Some properties of the spinor Fourier transform

T. Batard and T. Raeymaekers

Abstract. In this paper, the theory of the spinor Fourier transform introduced in [Batard, T., Berthier, M., Saint-Jean, C., *Clifford-Fourier Transform for Color Image Processing*, Geometric Algebra Computing for Engineering and Computer Science (E. Bayro-Corrochano and G. Scheuermann Eds.), Springer Verlag, London, 2010, pp. 135-161] is further developed. While in the original paper, the transform was determined for vector-valued functions only, it now will be extended to functions taking values in the entire Clifford algebra. Next, two bases are determined under which this Fourier transform is diagonalizable. A main stumbling block for further applications, in particular concerning filter design in the Fourier domain, is the lack of a proper convolution theorem. This problem will be tackled in the final section of this paper.

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1. Introduction

During the last years, there has been an increased interest in hypercomplex Fourier transforms and their applications in various aspects of signal processing. The main idea behind these applications is the representation of a signal (say, a color image) as an element of a suitable Clifford algebra. Successful developments of the hypercomplex approach can be found in the work of Sangwine (e.g. [15]). Examples of Fourier transforms for which eigenfunctions are used to construct them, can be found in e.g. [1, 2, 3, 4, 11, 12, 13]. The main issue that hinders further development of applications is the lack of a suitable convolution theorem for such hypercomplex Fourier transforms. This paper however, is based on another generalization of Fourier transforms. In [7, 8, 9], a spinor Fourier transform was developed, that is suitable for color image spectral analysis. In this work, a color image is considered to be a vector-valued function. As the link with group representations is often important when dealing with Fourier transforms, the spinor Fourier transform was constructed using group morphisms from $\mathbb{R}^2$ to Spin(4). In fact, this transform can also be compared to the general geometric Fourier transform introduced by Bujack, Scheuermann and Hitzer in [5]. Unfortunately, the restriction to vector-valued functions obliges us to tackle this transform in a different way.

In this paper, we build further upon the theory of this spinor Fourier transform. We have 3 large aims.

- A_1: An extension of the domain of the spinor Fourier transform from vector-valued functions to functions taking values in the entire Clifford algebra.
- A_2: Construction of two bases of eigenfunctions, diagonalizing the spinor Fourier transform.
- A_3: Defining a convolution product for the spinor Fourier transform.

Before getting to these goals, we introduce the used notations in Section 2. Here, we also briefly recapitulate the construction of the spinor Fourier transform, done in [7], as we will define it slightly differently from the original article for calculational ease. The morphisms from $\mathbb{R}^2$ to Spin(4) are constructed, and it is proven that they are characterized by four real numbers and a special type of bivector. As these morphisms are important in our construction, the properties of these bivectors are crucial as well. In Section 3, we explain these properties and describe this set of special bivectors in detail.

In Section 4, we are finally able to define our spinor Fourier transform for vector-valued functions and we come to aim A_1, a suitable extension to general Clifford-valued functions. In this section, some calculation rules are also developed, that are useful for reaching our second goal A_2, determining two sets of functions which diagonalize the spinor Fourier transform in Section 5.

Finally, this brings us to the last section of this article, the determination of a convolution product (A_3). We base our ideas on those of Mustard [18] which have also been used in e.g. [4].

2. The set-up

Before we explain the construction of the spinor Fourier transform (SFT), let us introduce some notations that will be used (see e.g. [14, 17]). The m-dimensional real space is denoted by $\mathbb{R}^m$, with standard orthogonal basis $(e_1, \ldots, e_m)$. The real Clifford algebra $\mathbb{R}_{m,0}$ over $\mathbb{R}^m$ is governed by the relations $e_i e_j + e_j e_i = 2\delta_{ij}$ for all $1 \leq i, j \leq m$. We will denote a vector $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ by $\underline{x}$. Within the Clifford algebra, the set of k-vectors is defined as

$$\mathbb{R}_{m,0}^{(k)} = \mathrm{span}_{\mathbb{R}}\{e_{i_1} \cdots e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq m\}.$$

The subspace of $\mathbb{R}_{m,0}$ of 2-vectors (also called bivectors) that square to -1 is denoted by $\mathcal{S}^{(2)}_{m,0}$. The pseudoscalar $e_1e_2e_3e_4$ in $\mathbb{R}_{4,0}$ is denoted by I.
We will regularly speak about *unit bivectors*, so we need these objects to be well-defined. In order to do this, we will use the *Clifford inner product.*

Definition 1. *The conjugation $a \mapsto \overline{a}$ is defined on the basis elements $e_A = e_{i_1} \dots e_{i_h}$, $i_1 < \dots < i_h$ by means of*

$$\overline{e}_A = (-1)^h e_{i_h} \dots e_{i_1} = (-1)^{\frac{h(h+1)}{2}} e_A,$$

and it is then linearly extended:

$$\overline{(a_A e_A + a_B e_B)} = a_A \overline{e}_A + a_B \overline{e}_B,$$

for all $a_A, a_B \in \mathbb{R}$.

With this knowledge, we can define the Clifford inner product as follows:

Definition 2. *For two Clifford numbers $a, b \in \mathbb{R}_{m,0}$, we define the Clifford inner product $\langle \cdot, \cdot \rangle$ as follows:*

$$\langle a, b \rangle = [\overline{a}b]_0,$$

where $[\overline{a}b]_0$ is the scalar part of the product $\overline{a}b$.

Remark 1. The Clifford inner product of two bivectors $A, B \in \mathbb{R}^{(2)}_{m,0}$, with $A = \sum_{i<j} a_{ij} e_{ij}$ and $B = \sum_{i<j} b_{ij} e_{ij}$ is then given by

$$\langle A, B \rangle = \sum_{i<j} a_{ij} b_{ij}.$$

The Clifford inner product for vectors corresponds to the standard Euclidean inner product.

From this definition, we can obtain a norm on $\mathbb{R}_{m,0}$:

Definition 3. *For any $a = \sum_A a_A e_A \in \mathbb{R}_{m,0}$, the Clifford norm $|\cdot|$ is defined as follows:*

$$|a| = \sqrt{\langle a, a \rangle} = \sqrt{[\overline{a}a]_0} = \sqrt{\sum_A a_A^2}.$$

The exact definition of a unit bivector is then the following:

Definition 4. *A bivector A in $\mathbb{R}^{(2)}_{m,0}$ is a unit bivector if $|A| = 1$. We will denote the space of unit bivectors in $\mathbb{R}^{(2)}_{m,0}$ with $\mathbb{S}^{(2)}_{m,0}$.*

Remark 2. Note that the space of unit bivectors $\mathbb{S}^{(2)}_{m,0}$ is in general different from the space $\mathcal{S}^{(2)}_{m,0}$ of bivectors that square to -1. This will be proven in Section 3.

In order to construct the SFT, we make use of group morphisms. As we will make slightly different assumptions then the original article, we briefly repeat the construction procedure of the SFT. First of all, we have the following result which was proven in [7].

