{T}(x, \sup(y, z)) = \sup(\mathcal{T}(x, y), \mathcal{T}(x, z))$,

(G.3) $\mathcal{T}(x, \inf(y, z)) = \inf(\mathcal{T}(x, y), \mathcal{T}(x, z))$,

(G.4) $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$,

(G.5) $\mathcal{T}([0, 1], x) = [0, x_2]$.

They showed that such a t-norm is increasing, so their t-norms are a special case of the t-norms on $\mathcal{L}^I$ as defined in Definition 2.4.

Clearly, commutative, associative binary operations on $\mathcal{L}^I$ satisfying (G.1)–(G.5) are t-norms on $\mathcal{L}^I$ which are join- and meet-morphisms. The two additional conditions (G.1) and (G.5) ensure that these t-norms are t-representable, as is shown in the next theorem.

**Theorem 3.2** [10] *For every commutative, associative binary operation $\mathcal{T}$ on $\mathcal{L}^I$ satisfying (G.1)–(G.5) there exists a t-norm $T$ on $([0,1],\leq)$ such that, for all $x,y$ in $L^I$,*

$$\mathcal{T}(x,y) = [T(x_1,y_1), T(x_2,y_2)].$$

We can extend this result as follows. First we need a lemma.

**Lemma 3.3** [5] *Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$ which is a join-morphism. Then there exists a t-norm $T$ on $([0,1],\leq)$ such that, for all $x,y$ in $L^I$,*

$$(\mathcal{T}(x,y))_1 = T(x_1,y_1).$$

**Theorem 3.4** *For any t-norm $\mathcal{T}$ on $\mathcal{L}^I$ satisfying (G.2) and (G.5) there exist t-norms $T_1$ and $T_2$ on $([0,1],\leq)$ such that, for all $x,y$ in $L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), T_2(x_2,y_2)].$$

*Proof.* From Lemma 3.3 it follows that there exist a t-norm $T_1$ on $([0,1],\leq)$ such that $(\mathcal{T}(x,y))_1 = T_1(x_1,y_1)$, for all $x,y$ in $L^I$. From (G.5) it follows that, for all $x,y$ in $L^I$,

$$\begin{aligned}(\mathcal{T}(x,y))_2 &= (\mathcal{T}([0,1],\mathcal{T}(x,y)))_2\\ &= (\mathcal{T}(\mathcal{T}([0,1],x),\mathcal{T}([0,1],y)))_2\\ &= (\mathcal{T}([0,x_2],[0,y_2]))_2.\end{aligned}$$

Hence $(\mathcal{T}(x,y))_2$ is independent of $x_1$ and $y_1$, for all $x,y$ in $L^I$. Let now $T_2(x_2,y_2) = (\mathcal{T}([x_2,x_2],[y_2,y_2]))_2$, for all $x_2,y_2$ in $[0,1]$. Similarly as in the proof of Lemma 3.3 given in [5] it is shown that $T_2$ is a t-norm on $([0,1],\leq)$. $\square$

Clearly, (G.5) is a rather restrictive condition. We will show that if this condition is not imposed, then the class of t-norms on $\mathcal{L}^I$ satisfying the other conditions is much larger.

For continuous t-norms on $\mathcal{L}^I$ we have the following relationship between sup- and join-morphism, and between inf- and meet-morphisms.

**Theorem 3.5** *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$. Then*

*(i) $\mathcal{T}$ is a sup-morphism if and only if $\mathcal{T}$ is a join-morphism;*

*(ii) $\mathcal{T}$ is an inf-morphism if and only if $\mathcal{T}$ is a meet-morphism.*

*Proof.* Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$. We prove the first statement, the second equivalence is proven in a similar way. Clearly, if $\mathcal{T}$ is a sup-morphism, then $\mathcal{T}$ is a join-morphism.

Assume conversely that $\mathcal{T}$ is a join-morphism. Let $x \in L^I$, $A$ be an arbitrary non-empty subset of $L^I$ and $a = \sup A$. Since $\mathcal{T}$ is increasing, we have that $\mathcal{T}(x,y) \leq_{L^I} \mathcal{T}(x,a)$, for all $y \in A$.

From Theorem 2.1 it follows that there exists a sequence $(y_n)_{n\in\mathbb{N}^*}$ in $A$ such that $(y_n)_1 > a_1 - \frac{1}{n}$, for all $n \in \mathbb{N}^*$. Let $y^* = \lim_{n\to+\infty} y_n$, then clearly $y_1^* = a_1$ and $y_2^* \leq a_2$. Similarly, there exists a sequence $(z_n)_{n\in\mathbb{N}^*}$ in $A$ such that $(z_n)_2 > a_2 - \frac{1}{n}$, for all $n \in \mathbb{N}^*$. Let $z^* = \lim_{n\to+\infty} z_n$, then $z_2^* = a_2$ and $z_1^* \leq a_1$. Since $\mathcal{T}$ is a join-morphism, $\mathcal{T}(x,a) = \sup(\mathcal{T}(x,y^*), \mathcal{T}(x,z^*)) = [\max((\mathcal{T}(x,y^*))_1, (\mathcal{T}(x,z^*))_1), \max((\mathcal{T}(x,y^*))_2, (\mathcal{T}(x,z^*))_2)]$.

