# Dual embeddings of dense near polygons 

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#### Abstract

Let $e: \mathcal{S} \rightarrow \Sigma$ be a full polarized projective embedding of a dense near polygon $\mathcal{S}$, i.e., for every point $p$ of $\mathcal{S}$, the set $H_{p}$ of points at non-maximal distance from $p$ is mapped by $e$ into a hyperplane $\Pi_{p}$ of $\Sigma$. We show that if every line of $\mathcal{S}$ is incident with precisely three points or if $\mathcal{S}$ satisfies a certain property $\left(P_{d e}\right)$ then the map $p \mapsto \Pi_{p}$ defines a full polarized embedding $e^{*}$ (the so-called dual embedding of $e$ ) of $\mathcal{S}$ into a subspace of the dual $\Sigma^{*}$ of $\Sigma$. This generalizes a result of [6] where it was shown that every embedding of a thick dual polar space has a dual embedding. We determine which known dense near polygons satisfy property $\left(P_{d e}\right)$. This allows us to conclude that every full polarized embedding of a known dense near polygon has a dual embedding.


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## 1 Introduction

A near polygon is a connected partial linear space $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I}), \mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, satisfying the property that for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$, there exists a unique point $\pi_{L}(p)$ on $L$ nearest to $p$. Here distances are measured in the collinearity graph $\Gamma$ of $\mathcal{S}$. If $d$ is the diameter of $\Gamma$, then the near polygon is called a near $2 d$-gon. A near 0 -gon is just a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles (GQ's). We refer to Payne and Thas [10] for the basic notions on generalized quadrangles to be used in this paper.

A near polygon is called slim if every line is incident with precisely three points. A near polygon is called dense if every line is incident with

[^0]at least three points and if every two points at distance 2 have at least two common neighbours. If $x$ and $y$ are two points of a dense near polygon at distance $k$ from each other, then by Theorem 4 of Brouwer and Wilbrink [4], $x$ and $y$ are contained in a unique convex subspace of diameter $k$. These convex subspaces are called quads if $k=2$ and hexes if $k=3$. The points and lines contained in a convex subspace of diameter $k$ define a sub-near$2 k$-gon. Convex subspaces of diameter $k$ are therefore also called convex sub- $2 k$-gons. We will now introduce two properties of dense near polygons.
(I) Let $\mathcal{S}$ be a dense near $2 n$-gon, $n \geq 2$, let $x$ be a point of $\mathcal{S}$, let $B$ be a convex sub-2( $n-2$ )-gon through $x$ and let $A$ be a convex sub-2( $n-1$ )-gon through $B$. Define the following graph $\Gamma(x, B, A)$ :

- the vertices of $\Gamma(x, B, A)$ are the convex subspaces of diameter $n-1$ through $B$ distinct from $A$;
- two distinct vertices $A_{1}$ and $A_{2}$ of $\Gamma(x, B, A)$ are adjacent if there exists a quad $Q$ satisfying (i) $Q \cap B$ is a point at distance $n-2$ from $x$, (ii) $Q \cap A, Q \cap A_{1}$ and $Q \cap A_{2}$ are lines.

We say that $\mathcal{S}$ satisfies property $\left(P_{d e}\right)$ if the graph $\Gamma(x, B, A)$ is connected for all points $x$, for all convex subspaces $B$ of diameter $n-2$ and all convex subspaces $A$ of diameter $n-1$ satisfying $\{x\} \subseteq B \subseteq A$.
(II) We say that a dense near polygon $\mathcal{S}$ satisfies property $\left(P_{d e}^{\prime}\right)$ if for every convex subspace $B$ of diameter $n-2$, for every point $x$ of $B$ and for every three distinct convex subspaces $A_{1}, A_{2}$ and $A_{3}$ of diameter $n-1$ through $B$, there exists a quad $Q$ satisfying (i) $Q \cap B=\{x\}$, (ii) $Q \cap A_{i}$ is a line for every $i \in\{1,2,3\}$. A dense near polygon which satisfies property $\left(P_{d e}^{\prime}\right)$ also satisfies property $\left(P_{d e}\right)$.
Let $\mathcal{S}$ be a partial linear space. A hyperplane of $\mathcal{S}$ is a proper subspace meeting each line of $\mathcal{S}$. A full (projective) embedding of $\mathcal{S}$ is an injective mapping $e$ from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space $\Sigma$ satisfying (i) $\langle e(\mathcal{P})\rangle=\Sigma$ and (ii) $e(L):=\{e(x) \mid x \in L\}$ is a line of $\Sigma$ for every line $L$ of $\mathcal{S}$. If $e: \mathcal{S} \rightarrow \Sigma$ is a full embedding, then for every hyperplane $\Pi$ of $\Sigma, e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \Pi)$ arises from $e$.

If $\mathcal{S}$ is a dense near $2 n$-gon and $x$ is a point of $\mathcal{S}$, then the set $H_{x}$ of points at distance at most $n-1$ from $x$ is a hyperplane of $\mathcal{S}$, called the singular hyperplane of $\mathcal{S}$ with deepest point $x$. [It follows from the theory of dense near polygons that $H_{x}$ cannot coincide with the whole set of points of $\mathcal{S}$. If $y$ is a point of $\mathcal{S}$ at maximal distance $k$ from $x$, then the unique convex sub- $2 k$-gon of $\mathcal{S}$ containing $x$ and $y$ must coincide with $\mathcal{S}$, see e.g. Theorem 2.14 of [8]. This forces $k$ to be equal to $n$.] A full embedding $e: \mathcal{S} \rightarrow \Sigma$ of a dense near polygon $\mathcal{S}$ is called polarized if every singular
hyperplane of $\mathcal{S}$ arises from $e$, or equivalently, if every singular hyperplane of $\mathcal{S}$ is mapped by $e$ into a necessarily unique hyperplane of $\Sigma$.

The following is the main result of this paper.
Main Theorem. Let $e: \mathcal{S} \rightarrow \Sigma$ be a full polarized embedding of a dense near $2 n$-gon $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})(n \geq 1)$. For every point $p$ of $\mathcal{S}$, let $e^{*}(p)$ be the point $\left\langle e\left(H_{p}\right)\right\rangle$ of the dual $\Sigma^{*}$ of $\Sigma$. Let $\Sigma^{(*)}$ be the subspace of $\Sigma^{*}$ generated by all points $\left\langle e\left(H_{p}\right)\right\rangle, p \in \mathcal{P}$. If $\mathcal{S}$ is slim or if $\mathcal{S}$ satisfies property $\left(P_{d e}\right)$, then $e^{*}$ defines a full polarized embedding of $\mathcal{S}$ in $\Sigma^{(*)}$.

Definition. The embedding $e^{*}$ defined in the Main Theorem is called the dual embedding of $e$.

The subscript "de" in property $\left(P_{d e}\right)$ refers to the word "dual embedding". The Main Theorem generalizes a result of Cardinali, De Bruyn and Pasini [6, Theorem 1.7] who showed that every full polarized embedding of a thick dual polar space admits a dual embedding. This fact also follows from our Main Theorem: in Section 5.1, we will show that every dual polar space with at least 3 points on every line satisfies property $\left(P_{d e}\right)$. Theorem 1.7 of [6] has already found applications in the study of the structure of full polarized embeddings of dual polar spaces, see Section 2 of Cardinali and De Bruyn [5]. Many dense near polygons admit a full embedding. This is certainly the case for all slim dense near polygons (see e.g. Proposition 3.11) and also for many examples of dual polar spaces and so-called product and glued near polygons.

Our paper is organized as follows. In Section 2, we will define additional notions regarding near polygons and embeddings of point-line geometries which we will use throughout this paper. In Section 3, we will consider the problem of the existence of dual embeddings for a more general class of point-line geometries. We will also prove there (see Proposition 3.11) that every full polarized embedding of a slim dense near polygon admits a dual embedding. In Section 4, we will extend this result to arbitrary full embeddings of dense near polygons satisfying property $\left(P_{d e}\right)$. In Section 5 , we will show that every known dense near $2 n$-gon, $n \geq 2$, satisfies property $\left(P_{d e}\right)$ except for the ones containing a so-called $\mathbb{E}_{1}$-hex or $\mathbb{E}_{2}$-hex. This shows that having property $\left(P_{d e}\right)$ is not an uncommon property. The whole discussion allows us to conclude that every full embedding of every known dense near polygon has a dual embedding.

## 2 Additional notions regarding near polygons and embeddings of point-line geometries

Let $\mathcal{S}$ be a given point-line geometry. Two full embeddings $e_{1}: \mathcal{S} \rightarrow \Sigma_{1}$ and $e_{2}: \mathcal{S} \rightarrow \Sigma_{2}$ of $\mathcal{S}$ are called isomorphic ( $e_{1} \cong e_{2}$ ) if there exists an isomorphism $f: \Sigma_{1} \rightarrow \Sigma_{2}$ such that $e_{2}=f \circ e_{1}$. If $e: \mathcal{S} \rightarrow \Sigma$ is a full embedding of $\mathcal{S}$ and if $U$ is a subspace of $\Sigma$ satisfying (C1) $\langle U, e(p)\rangle \neq U$ for every point $p$ of $\mathcal{S}$, and $(\mathrm{C} 2)\left\langle U, e\left(p_{1}\right)\right\rangle \neq\left\langle U, e\left(p_{2}\right)\right\rangle$ for any two distinct points $p_{1}$ and $p_{2}$ of $\mathcal{S}$, then there exists a full embedding $e / U$ of $\mathcal{S}$ in the quotient space $\Sigma / U$, mapping each point $p$ of $\mathcal{S}$ to $\langle U, e(p)\rangle$. If $e_{1}: \mathcal{S} \rightarrow \Sigma_{1}$ and $e_{2}: \mathcal{S} \rightarrow \Sigma_{2}$ are two full embeddings, then we say that $e_{1} \geq e_{2}$, if there exists a subspace $U$ in $\Sigma_{1}$ satisfying (C1), (C2) and $e_{1} / U \cong e_{2}$. If $e: \mathcal{S} \rightarrow \Sigma$ is a full embedding of $\mathcal{S}$, then by Ronan [11], there exists up to isomorphism a unique full embedding $\widetilde{e}: \mathcal{S} \rightarrow \widetilde{\Sigma}$ satisfying the following: (i) $\widetilde{e} \geq e$; (ii) if $e^{\prime} \geq e$ for some embedding $e^{\prime}$ of $\mathcal{S}$, then $\widetilde{e} \geq e^{\prime}$. We say that $\widetilde{e}$ is universal relative to $e$. If $\widetilde{e^{\prime}} \cong \widetilde{e}$ for any other embedding $e^{\prime}$ of $\mathcal{S}$ with the same underlying division ring, then $\widetilde{e}$ is called absolutely universal.

Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near polygon. We will denote the distance between two points $x$ and $y$ of $\mathcal{S}$ by $\mathrm{d}(x, y)$. For every point $x$ of $\mathcal{S}$, for every nonempty subset $X$ of $\mathcal{P}$ and every $i \in \mathbb{N}$, we define $\Gamma_{i}(x)=$ $\{y \in \mathcal{P} \mid \mathrm{d}(x, y)=i\}, \mathrm{d}(x, X)=\min \{\mathrm{d}(x, y) \mid y \in X\}$ and $\Gamma_{i}(X)=\{y \in$ $\mathcal{P} \mid \mathrm{d}(y, X)=i\}$. The maximal distance between two points of $X$ is called the diameter of $X$ and is denoted as $\operatorname{diam}(X)$. If $X_{1}$ and $X_{2}$ are two nonempty sets of points, then we define $\mathrm{d}\left(X_{1}, X_{2}\right)=\min \left\{\mathrm{d}\left(x_{1}, x_{2}\right) \mid x_{1} \in\right.$ $X_{1}$ and $\left.x_{2} \in X_{2}\right\}$. We will denote by $\left\langle *_{1}, *_{2}, \ldots, *_{k}\right\rangle$ the smallest convex subspace of $\mathcal{S}$ containing the objects $*_{1}, *_{2}, \ldots, *_{k}$. Here, $*_{i}, i \in$ $\{1, \ldots, k\}$, can be either a point or a nonempty set of points of $\mathcal{S}$. Obviously, $\left\langle *_{1}, *_{2}, \ldots, *_{k}\right\rangle$ is the intersection of all convex subspaces of $\mathcal{S}$ containing $*_{1}, *_{2}, \ldots, *_{k}$.

Suppose now that $\mathcal{S}$ is a dense near $2 n$-gon. If $x$ is a point of $\mathcal{S}$, then the lines and quads through $x$ define a linear space $\mathcal{L}(\mathcal{S}, x)$ called the local space at $x$. The convex sub-2i-gons, $i \in\{1, \ldots, n-1\}$, through $x$ define a rank- $(n-1)$-geometry $\mathcal{G}(\mathcal{S}, x)$ which is called the local geometry at $x$. A convex sub- $2 \delta$-gon $A$ of $\mathcal{S}$ is called big in $\mathcal{S}$ if every point of $\mathcal{S}$ outside $A$ is collinear with a necessarily unique point of $A$. We then necessarily have $\delta=n-1$. A convex subpolygon $A$ is called classical in $\mathcal{S}$ if for every point $x$ of $\mathcal{S}$, there exists a necessarily unique point $\pi_{A}(x)$ in $A$ such that $\mathrm{d}(x, y)=\mathrm{d}\left(x, \pi_{A}(x)\right)+\mathrm{d}\left(\pi_{A}(x), y\right)$ for every point $y$ of $A$. If $A$ is big, then $A$ is also classical in $\mathcal{S}$. If $A$ is a classical convex sub- $2 \delta$-gon of $\mathcal{S}$ and if $B$ is a convex sub- $2 \delta^{\prime}$-gon of $\mathcal{S}$ meeting $A$, then the diameter of $A \cap B$ is at least $\delta+\delta^{\prime}-n$ by Theorem 2.32 of [8].

If $H$ is a hyperplane of a dense near polygon $\mathcal{S}$ and if $Q$ is a quad of $\mathcal{S}$, then either (i) $Q \subseteq H$, (ii) $Q \cap H$ consists of those points of $Q$ which are collinear with a given point $x$ of $Q$, (iii) $Q \cap H$ is a subquadrangle of $Q$, (iv) $Q \cap H$ is an ovoid of $Q$, i.e., a set of points of $Q$ meeting each line in a unique point. If case (i), case (ii), case (iii), respectively case (iv), occurs, then we say that $Q$ is deep, singular, subquadrangular, respectively ovoidal, with respect to $H$. If case (ii) occurs, then we call $x$ the deep point of $Q$ with respect to $H$.

## 3 Embeddings of a class of parapolar spaces

Definitions. (1) A partial linear space is called a polar space if for every point $p$ and every line $L$, either 1 or all points of $L$ are collinear with $p$. The radical of a polar space is the set of points collinear with all points. A polar space is called nondegenerate if its radical is empty. A subspace of a polar space is said to be singular if any two points of it are collinear. The rank $r$ of a nondegenerate polar space is the maximal length $r$ of a chain $S_{0} \subset S_{1} \subset \cdots \subset S_{r}$ of singular subspaces where $S_{0}=\emptyset$ and $S_{i} \neq S_{i+1}$ for all $i \in\{0, \ldots, r-1\}$. A nondegenerate polar space of rank 2 is just a nondegenerate generalized quadrangle.
(2) A partial linear space is called a gamma space if for every point $p$ and every line $L$, either 0,1 or all points of $L$ are collinear with $p$.
(3) A parapolar space is a connected partial linear gamma space possessing a collection of convex subspaces, called symplecta, isomorphic to nondegenerate polar spaces of rank at least 2, with the property that each line is contained in a symplecton and that each quadrangle is contained in a unique symplecton. A parapolar space in which every pair of points at distance 2 are contained in a necessarily unique symplecton is called a strong parapolar space.

Shult introduced in [13] a class $\mathcal{E}$ of parapolar spaces. The class $\mathcal{E}$ is equal to $\mathcal{E}_{0} \cup \mathcal{E}_{1} \cup \mathcal{E}_{2} \cdots$, where the subclasses $\mathcal{E}_{i}, i \in \mathbb{N}$, are defined inductively in the following way. The subclass $\mathcal{E}_{0}$ contains one member, namely the point, and the subclass $\mathcal{E}_{1}$ consists of the lines which are incident with at least three points. The subclass $\mathcal{E}_{n}, n \geq 2$, contains those geometries $\mathcal{S}$ which satisfy the following properties:
$\left(E_{1}\right) \mathcal{S}$ is connected and its diameter is equal to $n$;
( $E_{2}$ ) every line of $\mathcal{S}$ is incident with at least three points;
$\left(E_{3}\right)$ any geodesic in $\mathcal{S}$ completes to a geodesic of length $n$;
$\left(E_{4}\right)$ for every point $x$ of $\mathcal{S}$, the set $H_{x}$ of points of $\mathcal{S}$ at distance at most $n-1$ from $x$ is a hyperplane of $\mathcal{S}$;
$\left(E_{5}\right)$ if $x_{1}$ and $x_{2}$ are two points of $\mathcal{S}$ with $k:=\mathrm{d}\left(x_{1}, x_{2}\right)<n$, then the convex closure $\left\langle x_{1}, x_{2}\right\rangle$ is a member of $\mathcal{E}_{k}$.

The elements of $\mathcal{E}_{2}$ are precisely the nondegenerate polar spaces in which each line is incident with at least three points. Every member of $\mathcal{E}_{n}, n \geq 2$, is a strong parapolar space. The symplecta are the convex closures of the pairs of points at distance 2 from each other. The class $\mathcal{E}$ contains every thick dual polar space and more generally every dense near polygon. The class $\mathcal{E}$ also contains some half-spin geometries, some Grassmann spaces and some exceptional geometries, see Shult [13, Section 6].

The geometric hyperplane $H_{x}$ defined in $\left(E_{4}\right)$ is called the singular hyperplane of $\mathcal{S}$ with deepest point $x$. By Shult [13, Lemma 6.1 (ii)], every geometric hyperplane of an element of $\mathcal{E}$ is a maximal subspace. In particular, this holds for the singular hyperplanes.

Definition. Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space. A set $\mathcal{H}$ of hyperplanes of $\mathcal{S}$ is called a pencil of hyperplanes if $\bigcup_{H \in \mathcal{H}} H=\mathcal{P}$ and $H_{1} \cap H_{2}=H_{1} \cap H_{3}=H_{2} \cap H_{3}$ for any three distinct hyperplanes $H_{1}, H_{2}$ and $H_{3}$ of $\mathcal{H}$.

Lemma 3.1 Let $\mathcal{S}$ be an element of $\mathcal{E}_{n}, n \geq 1$, and let $L$ be a line of $\mathcal{S}$. Then the set $\mathcal{H}=\left\{H_{x} \mid x \in L\right\}$ is a pencil of hyperplanes of $\mathcal{S}$.

Proof. Let $y$ be a point of $\mathcal{S}$. If $y$ has distance at most $n-1$ from each point of $L$, then $y$ is contained in every hyperplane of $\mathcal{H}$. If $y$ has distance $n$ from a point of $L$, then by property $\left(E_{4}\right), y$ is contained in precisely one hyperplane of $\mathcal{H}$. This proves the lemma.

Let $\mathcal{S}=\left(\mathcal{P}, \mathcal{L}\right.$, I) be an element of $\mathcal{E}_{n}, n \geq 1$, and let $e: \mathcal{S} \rightarrow \Sigma$ be a full embedding of $\mathcal{S}$. For every point $x$ of $\mathcal{S}$, the singular hyperplane $H_{x}$ is a maximal subspace of $\mathcal{S}$ and hence $\left\langle e\left(H_{x}\right)\right\rangle$ is either $\Sigma$ or a hyperplane of $\Sigma$. We call the embedding $e$ polarized if $\left\langle e\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma$ for every point $x$ of $\mathcal{S}$. If $e$ is a polarized embedding, then since $H_{x}$ is a maximal subspace, $\left\langle e\left(H_{x}\right)\right\rangle \cap e(\mathcal{P})=e\left(H_{x}\right)$ for every point $x$ of $\mathcal{S}$.

Proposition 3.2 Let $\mathcal{S} \in \mathcal{E}_{n}, n \geq 1$. Let $e_{1}: \mathcal{S} \rightarrow \Sigma_{1}$ and $e_{2}: \mathcal{S} \rightarrow \Sigma_{2}$ be two full embeddings of $\mathcal{S}$ such that $e_{1} \geq e_{2}$. If $e_{2}$ is polarized, then also $e_{1}$ is polarized.