Proposition 1. *The differentiable group morphisms from* $\mathbb{R}^2$ *to* Spin(3) *are given by*

$$\tilde{\varphi} : (x_1, x_2) \mapsto e^{(x_1 y_1 + x_2 y_2)B}$$

where B *is an element of* $\mathbb{S}^{(2)}_{3,0}$, *the set of unit bivectors in* $\mathbb{R}_{3,0}$, *and* $y_1, y_2 \in \mathbb{R}$.

It can easily be shown that every τ in Spin(4) is of the form

$$\tau = u + Iv = (a + be_1e_2 + ce_2e_3 + de_3e_1) + I(a' + b'e_1e_2 + c'e_2e_3 + d'e_3e_1),$$

where the following relations hold:

$$u\bar{u} + v\bar{v} = 1, u\bar{v} + v\bar{u} = 0.$$

The morphism χ: Spin(4) $\rightarrow$ Spin(3)$\times$ Spin(3) with

$$\chi(u + Iv) = (u + v, u - v)$$

is an isomorphism, with inverse: $\chi^{-1}(a, b) = \frac{a+b}{2} + I\frac{a-b}{2}$.

Proposition 2. *The differentiable group morphisms from* $\mathbb{R}^2$ *to* Spin(4) *are the morphisms* $\tilde{\phi}$ *of the form*

$$\tilde{\phi} : (x_1, x_2) \rightarrow e^{\frac{1}{4}[x_1(y_1+z_1)+x_2(y_2+z_2)][B+C+I(B-C)]} \times e^{\frac{1}{4}[x_1(y_1-z_1)+x_2(y_2-z_2)][B-C+I(B+C)]}$$

with $y_1, y_2, z_1, z_2 \in \mathbb{R}$ *and* $B, C \in \mathbb{S}^{(2)}_{3,0}$.

The proof can be found in [7]. Proposition 2 is a more convenient form to describe group morphisms from $\mathbb{R}^2$ to Spin(4). We can write the expression more elegantly by defining the bivector $D \in \mathbb{R}_{4,0}$:

$$D = \frac{1}{2}(B + C + I(B - C)). \tag{1}$$

Also, we have that

$$ID = \frac{1}{2}(B - C + I(B + C))$$

Thus, the morphisms $\tilde{\phi}$ are parametrized by four real numbers y_1, y_2, z_1, z_2 and one bivector $D \in \mathbb{R}^{(2)}_{4,0}$, so $\tilde{\phi}$ may be written as

$$\tilde{\phi} = e^{\frac{1}{2}[(x_1(y_1+z_1)+x_2(y_2+z_2))D]} e^{\frac{1}{2}[(x_1(y_1-z_1)+x_2(y_2-z_2))ID]}$$

At the moment, the properties of the bivectors $D = \frac{1}{2}(B + C + I(B - C))$ are not clear. This will be the topic of the next section.

3. Some remarks on bivectors

First of all, we will prove that the space of unit bivectors $\mathbb{S}^{(2)}_{m,0}$ is in general not equal to the space $\mathcal{S}^{(2)}_{m,0}$ of bivectors that square to -1. Let us define the exterior product or wedge product of two vectors:

Definition 5. *The wedge product of two vectors* $a = \sum_{j=1}^m a_i e_i$ *and* $b = \sum_{j=1}^m b_i e_i$ *in* $\mathbb{R}_{m,0}$ *is defined as*

$$a \wedge b = \sum_{i<j} e_i e_j (a_i b_j - a_j b_i) = \frac{1}{2}(ab - ba)$$

Definition 6. *Any bivector that can be written as the wedge product of two vectors is called a blade.*

In this section, the aim is to prove the following proposition

Proposition 3. *The space* $\mathcal{S}_{4,0}^{(2)}$ *in* $\mathbb{R}_{4,0}^{(2)}$ *coincides with the space of bivectors of type* $\frac{1}{2}(B + C + I(B - C))$*, where* $B, C \in \mathbb{S}_{3,0}^{(2)}$*. Moreover,* $\mathcal{S}_{4,0}^{(2)} \subset \mathbb{S}_{4,0}^{(2)}$*.* $\mathcal{S}_{4,0}^{(2)}$ *is also exactly the set of unit blades in* $\mathbb{R}_{4,0}^{(2)}$*.*

From e.g. [14], we know that bivectors in $\mathbb{R}_{4,0}^{(2)}$ can always be written as the sum of at most two blades, thus we can group bivectors in $\mathbb{R}_{4,0}^{(2)}$ in two categories:

- **Type** 1: bivectors that can be written as a blade: $a \wedge b$
- **Type** 2: bivectors of the form $a \wedge b + c \wedge d$, the sum of two blades, that cannot be written as a single blade.

Bivectors in $\mathbb{R}_{3,0}^{(2)}$ on the other hand are always of Type 1. We have the following lemmas:

Lemma 1. *A blade* $a \wedge b \in \mathbb{R}_{m,0}^{(2)}$ *always squares to a negative real number.*

Proof. Without loss of generality, we may assume that a and b are orthogonal vectors. Then

$$(a \wedge b)^2 = abab = -a^2 b^2 = -|a|^2|b|^2,$$

which finishes the proof. □

Corollary 1. *If* a *and* b *are orthogonal unit vectors in* $\mathbb{R}_{m,0}^{(2)}$*, then* $a \wedge b \in \mathcal{S}_{m,0}^{(2)}$*.*

Corollary 2. *We have that* $\mathbb{S}_{3,0}^{(2)} = \mathcal{S}_{3,0}^{(2)}$*.*

Opposite to Lemma 1, we have

Lemma 2. *A Type 2 bivector* $b_1 \wedge b_2 + b_3 \wedge b_4 \in \mathbb{R}_{4,0}^{(2)}$ *never squares to a scalar.*

Proof. Without any loss of generality, we may assume that $\{b_1, b_2, b_3, b_4\}$ is an orthogonal basis of $\mathbb{R}^4$. We then have that

$$\begin{aligned}
&(b_1 \wedge b_2 + b_3 \wedge b_4)^2 \\
&\quad = \quad (b_1 \wedge b_2)^2 + (b_3 \wedge b_4)^2 + (b_1 \wedge b_2)(b_3 \wedge b_4) + (b_3 \wedge b_4)(b_1 \wedge b_2) \\
&\quad = \quad -|b_1|^2|b_2|^2 - |b_3|^2|b_4|^2 + 2b_1 \wedge b_2 \wedge b_3 \wedge b_4.
\end{aligned}$$

The last term is a 4-vector, and can therefore never be scalar. □

Lemma 3. *The bivector* D *defined in* (1) *is an element of* $\mathcal{S}_{4,0}^{(2)}$*.*

Proof. Making use of the facts that $I^2 = 1$, $B^2 = C^2 = -1$ and B and C commute with the pseudoscalar I, a straightforward calculation gives us that $D^2 = -1$. $\square$

From these two lemmas, we find that the set of bivectors of the form $\frac{1}{2}(B + C + I(B - C))$, where B and C are unit bivectors in $\mathbb{R}^{(2)}_{3,0}$, must all be of Type 1, due to Lemma 2 and Lemma 3. However, we still need to prove that both sets are actually equal.