Assume that $(\mathcal{T}(x,a))_1 = (\mathcal{T}(x,y^*))_1$ (the case $(\mathcal{T}(x,a))_1 = (\mathcal{T}(x,z^*))_1$ is similar). Since $\mathcal{T}$ is continuous, we have in particular that

$$\begin{aligned}&(\forall \varepsilon_1 > 0)(\exists N \in \mathbb{N}^*)(\forall n \in \mathbb{N}^*)\\ &(n > N \implies |(\mathcal{T}(x,y_n))_1 - (\mathcal{T}(x,y^*))_1| + |(\mathcal{T}(x,y_n))_2 - (\mathcal{T}(x,y^*))_2| < \varepsilon_1).\end{aligned}$$

So, for any $\varepsilon_1 > 0$, there exists an $n \in \mathbb{N}^*$ such that $(\mathcal{T}(x,y^*))_1 - \varepsilon_1 < (\mathcal{T}(x,y_n))_1 \leq (\mathcal{T}(x,y^*))_1 = (\mathcal{T}(x,a))_1$. Hence, for any $\varepsilon_1 > 0$, there exists an element $y \in A$ such that $(\mathcal{T}(x,y))_1 > (\mathcal{T}(x,a))_1 - \varepsilon_1$. Similarly, for any $\varepsilon_2 > 0$, there exists a $z \in A$ such that $(\mathcal{T}(x,z))_2 > (\mathcal{T}(x,a))_2 - \varepsilon_2$. From Theorem 2.1 it follows that $\mathcal{T}(x,a) = \sup_{y\in A} \mathcal{T}(x,y)$. $\square$

In the following theorem the t-norms on $\mathcal{L}^I$ which satisfy the residuation principle and an additional border condition are characterized in terms of the class of t-norms $\mathcal{T}_{T_1,T_2,t}$ given in Example 2.1.

**Theorem 3.6** [7] *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ satisfies the residuation principle if and only if there exist two left-continuous t-norms $T_1$ and $T_2$ on $([0,1],\leq)$ and a real number $t \in [0,1]$ such that, for all $x,y \in L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)), T_2(x_1,y_2), T_2(y_1,x_2))],$$

*i.e. $\mathcal{T} = \mathcal{T}_{T_1,T_2,t}$, and, for all $x_1, y_1$ in $[0,1]$,*

$$\begin{cases} T_1(x_1,y_1) = T_2(x_1,y_1), & \textit{if } T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)),\\ T_1(x_1,y_1) \leq T_2(x_1,y_1), & \textit{else.}\end{cases}$$

We extend Theorem 3.6 to t-norms on $\mathcal{L}^I$ which are join-morphisms. The proof of the following theorem is analogous to the proof of Theorem 3.6 given in [7].

**Theorem 3.7** *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ is a join-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0,1],\leq)$ and a real number $t \in [0,1]$ such that, for all $x,y \in L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)), T_2(x_1,y_2), T_2(y_1,x_2))],$$

*i.e. $\mathcal{T} = \mathcal{T}_{T_1,T_2,t}$, and, for all $x_1, y_1$ in $[0,1]$,*

$$\begin{cases} T_1(x_1,y_1) = T_2(x_1,y_1), & \textit{if } T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)),\\ T_1(x_1,y_1) \leq T_2(x_1,y_1), & \textit{else.}\end{cases}$$

Now we characterize the t-norms on $\mathcal{L}^I$ belonging to the class $\mathcal{T}_{T_1,T_2,t}$ which are meet-morphisms. First we need some lemmas.

**Lemma 3.8** *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then $T_2(t, y_1) = \min(t, y_1)$, for all $y_1 \in [0, 1]$.*

*Proof.* Let arbitrarily $y_1 \in [0, 1]$. Then

$$\begin{aligned}\mathcal{T}_{T_1,T_2,t}([0,1], \inf([y_1, y_1], [0, 1])) &= \mathcal{T}_{T_1,T_2,t}([0, 1], [0, y_1])\\ &= [0, T_2(t, T_2(1, y_1))]\\ &= [0, T_2(t, y_1)].\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{T}_{T_1,T_2,t}([0,1], \inf([y_1, y_1], [0, 1])) &= \inf(\mathcal{T}_{T_1,T_2,t}([0, 1], [y_1, y_1]), \mathcal{T}_{T_1,T_2,t}([0, 1], [0, 1]))\\ &= \inf([0, \max(T_2(t, y_1), y_1)], [0, t])\\ &= \inf([0, y_1], [0, t])\\ &= [0, \min(y_1, t)].\end{aligned}$$

