Generalized Taylor Series in Hermitian Clifford Analysis

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Abstract

An orthogonal Appell basis of homogeneous polynomials is constructed for the Bergman space of square–integrable hermitian monogenic functions in the unit ball of $\mathbb{C}^n$, with values in a homogeneous subspace of spinor space, leading to a generalized Taylor series expansion for spherical monogenics.

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1 Introduction

Let $\mathbb{B}_2$ be the open unit disc in the complex plane. The Bergman space

$$\mathcal{A}^2(\mathbb{B}_2) = \{f(z) \in L^2(\mathbb{B}_2) : f \text{ is holomorphic in } \mathbb{B}_2\}$$

is a separable Hilbert space for the traditional inner product

$$\langle f, g\rangle = \int_{\mathbb{B}_2} f(\zeta)\,\overline{g(\zeta)}\,dV(\zeta)$$

which possesses a reproducing kernel

$$K(z,\zeta) = \frac{1}{\pi}\,\frac{1}{(1-z\overline{\zeta})^2}$$

i.e. for all $f \in \mathcal{A}^2(\mathbb{B}_2)$ it holds

$$f(z) = \int_{\mathbb{B}_2} K(z,\zeta)\,f(\zeta)\,dV(\zeta)$$

For this Bergman space $\mathcal{A}^2(\mathbb{B}_2)$ there exists a countable orthonormal basis $(\phi_j)_{j=0}^{\infty}$, and on each $E \times E$, E being a compact subset of $\mathbb{B}_2$, the series

$$\sum_{j=0}^{\infty} \phi_j(z)\,\overline{\phi_j(\zeta)}$$

converges uniformly to the reproducing kernel $K(z,\zeta)$. As is well–known, the polynomials $\left(z^j\right)_{j=0}^{\infty}$ form such a countable orthogonal basis for $\mathcal{A}^2(\mathbb{B}_2)$, and it can directly be verified that, indeed,

$$\sum_{j=0}^{\infty} \frac{j+1}{\pi}\, z^j\, \zeta^j = \frac{1}{\pi}\, \frac{1}{(1-z\overline{\zeta})^2} = K(z,\zeta)$$

This orthogonal basis for $\mathcal{A}^2(\mathbb{B}_2)$ shows another important property: the derivative (with respect to the complex variable z) of a basis polynomial, reproduces, up to a constant, another basis polynomial:

$$(z^j)' = j\, z^{j-1}$$

This property is known as the Appell property, see [1] for the original paper and [31, 32, 41] for the first contributions to Appell theory for systems of polynomials in a Clifford analysis setting. This Appell property is, quite naturally, important for numerical applications, since it allows for manipulating the orthogonal basis polynomials without having to refer to their explicit form.

In this paper we will construct an orthogonal Appell basis, consisting of homogeneous polynomials, for a Bergman space of hermitian monogenic functions. Hermitian monogenic functions form one of the current research topics in Clifford analysis, which, in its most basic form, is a higher dimensional generalization of holomorphic function theory in the complex plane, and, at the same time, a refinement of harmonic analysis, see e.g. [9, 34, 27, 36, 35]. At the heart of Clifford analysis lies the notion of a monogenic function, i.e. a Clifford algebra valued null solution of the Dirac operator $\underline{\partial} = \sum_{\alpha=1}^{m} e_\alpha\, \partial_{X_\alpha}$, where $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$ which underlies the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. We refer to this setting as the Euclidean one, since the fundamental group leaving the Dirac operator $\underline{\partial}$ invariant is the special orthogonal group $\mathrm{SO}(m;\mathbb{R})$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. In case the dimension m is even, say $m = 2n$, so–called hermitian Clifford analysis was recently introduced as a refinement of Euclidean Clifford analysis (see the books [44, 25] and the series of papers [45, 28, 5, 6, 18, 29, 11]). The considered functions now take values in the complex Clifford algebra $\mathbb{C}_{2n}$ or in complex spinor space $\mathbb{S}_n$. Hermitian Clifford analysis is based on the introduction of an additional datum, a (pseudo) complex structure J, inducing an associated Dirac operator $\underline{\partial}_J$; it then focusses on the simultaneous null solutions of both operators $\underline{\partial}$ and $\underline{\partial}_J$, called hermitian monogenic functions. The fundamental group in this function theory, which is still in full development (see also [10, 19, 46, 8, 7, 30]), is isomorphic with the unitary group $\mathrm{U}(n)$. It is worth mentioning that the traditional holomorphic functions of several complex variables are a special case of hermitian monogenic functions when the latter take their values in a specific homogeneous part of complex spinor space $\mathbb{S}_n$.

To meet the needs for numerical calculations, recently much effort has been put into the construction of orthogonal bases for spaces of homogeneous monogenic polynomials, mostly called (solid) spherical monogenics, in the framework of both Euclidean and hermitian Clifford analysis. Indeed, the basis polynomials appearing in the Taylor series expansion of (standard) monogenic functions, sometimes called Fueter polynomials, are not useful for that purpose since they are not orthogonal with respect to the Fischer inner product, which is the natural inner product used for Clifford algebra valued polynomials. Explicit constructions of orthogonal polynomial bases in the Euclidean Clifford analysis context were carried out in e.g. [24, 21, 22, 23] in a direct analytic way starting from spherical harmonics, and in e.g. [47, 37, 4, 38, 40, 39] by the so–called Gel'fand–Tsetlin [GT] approach. The notion of GT–basis stems from group representation theory: every irreducible finite dimensional module over a classical Lie group has its GT–basis (see e.g. [42], or [33] for the original paper), the construction of which is based on the so–called branching of the corresponding function spaces. Classical branching rules govern the decomposition of the restriction of an irreducible representation of a classical group into irreducible representations of a classical subgroup; they have important applications in physics. Branching for e.g. $\mathrm{SO}(m)$–irreducibles of spherical monogenics in $\mathbb{R}^m$ is, de facto, a direct sum decomposition into $\mathrm{SO}(m-1)$–irreducibles which are products of spherical monogenics in $\mathbb{R}^{m-1}$ multiplied by certain embedding factors. This step 1 branching from $\mathrm{SO}(m)$ to $\mathrm{SO}(m-1)$ having multiplicity one, the irreducible pieces corresponding to ever becoming smaller dimension m will eventually end up in one dimensional subspaces, enabling the construction of a basis of any irreducible representation of $\mathrm{SO}(m)$ and in particular of spaces of spherical monogenics. The advantage of this GT–approach for the construction of orthogonal bases for spaces of spherical monogenics, is that it fully uses the structure present in the Clifford setting in order to devise a method which remains applicable in all dimensions, due to its inductive nature, as opposed to the direct and explicit calculations, which do become cumbersome when the dimension increases. The branching rules from $\mathrm{U}(n)$ to $\mathrm{U}(n-1)$ also having multiplicity one, the GT–approach could also be used for designing orthogonal bases for spaces of homogeneous hermitian monogenic polynomials, mostly called (solid) spherical hermitian monogenics, as done in [13, 14, 15, 16]. A nice historical overview of all contributions to the construction of orthogonal bases of polynomials in the framework of Clifford analysis, can be found in the introduction of the paper [4]; we refer the interested reader to that paper and the references therein. In particular, in [2] and [3] the computed orthogonal bases are used for solving problems in elasticity.