Lemma 4. *Any bivector $A \in \mathbb{R}^{(2)}_{4,0}$ can be written in a unique way as $D + IF$, with $D, F \in \mathbb{R}^{(2)}_{3,0}$, and I the pseudoscalar in $\mathbb{R}_{4,0}$.*

Proof. Any bivector A in $\mathbb{R}_{4,0}$ is of the form

$$A = a_{12}e_1e_2 + a_{13}e_1e_3 + a_{14}e_1e_4 + a_{23}e_2e_3 + a_{24}e_2e_4 + a_{34}e_3e_4.$$

Since $D \in \mathbb{R}^{(2)}_{3,0}$ cannot contain any term with e_4, it must hold that D and F are given by

$$D = a_{12}e_1e_2 + a_{13}e_1e_3 + a_{23}e_2e_3$$

and

$$F = -a_{34}e_1e_2 + a_{24}e_1e_3 - a_{14}e_2e_3.$$

Each coefficient a_{ij} corresponds to a different basis-bivector, making this decomposition unique. $\square$

Note that the lemma above gives the explicit isomorphism between $\mathfrak{so}(4)$ and $\mathfrak{so}(3) \times \mathfrak{so}(3)$, or equivalently $\mathfrak{su}(2) \times \mathfrak{su}(2)$. Regarding the special type of bivectors that are used in this article, namely the ones of the form $\frac{1}{2}(B + C + I(B - C))$, we also have the following lemma.

Lemma 5. *Any unit bivector $D + IF$ in $\mathbb{R}^{(2)}_{4,0}$, with $D, F \in \mathbb{R}^{(2)}_{3,0}$ can be written as $\frac{1}{2}(B + C + I(B - C))$ with $B, C \in \mathbb{R}^{(2)}_{3,0}$. Moreover, B and C themselves are unit bivectors if and only if $\langle D, F\rangle = 0$.*

Proof. The first part of the lemma is fairly easy to see, by simply setting $B = D + F$ and $C = D - F$, or equivalently $D = \frac{B+C}{2}$ and $F = \frac{B-C}{2}$. Now, suppose

$$D = d_1e_2e_3 + d_2e_1e_3 + d_3e_1e_2$$

and

$$F = f_1e_2e_3 + f_2e_1e_3 + f_3e_1e_2.$$

Then $D + IF$ is a unit bivector if

$$|D + IF|^2 = d_1^2 + d_2^2 + d_3^2 + f_1^2 + f_2^2 + f_3^2 = 1.$$

The norms of B and C are given by

$$|B|^2 = (d_1 + f_1)^2 + (d_2 + f_2)^2 + (d_3 + f_3)^2 = 1 + 2(d_1f_1 + d_2f_2 + d_3f_3)$$

and

$$|C|^2 = (d_1 - f_1)^2 + (d_2 - f_2)^2 + (d_3 - f_3)^2 = 1 - 2(d_1f_1 + d_2f_2 + d_3f_3)$$

Both expressions equal 1 if and only if $d_1 f_1 + d_2 f_2 + d_3 f_3 = 0$. This proves the lemma. □

The previous lemma also shows that there exist unit bivectors in $\mathbb{R}^{(2)}_{4,0}$ for which B and C cannot be chosen unit. For instance, take $F + IF$, for any $F \in \mathbb{R}^{(2)}_{3,0}$, with $f_1^2 + f_2^2 + f_3^2 = \frac{1}{2}$.

Lemma 6. *For any blade (not necessarily unit) $a \wedge b = D + IF \in \mathbb{R}^{(2)}_{4,0}$, we have that $\langle D, F\rangle = 0$.*

Proof. Set

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$$

and

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4.$$

Then we have that

$$D = \frac{1}{2}((a_1 b_2 - a_2 b_1) e_1 e_2 + (a_1 b_3 - a_3 b_1) e_1 e_3 + (a_2 b_3 - a_3 b_2) e_2 e_3)$$

and

$$F = \frac{1}{2}(-(a_3 b_4 - a_4 b_3) e_1 e_2 + (a_2 b_4 - a_4 b_2) e_1 e_3 - (a_1 b_4 - a_4 b_1) e_2 e_3).$$

If we calculate the inner product of these two bivectors, we notice that this always equals 0. □

These last two lemmas finish the proof of Proposition 3.

4. Properties of the spinor Fourier transform

With the knowledge of the previous section, we can define the SFT for vector-valued functions. The idea is to define it as

$$\widetilde{\mathcal{F}}(f(\underline{x})) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{\phi}(\underline{x}) f(\underline{x}) \tilde{\phi}(-\underline{x}) d\underline{x}.$$

Definition 7. *Let $f : \mathbb{R}^2 \to \mathbb{R}^{(1)}_{4,0}$ be a vector-valued function in the Schwartz space $\mathcal{S}(\mathbb{R}^2, \mathbb{R}) \otimes \mathbb{R}_{4,0}$. The SFT is then defined as follows for each $D \in \mathcal{S}^{(2)}_{4,0}$:*

$$\widetilde{\mathcal{F}}_D(f) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\frac{1}{2}[(x_1(y_1+z_1)+x_2(y_2+z_2))D]} e^{\frac{1}{2}[(x_1(y_1-z_1)+x_2(y_2-z_2))ID]} f(x_1, x_2)$$
$$\times\, e^{-\frac{1}{2}[(x_1(y_1+z_1)+x_2(y_2+z_2))D]} e^{-\frac{1}{2}[(x_1(y_1-z_1)+x_2(y_2-z_2))ID]} dx_1 dx_2.$$

Throughout this paper, we restrict ourselves to the specific case of $z_1 = z_2 = 0$:

$$\mathcal{F}_D(f) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{\frac{1}{2}(x_1 y_1 + x_2 y_2) D} e^{\frac{1}{2}(x_1 y_1 + x_2 y_2) ID} f(x_1, x_2)$$
$$\times\, e^{-\frac{1}{2}(x_1 y_1 + x_2 y_2) D} e^{-\frac{1}{2}(x_1 y_1 + x_2 y_2) ID} dx_1 dx_2.$$

Remark 3. Note that this definition only makes sense for vector valued functions. if f is, say, scalar, then the transform reduces to the integral of f. The normalization factor is different from the one in [7], but it will prove to be more convenient in our further calculations. We will look at this last, simplified definition a bit closer and determine how it acts on a function f. In order for the SFT to be well-defined, we assume from now on that f is an element of the Schwartz space $\mathcal{S}(\mathbb{R}^2, \mathbb{R}_{4,0}) := \mathcal{S}(\mathbb{R}^2, \mathbb{R}) \otimes \mathbb{R}_{4,0}$.