Hence $T_2(t, y_1) = \min(t, y_1)$, for all $y_1 \in [0, 1]$. □

**Corollary 3.9** *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then there exists two t-norms $\hat{T}_1$ and $\hat{T}_2$ on $([0, 1], \leq)$ such that*

$$T_2 = (\langle 0, t, \hat{T}_1\rangle, \langle t, 1, \hat{T}_2\rangle).$$

*Proof.* Define, for all $x, y$ in $[0, 1]$,

$$\begin{aligned}\hat{T}_1(x, y) &= \frac{T_2(tx, ty)}{t},\\ \hat{T}_2(x, y) &= \frac{T_2(t + (1-t)x, t + (1-t)y) - t}{1-t}.\end{aligned} \tag{3}$$

Then it is easy to see that $\hat{T}_1$ is commutative, associative and increasing. Since from Lemma 3.8 it follows that $T_2(t, y) = \min(t, y)$, for all $y \in [0, 1]$, we obtain that $\hat{T}_1(1, y) = y$, for all $y \in [0, 1]$. So $\hat{T}_1$ is a t-norm. Similarly, we obtain that $\hat{T}_2$ is a t-norm on $([0, 1], \leq)$.

Let arbitrarily $x, y$ in $[0, 1]$ such that $x < t < y$ (the case $y < t < x$ is similar). Then we obtain that $x = \min(t, x) = T_2(t, x) \leq T_2(x, y) \leq T_2(1, x) = x$, so $T_2(x, y) = \min(x, y)$. It now easily follows that $T_2$ is equal to the ordinal sum of $\langle 0, t, \hat{T}_1\rangle$ and $\langle t, 1, \hat{T}_2\rangle$. □

**Lemma 3.10** *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then the t-norm $\hat{T}_2$ in the representation of $T_2$ given in Corollary 3.9 is equal to the minimum.*

*Proof.* Let arbitrarily $x_1, z_1$ in $[t, 1]$. From Lemma 3.8 it follows that $T_2(t, z_1) = \min(t, z_1) = t$. Furthermore, from Corollary 3.9 it follows that $T_2(x_1, z_1) \geq t$. So, we obtain

$$\begin{aligned}\mathcal{T}_{T_1,T_2,t}([x_1, 1], \inf([0, 1], [z_1, z_1])) &= \mathcal{T}_{T_1,T_2,t}([x_1, 1], [0, z_1]) \\ &= [0, \max(T_2(t, z_1), T_2(x_1, z_1))] \\ &= [0, \max(t, T_2(x_1, z_1))] \\ &= [0, T_2(x_1, z_1)]\end{aligned}$$

and

$$\begin{aligned}&\mathcal{T}_{T_1,T_2,t}([x_1, 1], \inf([0, 1], [z_1, z_1])) \\ &= \inf(\mathcal{T}_{T_1,T_2,t}([x_1, 1], [0, 1]), \mathcal{T}_{T_1,T_2,t}([x_1, 1], [z_1, z_1])) \\ &= \inf([0, \max(t, x_1)], [T_1(x_1, z_1), \max(T_2(t, z_1), T_2(x_1, z_1), z_1)]) \\ &= \inf([0, x_1], [T_1(x_1, z_1), z_1]) \\ &= [0, \min(x_1, z_1)].\end{aligned}$$

So $T_2(x_1, z_1) = \min(x_1, z_1)$. From (3) it easily follows that $\hat{T}_2 = \min$. □

**Corollary 3.11** *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that*

$$T_2 = (\langle 0, t, \hat{T}_1\rangle, \langle t, 1, \min\rangle).$$

**Lemma 3.12** *Assume that there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that $T_2 = (\langle 0, t, \hat{T}_1\rangle, \langle t, 1, \min\rangle)$, then $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism.*

*Proof.* Let arbitrarily $x, y, z$ in $L^I$. If $y \leq_{L^I} z$ (the case $y \geq_{L^I} z$ is similar), then $\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = \mathcal{T}_{T_1,T_2,t}(x, y) = \inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z))$. So, let $y_1 < z_1$ and $y_2 > z_2$ (the case $y_1 > z_1$ and $y_2 < z_2$ is similar). Then we have the following cases:

- $\max(x_1, y_1, z_1) \leq t$:

  From the fact that $T_2 \leq \min$ it follows that $T_2(x_1, z_2) \leq t$ and $T_2(x_2, y_1) \leq t$, so $T_2(x_1, z_2) \leq \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. Since $T_2(x_2, y_1) \leq T_2(x_2, z_1) \leq T_2(x_2, z_2)$, we obtain similarly that $T_2(x_2, y_1) \leq T_2(t, T_2(x_2, z_2))$. Thus,

  $$\begin{aligned}\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) &= \mathcal{T}_{T_1,T_2,t}(x, [y_1, z_2]) \\ &= [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2), T_2(x_2, y_1))] \\ &= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].\end{aligned}$$

  On the other hand, we obtain similarly that

  $$\begin{aligned}&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\ &= \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), T_2(t, T_2(x_2, z_2))]) \\ &= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))],\end{aligned}$$

  using the fact that $T_2$ is increasing, $y_1 < z_1$ and $y_2 > z_2$.