As already announced above, in this paper we will construct, via the GT–approach, an orthogonal Appell basis of homogeneous polynomials, for the Bergman space of hermitian monogenic functions defined in the open unit ball $\mathbb{B}_{2n}$ of $\mathbb{C}^n$ and taking their values in a homogeneous subspace $\mathbb{S}_n^r$ of spinor space $\mathbb{S}_n$. Starting point in our method is the construction, in Section 3, of an orthogonal Appell basis for the space $\mathcal{H}_{a,b}$ of complex–valued bi–homogeneous spherical harmonics in $\mathbb{C}^n$, based on branching for $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$, leading to the decomposition of $\mathcal{H}_{a,b}$ into $\mathrm{U}(n-1)$–irreducibles. In Section 4, a similar decomposition is obtained for the space $\mathcal{H}_{a,b}^{n,r}$ of $\mathbb{S}_n^r$–valued spherical harmonics in $\mathbb{C}^n$, which, alternatively, may also be decomposed in terms of spaces $\mathcal{M}_{a,b}^{n,r}$ of $\mathbb{S}_n^r$–valued spherical hermitian monogenics. Comparison of both decompositions then leads to expressions for the embedding factors in the decomposition of $\mathcal{M}_{a,b}^{n,r}$ into $\mathrm{U}(n-1)$–irreducibles, which show exactly the necessary properties to guarantee the Appell property of the orthogonal basis generated in this way. To make the paper self–contained a brief introduction on Clifford analysis is included.

2 Preliminaries on Clifford analysis

For a detailed description of the structure of a Clifford algebra we refer to e.g. [43]. Here we only recall the necessary basic notions. The real Clifford algebra $\mathbb{R}_{0,m}$ is constructed over the vector space $\mathbb{R}^{0,m}$ endowed with a non–degenerate quadratic form of signature $(0,m)$ and generated by the orthonormal basis $(e_1,\ldots,e_m)$. The non–commutative Clifford or geometric multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_\alpha e_\beta + e_\beta e_\alpha = -2\delta_{\alpha\beta} \quad , \quad \alpha,\beta = 1,\ldots,m \tag{1}$$

As a basis for $\mathbb{R}_{0,m}$ one takes for any set $A = \{j_1,\ldots,j_h\} \subset \{1,\ldots,m\}$, the element $e_A = e_{j_1}\ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \cdots < j_h \leq m$, together with $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so–called k–vector part of a. Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(X_1,\ldots,X_m)$ with the Clifford vector $X = \sum_{\alpha=1}^m e_\alpha X_\alpha$, for which it holds that

$X^2 = -|X|^2$. The vector valued first order differential operator $\partial = \sum_{\alpha=1}^m e_\alpha \, \partial_{X_\alpha}$, called Dirac operator, is the Fourier or Fischer dual of the Clifford variable X. It is this operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. More explicitly, a function f defined and continuously differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in (a subspace of) the Clifford algebra $\mathbb{R}_{0,m}$, is called (left) monogenic in Ω if $\partial[f] = 0$ in Ω. As the Dirac operator factorizes the Laplacian: $\Delta_m = -\partial^2$, monogenicity can be regarded as a refinement of harmonicity. The Dirac operator being rotationally invariant, this framework is usually referred to as Euclidean Clifford analysis.

When allowing for complex constants, the generators $(e_1, \ldots, e_m)$, still satisfying (1), produce the complex Clifford algebra $\mathbb{C}_m = \mathbb{R}_{0,m} \oplus i\,\mathbb{R}_{0,m}$. Any complex Clifford number $\lambda \in \mathbb{C}_m$ may thus be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,m}$, leading to the definition of the hermitian conjugation $\lambda^\dagger = (a+ib)^\dagger = \overline{a} - i\overline{b}$, where the bar notation stands for the Clifford conjugation in $\mathbb{R}_{0,m}$, i.e. the main anti–involution for which $\overline{e}_\alpha = -e_\alpha$, $\alpha = 1, \ldots, m$. This hermitian conjugation leads to a hermitian inner product on $\mathbb{C}_m$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and its associated norm $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$. This is the framework for hermitian Clifford analysis, which emerges from Euclidean Clifford analysis by introducing a so–called complex structure, i.e. an $\mathrm{SO}(m;\mathbb{R})$–element J with $J^2 = -\mathbf{1}$ (see [5, 6]), forcing the dimension to be even; from now on we put $m = 2n$. Usually J is chosen to act upon the generators of $\mathbb{C}_{2n}$ as $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1, \ldots, n$. By means of the projection operators $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ associated to J, first the Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j}), \qquad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j}), \qquad j = 1, \ldots, n \end{aligned}$$

The Witt basis elements satisfy the respective Grassmann and duality identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0, \quad \mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk}, \quad j, k = 1, \ldots, n$$

whence they are isotropic: $(\mathfrak{f}_j)^2 = 0, (\mathfrak{f}_j^\dagger)^2 = 0, j = 0, \ldots, n$. Next, denoting a vector in $\mathbb{R}^{0,2n}$ by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, which is identified with the Clifford vector $X = \sum_{j=1}^n (e_j\, x_j + e_{n+j}\, y_j)$, the hermitian Clifford variables z and $z^\dagger$ are produced similarly:

$$z = \frac{1}{2}(\mathbf{1} + iJ)[X] = \sum_{j=1}^n \mathfrak{f}_j\, z_j, \quad z^\dagger = -\frac{1}{2}(\mathbf{1} - iJ)[X] = \sum_{j=1}^n \mathfrak{f}_j^\dagger\, \overline{z}_j$$

where complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $\overline{z}_j = x_j - iy_j$, $j = 1, \ldots, n$. Finally, the Euclidean Dirac operator ∂ gives rise to the hermitian Dirac operators ∂_z and $\partial_z^\dagger$:

$$\partial_z^\dagger = \frac{1}{4}(\mathbf{1} + iJ)[\partial] = \sum_{j=1}^n \mathfrak{f}_j\, \partial_{\overline{z}_j}, \quad \partial_z = -\frac{1}{4}(\mathbf{1} - iJ)[\partial] = \sum_{j=1}^n \mathfrak{f}_j^\dagger\, \partial_{z_j}$$

involving the Cauchy–Riemann operators $\partial_{\overline{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and their complex conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ in the z_j–planes, $j = 1, \ldots, n$. Observe that hermitian vector variables and hermitian Dirac operators are isotropic, i.e. $z^2 = (z^\dagger)^2 = 0$ and $(\partial_z)^2 = (\partial_z^\dagger)^2 = 0$, whence the Laplacian allows for the decomposition and factorization

$$\Delta_{2n} = 4(\partial_z \partial_z^\dagger + \partial_z^\dagger \partial_z) = 4(\partial_z + \partial_z^\dagger)^2 = -4(\partial_z^\dagger - \partial_z)^2$$

while, dually,

$$-(z - z^\dagger)^2 = (z + z^\dagger)^2 = z\, z^\dagger + z^\dagger z = |z|^2 = |z^\dagger|^2 = |X|^2$$