Recall aim A_1 of this paper, finding a way to generalize the SFT to functions taking values in the entire Clifford algebra. Therefore, we introduce the following notations. For each $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{R}_{m,0}^{(k)})$, set

$$f^+ = \frac{1}{2}(f - DfD) \qquad \text{and} \qquad f^- = \frac{1}{2}(f + DfD).$$

Let $[\cdot,\cdot]$ be the commutator and $\{\cdot,\cdot\}$ the anticommutator. If k is *odd*, we have the relations

$$[D, f^+] = 0, \qquad \{D, f^-\} = 0, \qquad \{ID, f^+\} = 0 \qquad \text{and} \qquad [ID, f^-] = 0. \tag{2}$$

On the other hand, if k is *even*, we get

$$[D, f^+] = 0, \qquad \{D, f^-\} = 0, \qquad [ID, f^+] = 0 \qquad \text{and} \qquad \{ID, f^-\} = 0. \tag{3}$$

These relations can easily be calculated using the properties that $D^2 = (ID)^2 = -1$, where I is again the pseudoscalar defined in Section 2. This pseudoscalar (anti-)commutes with $f^\pm$, depending on the parity of k. Obviously, we have that $f = f^+ + f^-$. Consequently, the SFT for vector-valued functions can be rewritten as

$$\begin{aligned} \mathcal{F}_D(f) &= \mathcal{F}_D(f^+) + \mathcal{F}_D(f^-) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)ID} f^+(x_1,x_2)dx_1dx_2 \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)D} f^-(x_1,x_2)dx_1dx_2. \end{aligned}$$

The latter expression does not trivialize when applying to $\mathbb{R}_{4,0}$-valued functions, so we can use it to extend the SFT to the entire Clifford algebra.

Definition 8. *For any function $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{R}_{4,0})$, and any bivector $D \in \mathcal{S}_{4,0}^{(2)}$, we define the SFT $\mathcal{F}_D$ as*

$$\mathcal{F}_D(f) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)ID} f^+(x_1,x_2)dx_1dx_2 + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)D} f^-(x_1,x_2)dx_1dx_2.$$

Notice that $e_iI = -Ie_i$, hence the pseudoscalar I commutes with all bivectors D. Furthermore, we have that

$$e^{(x_1y_1+x_2y_2)D} = \sum_{k=0}^{\infty} \frac{(x_1y_1+x_2y_2)^k D^k}{k!}.$$

The derivative with respect to x_1 then equals

$$\partial_{x_1} e^{(x_1y_1+x_2y_2)D} = y_1 D e^{(x_1y_1+x_2y_2)D}.$$

This means we can derive the exponential function as we would expect to. Set $g = g^+ + g^-$, where

$$g^+ := e^{(x_1y_1+x_2y_2)ID} f^+ \qquad \text{and} \qquad g^- := e^{(x_1y_1+x_2y_2)D} f^-.$$

Then, since D and ID commute,

$$\partial_{x_1} g = y_1 ID g^+ + y_1 D g^- + e^{(x_1y_1+x_2y_2)ID} \partial_{x_1} f^+ + e^{(x_1y_1+x_2y_2)D} \partial_{x_1} f^-.$$

Provided f is a function in the Schwartz space, we get

$$0 = \int_{\mathbb{R}^2} \partial_{x_1} g dx_1 dx_2 = y_1 ID \mathcal{F}_D(f^+) + y_1 D \mathcal{F}_D(f^-) + \mathcal{F}_D(\partial_{x_1} f)$$

or

$$\mathcal{F}_D(\partial_{x_1} f) = -y_1 \left(ID\mathcal{F}_D(f^+) + D\mathcal{F}_D(f^-) \right).$$

Analogously,

$$\mathcal{F}_D(\partial_{x_2} f) = -y_2 \left(ID\mathcal{F}_D(f^+) + D\mathcal{F}_D(f^-) \right).$$

We also have that

$$\partial_{y_1} g^+ = IDx_1 g^+ \qquad \text{and} \qquad \partial_{y_1} g^- = Dx_1 g^-$$

or after multiplication with ID and D respectively,

$$x_1 g^+ = -ID\partial_{y_1} g^+ \qquad \text{and} \qquad x_1 g^- = -D\partial_{y_1} g^-.$$

Integrating over x_1 and x_2 and adding both equations yields

$$\mathcal{F}_D(x_1 f) = -ID\partial_{y_1} \left(\mathcal{F}_D(f^+) \right) - D\partial_{y_1} \left(\mathcal{F}_D(f^-) \right). \tag{4}$$

Analogously one obtains

$$\mathcal{F}_D(x_2 f) = -ID\partial_{y_2} \left(\mathcal{F}_D(f^+) \right) - D\partial_{y_2} \left(\mathcal{F}_D(f^-) \right). \tag{5}$$

Since $\mathcal{F}_D$ acts independently on f^+ and f^-, we get from the above equations

$$\mathcal{F}_D\left(\partial^2_{x_1} f\right) = -y_1^2 \mathcal{F}_D(f) \text{ and } \mathcal{F}_D\left(\partial^2_{x_2} f\right) = -y_2^2 \mathcal{F}_D(f). \tag{6}$$

Analogously we find from the dual formulas that

$$\mathcal{F}_D\left(x_1^2 f\right) = -\partial^2_{y_1} \mathcal{F}_D(f) \text{ and } \mathcal{F}_D\left(x_2^2 f\right) = -\partial^2_{y_2} \mathcal{F}_D(f). \tag{7}$$

Let us introduce the notations $|\underline{x}|^2 = x_1^2 + x_2^2$ and $\Delta_{\underline{x}} = \partial^2_{x_1} + \partial^2_{x_2}$ for any vector $\underline{x}$, and similar notations for $\underline{y}$. Taking the sum of the equations in (6) gives us the Fourier transform of $\Delta_{\underline{x}} f$.

$$\mathcal{F}_D\left(\Delta_{\underline{x}} f\right) = -|\underline{y}|^2 \mathcal{F}_D(f). \tag{8}$$

Similar calculations with (7) also give us the dual formula

$$\mathcal{F}_D\left(|\underline{x}|^2 f\right) = -\Delta_{\underline{y}} \mathcal{F}_D(f). \tag{9}$$

Also, we have the relations

$$\begin{aligned} \mathcal{F}_D\left((x_1 - \partial_{x_1}) f\right) &= ID(y_1 - \partial_{y_1}) \mathcal{F}_D(f^+) + D(y_1 - \partial_{y_1}) \mathcal{F}_D(f^-) \\ \mathcal{F}_D\left((x_2 - \partial_{x_2}) f\right) &= ID(y_2 - \partial_{y_2}) \mathcal{F}_D(f^+) + D(y_2 - \partial_{y_2}) \mathcal{F}_D(f^-), \end{aligned} \tag{10}$$

which will be useful in the next section.