Proof. We know that $e_{2} \cong e_{1} / U$ for some subspace $U$ of $\Sigma_{1}$. Let $x$ be an arbitrary point of $\mathcal{S}$. Since $e_{2}$ is polarized, $\left\langle e_{2}\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma_{1} / U$, i.e. a hyperplane $\Pi$ of $\Sigma_{1}$ through $U$. Obviously, $\left\langle e_{1}\left(H_{x}\right)\right\rangle=\Pi$. So, also $e_{1}$ is polarized.

Corollary 3.3 Let $\mathcal{S} \in \mathcal{E}_{n}, n \geq 1$, and let $e$ be a full polarized embedding of $\mathcal{S}$. Then also $\tilde{e}$ is a full polarized embedding.

Proposition 3.4 Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an element of $\mathcal{E}_{n}, n \geq 1$, and let $e$ : $\mathcal{S} \rightarrow \Sigma$ be a full polarized embedding of $\mathcal{S}$. Put $R_{e}:=\bigcap_{p \in \mathcal{P}}\left\langle e\left(H_{p}\right)\right\rangle$. Then $R_{e}$ satisfies the conditions (C1) and (C2) of Section 2 and the embedding $\bar{e}:=e / R_{e}$ is polarized.

Proof. (i) We show that $R_{e}$ satisfies property (C1). Let $x$ be an arbitrary point of $\mathcal{S}$. By property $\left(E_{3}\right)$, there exists a point $p$ at distance $n$ from $x$. Since $x \notin H_{p}$ and $e(\mathcal{P}) \cap\left\langle e\left(H_{p}\right)\right\rangle=e\left(H_{p}\right), e(x) \notin\left\langle e\left(H_{p}\right)\right\rangle$. Hence, $e(x) \notin R_{e}$.
(ii) We show that $R_{e}$ satisfies property (C2). Let $x_{1}$ and $x_{2}$ be two distinct points of $\mathcal{S}$. By property $\left(E_{3}\right)$, there exists a point $x$ at distance $n$ from $x_{1}$ such that $x_{2}$ is on a geodesic from $x_{1}$ to $x$. Then $e\left(x_{2}\right) \in\left\langle e\left(H_{x}\right)\right\rangle$ and hence $\left\langle R_{e}, e\left(x_{2}\right)\right\rangle \subseteq\left\langle e\left(H_{x}\right)\right\rangle$. On the other hand, since $x_{1} \notin H_{x}$, $e\left(x_{1}\right) \notin\left\langle e\left(H_{x}\right)\right\rangle$. Hence, $\left\langle R_{e}, e\left(x_{2}\right)\right\rangle \neq\left\langle R_{e}, e\left(x_{1}\right)\right\rangle$.
(iii) We show that $\bar{e}$ is polarized. Let $x$ denote an arbitrary point of $\mathcal{S}$. Since $e$ is polarized, $\left\langle e\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma$. Since $R_{e} \subseteq$ $\left\langle e\left(H_{x}\right)\right\rangle,\left\langle e\left(H_{x}\right)\right\rangle$ determines a hyperplane of $\Sigma / R_{e}$ which contains all points of $\bar{e}\left(H_{x}\right)$. This proves that $\bar{e}$ is polarized.

Suppose that $e: \mathcal{S} \rightarrow \Sigma$ is a full polarized embedding of an element $\mathcal{S}=$ $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ of $\mathcal{E}_{n}, n \geq 1$. Then $e^{*}(x):=\left\langle e\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma$ for every point $x$ of $\mathcal{S}$. Suppose that for every line $L$ of $\mathcal{S},\left\{e^{*}(x) \mid x \in L\right\}$ is a line in the dual $\Sigma^{*}$ of $\Sigma$. Then $e^{*}$ defines an embedding of $\mathcal{S}$ in a subspace $\Sigma^{(*)}$ of $\Sigma^{*}$, which we call the dual embedding of $e$. Notice that if $\operatorname{dim}(\Sigma)$ is finite, then there exists a natural bijective correspondence between the subspaces of $\Sigma$ and those of $\Sigma^{*}$. In this case, the subspace $R_{e}$ (as defined in Proposition 3.4) is the subspace of $\Sigma$ corresponding with the subspace $\Sigma^{(*)}$ of $\Sigma^{*}$.

Proposition 3.5 The dual embedding $e^{*}$ (if it exists) is polarized.
Proof. Let $x$ be an arbitrary point of $\mathcal{S}$ and let $H_{x}$ be the singular hyperplane with deepest point $x$. Since $H_{x}$ is a maximal subspace, the subspace of $\Sigma^{(*)}$ generated by $e^{*}\left(H_{x}\right)$ is either $\Sigma^{(*)}$ or a hyperplane of $\Sigma^{(*)}$. But since $e(x) \in e^{*}(y)$ for every $y \in H_{x}$ and $e(x) \notin e^{*}(y)$ for every point $y \notin H_{x}$, the subspace of $\Sigma^{(*)}$ generated by $e^{*}\left(H_{x}\right)$ must be a hyperplane of $\Sigma^{(*)}$. Hence, $e^{*}$ is polarized.

Proposition 3.6 Let $\mathcal{S} \in \mathcal{E}_{n}, n \geq 1$, and let $e_{1}$ and $e_{2}$ be two full polarized embeddings of $\mathcal{S}$ such that $e_{1} \geq e_{2}$. If one of $e_{1}, e_{2}$ has a dual embedding, then both $e_{1}$ and $e_{2}$ have dual embeddings and $e_{1}^{*} \cong e_{2}^{*}$.

Proof. Let $U$ denote the subspace of $\Sigma_{1}$ satisfying $(C 1),(C 2)$ and $e_{2} \cong$ $e_{1} / U$. Since $e_{2}$ is polarized, also $e_{1} / U$ is polarized. Let $x$ denote an arbitrary point of $\mathcal{S}$ and let $H_{x}$ denote the singular hyperplane with deepest point $x$. Since $e_{1} / U$ is polarized, $\left\langle e_{1} / U\left(H_{x}\right)\right\rangle=\left\langle\bigcup_{y \in H_{x}}\left\langle U, e_{1}(y)\right\rangle\right\rangle=$ $\left\langle U, e_{1}\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma_{1}$. Now, $\left\langle e_{1}\left(H_{x}\right)\right\rangle$ is a hyperplane of $\Sigma_{1}$ and it follows that $\left\langle e_{1} / U\left(H_{x}\right)\right\rangle=\left\langle e_{1}\left(H_{x}\right)\right\rangle / U$. The proposition now readily follows.

Proposition 3.7 Let $\mathcal{S} \in \mathcal{E}_{n}, n \geq 1$, and let $e: \mathcal{S} \rightarrow \Sigma$ be a full polarized embedding with $\operatorname{dim}(\Sigma)<\infty$ which has a dual embedding $e^{*}$. Then also $e^{*}$ has a dual embedding and $e^{* *} \cong \bar{e}$, with $\bar{e}$ the full polarized embedding of $\mathcal{S}$ as defined in Proposition 3.4.

Proof. Let $e^{*}: \mathcal{S} \rightarrow \Sigma^{(*)}$ denote the dual embedding of $e$. Since $\operatorname{dim}(\Sigma)<\infty$, we may identify the subspaces of $\Sigma$ with those of $\Sigma^{*}$. Recall that $R_{e}$ (as defined in Proposition 3.4) is the subspace of $\Sigma$ corresponding with $\Sigma^{(*)}$. Define $e^{* *}(x):=\bigcap_{y \in H_{x}} e^{*}(y)$ for every point $x$ of $\mathcal{S}$. By the proof of Proposition 3.5, $e^{* *}(x)=\left\langle R_{e}, e(x)\right\rangle$. If $L$ is a line of $\mathcal{S}$, then $\left\{\left\langle R_{e}, e(x)\right\rangle \mid x \in L\right\}$ is a line of the dual of $\Sigma^{(*)}$. Hence, $e^{*}$ has $e^{* *}$ as dual embedding. From the above treatment it is also clear that $e^{* *} \cong \bar{e}$.

Proposition 3.8 Let $\mathcal{S} \in \mathcal{E}_{n}, n \geq 1$, and let $e_{1}: \mathcal{S} \rightarrow \Sigma_{1}$ and $e_{2}: \mathcal{S} \rightarrow \Sigma_{2}$ be two full polarized embeddings of $\mathcal{S}$ with $\operatorname{dim}\left(\Sigma_{1}\right), \operatorname{dim}\left(\Sigma_{2}\right)<\infty$ having respective dual embeddings $e_{1}^{*}$ and $e_{2}^{*}$. Then $e_{1}^{*} \cong e_{2}^{*}$ if and only if $\widetilde{e_{1}} \cong \widetilde{e_{2}}$.

Proof. If $\widetilde{e_{1}} \cong \widetilde{e_{2}}$, then $e_{1}^{*} \cong{\widetilde{e_{1}}}^{*} \cong{\widetilde{e_{2}}}^{*} \cong e_{2}^{*}$ by Proposition 3.6. Conversely, suppose that $e_{1}^{*} \cong e_{2}^{*}$. Then $\bar{e}_{1} \cong \bar{e}_{2}$ by Proposition 3.7 and hence $\widetilde{e_{1}} \cong \widetilde{e_{1}} \cong \widetilde{\overline{e_{2}}} \cong \widetilde{e_{2}}$.

Consider now the following question for a certain geometry $\mathcal{S} \in \mathcal{E}_{n}, n \geq 2$ :
$(*)$ Has every full polarized embedding of $\mathcal{S}$ a dual embedding?
By Cardinali, De Bruyn and Pasini [6], we know that the answer to the above question is affirmative for thick dual polar spaces. Our Main Theorem, which we will prove in the following section, generalizes this result to arbitrary dense near polygons satisfying property $\left(P_{d e}\right)$. From Shult [13, Theorem 7.1], it readily follows (see Proposition 3.9 below) that the answer to question $(*)$ is also affirmative for all geometries $\mathcal{S} \in \mathcal{E}$ whose symplecta have polar rank at least 3 . Notice that for the nondegenerate polar spaces, this already follows from the work of Veldkamp [17].