We consider functions with values in an irreducible representation $\mathbb{S}_n$ of $\mathbb{C}_{2n}$, called spinor space, which is realized within $\mathbb{C}_{2n}$ using a primitive idempotent $I = I_1 \ldots I_n$, with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger$, $j = 1, \ldots, n$. With that choice, $\mathfrak{f}_j I = 0, j = 1, \ldots, n$, and so $\mathbb{S}_n \equiv \mathbb{C}_{2n} I \cong \bigwedge_n^\dagger I$, where $\bigwedge_n^\dagger = \bigwedge(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$ denotes the Grassmann algebra generated by $(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_n^\dagger)$. Hence $\mathbb{S}_n$ decomposes into so–called homogeneous parts

$$\mathbb{S}_n = \bigoplus_{r=0}^{n} \mathbb{S}_n^r = \bigoplus_{r=0}^{n} (\bigwedge_n^\dagger)^r I$$

with $(\bigwedge_n^\dagger)^r = \mathrm{span}_{\mathbb{C}}(\mathfrak{f}_{k_1}^\dagger \wedge \mathfrak{f}_{k_2}^\dagger \wedge \cdots \wedge \mathfrak{f}_{k_r}^\dagger : \{k_1, \ldots, k_r\} \subset \{1, \ldots, n\})$. By singling out one of the Witt basis vectors, viz. $\mathfrak{f}_n^\dagger$, and still using the same idempotent I, we can consider $\bigwedge(\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_{n-1}^\dagger) I$, which is isomorphic with spinor space $\mathbb{S}_{n-1}$ in $\mathbb{C}_{2(n-1)}$. This leads to the direct sum decomposition of complex spinor space $\mathbb{S}_n$:

$$\mathbb{S}_n = \mathbb{S}_{n-1} \oplus \mathfrak{f}_n^\dagger\, \mathbb{S}_{n-1}$$

and that of its homogeneous parts $\mathbb{S}_n^r$:

$$\mathbb{S}_n^r = (\bigwedge_{n-1}^\dagger)^r (\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_{n-1}^\dagger) I \oplus \mathfrak{f}_n^\dagger\, (\bigwedge_{n-1}^\dagger)^{r-1} (\mathfrak{f}_1^\dagger, \ldots, \mathfrak{f}_{n-1}^\dagger) I = \mathbb{S}_{n-1}^r \oplus \mathfrak{f}_n^\dagger\, \mathbb{S}_{n-1}^{r-1} \tag{2}$$

A continuously differentiable function F in an open region Ω of $\mathbb{R}^{2n}$ with values in (a subspace of) $\mathbb{C}_{2n}$ then is called (left) hermitian monogenic in Ω if and only if F satisfies in Ω the system $\partial_z F = 0 = \partial_z^\dagger F$, or, equivalently, the system $\partial F = \partial_J F$, with $\partial_J = J[\partial]$. A major difference between hermitian and Euclidean Clifford analysis concerns the underlying group invariance, which for $(\partial_z, \partial_z^\dagger)$ breaks down to the group $\mathrm{U}(n)$, see e.g. [5, 6]. This plays a fundamental role in the construction of orthogonal bases for spaces of hermitian monogenic polynomials.

3 Spherical harmonics

In this section we will construct an orthogonal Appell basis for the harmonic Bergman space in the unit ball $\mathbb{B}_{2n}$ of $\mathbb{C}^n \cong \mathbb{R}^{2n}$

$$\mathcal{B}^2(\mathbb{B}_{2n}; \mathbb{C}) = \{F \in L^2(\mathbb{B}_{2n}; \mathbb{C}) : F \text{ is harmonic in } \mathbb{B}_{2n}\}$$

which is a Hilbert space with reproducing kernel for the traditional inner product

$$\langle F, G \rangle = \int_{\mathbb{B}_{2n}} F(X)\, \overline{G(X)}\, dV(X) \quad \text{for} \quad F, G \in \mathcal{B}^2(\mathbb{B}_{2n}; \mathbb{C})$$

Functions defined in domains of $\mathbb{R}^{2n}$ can always be expressed in terms of the complex variables $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ and their complex conjugates $\overline{z} = (\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n)$. The hermitian inner product on $\mathbb{C}^n$ reads

$$\langle z, \zeta \rangle = z_1 \overline{\zeta}_1 + z_2 \overline{\zeta}_2 + \cdots + z_n \overline{\zeta}_n$$

with corresponding normsquared

$$|z|^2 = \langle z, z \rangle$$

and the Laplace operator takes the form

$$\Delta_{2n} = \partial_{z_1} \partial_{\overline{z}_1} + \partial_{z_2} \partial_{\overline{z}_2} + \cdots + \partial_{z_n} \partial_{\overline{z}_n}$$

The reproducing kernel for $\mathcal{B}^2(\mathbb{B}_{2n};\mathbb{C})$ then is given by

$$\mathcal{R}(z,\zeta) = \frac{\Gamma(n)}{2\pi^n} \frac{1}{(1-\langle z,\zeta\rangle - \langle \zeta,z\rangle + \langle z,z\rangle\langle\zeta,\zeta\rangle)^n} \left(\frac{2n(1-\langle z,z\rangle\langle\zeta,\zeta\rangle)^2}{1-\langle z,\zeta\rangle - \langle \zeta,z\rangle + \langle z,z\rangle\langle\zeta,\zeta\rangle} - 4\langle z,z\rangle\langle\zeta,\zeta\rangle \right)$$

An orthogonal Appell basis for the harmonic Bergman space $\mathcal{B}^2(\mathbb{B}_{2n};\mathbb{C})$ will be given in terms of bi–homogeneous harmonic polynomials; we denote by $\mathcal{H}_{a,b}(\mathbb{C}^n)$ the space of scalar–valued harmonic polynomials which are a–homogeneous in the variables z and b–homogeneous in the variables $\overline{z}$. The (complex) dimension of $\mathcal{H}_{a,b}(\mathbb{C}^n)$ is

$$\dim \mathcal{H}_{a,b}(\mathbb{C}^n) = \binom{n+a-1}{n-1}\binom{n+b-1}{n-1} - \binom{n+a-2}{n-1}\binom{n+b-2}{n-1}$$