5. Eigenfunctions for the spinor Fourier transform

In this section, we tackle aim A_2, finding two bases of eigenfunctions for the SFT. We start with the cartesian basis.

5.1. The cartesian basis

It is well-known that the space of Schwartz functions $\mathcal{S}(\mathbb{R})$ has a basis formed by the 1-dimensional Hermite functions, defined as follows (see [20]):

Definition 9. *The one-dimensional Hermite functions are defined as follows, for $k \in \mathbb{N}$*

$$\psi_k(x) = (x - \partial_x)^k e^{-x^2/2} = H_k(x)e^{-x^2/2}.$$

Then a basis for $\mathcal{S}(\mathbb{R}^2) \otimes \mathbb{R}_{4,0}$ is given by the functions

$$\psi_{k,l}(x_1, x_2) = \psi_k(x_1)\psi_l(x_2),$$

for all $k, l \in \mathbb{N}$, with coefficients in $\mathbb{R}_{4,0}$. The next step in our reasoning is to determine the Fourier transforms of these basis functions. First note that for each real number c, we have

$$\begin{aligned} e^{cD} &= 1 + \frac{cD}{1!} + \frac{c^2D^2}{2!} + \frac{c^3D^3}{3!} + \cdots \\ &= \left(1 - \frac{c^2}{2!} + \frac{c^4}{4!} - \cdots\right) + D\left(\frac{c}{1!} - \frac{c^3}{3!} + \frac{c^5}{5!} - \cdots\right) \\ &= \cos(c) + D\sin(c). \end{aligned}$$

For each element $a = a^+ + a^- \in \mathbb{R}_{4,0}$, we have

$$\begin{aligned} \mathcal{F}_D(a\psi_{0,0}) = &\frac{1}{2\pi}\int_{\mathbb{R}^2} (\cos(x_1y_1+x_2y_2)+ID\sin(x_1y_1+x_2y_2))a^+e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\,dx_1dx_2 \\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^2} (\cos(x_1y_1 + x_2y_2) + D\sin(x_1y_1 + x_2y_2))a^-e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\,dx_1dx_2. \end{aligned}$$

We calculate the first integral:

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}^2} (\cos(x_1y_1 + x_2y_2) + ID\sin(x_1y_1 + x_2y_2))a^+e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\,dx_1dx_2 \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} ((\cos(x_1y_1)\cos(x_2y_2) - \sin(x_1y_1)\sin(x_2y_2)) \\ &\qquad +ID(\sin(x_1y_1)\cos(x_2y_2) - \cos(x_1y_1)\sin(x_2y_2)))\,a^+e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\,dx_1dx_2. \end{aligned}$$

The sine function is an odd function, so only the first term remains. This equals

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos(x_1y_1)e^{-\frac{x_1^2}{2}}\left(\int_{-\infty}^{+\infty} \cos(x_2y_2)e^{-\frac{x_2^2}{2}}a^+dx_2\right)dx_1 \\ &= \frac{1}{2\pi}\sqrt{2\pi}e^{-\frac{y_1^2}{2}}\sqrt{2\pi}e^{-\frac{y_2^2}{2}}a^+ = a^+\psi_{0,0}. \end{aligned}$$

Similar calculations can be done for the second integral, resulting in the equality

$$\mathcal{F}_D(a\psi_{0,0}) = a\psi_{0,0}.$$

In general, we get that

$$\begin{aligned}\mathcal{F}_D(a\psi_{k,l}) &= \mathcal{F}_D(a^+\psi_{k,l}) + \mathcal{F}_D(a^-\psi_{k,l}) \\ &= \mathcal{F}_D\left((x_1-\partial_{x_1})^k(x_2-\partial_{x_2})^l a^+ e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\right) \\ &\quad +\mathcal{F}_D\left((x_1-\partial_{x_1})^k(x_2-\partial_{x_2})^l a^- e^{-\frac{x_1^2}{2}}e^{-\frac{x_2^2}{2}}\right).\end{aligned}$$

Using formula (10) repeatedly, we find that

$$\mathcal{F}_D(a\psi_{k,l}(x_1,x_2)) = (ID)^{k+l}a^+\psi_{k,l}(y_1,y_2) + (D)^{k+l}a^-\psi_{k,l}(y_1,y_2).$$

We can summarize our results in the following proposition.

Proposition 4. *The SFT acts on the cartesian basis elements $\psi_{k,l}$ as*

$$\mathcal{F}_D(a\psi_{k,l}(x_1,x_2)) = (ID)^{k+l}a^+\psi_{k,l}(y_1,y_2) + (D)^{k+l}a^-\psi_{k,l}(y_1,y_2).$$

where $a \in \mathbb{R}_{4,0}$.

Since $D^2 = (ID)^2 = -1$, this means that we have 4 possible situations:

- $k+l = 0 \mod 4$:
$$\mathcal{F}_D(a\psi_{k,l}) = a\psi_{k,l}$$
- $k+l = 1 \mod 4$:
$$\mathcal{F}_D(a\psi_{k,l}) = IDa^+\psi_{k,l} + Da^-\psi_{k,l}$$
- $k+l = 2 \mod 4$:
$$\mathcal{F}_D(a\psi_{k,l}) = -a\psi_{k,l}$$
- $k+l = 3 \mod 4$:
$$\mathcal{F}_D(a\psi_{k,l}) = -IDa^+\psi_{k,l} - Da^-\psi_{k,l}.$$

It follows that

$$\mathcal{F}_D^2(a\psi_{k,l}) = (-1)^{k+l}a\psi_{k,l} \qquad \text{and} \qquad \mathcal{F}_D^4(a\psi_{k,l}) = a\psi_{k,l}.$$

This shows that applying the SFT four times equals the identity, as is also the case for the classical FT. Now it also becomes easy to formulate the inversion theorem for the SFT.