Proposition 3.9 Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an element of $\mathcal{E}_{n}, n \geq 2$, whose symplecta have polar rank at least 3. Then every full polarized embedding $e: \mathcal{S} \rightarrow \Sigma$ has a dual embedding.

Proof. By Shult [13, Theorem 7.1], $\mathcal{S}$ admits Veldkamp lines. This means the following:
(i) If $A$ and $B$ are two distinct hyperplanes of $\mathcal{S}$, then $A$ is not contained in $B$.
(ii) If $A, B$ and $C$ are three distinct hyperplanes such that $A \cap B \subseteq C$, then $A \cap B=A \cap C=B \cap C$.

We will make use of the following property.
Property. Let $\mathcal{H}$ be a pencil of hyperplanes of $\mathcal{S}$ and let $H_{1}$ and $H_{2}$ be two distinct elements of $\mathcal{H}$. Then $\mathcal{H}$ coincides with the set $\mathcal{H}^{\prime}$ of all hyperplanes through $H_{1} \cap H_{2}$.
Proof. Obviously, $\mathcal{H} \subseteq \mathcal{H}^{\prime}$. Suppose that there exists a hyperplane $H^{\prime} \in \mathcal{H}^{\prime} \backslash \mathcal{H}$. Since $H^{\prime}$ is a maximal subspace and $H_{1} \cap H_{2}$ is not a maximal subspace, there exists a point $y \in H^{\prime} \backslash\left(H_{1} \cap H_{2}\right)$. Let $H$ denote the unique element of $\mathcal{H}$ through $y$ and let $i \in\{1,2\}$ such that $H_{i} \neq H$. Then the triple $\left\{H, H^{\prime}, H_{i}\right\}$ contradicts property (ii) in the definition of Veldkamp line. (qed)

Now, let $L$ be a line of $\mathcal{S}$ through two distinct collinear points $x_{1}$ and $x_{2}$, and let $\Pi_{i}:=\left\langle e\left(H_{x_{i}}\right)\right\rangle, i \in\{1,2\}$. Consider the following two pencils of hyperplanes of $\mathcal{S}$ :

- $\mathcal{H}_{1}:=\left\{H_{x} \mid x \in L\right\}$ (see Lemma 3.1);
- $\mathcal{H}_{2}:=\left\{e^{-1}(e(\mathcal{P}) \cap \Pi) \mid \Pi\right.$ is a hyperplane of $\Sigma$ through $\left.\Pi_{1} \cap \Pi_{2}\right\}$.

The hyperplanes $H_{x_{1}}$ and $H_{x_{2}}$ are contained in $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$. Hence $\mathcal{H}_{1}=$ $\mathcal{H}_{2}$ by the previous property. It now readily follows that $e$ has a dual embedding.

The answer to question $(*)$ is also affirmative for geometries of $\mathcal{E}$ containing only lines with three points, as we will prove now. Notice first that if $H_{1}$ and $\mathrm{H}_{2}$ are two distinct hyperplanes of a partial linear space $\mathcal{S}$ with three points on each line, then the complement $\overline{H_{1} \Delta H_{2}}$ of the symmetric difference of $H_{1}$ and $H_{2}$ is again a hyperplane.

Lemma 3.10 Let $\mathcal{S}$ be an element of $\mathcal{E}_{n}, n \geq 1$, with three points on every line and let $L=\left\{x_{1}, x_{2}, x_{3}\right\}$ be a line of $\mathcal{S}$. Then the singular hyperplane $H_{x_{3}}$ is equal to the hyperplane $\overline{H_{x_{1}} \Delta H_{x_{2}}}$.

Proof. This is an immediate corollary of Lemma 3.1.
Proposition 3.11 Let $\mathcal{S}$ be an element of $\mathcal{E}_{n}, n \geq 1$, and suppose that every line of $\mathcal{S}$ contains precisely three points. Then
(i) there exists at least one full polarized embedding of $\mathcal{S}$;
(ii) every full polarized embedding of $\mathcal{S}$ has a dual embedding.

Proof. (i) Let $W$ denote the set of all hyperplanes of $\mathcal{S}$ union $\{\mathcal{S}\}$. Then $W$ has the structure of a $\mathbb{F}_{2}$-vectorspace if we take the following addition and scalar multiplication $\left(w_{1}, w_{2} \in W\right): 0 \cdot w_{1}=\mathcal{S}, 1 \cdot w_{1}=w_{1}, w_{1}+w_{2}:=$ $\overline{w_{1} \Delta w_{2}}$. For every point $x$ of $\mathcal{S}$, let $H_{x}$ denote the singular hyperplane with deepest point $x$. Let $V$ denote the subspace of $W$ generated by all singular hyperplanes. For every point $x$ of $\mathcal{S}$, the set of all elements of $V$ containing the point $x$ obviously determines a hyperplane $V_{x}$ of $V$. By Lemma 3.10, the map $p \rightarrow H_{p}$ defines a full projective embedding $e$ of $\mathcal{S}$ in $\mathrm{PG}(V)$. Notice that this map is injective by $\left(E_{3}\right)$. The embedding $e$ is polarized since $e\left(H_{p}\right) \subseteq V_{p}$ for every point $p$ of $\mathcal{S}$.
(ii) Let $e: \mathcal{S} \rightarrow \Sigma$ be a full polarized embedding of $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$. Put $e^{*}(x):=\left\langle e\left(H_{x}\right)\right\rangle$ for every point $x$ of $\mathcal{S}$. Let $L=\left\{x_{1}, x_{2}, x_{3}\right\}$ be an arbitrary line of $\mathcal{S}$ and let $\Pi$ be the unique hyperplane of $\Sigma$ through $e^{*}\left(x_{1}\right) \cap$ $e^{*}\left(x_{2}\right)$ different form $e^{*}\left(x_{1}\right)$ and $e^{*}\left(x_{2}\right)$. Then $e^{-1}(\Pi \cap e(\mathcal{P}))$ coincides with $\overline{H_{x_{1}} \Delta H_{x_{2}}}=H_{x_{3}}$. Hence, $e^{*}\left(x_{1}\right), e^{*}\left(x_{2}\right)$ and $e^{*}\left(x_{3}\right)$ form a line of $\Sigma^{*}$. This proves that $e^{*}$ is an embedding.

Remarks. By Ronan [11] and Proposition 3.11, every element $\mathcal{S}$ of $\mathcal{E}_{n}$, $n \geq 1$, with three points on each line has an absolutely universal embedding. Suppose now that the absolutely universal embedding space is finitedimensional. By Proposition 3.8, it then follows that all dual embeddings are isomorphic to a certain embedding $e_{m}$. This embedding $e_{m}$ is called the minimal full polarized embedding of $\mathcal{S}$. If $e$ is a full polarized embedding of $\mathcal{S}$, then $e \geq e_{m}$ by Proposition 3.7. The minimal full polarized embedding of a slim dense near polygon is also called the near polygon embedding, see Brouwer and Shpectorov [3].

## 4 Completion of the proof of the Main Theorem

Lemma 4.1 Let $L$ be a line of a dense near $2 n$-gon $\mathcal{S}(n \geq 1)$, let $x$ be a point of $L$ and let $V$ denote the set of all convex subspaces of diameter $n-1$ through $x$ not containing L. Let $\Gamma_{x, L}$ be the graph with vertex set $V$, with two vertices adjacent whenever they intersect in a convex subspace of diameter $n-2$. Then the graph $\Gamma_{x, L}$ is connected.

Proof. Let $A_{1}$ and $A_{2}$ be two arbitrary vertices of $\Gamma_{x, L}$. We will prove by downwards induction on $\operatorname{diam}\left(A_{1} \cap A_{2}\right)$ that $A_{1}$ and $A_{2}$ are connected by a path. Obviously, this holds if $\operatorname{diam}\left(A_{1} \cap A_{2}\right) \geq n-2$. So, suppose
$\operatorname{diam}\left(A_{1} \cap A_{2}\right)=n-\delta$ with $\delta \geq 3$. Let $L_{1}$ denote a line of $A_{1}$ through $x$ not contained in $A_{1} \cap A_{2}$. [Such a line exists since a convex subspace $F$ of a dense near polygon is completely determined by the neighbourhood $\Gamma_{1}(u) \cap F$ of one its points $u \in F$, see e.g. Theorem 2.14 of [8]. Notice also that $A_{1} \cap A_{2}$ is properly contained in $A_{1}$.] Then $\left\langle A_{1} \cap A_{2}, L_{1}, L\right\rangle$ is a convex subspace of diameter $n-\delta+2 \leq n-1$ different from $A_{2}$. Hence, there exists a line $L_{2}$ in $A_{2}$ through $x$ not contained in $\left\langle A_{1} \cap A_{2}, L_{1}, L\right\rangle$. [Similar explanation as above. Notice also that $\left\langle A_{1} \cap A_{2}, L_{1}, L\right\rangle \cap A_{2}$ is properly contained in $A_{2}$.] Now, the convex subspace $\left\langle A_{1} \cap A_{2}, L_{1}, L_{2}\right\rangle$ has diameter $n-\delta+2 \leq n-1$ not containing the line $L$. We will now show that there exists a convex subspace $A_{3}$ of diameter $n-1$ through $\left\langle A_{1} \cap A_{2}, L_{1}, L_{2}\right\rangle$ not containing the line $L$. Take a point $u \in L \backslash\{x\}$ and a point $v$ of $\left\langle A_{1} \cap A_{2}, L_{1}, L_{2}\right\rangle$ at maximal distance $n-\delta+2$ from $x$. Then there exists a point $w$ at maximal distance $n$ from $u$ such that $v$ is on a geodesic from $u$ to $w$. Since $\mathrm{d}(u, v)=\mathrm{d}(u, x)+\mathrm{d}(x, v)=1+\mathrm{d}(x, v)$, we necessarily have $\mathrm{d}(x, w)=\mathrm{d}(x, v)+\mathrm{d}(v, w)=n-1$. Clearly, $A_{3}:=\langle x, w\rangle$ has diameter $n-1$, contains $\left\langle A_{1} \cap A_{2}, L_{1}, L_{2}\right\rangle=\langle x, v\rangle$ (since $v$ is on a geodesic from $x$ to $w$ ), but not the line $L$ (since $\mathrm{d}(w, u)=n)$. Now, by the induction hypothesis, $A_{3}$ and $A_{i}(i \in\{1,2\})$ are connected by a path. Hence, also $A_{1}$ and $A_{2}$ are connected by a path.