The space $\mathcal{H}_{a,b}(\mathbb{C}^n)$ is an irreducible module over the unitary group $\mathrm{U}(n)$. By singling out one of the variables, say z_n, and considering the subgroup $\mathrm{U}(n-1)$ whose action leaves z_n unaltered, the space $\mathcal{H}_{a,b}(\mathbb{C}^n)$ becomes reducible for $\mathrm{U}(n-1)$, leading to the branching of $\mathcal{H}_{a,b}(\mathbb{C}^n)$ for $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$, where $\mathcal{H}_{a,b}(\mathbb{C}^n)$ is decomposed into $\mathrm{U}(n-1)$–irreducibles (see [40]):

$$\mathcal{H}_{a,b}(\mathbb{C}^n) = \bigoplus_{c=0}^{a} \bigoplus_{d=0}^{b} F^n_{a,b,c,d}(z,\overline{z})\, \mathcal{H}_{c,d}(\mathbb{C}^{n-1}) \tag{3}$$

The embedding factors $F^n_{a,b,c,d}(z,\overline{z})$ may be expressed in terms of Jacobi polynomials:

$$F^n_{a,b,c,d}(z,\overline{z}) = C\, |z|^{2s}\, P_s^{c+d+n-2,S-s}(t)\, z_n^{\frac{S-s+k}{2}}\, \overline{z}_n^{\frac{S-s-k}{2}}$$

with $s = min(a-c, b-d), S = max(a-c, b-d), k = (a-c)-(b-d), C = \frac{(c+1)_{(a-c)}(d+1)_{(b-d)}}{S!(c+d+n-1)_s}$ and t the dimensionless variable

$$t = \frac{|z_n|^2 - |\underline{z}|^2}{|z|^2}$$

where $\underline{z} = (z_1,\ldots,z_{n-1})$ and $\underline{\overline{z}} = (\overline{z}_1,\ldots,\overline{z}_{n-1})$. More explicitly, these embedding factors can be written as

$$F^n_{a,b,c,d}(z,\overline{z}) = C \sum_{\sigma=0}^{s} \binom{s+c+d+n-2}{\sigma}\binom{S}{s-\sigma}(-|\underline{z}|^2)^{s-\sigma}\, z_n^{\frac{S-s+k+2\sigma}{2}}\, \overline{z}_n^{\frac{S-s-k+2\sigma}{2}}$$

Note that $F^n_{a,b,a,b} = 1$, for all a,b,n.

A straightforward computation leads to the following results concerning derivatives of the functions $F^n_{a,b,c,a}(z,\overline{z})$, which will be used in the sequel.

Lemma 1. *For $j = 1,\ldots,n-1$ one has*

$$\begin{aligned}
\partial_{z_j} F^n_{a,b,c,d} &= -\frac{b(c+1)}{c+d+n-1}\, \overline{z}_j\, F^n_{a,b-1,c+1,d} \quad \text{if} \quad a \geq c+1 \text{ and } b \geq d+1 && (4)\\
\partial_{\overline{z}_j} F^n_{a,b,c,d} &= -\frac{a(d+1)}{c+d+n-1}\, z_j\, F^n_{a-1,b,c,d+1} \quad \text{if} \quad a \geq c+1 \text{ and } b \geq d+1 && (5)\\
\partial_{z_j} F^n_{a,b,c,d} &= 0 \quad \text{if} \quad a = c \text{ or } b = d\\
\partial_{\overline{z}_j} F^n_{a,b,c,d} &= 0 \quad \text{if} \quad a = c \text{ or } b = d
\end{aligned}$$

while also

$$\begin{aligned}
\partial_{z_n} F^n_{a,b,c,d} &= a\, F^n_{a-1,b,c,d} \quad \text{if} \quad a \geq c+1 & (6)\\
\partial_{z_n} F^n_{a,b,a,d} &= 0 \\
\partial_{\overline{z}_n} F^n_{a,b,c,d} &= b\, F^n_{a,b-1,c,d} \quad \text{if} \quad b \geq d+1 & (7)\\
\partial_{\overline{z}_n} F^n_{a,b,c,b} &= 0
\end{aligned}$$

A straightforward, but tedious, computation leads to the following identities involving the functions $F^n_{a,b,c,d}$, which will show to be crucial in the proof of Proposition 6.

Lemma 2. *One has*

$$a\, F^n_{a-1,b,c,d} - (c+1)\, F^n_{a,b,c+1,d} + \frac{a(c+1)(d+1)}{(c+d+n-1)(c+d+n)} |\underline{z}|^2\, F^n_{a-1,b,c+1,d+1} = 0 \tag{8}$$

$$b\, F^n_{a,b-1,c,d} - (d+1)\, F^n_{a,b,c,d+1} + \frac{b(c+1)(d+1)}{(c+d+n-1)(c+d+n)} |\underline{z}|^2\, F^n_{a,b-1,c+1,d+1} = 0 \tag{9}$$

$$b(c+1)\, F^n_{a,b-1,c+1,d} - a(d+1)\, F^n_{a-1,b,c,d+1} = 0 \tag{10}$$

Recursive use of the decomposition (3) through the chain of subgroups $\mathrm{U}(n-1) \supset \mathrm{U}(n-2) \supset \ldots \supset \mathrm{U}(1)$, will eventually lead to an orthogonal basis for the space $\mathcal{H}_{a,b}(\mathbb{C}^n)$; as explained in Section 1, this is the so–called Gel'fand–Tsetlin basis. This construction itself is, quite naturally, well–known, see e.g. [48], section 11.3. What is new, is that the orthogonal basis obtained, enjoys the Appell property, as expressed in Proposition 2.

Bearing in mind that in the complex plane ($n = 1$) an orthogonal basis of $\mathcal{H}_{a,b}(\mathbb{C})$ is given by

$$\{z_1^{a_1}, \overline{z}_1^{b_1} : a_1, b_1 = 0, 1, \ldots\}$$

we obtain the following result (see [40]).

Proposition 1. *An orthogonal basis, with respect to the L^2–inner product, of the space of scalar–valued (solid) spherical harmonics $\mathcal{H}_{a,b}(\mathbb{C}^n)$ is given by*

$$h_\mu(z, \overline{z}) = \left(\prod_{j=2}^{n} F^j_{a_j,b_j,a_{j-1},b_{j-1}} \right) z_1^{a_1}\, \overline{z}_1^{b_1} \tag{11}$$

where $\mu = (a_n, b_n)(a_{n-1}, b_{n-1}) \ldots (a_1, b_1)$, with $(a_n, b_n) = (a, b)$, $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq 0$, $b_n \geq b_{n-1} \geq \cdots \geq b_1 \geq 0$, and a_1 and b_1 not being different from zero at the same time.