Theorem 1. *The inverse of the SFT is given by*

$$\mathcal{F}_D^{-1}(f) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-(x_1y_1+x_2y_2)ID}f^+(x_1,x_2)dx_1dx_2 + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-(x_1y_1+x_2y_2)D}f^-(x_1,x_2)dx_1dx_2.$$

Proof. A straightforward calculation gives us that

$$\begin{aligned}
&\mathcal{F}_D^{-1}(\mathcal{F}_D(f^-))\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(u_1y_1+u_2y_2)D}e^{(x_1y_1+x_2y_2)D}f^-(x_1,x_2)dx_1dx_2dy_1dy_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{((x_1-u_1)y_1+(x_2-u_2)y_2)D}f^-(x_1,x_2)dx_1dx_2dy_1dy_2\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (\cos((x_1-u_1)y_1)\cos((x_2-u_2)y_2)f^-(x_1,x_2)dx_1dx_2dy_1dy_2\\
&= \int_{\mathbb{R}^2}\delta(u_1-x_1)\delta(u_2-x_2)f^-(x_1,x_2)dx_1dx_2\\
&= f^-(u_1,u_2)
\end{aligned}$$

Analogously, we find that

$$\mathcal{F}_D^{-1}(\mathcal{F}_D(f^+)) = f^+(u_1,u_2).$$

□

5.2. The spherical basis

We also take a look at a second basis for $\mathcal{S}(\mathbb{R}^2)\otimes\mathbb{R}_{4,0}$. Define the functions $\psi_{j,k,\ell}$ by

$$\psi_{j,k,\ell} := L_j^{\frac{m}{2}+k-1}(|\underline{x}|^2)H_k^{(\ell)}e^{-|\underline{x}|^2/2} \tag{11}$$

where $j,k\in\mathbb{N}$, $\{H_k^{(\ell)}\in\mathcal{H}_k : \ell=1,\ldots,\dim\mathcal{H}_k\}$ is a basis for the space $\mathcal{H}_k = \mathcal{P}_k\cap\ker\Delta_{\underline{x}}$ of solid spherical harmonics of degree k, and L_j^α are the Laguerre polynomials. The set $\{\psi_{j,k,\ell}\}$ forms a basis of $\mathcal{S}(\mathbb{R}^m)$, see e.g. [10]. This reduces in our situation ($m=2$) to

$$\psi_{j,k,\ell} := L_j^k(x_1^2+x_2^2)H_k^{(\ell)}e^{-(x_1^2+x_2^2)/2} \tag{12}$$

where $j,k\in\mathbb{N}$, $\{H_k^{(\ell)}\in\mathcal{H}_k : \ell = 1 \text{ or } 2\}$ is a basis for the, now 2 dimensional, space $\mathcal{H}_k$. Taking coefficients in $\mathbb{R}_{4,0}$, this set forms a basis (as a right module) for $\mathcal{S}(\mathbb{R}^2)\otimes\mathbb{R}_{4,0}$.

Computing the SFT of the spherical basis is a bit more involved than in the case of the cartesian basis. First of all, it would in principle be possible to expand the spherical basis into the cartesian basis, and use the result of the previous section. However, the change of basis is quite involved (see e.g. [16]). Therefore, we follow a different strategy, based again on an explicit computation. To that aim, we will need the following derivation formula

$$H_k(\partial_{x_1},\partial_{x_2})e^{-(x_1^2+x_2^2)/2} = (-1)^kH_k(x_1,x_2)e^{-(x_1^2+x_2^2)/2}, \qquad \forall H_k\in\mathcal{H}_k \tag{13}$$

which is well-known (see e.g. [19]) and holds in arbitrary dimension. We will however only need the two-dimensional version.

Proposition 5. *The SFT acts on the spherical basis $\{\psi_{j,k,\ell}\}$ as*

$$\mathcal{F}_D(a\psi_{j,k,\ell}) = (-1)^j\left((D)^ka^- + (ID)^ka^+\right)\psi_{j,k,\ell}$$

when $k>0$ and where $a\in\mathbb{R}_{4,0}$.

Proof. We first deal with the case of $\psi_{0,k,\ell}$. Notice that $H_k^{(\ell)}$ can be expanded as

$$H_k^{(\ell)} = \sum_{p=0}^{k} \alpha_p^k x_1^{k-p} x_2^p$$

for suitably chosen constants $\alpha_p^k \in \mathbb{R}$. This expansion allows us to compute the SFT as

$$\begin{aligned}
\mathcal{F}_D(a^+\psi_{0,k,\ell}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)ID} a^+\psi_{0,k,\ell}(x_1,x_2)dx_1dx_2 \\
&= \frac{1}{2\pi}\sum_{p=0}^{k}\alpha_p^k \int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)ID} a^+ x_1^{k-p}x_2^p e^{-(x_1^2+x_2^2)/2}dx_1dx_2 \\
&= \sum_{p=0}^{k}\alpha_p^k \mathcal{F}_D\left(a^+x_1^{k-p}x_2^pe^{-(x_1^2+x_2^2)/2}\right).
\end{aligned}$$

Using (4) and (5) we subsequently obtain

$$\begin{aligned}
\mathcal{F}_D\left(a^+x_1^{k-p}x_2^pe^{-(x_1^2+x_2^2)/2}\right) &= (-ID)^k\partial_{y_1}^{k-p}\partial_{y_2}^p\mathcal{F}_D\left(a^+e^{-(x_1^2+x_2^2)/2}\right) \\
&= (-ID)^k\partial_{y_1}^{k-p}\partial_{y_2}^p a^+e^{-(y_1^2+y_2^2)/2}.
\end{aligned}$$

Consequently, using (13) we find

$$\begin{aligned}
\mathcal{F}_D\left(a^+\psi_{0,k,\ell}\right) &= \sum_{p=0}^{k}\alpha_p^k\mathcal{F}_D\left(ax_1^{k-p}x_2^pe^{-(x_1^2+x_2^2)/2}\right) \\
&= \sum_{p=0}^{k}\alpha_p^k(-1)^k(ID)^k\partial_{y_1}^{k-p}\partial_{y_2}^p a^+e^{-(y_1^2+y_2^2)/2} \\
&= \sum_{p=0}^{k}\alpha_p^k(ID)^k{y_1}^{k-p}{y_2}^p a^+e^{-(y_1^2+y_2^2)/2} \\
&= (ID)^k a^+\psi_{0,k,\ell}.
\end{aligned}$$

Analogously, we get

$$\mathcal{F}_D\left(a^-\psi_{0,k,\ell}\right) = (D)^k a^-\psi_{0,k,\ell}$$

Following [10], we observe that

$$\begin{aligned}
\psi_{j,k,\ell} &= L_j^k(x_1^2+x_2^2)H_k^{(\ell)}e^{-|\underline{x}|^2/2} \\
&= c_{j,k,\ell}\left(\Delta_{\underline{x}} + |\underline{x}|^2 - \frac{1}{2}[\Delta_{\underline{x}}, |\underline{x}|^2]\right)^j H_k^{(\ell)}e^{-(x^2+y^2)/2}
\end{aligned}$$

where $c_{j,k,\ell}$ is a real constant and $\Delta_{\underline{x}} = \partial_{x_1}^2 + \partial_{x_2}^2$. Hence, using formulas (8) and (9) we may compute

$$\begin{aligned}\mathcal{F}_D(a^+\psi_{j,k,\ell}) &= c_{j,k,\ell}\mathcal{F}_D\left(\left(\Delta_{\underline{x}} + |\underline{x}|^2 - \frac{1}{2}[\Delta_{\underline{x}}, |\underline{x}|^2]\right)^j a^+\psi_{0,k,\ell}\right)\\ &= (-1)^j c_{j,k,\ell}\left(\Delta_{\underline{y}} + |\underline{y}|^2 - \frac{1}{2}[\Delta_{\underline{y}}, |\underline{y}|^2]\right)^j \mathcal{F}_D\left(a^+\psi_{0,k,\ell}\right)\\ &= (-1)^j (ID)^k a^+\psi_{0,k,\ell}.\end{aligned}$$