Lemma 4.2 Let $a$ and $b$ be two distinct collinear points of $a$ dense near $2 n$-gon $\mathcal{S}(n \geq 1)$ satisfying property $\left(P_{d e}\right)$. If $H$ is a hyperplane of $\mathcal{S}$ such that $H \cap\left(H_{a} \cup H_{b}\right)=H_{a} \cap H_{b}$, then $H=H_{c}$ for some point $c$ on the line $L=a b$ different from $a$ and $b$.

Proof. Notice first that:
(i) Every hyperplane of $\mathcal{S}$ is a maximal subspace. In particular, this property holds for the singular hyperplanes (recall [13, Lemma 6.1]).
(ii) $H_{a} \cap H_{b}$ consists of those points of $\mathcal{S}$ at distance at most $n-2$ from $L$.
(iii) $H_{a} \cap H_{b}$ is not a maximal subspace, since it is contained in the two distinct maximal subspaces $H_{a}$ and $H_{b}$.

By (i) and (iii), there exists a point $u \in H \backslash\left(H_{a} \cap H_{b}\right)$ and by (ii), $\mathrm{d}(u, L)=$ $n-1$. Let $c$ denote the unique point of $L$ nearest to $u$. Since $u \notin H_{a} \cup H_{b}$, $a \neq c \neq b$.

Step 1: $\langle u, c\rangle \subseteq H$.
Let $v$ denote an arbitrary point of $\langle u, c\rangle$. If $\mathrm{d}(c, v) \leq n-2$, then $v \in$ $H$ since $v \in H_{a} \cap H_{b}$. Suppose now that $\mathrm{d}(c, v)=n-1$. Since the hyperplane $\Gamma_{\leq n-2}(c) \cap\langle u, c\rangle$ of $\langle u, c\rangle$ is a maximal subspace, $\Gamma_{n-1}(c) \cap\langle u, c\rangle$
is connected. So, there exists a path $w_{0}, w_{1}, w_{2}, \ldots, w_{k}$ in $\Gamma_{n-1}(c) \cap\langle u, c\rangle$ connecting the points $u=w_{0}$ and $v=w_{k}$. If $w_{i} \in H$ for a certain $i \in$ $\{0, \ldots, k-1\}$, then also $w_{i+1} \in H$ since $w_{i} \in H$ and $w_{i} w_{i+1} \cap \Gamma_{n-2}(c) \subseteq H$. Since $w_{0} \in H$, it follows that also $v=w_{k} \in H$.

Step 2: If $A_{1}$ and $A_{2}$ are two adjacent vertices of $\Gamma_{c, L}$ such that $A_{1} \subseteq H$, then also $A_{2} \subseteq H$.
Put $A_{3}:=\left\langle A_{1} \cap A_{2}, L\right\rangle$. Since $\mathcal{S}$ satisfies property $\left(P_{d e}\right)$, the graph $\Gamma\left(c, A_{1} \cap\right.$ $\left.A_{2}, A_{3}\right)$ is connected. So, it suffices to prove the claim in the case that $A_{1}$ and $A_{2}$ are adjacent vertices of $\Gamma\left(c, A_{1} \cap A_{2}, A_{3}\right)$. There exists then a quad $Q$ such that: (i) $Q \cap\left(A_{1} \cap A_{2}\right)$ is a point $y$ at distance $n-2$ from $c$; (ii) for every $i \in\{1,2,3\}, L_{i}:=Q \cap A_{i}$ is a line. Notice that the lines $L_{1}$ and $L_{3}$ of $Q$ are contained in $H$. There are two possibilities:
(1) $Q$ is subquadrangular or deep with respect to $H$;
(2) $Q$ is singular with respect to $H$.

Suppose case (1) occurs. Put $a^{*}:=\pi_{L_{3}}(a)$ and let $x$ denote an arbitrary point of $\left(\Gamma_{1}\left(a^{*}\right) \cap Q\right) \backslash L_{3}$ contained in $H$. Since $\mathrm{d}\left(x, A_{3}\right)=\mathrm{d}\left(x, a^{*}\right)=1$, $\Gamma_{n-1}(x) \cap L=\Gamma_{n-2}\left(a^{*}\right) \cap L=\{a\}$. So, $x$ is a point of $H \cap\left(H_{a} \cup H_{b}\right)$ not contained in $H_{a} \cap H_{b}$, a contradiction.

Hence, case (2) occurs. Then $Q$ is singular with deep point $L_{1} \cap L_{3}$. It follows that $L_{2} \subseteq H$. Hence, $\Gamma_{n-1}(c) \cap A_{2}$ contains a point of $H$. By Step 1, it then follows that $A_{2} \subseteq H$.
Step 3: $H=H_{c}$.
Since $H_{c}$ is a maximal subspace, it suffices to show that $H_{c} \subseteq H$ or that any convex subspace $A$ of diameter $n-1$ through $c$ is contained in $H$. If $L \subseteq A$, then $A \subseteq H_{a} \cap H_{b} \subseteq H$. Hence, we may suppose that $L \nsubseteq A$, or that $A$ is a vertex of $\Gamma_{c, L}$. By Step 2, Lemma 4.1 and the fact that $\langle u, c\rangle \subseteq H$, it follows that $A \subseteq H$. This proves the statement.

We can now complete the proof of the Main Theorem. Let $L$ be an arbitrary line of a dense near polygon $\mathcal{S}$ which satisfies property $\left(P_{d e}\right)$ and let $e: \mathcal{S} \rightarrow$ $\Sigma$ be a full polarized embedding of $\mathcal{S}$. We must show that $\left\{e^{*}(x) \mid x \in L\right\}$ is a line of the dual $\Sigma^{*}$ of $\Sigma$. Let $x_{1}$ and $x_{2}$ denote two distinct points of $L$. If $\Pi$ is a hyperplane of $\Sigma$ through $e^{*}\left(x_{1}\right) \cap e^{*}\left(x_{2}\right)$, then by Lemma $4.2, e^{-1}(\Pi \cap e(\mathcal{P}))=H_{c}$ for some point $c$ of $L$ different from $x_{1}$ and $x_{2}$. Conversely, take an arbitrary point $c^{\prime}$ on $L$ different from $x_{1}$ and $x_{2}$ and let $d$ be an arbitrary point of $\Gamma_{n-1}(L) \cap \Gamma_{n-1}\left(c^{\prime}\right)$. Then $e(d)$ is not contained in $e^{*}\left(x_{1}\right) \cap e^{*}\left(x_{2}\right)$, since $\left(e^{*}\left(x_{1}\right) \cap e^{*}\left(x_{2}\right)\right) \cap e(\mathcal{P})=\left(e^{*}\left(x_{1}\right) \cap e(\mathcal{P})\right) \cap\left(e^{*}\left(x_{2}\right) \cap\right.$ $e(\mathcal{P}))=e\left(H_{x_{1}}\right) \cap e\left(H_{x_{2}}\right)=e\left(H_{x_{1}} \cap H_{x_{2}}\right)$. So, $\Pi^{\prime}:=\left\langle e(d), e^{*}\left(x_{1}\right) \cap e^{*}\left(x_{2}\right)\right\rangle$ is a hyperplane of $\Sigma$. As before, $e^{-1}\left(\Pi^{\prime} \cap e(\mathcal{P})\right)$ is a singular hyperplane of $\mathcal{S}$ with deepest point on $L$. It follows that $e^{-1}\left(\Pi^{\prime} \cap e(\mathcal{P})\right)=H_{c^{\prime}}$ since $d \in e^{-1}\left(\Pi^{\prime} \cap e(\mathcal{P})\right)$. This proves that $\left\{e^{*}(x) \mid x \in L\right\}$ is a line of $\Sigma^{*}$.

## 5 Dense near polygons satisfying property ( $P_{d e}$ )

Every known dense near polygon either belongs to a certain list of near polygons or is obtained by applying the so-called direct-product and glueing constructions to near polygons of that list. The list includes the dual polar spaces, the near hexagons $\mathbb{E}_{1}, \mathbb{E}_{2}, \mathbb{E}_{3}$ and the infinite classes $\mathbb{G}_{n}, \mathbb{H}_{n}, \mathbb{I}_{n}$ $(n \geq 3)$. For each of these near polygons we will now determine whether they satisfy property $\left(P_{d e}\right)$ or not. From Propositions 5.1, 5.2, 5.3, 5.4, 5.5, $5.6,5.7,5.8$ and 5.9 below, it follows that every known dense near $2 n$-gon, $n \geq 2$, satisfies property $\left(P_{d e}\right)$, except for the ones containing an $\mathbb{E}_{1}$-hex or an $\mathbb{E}_{2}$-hex. From the Main Theorem, it then readily follows that every full polarized embedding of a known dense near polygon has a dual embedding.

### 5.1 Dual polar spaces

With every nondegenerate polar space $\Pi$ of rank $n \geq 2$, there is associated a point-line geometry $\Delta$, whose points are the maximal singular subspaces of $\Pi$, whose lines are the next-to-maximal singular subspaces of $\Pi$ and whose incidence relation is reverse containment. The geometry $\Delta$ is called a dual polar space. The dual polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles. Every convex subspace of a dual polar space $\Delta$ is classical in $\Delta$.

Proposition 5.1 Let $\Delta$ be a dual polar space of rank at least 2 with at least 3 points on every line. Then $\Delta$ satisfies property $\left(P_{d e}\right)$. In particular, every nondegenerate generalized quadrangle with at least 3 points per line satisfies property ( $P_{d e}$ ).

Proof. We will show that $\Delta$ satisfies property $\left(P_{d e}^{\prime}\right)$. So, let $B$ be an arbitrary convex subspace of diameter $n-2$, let $x$ denote an arbitrary point of $B$ and let $A_{1}, A_{2}$ and $A_{3}$ be three convex subspaces of diameter $n-1$ through $B$. Take a quad $Q$ through $x$ intersecting $B$ only in the point $x$. Since $A_{i}, i \in\{1,2,3\}$, is big in $\Delta, A \cap B_{i}$ is a line (recall Theorem 2.32 of [8]). This proves that $\Delta$ satisfies property $\left(P_{d e}^{\prime}\right)$ and hence also property $\left(P_{d e}\right)$.