- $\max(x_1, y_1) \le t < z_1$:

  Similarly as in the previous case, we have that

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))]$$

  and

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_1))]) \\
&= [T_1(x_1, y_1), \min(T_2(t, T_2(x_2, y_2)), \max(\min(t, T_2(x_2, z_2)), T_2(x_2, z_1)))].
\end{aligned}$$

  We have two cases:

  1. $x_2 \le t$: in this case, we have that $T_2(x_2, z_1) = \min(x_2, z_1) = x_2 \le t$, so $T_2(x_2, z_1) \le \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. Hence

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= [T_1(x_1, y_1), \min(T_2(t, T_2(x_2, y_2)), T_2(t, T_2(x_2, z_2)))] \\
&= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].
\end{aligned}$$

  2. $x_2 > t$: in this case, $T_2(x_2, z_1) = \min(x_2, z_1) > t$, so $T_2(x_2, y_2) \ge T_2(x_2, z_2) \ge T_2(x_2, z_1) > t$. Thus,

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= [T_1(x_1, y_1), \min(\min(t, T_2(x_2, y_2)), T_2(x_2, z_1)))] \\
&= [T_1(x_1, y_1), t]
\end{aligned}$$

     and

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), \min(t, T_2(x_2, z_2))] = [T_1(x_1, y_1), t].$$

- $x_1 \le t < y_1$ $(< z_1)$:

  We have that $T_2(x_1, z_2) \le x_1 \le t$, so $T_2(x_1, z_2) \le \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. We obtain

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, y_1))]$$

  and similarly

$$\begin{aligned}
\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) = \inf(&[T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_2, y_1))], \\
&[T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_2, z_1))]).
\end{aligned}$$

  We have two cases:

  1. $x_2 \le t$: in this case, we have that $T_2(x_2, y_1) \le t$, so, using the fact that $y_1 < z_1 \le z_2$, $T_2(x_2, y_1) \le \min(t, T_2(x_2, z_2)) = T_2(t, T_2(x_2, z_2))$. Thus,

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].$$

     Similarly, we obtain that $\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))]$.

2. $x_2 > t$: from the representation of $T_2$ it follows that $T_2(x_2, y_2) \geq T_2(x_2, z_2) \geq T_2(x_2, z_1) \geq t$. So, using the fact that $T_2(t, a) = \min(t, a)$ for all $a \in [0, 1]$, we obtain

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(t, T_2(x_2, y_1))]$$

and

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= [T_1(x_1, y_1), \min(\max(t, T_2(x_2, y_1)), \max(t, T_2(x_2, z_1)))] \\
&= [T_1(x_1, y_1), \max(t, T_2(x_2, y_1))].
\end{aligned}$$

- $(y_1 <)\ z_1 \leq t < x_1$:
  Similarly as in the previous case, we obtain that

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2))]$$

and

$$\begin{aligned}
\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) = \inf([&T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2))], \\
&[T_1(x_1, z_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2))]).
\end{aligned}$$

We have two cases:

1. $y_2 \leq t$: we obtain that $T_2(x_1, z_2) \leq T_2(x_1, y_2) \leq t$, so $T_2(x_1, y_2) \leq \min(t, T_2(x_2, y_2)) = T_2(t, T_2(x_2, y_2))$ and similarly for $T_2(x_1, z_2)$. Thus

$$\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) = [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))]$$

and

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= \inf([T_1(x_1, y_1), T_2(t, T_2(x_2, y_2))], [T_1(x_1, z_1), T_2(t, T_2(x_2, z_2))]) \\
&= [T_1(x_1, y_1), T_2(t, T_2(x_2, z_2))].
\end{aligned}$$

2. $y_2 > t$: we have that $T_2(x_1, y_2) \geq t \geq \min(t, T_2(x_2, z_2))$ and $T_2(x_1, y_2) \geq T_2(x_1, z_2)$, so

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)) \\
&= [T_1(x_1, y_1), \min(T_2(x_1, y_2), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2)))] \\
&= [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, z_2)), T_2(x_1, z_2))] \\
&= \mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)).
\end{aligned}$$

- $y_1 \leq t < \min(x_1, z_1)$:
  We have that $T_2(x_2, y_1) \leq y_1 \leq t \leq T_2(x_1, z_2) \leq T_2(x_1, y_2)$, so

$$\begin{aligned}
\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)) &= [T_1(x_1, y_1), \max(\min(t, T_2(x_2, z_2)), T_2(x_1, z_2))] \\
&= [T_1(x_1, y_1), T_2(x_1, z_2)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x,y),\mathcal{T}_{T_1,T_2,t}(x,z))\\
&=\inf([T_1(x_1,y_1),\max(T_2(x_1,y_2),T_2(x_2,y_1))],\\
&\qquad [T_1(x_1,z_1),\max(T_2(x_1,z_2),T_2(x_2,z_1))])\\
&=[T_1(x_1,y_1),\min(T_2(x_1,y_2),\max(T_2(x_1,z_2),T_2(x_2,z_1)))].
\end{aligned}$$