As already mentioned, the importance of this basis (11) for the space $\mathcal{H}_{a,b}(\mathbb{C})$ not only resides in its orthogonality but also in its Appell property, as shown in the next proposition. We introduce the following notations:

$\partial_\mu = \partial_{z_n}^{a_n-a_{n-1}} \partial_{\overline{z}_n}^{b_n-b_{n-1}} \partial_{z_{n-1}}^{a_{n-1}-a_{n-2}} \partial_{\overline{z}_{n-1}}^{b_{n-1}-b_{n-2}} \cdots \partial_{z_2}^{a_2-a_1} \partial_{\overline{z}_2}^{b_2-b_1} \partial_{z_1}^{a_1} \partial_{\overline{z}_1}^{b_1}$ and $\mu! = a!\, b!$

Proposition 2. *The orthogonal basis of $\mathcal{H}_{a,b}(\mathbb{C}^n)$, given by (11), possesses the Appell property, i.e. for $\mu = (a,b)\underline{\mu}$ with $\underline{\mu} = (a_{n-1}, b_{n-1}) \ldots (a_1, b_1)$ there holds:*

(i) $\partial_{z_n} h_\mu = a\, h_{(a-1,b)\underline{\mu}}$ *if* $a = a_n > a_{n-1}$, *while* $\partial_{z_n} h_\mu = 0$ *if* $a = a_n = a_{n-1}$

(ii) $\partial_{\overline{z}_n} h_\mu = b\, h_{(a,b-1)\underline{\mu}}$ *if* $b = b_n > b_{n-1}$, *while* $\partial_{z_n} h_\mu = 0$ *if* $b = b_n = b_{n-1}$

(iii) $\partial_\mu h_\mu = \mu!$

Proof
First note that the differential operators ∂_{z_n} and $\partial_{\overline{z}_n}$ are invariant under the action of $\mathrm{U}(n-1)$, which allows for predicting the form of the outcome of their action on the embedding factors, however without knowledge of the exact values of the constants. Writing more explicitly the expression (11) for the basis polynomials

$$h_\mu(z,\overline{z}) = F^n_{a_n,b_n,a_{n-1},b_{n-1}} F^{n-1}_{a_{n-1},b_{n-1},a_{n-2},b_{n-2}} \cdots F^2_{a_2,b_2,a_1,b_1} z_1^{a_1}\, \overline{z}_1^{b_1}$$

it is easily seen that only the first factor contains the variables z_n and $\overline{z}_n$. Then (i) and (ii) follow from (6) and (7) respectively. To prove (iii) it suffices to remark that the differential operator ∂_μ is of order a with respect to the z–variables, and of order b with respect to the $\overline{z}$–variables, so that $\partial_\mu h_\mu$ is a constant and thus equals $\partial_\mu h_\mu|_{z=0}$, which turns out to be exactly $a!\, b!$. □

By way of illustration we list the orthogonal basis polynomials of the spaces $\mathcal{H}_{2,1}$, $\mathcal{H}_{2,0}$ and $\mathcal{H}_{1,1}$ in complex dimension $n = 2$; the Appell properties of Proposition 2 are easily verified:

- 3 basis polynomials for the space $H_{2,0}$
  $h_{2,0,0,0} = z_2^2$
  $h_{2,0,1,0} = z_1 z_2$
  $h_{2,0,2,0} = z_1^2$
- 3 basis polynomials for the space $H_{1,1}$
  $h_{1,1,0,0} = -z_1\overline{z}_1 + z_2\overline{z}_2$
  $h_{1,1,0,1} = \overline{z}_1 z_2$
  $h_{1,1,1,0} = z_1\overline{z}_2$
- 4 basis polynomials for the space $H_{2,1}$
  $h_{2,1,0,0} = -z_2(2z_1\overline{z}_1 - z_2\overline{z}_2)$
  $h_{2,1,1,0} = -z_1(z_1\overline{z}_1 - 2z_2\overline{z}_2)$
  $h_{2,1,0,1} = z_2^2\overline{z}_1$
  $h_{2,1,2,0} = z_1^2\overline{z}_2$

Since the space of all (solid) spherical harmonics is dense in the harmonic Bergman space $\mathcal{B}^2(\mathbb{B}_{2n};\mathbb{C})$, we have found in this way an orthogonal Appell basis $\{h_\mu(z,\overline{z})\}_\mu$ for this separable Hilbert space, leading to a generalized Taylor series for spherical harmonics, as expressed in the following theorem. Note that in [38] another GT–basis for the same Bergman space $\mathcal{B}^2(\mathbb{B}_{2n};\mathbb{C})$ has been obtained, using (real) one step branching, breaking the symmetry from $\mathrm{SO}(m)$ to $\mathrm{SO}(m-1)$, with $m = 2n$.

Theorem 1. *(Generalized Taylor series for spherical harmonics)*
Each function $G \in \mathcal{B}^2(\mathbb{B}_{2n};\mathbb{C})$ may be expanded into an orthogonal series

$$G(z,\overline{z}) = \sum_\mu \mathbf{t}_\mu[G]\, h_\mu(z,\overline{z})$$

where the coefficients are given by

$$\mathbf{t}_\mu[G] = \frac{1}{\mu!}\,\partial_\mu G(z,\overline{z})|_{z=0}$$

4 Spherical hermitian monogenics

We consider the space $\mathcal{M}_k^n$ of homogeneous monogenic polynomials defined in $\mathbb{C}^n$, taking values in complex spinor space $\mathbb{S}_n$ and with global degree of homogeneity k in the complex variables $(z, \overline{z})$. When specifying the bidegree (a, b) in the variables $(z, \overline{z})$ respectively, we denote the corresponding space by $\mathcal{M}_{a,b}^n$, and it holds that

$$\mathcal{M}_k^n = \bigoplus_{a+b=k} \mathcal{M}_{a,b}^n$$

We denote by $\mathcal{M}_{a,b}^{n,r}$ its subspace of (a, b)–homogeneous monogenic polynomials with values in the homogeneous part $\mathbb{S}_n^r$ of spinor space $\mathbb{S}_n$. Note that the polynomials in $\mathcal{M}_{a,b}^{n,r}$ are automatically *hermitian* monogenic, seen the action of the hermitian Dirac operators on the space $\mathcal{P}_{a,b}^{n,r}$ of (a, b)–homogeneous polynomials with values in $\mathbb{S}_n^r$,

$$\partial_z : \mathcal{P}_{a,b}^{n,r} \longrightarrow \mathcal{P}_{a-1,b}^{n,r+1} \quad \text{and} \quad \partial_z^\dagger : \mathcal{P}_{a,b}^{n,r} \longrightarrow \mathcal{P}_{a,b-1}^{n,r-1}$$

It is clear that $\mathcal{M}_{a,b}^{n,r}$ is a subspace of $\mathcal{H}_{a,b}^{n,r}$, the space of (a, b)–homogeneous harmonic polynomials with values in $\mathbb{S}_n^r$. Whereas the space $\mathcal{H}_{a,b}(\mathbb{C}^n)$ is a $\mathrm{U}(n)$–module (see Section 3), the space $\mathcal{H}_{a,b}^{n,r}$ is no longer $\mathrm{U}(n)$–irreducible, and it may be decomposed in terms of spaces of (solid) spherical hermitian monogenics, i.e. bi–homogeneous hermitian monogenic polynomials, those spaces being, indeed, irreducible modules over the unitary group $\mathrm{U}(n)$. This is the content of the following so–called Fischer decomposition (see e.g. [20]).