### 5.2 The near hexagon $\mathbb{E}_{1}$

Let $C$ be the extended ternary Golay code, i.e. the 6 -dimensional subspace of $\mathbb{F}_{3}^{12}$ generated by the rows of the following matrix:

$$
\left[\begin{array}{rrrrrrrrrrrr}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1
\end{array}\right]
$$

With $C$, there is associated a near hexagon $\mathbb{E}_{1}$, see Shult and Yanushka [14] or De Bruyn [8, Section 6.5]. The points of $\mathbb{E}_{1}$ are the cosets of $C$ in $\mathbb{F}_{3}^{12}$ and two cosets are collinear whenever they contain vectors which differ in only one position.

Proposition 5.2 The near hexagon $\mathbb{E}_{1}$ does not satisfy property $\left(P_{d e}\right)$.
Proof. The near hexagon $\mathbb{E}_{1}$ cannot satisfy property $\left(P_{d e}\right)$ since every quad is a grid.

### 5.3 The near hexagon $\mathbb{E}_{2}$

Let $S(5,8,24)$ be the unique 5 -( $24,8,1$ )-design. [Such a design is an incidence structure of points and blocks satisfying (i) there are precisely $\mathbf{2 4}$ points, (ii) each block contains precisely 8 points, (iii) every 5 distinct points are contained in precisely 1 block.] With the design $S(5,8,24)$, there is associated a near hexagon $\mathbb{E}_{2}$, see Shult and Yanushka [14] or [8, Section 6.6]. The points of $\mathbb{E}_{2}$ are the blocks of $S(5,8,24)$ and the lines are the triples of mutually disjoint blocks (natural incidence).

Proposition 5.3 The near hexagon $\mathbb{E}_{2}$ does not satisfy property $\left(P_{d e}\right)$.
Proof. Let $x$ be a point of $\mathbb{E}_{2}$, let $B$ be a line through $x$ and let $A$ be a quad through $B$. We will show that the graph $\Gamma(x, B, A)$ has three connected components. Put $B=\left\{x, x_{1}, x_{2}\right\}$
Claim. Suppose $A_{1}$ and $A_{2}$ are two quads through $B$ such that $A_{1}, A_{2}$ and $A$ are mutually different. If there exists a quad $Q_{1}$ through $x_{1}$ satisfying ( $i$ ) $Q_{1} \cap B=\left\{x_{1}\right\}$ and (ii) $Q_{1} \cap A, Q_{1} \cap A_{1}$ and $Q_{1} \cap A_{2}$ are lines, then there exists a quad $Q_{2}$ through $x_{2}$ satisfying (i) $Q_{2} \cap B=\left\{x_{2}\right\}$ and (ii) $Q_{2} \cap A$, $Q_{2} \cap A_{1}$ and $Q_{2} \cap A_{2}$ are lines.
Let $L_{i}, i \in\{1,2\}$, be a line of $A_{i}$ through $x_{2}$ different from $B$. Then the quad $Q_{2}:=\left\langle L_{1}, L_{2}\right\rangle$ does not contain $B$ and hence is disjoint from $Q_{1}$. Now, disjoint quads of $\mathbb{E}_{2}$ can only have one kind of mutual position, see Section 4 of [2]. It follows that $Q_{2} \cap \Gamma_{1}\left(Q_{1}\right)$ consists of a point of $Q_{2}$ together with all its neighbours. Since $L_{1}, L_{2} \subseteq Q_{2} \cap \Gamma_{1}\left(Q_{1}\right), Q_{2} \cap \Gamma_{1}\left(Q_{1}\right)=L_{1} \cup L_{2} \cup L_{3}$,
where $L_{3}$ is the third line of $Q_{2}$ through $x_{2}$. Now, let $L_{3}^{\prime}$ denote the unique line of $Q_{1}$ such that every point of $L_{3}^{\prime}$ is collinear with a unique point of $L_{3}$. Then $L_{3}^{\prime} \subseteq A$. Every point of $L_{3} \backslash\left\{x_{2}\right\}$ is collinear with the point $x_{2}$ and with a point of $L_{3}^{\prime}$. It follows that $L_{3} \subseteq A$. This proves the claim.
Now, consider the local space $\mathcal{L}=\mathcal{L}\left(\mathcal{S}, x_{2}\right)$ which is isomorphic to $\operatorname{PG}(3,2)$, see [2, Section 4.3]. For every convex subspace $F$ through $x_{2}$, let $\widetilde{F}$ denote the corresponding subspace of $\mathcal{L}$. Then $\widetilde{B}$ is a point of $\mathcal{L}$ and $\widetilde{A}$ is a line of $\mathcal{L}$ through $\widetilde{B}$. Now, suppose that $A_{1}$ and $A_{2}$ are two adjacent vertices of $\Gamma(x, B, A)$. Then by the previous claim, there exists a line $\widetilde{Q}$ in $\mathcal{L}$ which intersects the lines $\widetilde{A}, \widetilde{A_{1}}$ and $\widetilde{A_{2}}$ in points different from $\widetilde{B}$. This implies that the lines $\widetilde{A_{1}}$ and $\widetilde{A_{2}}$ are contained in the same plane of $\mathcal{L}$ through $\widetilde{A}$. It is now obvious that the graph $\Gamma(x, B, A)$ has three connected components.

### 5.4 The near hexagon $\mathbb{E}_{3}$

Consider in $\mathrm{PG}(6,3)$ a nonsingular parabolic quadric $Q(6,3)$ and a nontangent hyperplane $\pi$ intersecting $Q(6,3)$ in a nonsingular elliptic quadric $Q^{-}(5,3)$. There is a polarity associated with $Q(6,3)$ and we call two points orthogonal when one of them is contained in the polar hyperplane of the other. Let $N$ denote the set of 126 internal points of $Q(6,3)$ which are contained in $\pi$, i.e. the set of all 126 points in $\pi$ for which the polar hyperplane intersects $Q(6,3)$ in a nonsingular elliptic quadric. The following near hexagon $\mathbb{E}_{3}$ can now be constructed, see $[4$, Section (n)]. The points of $\mathbb{E}_{3}$ are the 6 -tuples of mutually orthogonal points of $N$, the lines of $\mathbb{E}_{3}$ are the pairs of mutually orthogonal points of $N$ and incidence is reverse containment. Group-theoretical constructions for $\mathbb{E}_{3}$ can be found in Aschbacher [1], Kantor [9] and Ronan and Smith [12]. Every local space of $\mathbb{E}_{3}$ is isomorphic to $\overline{W(2)}$, the linear space derived from $W(2)$ by adding its ovoids as extra lines.

Proposition 5.4 The near hexagon $\mathbb{E}_{3}$ satisfies property $\left(P_{d e}\right)$.
Proof. We will show that $\mathbb{E}_{3}$ satisfies property $\left(P_{d e}^{\prime}\right)$. So, let $x$ denote an arbitrary point of $\mathbb{E}_{3}$, let $L$ denote an arbitrary line through $x$ and let $Q_{1}, Q_{2}, Q_{3}$ be three distinct quads through $L$. Let $Q$ be one of the four $Q(5,2)$-quads through $x$ not containing the line $L$. Then $Q$ is big in $\mathbb{E}_{3}$ and hence $Q \cap Q_{i}$ is a line for every $i \in\{1,2,3\}$. This proves that $\mathbb{E}_{3}$ satisfies property $\left(P_{d e}^{\prime}\right)$ and hence also property $\left(P_{d e}\right)$.

### 5.5 The near polygon $\mathbb{I}_{n}, n \geq 3$

Let $Q(2 n, 2), n \geq 2$, be a nonsingular parabolic quadric of $\mathrm{PG}(2 n, 2)$ and let $\Pi$ be a hyperplane of $\operatorname{PG}(2 n, 2)$ intersecting $Q(2 n, 2)$ in a nonsingular hyperbolic quadric $Q^{+}(2 n-1,2)$. The generators of $Q(2 n, 2)$ are the points of a dual polar space $D Q(2 n, 2)$. The generators of $Q(2 n, 2)$ not contained in $Q^{+}(2 n-1,2)$ form a subspace of $D Q(2 n, 2)$. The points and lines contained in this subspace define a dense near $2 n$-gon which we will denote by $\mathbb{I}_{n}$. Let $V$ denote the set of all subspaces of $Q(2 n, 2)$ with exception of the $(n-2)$ - and $(n-1)$-dimensional subspaces contained in $\Pi$. There exists a bijective correspondence between the nonempty convex subspaces of $\mathbb{I}_{n}$ and the elements of $V$ : if $\alpha$ is a subspace of $V$, then the generators through $\alpha$ define a convex subspace of $\mathbb{I}_{n}$. For more details on the above-mentioned facts, we refer to $[8$, Chapter 6].

Proposition 5.5 The near $2 n$-gon $\mathbb{I}_{n}, n \geq 3$, satisfies property $\left(P_{d e}\right)$.
Proof. We will show that $\mathbb{I}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$. Let $x$ denote an arbitrary point of $\mathbb{I}_{n}$, let $B$ denote an arbitrary convex subspace of diameter $n-2$ through $x$ and let $A_{1}, A_{2}$ and $A_{3}$ be three distinct convex subspaces of diameter $n-1$ through $B$. For every convex subspace $F$ of $\mathbb{I}_{n}$, let $\widetilde{F}$ denote the corresponding subspace of $Q(2 n, 2)$. The point $x$ corresponds with a generator $\widetilde{x}$ of $Q(2 n, 2)$. The subspace $\widetilde{B}$ is a line of $\widetilde{x}$ and $\widetilde{A_{1}}, \widetilde{A_{2}}$ and $\widetilde{A_{3}}$ are three points of $\widetilde{B}$. Now, there always exists a subspace $\widetilde{Q}$ of dimension $n-3$ in $\widetilde{x}$ which is disjoint from $\widetilde{B}$ and which is not contained in the ( $n-2$ )-dimensional subspace $\widetilde{x} \cap \Pi$. The quad $Q$ corresponding with $\widetilde{Q}$ intersects $B$ in the point $x$ and $A_{1}, A_{2}$ and $A_{3}$ in the respective lines $L_{1}, L_{2}$ and $L_{3}$ with $\widetilde{L_{i}}=\left\langle\widetilde{Q}, \widetilde{A_{i}}\right\rangle, i \in\{1,2,3\}$. This proves that $\mathbb{I}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$ and hence also property $\left(P_{d e}\right)$.