We have two cases:

1. $x_1 < \min(x_2,z_1)$: in this case, we have that $T_2(x_1,z_2)=\min(x_1,z_2)=x_1<\min(x_2,z_1)=T_2(x_2,z_1)$ (using Corollary 3.11), so

$$\begin{aligned}
&\min(T_2(x_1,y_2),\max(T_2(x_1,z_2),T_2(x_2,z_1)))\\
&=\min(T_2(x_1,y_2),T_2(x_2,z_1))\\
&=\min(x_1,y_2,x_2,z_1)\\
&=x_1=\min(x_1,z_2)=T_2(x_1,z_2).
\end{aligned}$$

2. $x_1 \geq \min(x_2,z_1)$: since $z_2\geq z_1\geq \min(x_2,z_1)$, we have that $T_2(x_1,z_2)=\min(x_1,z_2)\geq \min(x_2,z_1)=T_2(x_2,z_1)$, so

$$\begin{aligned}
&\min(T_2(x_1,y_2),\max(T_2(x_1,z_2),T_2(x_2,z_1)))\\
&=\min(T_2(x_1,y_2),T_2(x_1,z_2))\\
&=T_2(x_1,z_2),
\end{aligned}$$

since $y_2>z_2$.

- $t\leq \min(x_1,y_1,z_1)$:
From Lemma 3.8 and Corollary 3.11 it follows that

$$\begin{aligned}
\mathcal{T}_{T_1,T_2,t}(x,\inf(y,z)) &= \mathcal{T}_{T_1,T_2,t}(x,[y_1,z_2])\\
&=[T_1(x_1,y_1),\max(\min(t,T_2(x_2,z_2)),\min(x_1,z_2),\min(x_2,y_1))]\\
&=[T_1(x_1,y_1),\max(\min(x_1,z_2),\min(x_2,y_1))].
\end{aligned}$$

On the other hand, we obtain similarly that

$$\begin{aligned}
&\inf(\mathcal{T}_{T_1,T_2,t}(x,y),\mathcal{T}_{T_1,T_2,t}(x,z))\\
&=\inf([T_1(x_1,y_1),\max(\min(x_1,y_2),\min(x_2,y_1))],\\
&\qquad [T_1(x_1,z_1),\max(\min(x_1,z_2),\min(x_2,z_1))]).
\end{aligned}$$

Clearly, it holds that $(\mathcal{T}_{T_1,T_2,t}(x,\inf(y,z)))_1=T_1(x_1,y_1)=\min(T_1(x_1,y_1),T_1(x_1,z_1))=(\inf(\mathcal{T}_{T_1,T_2,t}(x,y),\mathcal{T}_{T_1,T_2,t}(x,z)))_1$. For the second projection, we have two cases:

1. $x_1<\min(x_2,z_1)$: in this case, we have that $\min(x_1,z_2)=x_1<\min(x_2,z_1)\leq z_2<y_2$. So, $(\mathcal{T}_{T_1,T_2,t}(x,\inf(y,z)))_2=\max(x_1,\min(x_2,y_1))$. On the other hand

$$\begin{aligned}
(\inf(\mathcal{T}_{T_1,T_2,t}(x,y),\mathcal{T}_{T_1,T_2,t}(x,z)))_2 &=\min(\max(x_1,\min(x_2,y_1)),\min(x_2,z_1))\\
&=\max(x_1,\min(x_2,y_1))\\
&=(\mathcal{T}_{T_1,T_2,t}(x,\inf(y,z))_2
\end{aligned}$$

using the fact that $y_1<z_1$ and $x_1<\min(x_2,z_1)$.

2. $x_1 \geq \min(x_2, z_1)$: in this case, we have that $x_1 = x_2$ or $x_1 \geq z_1$, so $\min(x_1, z_2) \geq \min(x_2, z_1)$. If $x_1 = x_2$, then $(\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)))_2 = \min(x_1, z_2)$, because $z_2 \geq z_1 > y_1$. On the other hand, $(\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)))_2 = \min(\min(x_1, y_2), \min(x_1, z_2)) = \min(x_1, z_2)$.
If $x_1 \geq z_1$, then $(\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)))_2 = \max(\min(x_1, z_2), y_1) = \min(x_1, z_2)$, because $y_1 < z_1 \leq x_1 \leq x_2$. On the other hand, $(\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)))_2 = \min(\max(\min(x_1, y_2), y_1), \min(x_1, z_2)) = \min(x_1, z_2)$, using the fact that $z_2 < y_2$. So again $(\inf(\mathcal{T}_{T_1,T_2,t}(x, y), \mathcal{T}_{T_1,T_2,t}(x, z)))_2 = (\mathcal{T}_{T_1,T_2,t}(x, \inf(y, z)))_2$.