Proposition 3. *The space $\mathcal{H}_{a,b}^{n,r}$ of $\mathbb{S}_n^r$–valued spherical harmonics may be decomposed into $U(n)$–irreducibles as*

$$\begin{aligned} \mathcal{H}_{a,b}^{n,r} \quad = \quad & \mathcal{M}_{a,b}^{n,r} \oplus z\, \mathcal{M}_{a-1,b}^{n,r+1} \oplus z^\dagger\, \mathcal{M}_{a,b-1}^{n,r-1} \\ & \oplus \big((a-1+r)zz^\dagger - (b-1+n-r)z^\dagger z\big)\, \mathcal{M}_{a-1,b-1}^{n,r} \end{aligned} \tag{12}$$

Note that also the space $\mathcal{M}_k^n$ may be decomposed into $\mathrm{U}(n)$–irreducibles involving appropriate spaces $\mathcal{M}_{a,b}^{n,r}$. This Fischer decomposition, for which we refer to [14], where it is proved via analytic methods, and to [26] for a group representation approach, explicitly reads as follows.

Proposition 4. *The space $\mathcal{M}_k^n$ may be decomposed into* $\mathrm{U}(n)$*–irreducibles as*

$$\mathcal{M}_k^n = \bigoplus_{a+b=k} \bigoplus_{r=0}^{n} \mathcal{M}_{a,b}^{n,r} \oplus \bigoplus_{a+b=k-1} \bigoplus_{r=1}^{n-1} \big((b+n-r)z + (a+r)z^\dagger\big)\, \mathcal{M}_{a,b}^{n,r} \tag{13}$$

Now, following the same procedure as in Section 3, we may decompose the space $\mathcal{M}_{a,b}^{n,r}$ by branching for $\mathrm{U}(n) \downarrow \mathrm{U}(n-1)$ (see [17]). Putting

$$z = \mathfrak{f}_n z_n + \sum_{j=1}^{n-1} \mathfrak{f}_j z_j = \mathfrak{f}_n z_n + \underline{z} \quad \text{and} \quad z^\dagger = \mathfrak{f}_n^\dagger \overline{z}_n + \sum_{j=1}^{n-1} \mathfrak{f}_j^\dagger \overline{z}_j = \mathfrak{f}_n^\dagger \overline{z}_n + \underline{z}^\dagger$$

this orthogonal decomposition of $\mathcal{M}_{a,b}^{n,r}$ reads as follows.

Proposition 5. *The* $\mathrm{U}(n)$*–module $\mathcal{M}_{a,b}^{n,r}$ of bi–homogeneous $\mathbb{S}_n^r$–valued hermitian monogenic polynomials in the variables $(z, \overline{z})$ may be decomposed into* $\mathrm{U}(n-1)$*–irreducibles of polynomials in the variables $(\underline{z}, \overline{\underline{z}})$ as*

$$\mathcal{M}_{a,b}^{n,r} = \bigoplus_{c=0}^{a} \bigoplus_{d=0}^{b} \bigoplus_{s=r-1}^{r} X_{a,b,c,d}^{n,r,s}\, \mathcal{M}_{c,d}^{n-1,s} \tag{14}$$

Remark 1. *In the above decomposition (14) it is assumed that $0 < r < n$. When $r = 0$, hermitian monogenicity is nothing else but anti–holomorphy and the polynomials at stake only depend on the variables $(\overline{z}_1, \ldots, \overline{z}_n)$, while for $r = n$ the hermitian monogenic polynomials are holomorphic and only depend on the variables $(z_1, \ldots, z_n)$ (see e.g. [6]). In these two exceptional cases the branching decomposition formula (14) takes a specific form:*

$$
\begin{aligned}
\mathcal{M}^{n,0}_{0,b} &= \bigoplus_{d=0}^{b} X^{n,0,0}_{0,b,0,d}\, \mathcal{M}^{n-1,0}_{0,d} \quad \text{with} \quad X^{n,0,0}_{0,b,0,d} \approx \overline{z}_n^{\,b-d} \\
\mathcal{M}^{n,n}_{a,0} &= \bigoplus_{c=0}^{a} X^{n,n,n-1}_{a,0,c,0}\, \mathcal{M}^{n-1,n-1}_{c,0} \quad \text{with} \quad X^{n,n,n-1}_{a,0,c,0} \approx z_n^{a-c}\, \mathfrak{f}_n^{\dagger}
\end{aligned}
$$

Remark 2. *For completeness let us mention the (complex) dimension of these spaces of hermitian spherical monogenics. One has (see e.g. [20]):*

$$\dim \mathcal{M}^{n,r}_{a,b} = \frac{a+b+n}{a+r} \binom{b+n-r-1}{b} \binom{a+n-1}{a} \binom{b+n-1}{r-1}$$

whereas

$$\dim \mathcal{M}^{n,n}_{a,0} = \binom{a+n-1}{a} \quad \text{and} \quad \dim \mathcal{M}^{n,0}_{0,b} = \binom{b+n-1}{b}$$

In [17] the embedding factors $X^{n,r,s}_{a,b,c,d}$ appearing in the decomposition (14) were determined explicitly in terms of the polynomials appearing in the expressions for the orthogonal bases of the spaces $\mathcal{M}_k(\mathbb{R}^m; \mathbb{R}_{0,m})$ of monogenic k–homogeneous polynomials defined in Euclidean space $\mathbb{R}^m$ and taking values in the real Clifford algebra $\mathbb{R}_{0,m}$, obtained in [27, Lemma 4.4, pp.260–262]. Here, aiming at basis polynomials enjoying the Appell property, we follow another approach. Combining the decomposition (3) of the space $\mathcal{H}_{a,b}$ with the splitting (2) of the spinor subspace $\mathbb{S}^r_n$, we obtain

$$
\begin{aligned}
\mathcal{H}^{n,r}_{a,b} &= \mathcal{H}_{a,b}(\mathbb{C}^n) \otimes \mathbb{S}^r_n \\
&= \mathcal{H}_{a,b}(\mathbb{C}^n) \otimes \left(\mathbb{S}^r_{n-1} \oplus \mathfrak{f}_n^{\dagger}\, \mathbb{S}^{r-1}_{n-1} \right) \\
&= \bigoplus_{c=0}^{a} \bigoplus_{d=0}^{b} F^n_{a,b,c,d}(z, \overline{z}) \left(\mathcal{H}^{n-1,r}_{c,d} \oplus \mathfrak{f}_n^{\dagger}\, \mathcal{H}^{n-1,r-1}_{c,d} \right) \qquad (15)
\end{aligned}
$$