### 5.6 The near polygon $\mathbb{H}_{n}, n \geq 3$

Let $X$ be a set of size $2 n+2, n \geq 2$. The following near $2 n$-gon can then be constructed. The points of $\mathbb{H}_{n}$ are the partitions of $X$ in $n+1$ subsets of size 2 and the lines are the partitions of $X$ in $n-1$ subsets of size 2 and 1 subset of size 4. A point is incident with a line if and only if the partition corresponding with the point is a refinement of the partition corresponding with the line. There exists a bijective correspondence between the convex sub- $2 \delta$-gons of $\mathbb{H}_{n}$ and the partitions of $X$ in $n+1-\delta$ subsets of even size. If $P$ is such a partition, then the points of $\mathbb{H}_{n}$ which are a refinement of $P$ define a convex sub- $2 \delta$-gon of $\mathbb{H}_{n}$. For more details on the above-mentioned facts, we refer to $[8$, Chapter 6].

Proposition 5.6 The near $2 n$-gon $\mathbb{H}_{n}, n \geq 3$, satisfies property $\left(P_{d e}\right)$.

Proof. We will prove that $\mathbb{H}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$. Let $x$ denote an arbitrary point of $\mathbb{H}_{n}$, let $B$ denote an arbitrary sub-( $2 n-4$ )-gon through $x$ and let $A_{1}, A_{2}$ and $A_{3}$ denote (the) three distinct sub-( $2 n-2$ )-gons through $B$. Let $\left\{X_{1}, X_{2}, X_{3}\right\}$ denote the partition of $X$ corresponding with $B$. The partition $P_{x}$ corresponding with $x$ is a refinement of $\left\{X_{1}, X_{2}, X_{3}\right\}$. Without loss of generality, we may suppose that the convex sub- $(2 n-2)$ gon $A_{i}, i \in\{1,2,3\}$, corresponds with the partition $\left\{X_{i}, X_{i+1} \cup X_{i+2}\right\}$, where the indices are taken modulo 3 . Let $v_{1}, v_{2}, v_{3}$ be three subsets of size 2 of $P_{x}$ such that $v_{1} \subseteq X_{1}, v_{2} \subseteq X_{2}$ and $v_{3} \subseteq X_{3}$. Let $Q$ be the quad of $\mathbb{H}_{n}$ corresponding with the partition $P_{Q}:=P_{x} \backslash\left\{v_{1}, v_{2}, v_{3}\right\} \cup\left\{v_{1} \cup v_{2} \cup v_{3}\right\}$. It is obvious that:
(i) $P_{x}$ is the unique common refinement of $P_{Q}$ and $\left\{X_{1}, X_{2}, X_{3}\right\}$ in subsets of even size. Hence, $Q$ intersects $B$ in the point $x$.
(ii) The partition $P_{x} \backslash\left\{v_{i+1}, v_{i+2}\right\} \cup\left\{v_{i+1} \cup v_{i+2}\right\}$ is a refinement of $P_{Q}$ and $\left\{X_{i}, X_{i+1} \cup X_{i+2}\right\}(i \in\{1,2,3\})$. Hence, $Q$ and $A_{i}$ intersect in a line.

This proves that $\mathbb{H}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$ and hence also property $\left(P_{d e}\right)$.

### 5.7 The near polygon $\mathbb{G}_{n}, n \geq 3$

Let $H(2 n-1,4), n \geq 2$, denote a nonsingular hermitian variety in $\mathrm{PG}(2 n-$ $1,4)$. Without loss of generality, we may suppose that $H(2 n-1,4)$ has equation $X_{0}^{3}+X_{1}^{3}+\cdots+X_{2 n-1}^{3}=0$ with respect to a certain reference system. Let $X$ denote the set of points of $\operatorname{PG}(2 n-1,4)$ with precisely two nonzero coordinates.

The generators of $H(2 n-1,4)$ are the points of a dual polar space $D H(2 n-1,4)$. The generators of $H(2 n-1,4)$ containing precisely $n$ points of $X$ form a subspace of $D H(2 n-1,4)$. The points and lines which are contained in this subspace define a near $2 n$-gon which we will denote by $\mathbb{G}_{n}$. If $n \geq 3$, then $\operatorname{Aut}\left(\mathbb{G}_{n}\right)$ has two orbits on the set of lines. One orbit consists of those lines which are not contained in a $W(2)$-quad. We call these lines special. The other orbit consists of the so-called ordinary lines. Every point of $\mathbb{G}_{n}$ is contained in precisely $n$ special lines and if $L_{1}, \ldots, L_{k}$ are $k \in\{2, \ldots, n\}$ such lines then $\left\langle L_{1}, \ldots, L_{k}\right\rangle \cong \mathbb{G}_{k}$. Since $\mathbb{G}_{2} \cong Q(5,2)$, every two intersecting special lines generate a $Q(5,2)$-quad. For more details on the above-mentioned facts, we refer to [8, Chapter 6].

Proposition 5.7 The near $2 n$-gon $\mathbb{G}_{n}, n \geq 3$, satisfies property $\left(P_{d e}\right)$.
Proof. We will show that $\mathbb{G}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$. Let $x$ denote an arbitrary point of $\mathbb{G}_{n}$, let $B$ denote a convex sub- $(2 n-4)$-gon through $x$
and let $A_{1}, A_{2}$ and $A_{3}$ be three convex sub- $(2 n-2)$-gons through $B$. Let $L_{1}, \ldots, L_{n}$ denote the $n$ special lines through $x$. Without loss of generality, we may suppose that, for a certain $k \in\{0, \ldots, n\}$, the lines $L_{1}, \ldots, L_{k}$ are contained in $B$ and the lines $L_{k+1}, \ldots, L_{n}$ are not contained in $B$. If $k \geq n-1$, then $\left\langle L_{1}, \ldots, L_{k}\right\rangle \cong \mathbb{G}_{k}$ has diameter at at least $n-1$ and is contained in $B$, a contradiction. Hence, $k \leq n-2$. Suppose that all quads $\left\langle L_{i}, L_{j}\right\rangle, i, j \in\{k+1, \ldots, n\}$ with $i \neq j$, meet $B$ in a line. Then the sub$(2 n-2)$-gon $\left\langle B, L_{n}\right\rangle$ contains all special lines $L_{1}, \ldots, L_{n}$ and hence also $\mathbb{G}_{n}$, which is impossible. Hence, there exists a $Q(5,2)$-quad $Q$ through $x$ such that $Q \cap B=\{x\}$. Now, each $Q(5,2)$-quad is classical in $\mathbb{G}_{n}$ and it follows that $Q$ intersects $A_{i}, i \in\{1,2,3\}$, in a line (recall [8, Theorem 2.32]). This proves that $\mathbb{G}_{n}$ satisfies property $\left(P_{d e}^{\prime}\right)$ and hence also property $\left(P_{d e}\right)$.

### 5.8 Product near polygons

Any two near polygons $\mathcal{A}_{1}=\left(\mathcal{P}_{1}, \mathcal{L}_{1}, \mathrm{I}_{1}\right)$ and $\mathcal{A}_{2}=\left(\mathcal{P}_{2}, \mathcal{L}_{2}, \mathrm{I}_{2}\right)$ of diameter at least 1 give rise to a so-called product near polygon $\mathcal{A}_{1} \times \mathcal{A}_{2}$. The point set of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ is equal to $\mathcal{P}_{1} \times \mathcal{P}_{2}$ and the line set is equal to $\left(\mathcal{P}_{1} \times \mathcal{L}_{2}\right) \cup\left(\mathcal{L}_{1} \times \mathcal{P}_{2}\right)$. (We assume here that $\mathcal{P}_{1} \cap \mathcal{L}_{1}=\emptyset$ and $\mathcal{P}_{2} \cap \mathcal{L}_{2}=\emptyset$.) The point $(x, y)$ of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ is incident with the line $(z, L) \in \mathcal{P}_{1} \times \mathcal{L}_{2}$ if and only if $x=z$ and $y \mathrm{I}_{2} L$, the point $(x, y)$ of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ is incident with the line $(M, u) \in \mathcal{L}_{1} \times \mathcal{P}_{2}$ if and only if $x \mathrm{I}_{1} M$ and $y=u$. The near polygon $\mathcal{A}_{1} \times \mathcal{A}_{2}$ is called the direct product of $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ and its diameter is equal to the sum of the diameters of $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$. There exists a partition $T_{i}, i \in\{1,2\}$, of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ in convex subpolygons isomorphic to $\mathcal{A}_{i}$ such that the following holds: (1) every subpolygon of $T_{1}$ intersects every subpolygon of $T_{2}$ in a point; (2) every line of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ is contained in a unique subpolygon of $T_{1} \cup T_{2}$. Every subpolygon of $T_{1} \cup T_{2}$ is classical in $\mathcal{A}_{1} \times \mathcal{A}_{2}$ and if $F$ and $F^{\prime}$ are two subpolygons of $T_{i}, i \in\{1,2\}$, then the map $F^{\prime} \rightarrow F, x \mapsto \pi_{F}(x)$ defines an isomorphism between $F^{\prime}$ and $F$.

Let $x$ denote an arbitrary point of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ and let $F_{i}(x), i \in\{1,2\}$, denote the unique element of $T_{i}$ through $x$. For every convex subspace $U$ of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ through $x$, let $U_{i}, i \in\{1,2\}$, denote the convex subspace $F_{i}(x) \cap U$ of $F_{i}(x)$. It holds $\operatorname{diam}(U)=\operatorname{diam}\left(U_{1}\right)+\operatorname{diam}\left(U_{2}\right)$ and if $\operatorname{diam}\left(U_{1}\right), \operatorname{diam}\left(U_{2}\right) \geq 1$, then $U \cong U_{1} \times U_{2}$.

If $V_{1}$ is a convex subspace of $F_{1}(x)$ through $x$ and if $V_{2}$ is a convex subspace of $F_{2}(x)$ through $x$, then $U:=\left\langle V_{1}, V_{2}\right\rangle$ is a convex subspace of $\mathcal{A}_{1} \times \mathcal{A}_{2}$ through $x$ with $U_{1}=V_{1}$ and $U_{2}=V_{2}$.