□

Now we obtain the main result.

**Theorem 3.13** *For any t-norms $T_1$ and $T_2$ on $([0,1], \leq)$ and $t \in [0,1]$, $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism if and only if there exists a t-norm $\hat{T}_1$ on $([0,1], \leq)$ such that*

$$T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle).$$

*Proof.* This follows immediately from Corollary 3.11 and Lemma 3.12. □

If we assume that $T_1 = T_2$, then we do not only obtain that $T_1$ is the ordinal sum of two t-norms on $([0,1], \leq)$, but we can also write the t-norm $\mathcal{T}_{T_1,T_1,t} = \mathcal{T}_{T_1,t}$ as an ordinal sum of two t-norms on $\mathcal{L}^I$. This is shown in the next theorem.

**Theorem 3.14** *For any t-norm $T$ on $([0,1], \leq)$ and $t \in [0,1]$, $\mathcal{T}_{T,t}$ is a meet-morphism if and only if there exists a t-norm $\hat{T}_1$ on $([0,1], \leq)$ such that*

$$\mathcal{T}_{T,t} = (\varnothing \;/\; \langle 0_{\mathcal{L}^I}, [t,t], \mathcal{T}_{\hat{T}_1,\hat{T}_1} \rangle \;/\; \langle [t,t], 1_{\mathcal{L}^I}, \mathcal{T}_{\min} \rangle),$$

*where, for all $x, y$ in $L^I$,*

$$\mathcal{T}_{\hat{T}_1,\hat{T}_1}(x, y) = [\hat{T}_1(x_1, y_1), \hat{T}_1(x_2, y_2)],$$
$$\mathcal{T}_{\min}(x, y) = [\min(x_1, y_1), \max(\min(x_1, y_2), \min(x_2, y_1))].$$

*Proof.* Assume first that $\mathcal{T}_{T,t}$ is a meet-morphism. From Theorem 3.13 it follows that there exists a t-norm $\hat{T}_1$ on $([0,1], \leq)$ such that $T = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle)$.

Let $\phi : [0,t] \to [0,1] : x_1 \mapsto \frac{x_1}{t}$ and $\hat{T}'_1 = \phi^{-1} \circ \hat{T}_1 \circ (\phi \times \phi)$. Define for all $x, y$ in $L^I$,

$$\Phi_1(x) = [\phi(x_1), \phi(x_2)],$$
$$\Phi_2(x) = \left[\frac{x_1 - t}{1 - t}, \frac{x_2 - t}{1 - t}\right],$$
$$\mathcal{T}'_{\hat{T}_1,\hat{T}_1} = \Phi_1^{-1} \circ \mathcal{T}_{\hat{T}_1,\hat{T}_1} \circ (\Phi_1 \times \Phi_1),$$
$$\mathcal{T}'_{\min} = \Phi_2^{-1} \circ \mathcal{T}_{\min} \circ (\Phi_2 \times \Phi_2).$$

Note that $\mathcal{T}'_{\min}$ defined by the formula above is a transformation of $\mathcal{T}_{\min}$ and not a member of the class of t-norms $\mathcal{T}'_T$ given in Example 2.1. Then, for all $x, y, x', y'$ in $L^I$ such that $x \leq_{L^I} [t,t]$, $y \leq_{L^I} [t,t]$, $x' \geq_{L^I} [t,t]$ and $y' \geq_{L^I} [t,t]$,

$$\mathcal{T}'_{\hat{T}_1,\hat{T}_1}(x, y) = [\hat{T}'_1(x_1, y_1), \hat{T}'_1(x_2, y_2)],$$
$$\mathcal{T}'_{\min}(x', y') = [\min(x'_1, y'_1), \max(\min(x'_1, y'_2), \min(x'_2, y'_1))].$$

We consider the following cases:

1. $\max(x_2, y_2) \le t$: using Lemma 3.8, we obtain

$$\begin{aligned}\mathcal{T}_{T,t}(x,y) &= [T(x_1,y_1), \max(\min(t, T(x_2,y_2)), T(x_1,y_2), T(x_2,y_1))] \\ &= [T(x_1,y_1), \max(T(x_2,y_2), T(x_1,y_2), T(x_2,y_1))] \\ &= [\hat{T}_1'(x_1,y_1), \hat{T}_1'(x_2,y_2)].\end{aligned}$$

2. $\max(x_2, y_1) \le t < y_2$ (the case $\max(y_2, x_1) \le t < x_2$ is similar): we obtain in a completely similar way that $\mathcal{T}_{T,t}(x,y) = [\hat{T}_1'(x_1,y_1), \min(x_2,y_2)] = [\hat{T}_1'(x_1,y_1), x_2] = [\hat{T}_1'(x_1,y_1), \hat{T}_1'(x_2,t)]$.