Analogously, we have that

$$\mathcal{F}_D(a^-\psi_{j,k,\ell}) = (-1)^j (D)^k a^-\psi_{0,k,\ell}.$$

Combining both results then completes the proof. □

Since $D^2 = -1$, this means that we have again 4 possible situations:

- $k = 0 \mod 4$:
$$\mathcal{F}_D(a\psi_{j,k,l}) = (-1)^j a\psi_{j,k,l}$$
- $k = 1 \mod 4$:
$$\mathcal{F}_D(a\psi_{j,k,l}) = (-1)^j (IDa^+ + Da^-)\psi_{j,k,l}$$
- $k = 2 \mod 4$:
$$\mathcal{F}_D(a\psi_{j,k,l}) = (-1)^{j+1} a\psi_{j,k,l}$$
- $k = 3 \mod 4$:
$$\mathcal{F}_D(a\psi_{j,k,l}) = (-1)^{j+1} (IDa^+ + Da^-)\psi_{j,k,l}$$

Remark 4. The previous result is quite interesting: it shows that both the cartesian and the spherical basis are well-behaved under the SFT. This is rather surprising, as in all the other examples of hypercomplex Fourier transforms (see e.g. [4]), the transform is only diagonalized by one basis and not by both.

6. Mustard convolution product

In this section, we take a look at a possible convolution product for the SFT. The definition is based on the observation that in the classical case, the following interaction between the convolution and the Fourier transform holds:

$$\mathcal{F}(f * g) = (2\pi)^{m/2}\mathcal{F}(f)\mathcal{F}(g).$$

In order to keep this property, we define the Mustard convolution product as

$$f \circ g = 2\pi\mathcal{F}_D^{-1}\left(\mathcal{F}_D(f)\mathcal{F}_D(g)\right),$$

a technique that was also used in [4, 18].

Remark 5. Remark that aim A_3, the realization of a convolution product is the main reason for aim A_1 of this article. With the original definition of the SFT, the above definition of Mustard convolution product is not possible as the product of two vector-valued functions is no longer vector-valued.

In this section, we will explicitly calculate what this convolution product looks like. We have that

$$f \circ g = 2\pi \mathcal{F}_D^{-1}\left(\mathcal{F}_D(f^+)\mathcal{F}_D(g^+) + \mathcal{F}_D(f^+)\mathcal{F}_D(g^-) \right. \\ \left. + \mathcal{F}_D(f^-)\mathcal{F}_D(g^+) + \mathcal{F}_D(f^-)\mathcal{F}_D(g^-)\right). \quad (14)$$

Because the SFT is defined as a piecewise action, we have to develop a multiplication table. Take two $\mathbb{R}_{4,0}$-valued functions, $f = f^+ + f^-$ and $g = g^+ + g^-$. We have four possible combinations.

$$\begin{aligned} f^+g^+ &= \frac{1}{4}(f - DfD)(g - DgD) \\ &= \frac{1}{4}(fg - DfgD - fDgD - DfDg) \\ &= (fg)^+ - (DfDg)^+ \end{aligned}$$

$$\begin{aligned} f^-g^- &= \frac{1}{4}(f + DfD)(g + DgD) \\ &= \frac{1}{4}(fg - DfgD + fDgD + DfDg) \\ &= (fg)^+ + (DfDg)^+ \end{aligned}$$

$$\begin{aligned} f^+g^- &= \frac{1}{4}(f - DfD)(g + DgD) \\ &= \frac{1}{4}(fg + DfgD + fDgD - DfDg) \\ &= (fg)^- + (fDgD)^- \end{aligned}$$

$$\begin{aligned} f^-g^+ &= \frac{1}{4}(f + DfD)(g - DgD) \\ &= \frac{1}{4}(fg + DfgD - fDgD + DfDg) \\ &= (fg)^- + (DfDg)^- \end{aligned}$$

Hence, symbolically, we get a multiplication table of the form

	+	−
+	+	−
−	−	+

which we can use to determine how $\mathcal{F}_D^{-1}$ acts on the product in (14).

Remark 6. In the following calculations, we will make use of the (anti-)commutation relations (2) and (3). That is why, for the remainder of this section, we split a $\mathbb{R}_{4,0}$-valued function $f = f_o + f_e$, whereby f_o (resp. f_e) is the sum of the k-vectors where k is odd (resp. even). One can easily calculate that for any f, $(f_o)^+ = (f^+)_o$, $(f_e)^+ = (f^+)_e$, $(f_o)^- = (f^-)_o$ and $(f_e)^- = (f^-)_e$, so without loss of generality, we can decompose $f = f_e^+ + f_o^+ + f_e^- + f_o^-$.

If f_o is a k-vector with k odd, we get that

$$\begin{aligned}
f_o^+ \circ g^+ &= 2\pi\mathcal{F}_D^{-1}\left(\mathcal{F}_D(f_o^+)\mathcal{F}_D(g^+)\right) \\
&= \frac{1}{2\pi}\mathcal{F}_D^{-1}\left(\int_{\mathbb{R}^2} e^{(x_1y_1+x_2y_2)ID} f_o^+(\underline{x})d\underline{x}\int_{\mathbb{R}^2} e^{(u_1y_1+u_2y_2)ID} g^+(\underline{u})d\underline{u}\right) \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{-(y_1v_1+y_2v_2)ID}e^{(x_1y_1+x_2y_2)ID}e^{-(u_1y_1+u_2y_2)ID}f_o^+(\underline{x})g^+(\underline{u})d\underline{x}d\underline{y}d\underline{u} \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{(y_1(x_1-u_1-v_1)+y_2(x_2-u_2-v_2))ID}f_o^+(\underline{x})g^+(\underline{u})d\underline{x}d\underline{y}d\underline{u} \\
&= \int_{\mathbb{R}^4}\delta(u_1+v_1-x_1)\delta(u_2+v_2-x_2)f_o^+(\underline{x})g^+(\underline{u})d\underline{x}d\underline{u} \\
&= \int_{\mathbb{R}^2} f_o^+(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u}.
\end{aligned}$$

Similarly, if f_e is a k-vector with k even, we get

$$f_e^+ \circ g^+ = \int_{\mathbb{R}^2} f_e^+(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u}.$$