Proposition 5.8 Let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two dense near polygons.
(i) If $\operatorname{diam}\left(\mathcal{A}_{1}\right)=\operatorname{diam}\left(\mathcal{A}_{2}\right)=1$, then $\mathcal{A}_{1} \times \mathcal{A}_{2}$ satisfies property $\left(P_{\text {de }}\right)$.
(ii) If $\operatorname{diam}\left(\mathcal{A}_{1}\right)=1$ and $\operatorname{diam}\left(\mathcal{A}_{2}\right) \geq 2$, then $\mathcal{A}_{1} \times \mathcal{A}_{2}$ satisfies property ( $P_{\text {de }}$ ) if and only if $\mathcal{A}_{2}$ satisfies property $\left(P_{d e}\right)$.
(iii) If $\operatorname{diam}\left(\mathcal{A}_{1}\right), \operatorname{diam}\left(\mathcal{A}_{2}\right) \geq 2$, then $\mathcal{A}_{1} \times \mathcal{A}_{2}$ satisfies property $\left(P_{\text {de }}\right)$ if and only if $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ satisfy property $\left(P_{d e}\right)$.

Proof. Put $\operatorname{diam}\left(\mathcal{A}_{1} \times \mathcal{A}_{2}\right)=n$. Let $x$ denote an arbitrary point of $\mathcal{A}_{1} \times \mathcal{A}_{2}$, let $B$ denote an arbitrary convex sub- $(2 n-4)$-gon through $x$ and suppose $C, D$ and $E$ are three convex sub- $(2 n-2)$-gons through $B$. We will use the notations as introduced before this proposition. We have $\operatorname{diam}\left(B_{1}\right)+\operatorname{diam}\left(B_{2}\right)=\operatorname{diam}(B)=\operatorname{diam}\left(F_{1}(x)\right)+\operatorname{diam}\left(F_{2}(x)\right)-2$. If $\operatorname{diam}\left(B_{1}\right)=\operatorname{diam}\left(F_{1}(x)\right)-1$ and $\operatorname{diam}\left(B_{2}\right)=\operatorname{diam}\left(F_{2}(x)\right)-1$, then $B$ is contained in precisely two convex subpolygons of diameter $n-1$, namely $\left\langle F_{1}(x), B_{2}\right\rangle$ and $\left\langle B_{1}, F_{2}(x)\right\rangle$, a contradiction. Hence, there exists an $i \in$ $\{1,2\}$ such that $\operatorname{diam}\left(B_{i}\right)=\operatorname{diam}\left(F_{i}(x)\right)-2$ and $B_{3-i}=F_{3-i}(x)$. We now show that $D$ and $E$ are adjacent vertices of $\Gamma(x, B, C)$ if and only of $D_{i}$ and $E_{i}$ are adjacent vertices of $\Gamma\left(x, B_{i}, C_{i}\right)$.

If $Q$ is a quad meeting $B$ in a point of $\Gamma_{n-2}(x)$ and $C, D$ and $E$ in lines, then $Q$ is contained in a convex subspace of $T_{i}$ and $\pi_{F_{i}(x)}(Q)$ is a quad meeting $B_{i}$ in a point at distance $\operatorname{diam}\left(F_{i}(x)\right)-2$ from $x$ and $C_{i}, D_{i}$ and $E_{i}$ in lines. Conversely, suppose $Q$ is a quad meeting $B_{i}$ in a point at distance $\operatorname{diam}\left(F_{i}(x)\right)-2$ from $x$ and $C_{i}, D_{i}$ and $E_{i}$ in lines. Take a point $y$ in $B$ at distance $n-2$ from $x$ such that $\pi_{F_{i}(x)}(y)$ coincides with the point $Q \cap B_{i}$. Then $\pi_{F_{i}(y)}(Q)$ is a quad meeting $B$ in a point of $\Gamma_{n-2}(x)$ and $C$, $D$ and $E$ in lines.

The proposition now readily follows.

### 5.9 Glued near polygons

Let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two dense near polygons of diameter at least 2 . If $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ satisfy certain nice properties, then it is possible to construct a so-called glued near polygon of type $\mathcal{A}_{1} \otimes_{1} \mathcal{A}_{2}$ (or shortly of type $\mathcal{A}_{1} \otimes \mathcal{A}_{2}$ ) from $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$. We refer to De Bruyn [7] for the precise details. For the purposes of the present paper, it suffices to know that a glued near polygon $\mathcal{A}$ of type $\mathcal{A}_{1} \otimes \mathcal{A}_{2}$ satisfies the following properties: (1) there exists a partition $T_{i}, i \in\{1,2\}$, of $\mathcal{A}$ in convex subpolygons isomorphic to $\mathcal{A}_{i}$, (2) every subpolygon of $T_{1}$ intersects every subpolygon of $T_{2}$ in a line, (3) every line of $\mathcal{A}$ is contained in a convex subpolygon of $T_{1} \cup T_{2}$, (4) $\mathcal{A}$ has diameter $\operatorname{diam}\left(\mathcal{A}_{1}\right)+\operatorname{diam}\left(\mathcal{A}_{2}\right)-1$. Every subpolygon of $T_{1} \cup T_{2}$ is classical in $\mathcal{A}$ and if $F$ and $F^{\prime}$ are two subpolygons of $T_{i}, i \in\{1,2\}$, then the map $F^{\prime} \rightarrow F, x \mapsto \pi_{F}(x)$ defines an isomorphism from $F^{\prime}$ to $F$.

Let $x$ denote an arbitrary point of $\mathcal{A}$ and let $F_{i}(x), i \in\{1,2\}$, denote the unique element of $T_{i}$ through $x$. Put $L(x):=F_{1}(x) \cap F_{2}(x)$. For every convex subspace $U$ of $\mathcal{A}$ through $x$, let $U_{i}, i \in\{1,2\}$, denote
the convex subspace $F_{i}(x) \cap U$ of $F_{i}(x)$. If $L(x) \subseteq U$, then $\operatorname{diam}(U)=$ $\operatorname{diam}\left(U_{1}\right)+\operatorname{diam}\left(U_{2}\right)-1$ and $U$ is a glued near polygon of type $U_{1} \otimes U_{2}$ in case that $U_{1}$ and $U_{2}$ have diameters at least 2. If $L(x) \cap U=\{x\}$, then $\operatorname{diam}(U)=\operatorname{diam}\left(U_{1}\right)+\operatorname{diam}\left(U_{2}\right)$ and $U \cong U_{1} \times U_{2}$ in case that $U_{1}$ and $U_{2}$ have diameters at least 1.

If $V_{1}$ and $V_{2}$ are two convex subspaces of $F_{1}(x)$ and $F_{2}(x)$, respectively, satisfying either $L(x)=V_{1} \cap V_{2}$ or $L(x) \cap V_{1}=L(x) \cap V_{2}=\{x\}$, then $U:=\left\langle V_{1}, V_{2}\right\rangle$ is a convex subspace of $\mathcal{A}$ through $x$ with $U_{1}=V_{1}$ and $U_{2}=V_{2}$.

The above facts have been proved in Section 3 of [7], where the convex subpolygons of glued near polygons were studied.

Proposition 5.9 Let $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ be two dense near polygons of diameter at least 2 and let $\mathcal{A}$ be a glued near polygon of type $\mathcal{A}_{1} \otimes \mathcal{A}_{2}$, then $\mathcal{A}$ satisfies property $\left(P_{\text {de }}\right)$ if and only if $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ satisfy property $\left(P_{d e}\right)$.

Proof. Put $n=\operatorname{diam}(\mathcal{A})$. Let $x$ denote an arbitrary point of $\mathcal{A}$, let $B$ denote an arbitrary convex sub- $(2 n-4)$-gon through $x$ and suppose $C$, $D$ and $E$ are three convex sub- $(2 n-2)$-gons through $B$. We will use the notations as introduced before this proposition. We distinguish two cases.
(i) Suppose that $L(x) \subseteq B$. Then we have $\operatorname{diam}\left(B_{1}\right)+\operatorname{diam}\left(B_{2}\right)=$ $\operatorname{diam}(B)+1=\operatorname{diam}\left(F_{1}(x)\right)+\operatorname{diam}\left(F_{2}(x)\right)-2$. If $\operatorname{diam}\left(B_{1}\right)=\operatorname{diam}\left(F_{1}(x)\right)-$ 1 and $\operatorname{diam}\left(B_{2}\right)=\operatorname{diam}\left(F_{2}(x)\right)-1$, then $B$ is contained in precisely two convex subpolygons of diameter $n-1$, namely $\left\langle F_{1}(x), B_{2}\right\rangle$ and $\left\langle B_{1}, F_{2}(x)\right\rangle$, a contradiction. Hence, there exists an $i \in\{1,2\}$ such that $\operatorname{diam}\left(B_{i}\right)=$ $\operatorname{diam}\left(F_{i}(x)\right)-2$ and $B_{3-i}=F_{3-i}(x)$. As in Proposition 5.8, one can show that $D$ and $E$ are adjacent vertices of $\Gamma(x, B, C)$ if and only if $D_{i}$ and $E_{i}$ are adjacent vertices of $\Gamma\left(x, B_{i}, C_{i}\right)$.
(ii) Suppose that $B_{1} \cap L(x)=\{x\}=B_{2} \cap L(x)$. Then we have diam $\left(B_{1}\right)+$ $\operatorname{diam}\left(B_{2}\right)=\operatorname{diam}(B)=\operatorname{diam}\left(F_{1}(x)\right)+\operatorname{diam}\left(F_{2}(x)\right)-3$. Hence, there exists an $i \in\{1,2\}$ such that $\operatorname{diam}\left(B_{i}\right)=\operatorname{diam}\left(F_{i}(x)\right)-2$ and $\operatorname{diam}\left(B_{3-i}\right)=$ $\operatorname{diam}\left(F_{3-i}(x)\right)-1$. Every quad intersecting $B$ in a point and $C, D$ and $E$ in lines is contained in a subpolygon of $T_{i}$. As before, one can reason that $D$ and $E$ are collinear points of $\Gamma(x, B, C)$ if and only if $D_{i}$ and $E_{i}$ are collinear points of $\Gamma\left(x, B_{i}, C_{i}\right)$.

The proposition now readily follows.

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