3. $\max(x_1, y_1) \le t < \min(x_2, y_2)$: we obtain that $T(t, T(x_2,y_2)) = \min(t, x_2, y_2) = t$, $T(x_1,y_2) \le x_1 \le t$ and $T(x_2,y_1) \le y_1 \le t$. So $\mathcal{T}_{T,t}(x,y) = [T(x_1,y_1), t] = [\hat{T}_1'(x_1,y_1), \hat{T}_1'(t,t)]$.

4. $x_2 \le t < y_1$ (the case $y_2 \le t < x_1$ is similar): we obtain that $T(t, T(x_2,y_2)) = \min(t, x_2, y_2) = x_2$, $T(x_1,y_2) = \min(x_1,y_2) = x_1$ and $T(x_2,y_1) = \min(x_2,y_1) = x_2$. So $\mathcal{T}_{T,t}(x,y) = [T(x_1,y_1), x_2] = [\min(x_1,y_1), \min(x_2,y_1)] = [\min(x_1,y_1), \min(x_2,y_2)]$.

5. $x_1 \le t < \min(x_2, y_1)$ (the case $y_1 \le t < \min(y_2, x_1)$ is similar): we obtain that $T(t, T(x_2,y_2)) = \min(t, x_2, y_2) = t$, $T(x_1,y_2) = \min(x_1,y_2) = x_1$ and $T(x_2,y_1) = \min(x_2,y_1) > t$. So $\mathcal{T}_{T,t}(x,y) = [T(x_1,y_1), \min(x_2,y_1)] = [\min(x_1,y_1), \max(\min(t,y_2), \min(x_2,y_1))]$.

6. $t < \min(x_1, y_1)$: we obtain that $T(t, T(x_2,y_2)) = \min(t, x_2, y_2) = t$, so $\mathcal{T}_{T,t}(x,y) = [\min(x_1,y_1), \max(\min(x_1,y_2), \min(x_2,y_1))]$.

We see that

$$(\mathcal{T}_{T,t}(x,y))_1 = T(x_1,y_1) = \begin{cases}\hat{T}_1'(x_1,y_1), & \text{if } (x_1,y_1) \in [0,t]^2, \\ \min(x_1,y_1), & \text{else.}\end{cases}$$

So, the first projection of $\mathcal{T}_{T,t}$ is determined by the ordinal sum of $\langle 0, t, \hat{T}_1 \rangle$ and $\langle t, 1, \min \rangle$. The second projection of $\mathcal{T}_{T,t}$ is given by

$$\begin{aligned}&(\mathcal{T}_{T,t}(x,y))_2 \\ &= \begin{cases}(\mathcal{T}'_{\hat{T}_1,\hat{T}_1}([x_1, \min(x_2,t)], [y_1, \min(y_2,t)]))_2, \\ \qquad \text{if } x_2 > 0 \text{ and } x_1 \le t \text{ and } y_2 > 0 \text{ and } y_1 \le t, \\ (\mathcal{T}'_{\min}([\max(x_1,t), x_2], [\max(y_1,t), y_2]))_2, \\ \qquad \text{if } (x_1 \in ]t,1] \text{ and } y_2 > t \text{ and } y_1 \le 1) \\ \qquad \text{or } (y_1 \in ]t,1] \text{ and } x_2 > t \text{ and } x_1 \le 1), \\ \min(x_2,y_2), \text{ if the previous conditions do not hold} \\ \qquad \text{and } (x_2 \le 0 \text{ or } y_2 \le 0), \\ \min(x_2,y_1), \text{ if the previous conditions do not hold and } x_1 \le y_1, \\ \min(y_2,x_1), \text{ else.}\end{cases}\end{aligned}$$

This corresponds to the formula in Theorem 2.3, in which $A = \{1,2\}$, $a_1 = 0_{\mathcal{L}^I}$, $e_1 = a_2 = [t,t]$, $e_2 = 1_{\mathcal{L}^I}$, $k = 1$, $A_< = \varnothing$ and $A_> = \{2\}$. Hence $\mathcal{T}_{T,t}$ is the ordinal sum of the summands $\langle 0_{\mathcal{L}^I}, [t,t], \mathcal{T}_{\hat{T}_1,\hat{T}_1} \rangle$ and $\langle [t,t], 1_{\mathcal{L}^I}, \mathcal{T}_{\min} \rangle$, with $k = 1$.