For any f and g, we get that

$$\begin{aligned}
f^+ \circ g^- &= 2\pi\mathcal{F}_D^{-1}\left(\mathcal{F}_D(f^+)\mathcal{F}_D(g^-)\right) \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{-(y_1v_1+y_2v_2)D}e^{(x_1y_1+x_2y_2)ID}e^{(u_1y_1+u_2y_2)D}f^+(\underline{x})g^-(\underline{u})d\underline{x}d\underline{y}d\underline{u} \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{(y_1(u_1-v_1)+y_2(u_2-v_2))D}e^{(x_1y_1+x_2y_2)ID}f^+(\underline{x})g^-(\underline{u})d\underline{x}d\underline{y}d\underline{u} \\
&= \frac{1+I}{2}\int_{\mathbb{R}^4}\delta(-u_1+v_1-x_1)\delta(-u_2+v_2-x_2)f^+(\underline{x})g^-(\underline{u})d\underline{x}d\underline{u} \\
&\quad +\frac{1-I}{2}\int_{\mathbb{R}^4}\delta(u_1-v_1-x_1)\delta(u_2-v_2-x_2)f^+(\underline{x})g^-(\underline{u})d\underline{x}d\underline{u} \\
&= \frac{1+I}{2}\int_{\mathbb{R}^2} f^+(-\underline{u}+\underline{v})g^-(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f^+(\underline{u}-\underline{v})g^-(\underline{u})d\underline{u},
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
&e^{(y_1(u_1-v_1)+y_2(u_2-v_2))D}e^{(x_1y_1+x_2y_2)ID} \\
&\quad = (\cos(y_1(u_1-v_1)+y_2(u_2-v_2)) + D\sin(y_1(u_1-v_1)+y_2(u_2-v_2))) \\
&\qquad \times(\cos(x_1y_1+x_2y_2)+ID\sin(x_1y_1+x_2y_2))
\end{aligned}$$

which reduces to

$$\cos(y_1(u_1-v_1+x_1)+y_2(u_2-v_2+x_2))\left(\frac{1+I}{2}\right) + \cos(y_1(u_1-v_1-x_1)+y_2(u_2-v_2-x_2))\left(\frac{1-I}{2}\right)$$

under the integral. This can be proven by using classical trigonometric formulas. Next, for any f and g, we get that

$$
\begin{aligned}
f^- \circ g^- &= 2\pi \mathcal{F}_D^{-1}\left(\mathcal{F}_D(f^-)\mathcal{F}_D(g^-)\right)\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{-(y_1v_1+y_2v_2)ID} e^{(x_1y_1+x_2y_2)D} e^{-(u_1y_1+u_2y_2)D} f^-(\underline{x})g^-(\underline{u})d\underline{x}d\underline{y}d\underline{u}\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{-(y_1v_1+y_2v_2)ID} e^{(y_1(x_1-u_1)+y_2(x_2-u_2))D} f^-(\underline{x})g^-(\underline{u})d\underline{x}d\underline{y}d\underline{u}\\
&= \frac{1+I}{2}\int_{\mathbb{R}^4} \delta(v_1+u_1-x_1)\delta(v_2+u_2-x_2) f^-(\underline{x})g^-(\underline{u})d\underline{x}d\underline{u}\\
&\quad + \frac{1-I}{2}\int_{\mathbb{R}^4} \delta(v_1-u_1+x_1)\delta(v_2-u_2+x_2) f^-(\underline{x})g^-(\underline{u})d\underline{x}d\underline{u}\\
&= \frac{1+I}{2}\int_{\mathbb{R}^2} f^+(\underline{u}+\underline{v})g^-(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f^-(\underline{u}-\underline{v})g^-(\underline{u})d\underline{u},
\end{aligned}
$$

For the last part, we have to make a difference between f_o being a sum of k-vectors with k odd and f_e being a sum k-vectors with k even. In the odd case, we have

$$
\begin{aligned}
f_o^- \circ g^+ &= 2\pi \mathcal{F}_D^{-1}\left(\mathcal{F}_D(f_o^-)\mathcal{F}_D(g^+)\right)\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{-(y_1v_1+y_2v_2)D} e^{(x_1y_1+x_2y_2)D} f_o^-(\underline{x}) e^{(u_1y_1+u_2y_2)ID} g^+(\underline{u})d\underline{x}d\underline{y}d\underline{u}\\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^6} e^{(y_1(x_1-v_1)+y_2(x_2-v_2))D} e^{(u_1y_1+u_2y_2)ID} f_o^-(\underline{x})g^+(\underline{u})d\underline{x}d\underline{y}d\underline{u}\\
&= \frac{1+I}{2}\int_{\mathbb{R}^4} \delta(-u_1+v_1-x_1)\delta(-u_2+v_2-x_2) f_o^-(\underline{x})g^+(\underline{u})d\underline{x}d\underline{u}\\
&\quad + \frac{1-I}{2}\int_{\mathbb{R}^4} \delta(-u_1-v_1+x_1)\delta(-u_2-v_2+x_2) f_o^-(\underline{x})g^+(\underline{u})d\underline{x}d\underline{u}\\
&= \frac{1+I}{2}\int_{\mathbb{R}^2} f_o^-(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f_o^-(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u}.
\end{aligned}
$$

In the even case, we get

$$
f_e^- \circ g^+ = \frac{1+I}{2}\int_{\mathbb{R}^2} f_e^-(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f_e^-(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u}.
$$

Thus, in total, we can formulate our results in a theorem.

Theorem 2. *For any two $\mathbb{R}_{4,0}$-valued functions f and g, the Mustard convolution can be calculated as*

$$\begin{aligned}
f \circ g \quad = \quad & \int_{\mathbb{R}^2} f_o^+(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} + \int_{\mathbb{R}^2} f_e^+(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} \\
& + \frac{1+I}{2}\int_{\mathbb{R}^2} f^+(-\underline{u}+\underline{v})g^-(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f^+(\underline{u}-\underline{v})g^-(\underline{u})d\underline{u} \\
& + \frac{1+I}{2}\int_{\mathbb{R}^2} f^-(\underline{u}+\underline{v})g^-(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f^-(\underline{u}-\underline{v})g^-(\underline{u})d\underline{u} \\
& + \frac{1+I}{2}\int_{\mathbb{R}^2} f_o^-(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f_o^-(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} \\
& + \frac{1+I}{2}\int_{\mathbb{R}^2} f_e^-(\underline{u}+\underline{v})g^+(\underline{u})d\underline{u} + \frac{1-I}{2}\int_{\mathbb{R}^2} f_e^-(-\underline{u}+\underline{v})g^+(\underline{u})d\underline{u}.
\end{aligned}$$

7. Conclusion

In this article we have defined the spinor Fourier transform for functions taking values in the Clifford algebra $\mathbb{R}_{4,0}$. Also, two bases of eigenfunctions were determined for this transform. Finally, a convolution product was established, based on the idea of Mustard in [18]. We expect that this convolution product will find many applications in color image processing.

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T. Batard
Department of Information and Communication Technologies,
Universitat Pompeu Fabra Barcelona,
Roc Boronat 138,
08018 Barcelona, Spain
e-mail: `thomas.batard@upf.edu`

T. Raeymaekers
Department of Mathematical Analysis,
Faculty of Engineering and Architecture,
Ghent University, Galglaan 2,
9000 Gent, Belgium
e-mail: `tr@cage.ugent.be`