Conversely, assume that $\mathcal{T}_{T,t}$ is the ordinal sum of the summands $\langle 0_{\mathcal{L}^I}, [t,t], \mathcal{T}_{\hat{T}_1,\hat{T}_1}\rangle$ and $\langle [t,t], 1_{\mathcal{L}^I}, \mathcal{T}_{\min}\rangle$, with $k=1$. Then from Theorem 2.3 it follows that $T$ is the ordinal sum of $\langle 0, t, \hat{T}_1\rangle$ and $\langle t, 1, \min\rangle$. Using Theorem 3.13 we obtain that $\mathcal{T}_{T,t}$ is a meet-morphism. $\square$

**Corollary 3.15** *Let $T$ be a t-norm on $([0,1],\leq)$.*

- *If $t=0$, then $\mathcal{T}_{T,0}$ is a meet-morphism if and only if $\mathcal{T}_{T,0} = \mathcal{T}_{\min}$.*
- *If $t=1$, then $\mathcal{T}_{T,1} = \mathcal{T}_{T,T}$ is a meet-morphism for any $T$.*

By combining Theorems 3.6 and 3.13, we obtain the following result.

**Theorem 3.16** *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ is a join-morphism and a meet-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0,1],\leq)$ and a real number $t \in [0,1]$ such that, for all $x,y \in L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)), T_2(x_1,y_2), T_2(y_1,x_2))],$$

*$T_2$ is the ordinal sum $(\langle 0,t,\hat{T}_1\rangle, \langle t,1,\min\rangle)$, where $\hat{T}_1$ is a t-norm on $([0,1],\leq)$, and, for all $x_1, y_1$ in $[0,1]$,*

$$T_1(x_1,y_1) = T_2(x_1,y_1), \text{ if } T_2(x_1,y_1) > t.$$

# 4 Conclusion

In this paper we investigated t-norms in interval-valued fuzzy set theory which are meet-morphisms. First we showed that for continuous t-norms the notions of sup- and join-morphism, respectively the notions of inf- and meet-morphism, collapse. We considered a general class of t-norms (given in [7]) and investigated under which conditions t-norms belonging to this class are meet-morphisms. We also showed that there exist non-trivial examples of t-norms in this class, i.e. t-norms which belong to this class but not to the class investigated in [5, 18]. Finally we gave a characterization of the t-norms which are both a join- and a meet-morphism and which satisfy an additional border condition.

# References


[1] K. T. Atanassov, Intuitionistic fuzzy sets, Physica-Verlag, Heidelberg, New York, 1999.

[2] A. H. Clifford, Naturally totally ordered commutative semigroups, Amer. J. Math. 76 (1954) 631–646.

[3] B. De Baets, R. Mesiar, Triangular norms on product lattices, Fuzzy Sets and Systems 104 (1) (1999) 61–75.

[4] G. Deschrijver, Ordinal sums in interval-valued fuzzy set theory, New Mathematics and Natural Computation 1 (2) (2005) 243–259.

[5] G. Deschrijver, A representation of t-norms in interval-valued L-fuzzy set theory, Fuzzy Sets and Systems 159 (13) (2008) 1597–1618.

[6] G. Deschrijver, C. Cornelis, E. E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, IEEE Transactions on Fuzzy Systems 12 (1) (2004) 45–61.

[7] G. Deschrijver, E. E. Kerre, Classes of intuitionistic fuzzy t-norms satisfying the residuation principle, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 11 (6) (2003) 691–709.

[8] G. Deschrijver, E. E. Kerre, On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 133 (2) (2003) 227–235.

[9] G. Deschrijver, E. E. Kerre, Implicators based on binary aggregation operators in interval-valued fuzzy set theory, Fuzzy Sets and Systems 153 (2) (2005) 229–248.

[10] M. Gehrke, C. L. Walker, E. A. Walker, Some comments on interval valued fuzzy sets, International Journal of Intelligent Systems 11 (10) (1996) 751–759.

[11] M. B. Gorzałczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems 21 (1) (1987) 1–17.

[12] S. Jenei, A note on the ordinal sum theorem and its consequence for the construction of triangular norms, Fuzzy Sets and Systems 126 (2) (2002) 199–205.

[13] F. Karaçal, D. Khadjiev, $\vee$-distributive and infinitely $\vee$-distributive t-norms on complete lattices, Fuzzy Sets and Systems 151 (2) (2005) 341–352.

[14] C.-H. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189–212.

[15] R. Sambuc, Fonctions $\Phi$-floues. Application à l'aide au diagnostic en pathologie thyroidienne, Ph.D. thesis, Université de Marseille, France (1975).

[16] S. Saminger, On ordinal sums of triangular norms on bounded lattices, Fuzzy Sets and Systems 157 (10) (2006) 1403–1416.

[17] B. Schweizer, A. Sklar, Associative functions and abstract semigroups, Publ. Math. Debrecen 10 (1963) 69–81.

[18] B. Van Gasse, C. Cornelis, G. Deschrijver, E. E. Kerre, A characterization of interval-valued residuated lattices, International Journal of Approximate Reasoning 49 (2) (2008) 478–487.

[19] G. Wang, W. Wang, Left-continuity of t-norms on the $n$-dimensional Euclidean cube, Applied Mathematics Letters 23 (4) (2010) 479–483.

[20] D. Zhang, Triangular norms on partially ordered sets, Fuzzy Sets and Systems 153 (2) (2005) 195–209.