Comparing this decomposition (15) with the Fischer decomposition (12), it becomes clear that it must be possible to express the embedding factors $X^{n,r,s}_{a,b,c,d}$ appearing in (14), in terms of the embedding factors $F^n_{a,b,c,d}$ appearing in the branching (3) of $\mathcal{H}_{a,b}(\mathbb{C}^n)$. The result is the following.

Proposition 6. *For $0 < r < n$ there holds*

$$X^{n,r,r}_{a,b,c,d} = F^n_{a,b,c,d} - \frac{c+1}{c+r} F^n_{a,b,c+1,d}\, \underline{z}\, \mathfrak{f}_n^{\dagger} + C_1\, F^n_{a,b,c+1,d+1} \left(\underline{z}\, \underline{z}^{\dagger} - C_2\, \underline{z}^{\dagger} \underline{z} \right) \qquad (16)$$

$$X^{n,r,r-1}_{a,b,c,d} = F^n_{a,b,c,d}\, \mathfrak{f}_n^{\dagger} - \frac{d+1}{d+n-r} F^n_{a,b,c,d+1}\, \underline{z}^{\dagger} + C_1\, F^n_{a,b,c+1,d+1} \left(\underline{z}^{\dagger} \underline{z} - C_3\, \underline{z}\, \underline{z}^{\dagger} \right) \mathfrak{f}_n^{\dagger} \qquad (17)$$

with

$$C_1 = \frac{(c+1)(d+1)}{(c+d+n-1)(c+d+n)}, C_2 = \frac{d+n-r-1}{c+r}, C_3 = \frac{c+r-1}{d+n-r}$$

whereas

$$X^{n,0,0}_{0,b,0,d} = \frac{b!}{(b-d)!\, d!}\, \overline{z}_n^{\,b-d} \quad \text{and} \quad X^{n,n,n-1}_{a,0,c,0} = \frac{a!}{(a-c)!\, c!}\, z_n^{a-c}\, \mathfrak{f}_n^{\dagger}$$

Proof
Plugging in the decomposition (12) for the spaces $\mathcal{H}^{n-1,r}_{c,d}$ and $\mathcal{H}^{n-1,r-1}_{c,d}$ into the decomposition (15), gives rise to terms of the following eight types:

- $F^n_{a,b,c,d}\,\mathcal{M}^{n-1,r}_{c,d}$ and $F^n_{a,b,c,d}\,\mathfrak{f}^\dagger_n\,\mathcal{M}^{n-1,r-1}_{c,d}$
- $F^n_{a,b,c,d}\,\underline{z}\,\mathcal{M}^{n-1,r+1}_{c-1,d}$ and $F^n_{a,b,c,d}\,\underline{z}\,\mathfrak{f}^\dagger_n\,\mathcal{M}^{n-1,r}_{c-1,d}$
- $F^n_{a,b,c,d}\,\underline{z}^\dagger\,\mathcal{M}^{n-1,r-1}_{c,d-1}$ and $F^n_{a,b,c,d}\,\underline{z}^\dagger\,\mathfrak{f}^\dagger_n\,\mathcal{M}^{n-1,r-2}_{c,d-1}$
- $F^n_{a,b,c,d}\,((c+r-1)\underline{z}\underline{z}^\dagger-(d+n-r-2)\underline{z}^\dagger\underline{z})\,\mathcal{M}^{n-1,r}_{c-1,d-1}$ and

  $F^n_{a,b,c,d}\,((c+r-2)\underline{z}\underline{z}^\dagger-(d+n-r-1)\underline{z}^\dagger\underline{z})\,\mathfrak{f}^\dagger_n\,\mathcal{M}^{n-1,r-1}_{c-1,d-1}$

The embedding factor $X^{n,r,r}_{a,b,c,d}$ of $\mathcal{M}^{n-1,r}_{c,d}$ is an appropriate combination of three of them, viz.

$$X^{n,r,r}_{a,b,c,d}=F^n_{a,b,c,d}+A\,F^n_{a,b,c+1,d}\,\underline{z}\,\mathfrak{f}^\dagger_n+B\,F^n_{a,b,c+1,d+1}\,(\underline{z}\underline{z}^\dagger-\frac{d+n-r-1}{c+r}\underline{z}^\dagger\underline{z})$$

and a similar combination is needed for the embedding factor $X^{r,r-1,n}_{a,b,c,d}$ of $\mathcal{M}^{n-1,r-1}_{c,d}$:

$$X^{n,r,r-1}_{a,b,c,d}=F^n_{a,b,c,d}\,\mathfrak{f}^\dagger_n+A'\,F^n_{a,b,c,d+1}\,\underline{z}^\dagger+B'\,F^n_{a,b,c+1,d+1}\,(-\frac{c+r-1}{d+n-r}\underline{z}\underline{z}^\dagger+\underline{z}^\dagger\underline{z})\,\mathfrak{f}^\dagger_n$$

The coefficients A,B,A',B' are now determined by expressing that the products $X^{n,r,r}_{a,b,c,d}\,\mathcal{M}^{n-1,r}_{c,d}$ and $X^{n,r,r-1}_{a,b,c,d}\,\mathcal{M}^{n-1,r-1}_{c,d}$ have to be hermitian monogenic, i.e. have to be annihilated by the hermitian Dirac operators $\partial_{\underline{z}}$ and $\partial^\dagger_{\underline{z}}$. The action of the operator $\partial_{\underline{z}}$ on the product $X^{n,r,r}_{a,b,c,d}\,\mathcal{M}^{n-1,r}_{c,d}$ generates a system of three equations:

$$\begin{aligned}
a\,F^n_{a-1,b,c,d}\,\mathfrak{f}^\dagger_n+A\,(c+r)\,F^n_{a,b,c+1,d}\,\mathfrak{f}^\dagger_n+B\,a\,|\underline{z}|^2\,F^n_{a-1,b,c+1,d+1}\,\mathfrak{f}^\dagger_n &= 0\\
-\frac{b(c+1)}{c+d+n-1}\,\underline{z}^\dagger\,F^n_{a,b-1,c+1,d}+B\,(c+d+n)\,\underline{z}^\dagger F^n_{a,b,c+1,d+1}-B\,\frac{b(c+2)}{c+d+n+1}\,|\underline{z}|^2\,\underline{z}^\dagger\,F^n_{a,b-1,c+2,d+1} &= 0\\
-A\,\frac{b(c+2)}{c+d+n}\,\underline{z}^\dagger\,\underline{z}\,\mathfrak{f}^\dagger_n\,F^n_{a,b-1,c+2,d}-B\,\frac{a(d+n-r-1)}{c+r}\,\underline{z}^\dagger\,\underline{z}\,\mathfrak{f}^\dagger_n\,F^n_{a-1,b,c+1,d+1}-B\,a\,\underline{z}^\dagger\,\underline{z}\,\mathfrak{f}^\dagger_n\,F^n_{a-1,b,c+1,d+1} &= 0
\end{aligned}$$

which is solvable for the two unknowns A and B, viz. $A=-\frac{c+1}{c+r}$, $B=\frac{(c+1)(d+1)}{(c+d+n-1)(c+d+n)}$, in virtue of the identities (8), (9), (10). The action of the operator $\partial^\dagger_{\underline{z}}$ on the same product generates one additional equation:

$$\begin{aligned}
-B\,\frac{(d+n-r-1)(c+d+n-1)}{c+r}\,F^n_{a,b,c+1,d+1}\,\underline{z}-A\,b\,F^n_{a,b-1,c+1,d}\,\underline{z}-\frac{a(d+1)}{c+d+n-1}\,F^n_{a-1,b,c,d+1}\,\underline{z}&\\
+B\,\frac{a(d+2)(d+n-r-1)}{(c+r)(c+d+n+1)}\,F^n_{a-1,b,c+1,d+2}\,|\underline{z}|^2\,\underline{z}&=0
\end{aligned}$$

which is satisfied for the values of A and B found, in virtue of the identities (8) and (10).
For the action of the hermitian Dirac operators on the product $X^{n,r,r-1}_{a,b,c,d}\,\mathcal{M}^{n-1,r-1}_{c,d}$ the situation is completely similar. The action of $\partial^\dagger_{\underline{z}}$ leads to a system of three equations which can be solved for the two unknowns A' and B', viz. $A'=-\frac{d+1}{d+n-r}$, $B'=\frac{(c+1)(d+1)}{(c+d+n-1)(c+d+n)}$ in virtue of the same identities (8), (9), (10). The action of $\partial_{\underline{z}}$ results into one supplementary equation which is satisfied by the already found values of A' and B' in virtue of the identities (9) and (10). $\square$

Remark 3. *In the expressions (16) and (17) for the embedding factors it is tacitly assumed that the terms containing the factors* $F^n_{a,b,c+1,d}$*,* $F^n_{a,b,c,d+1}$ *and* $F^n_{a,b,c+1,d+1}$ *do not appear when* $c+1>a$ *or* $d+1>b$*.*

As explained in Section 1, recursive use of the branching (14) will lead to the Gel'fand–Tsetlin basis of $\mathcal{M}^{n,r}_{a,b}$. We recall some notations and introduce some new ones:

- $$\nu = (r_n, r_{n-1}, \ldots r_1) = (r_n, \underline{\nu}) \quad \text{with} \quad r_n = r, \quad j \geq r_j \geq r_{j-1} \geq r_j - 1 \quad \text{and} \quad r_1 = 0 \text{ or } 1$$

- $$\mu = (a_n, b_n)(a_{n-1}, b_{n-1}) \ldots (a_1, b_1) = (a_n, b_n)\ \underline{\mu}$$

  with
  $$(a_n, b_n) = (a, b), a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq 0, b_n \geq b_{n-1} \geq \cdots \geq b_1 \geq 0$$
  and $r_j = 0 \Longrightarrow a_j = 0$, $r_j = j \Longrightarrow b_j = 0$

- $$\partial_\mu = \partial^{a_n - a_{n-1}}_{z_n} \partial^{b_n - b_{n-1}}_{\overline{z}_n} \partial^{a_{n-1} - a_{n-2}}_{z_{n-1}} \partial^{b_{n-1} - b_{n-2}}_{\overline{z}_{n-1}} \cdots \partial^{a_2 - a_1}_{z_2} \partial^{b_2 - b_1}_{\overline{z}_2} \partial^{a_1}_{z_1} \partial^{b_1}_{\overline{z}_1} \quad \text{and} \quad \mu! = a!b!$$

- $$\mathfrak{f}^\dagger_\nu = (\mathfrak{f}^\dagger_n)^{r - r_{n-1}} \cdots (\mathfrak{f}^\dagger_2)^{r_2 - r_1} (\mathfrak{f}^\dagger_1)^{r_1}\, I$$

Proposition 7. *The spaces* $\mathcal{M}^{n,r}_{a,b}$ *of* $\mathbb{S}^r_n$*–valued spherical hermitian monogenics show the following orthogonal bases with respect to the* L^2*–inner product:*

- *if* $r = 0, a = 0$ *then* $\mathcal{M}^{n,0}_{0,b}$ *has the orthogonal basis* $\{\overline{z}_1^{k_1} \cdots \overline{z}_n^{k_n}\, I : k_1 + \cdots + k_n = b\}$*;*
- *if* $r = n, b = 0$ *then* $\mathcal{M}^{n,n}_{a,0}$ *has the orthogonal basis* $\{z_1^{k_1} \cdots z_n^{k_n}\, \mathfrak{f}^\dagger_1 \cdots \mathfrak{f}^\dagger_n\, I : k_1 + \cdots + k_n = a\}$*;*
- *in general,* $\mathcal{M}^{n,r}_{a,b}$ *has the orthogonal basis*

$$\left\{ g^\nu_\mu = X^{n, r_n, r_{n-1}}_{a_n, b_n, a_{n-1}, b_{n-1}} \cdots X^{2, r_2, r_1}_{a_2, b_2, a_1, b_1}\, g^{r_1}_{a_1, b_1} \right\}_{\underline{\mu}, \underline{\nu}} \tag{18}$$

  with $g^0_{0,b_1} = \overline{z}_1^{b_1}\, I$ *and* $g^1_{a_1,0} = z_1^{a_1}\, \mathfrak{f}^\dagger_1\, I$*.*

Moreover the above orthogonal bases show the Appell property.

Proposition 8. *The orthogonal basis of* $\mathcal{M}^{n,r}_{a,b}$*, given by (18), possesses the Appell property, i.e.*

(i) $\partial_{z_n} g^\nu_\mu = a\, g^\nu_{(a-1,b)\underline{\mu}}$ *if* $a = a_n > a_{n-1}$*, while* $\partial_{z_n} g^\nu_\mu = 0$ *if* $a = a_n = a_{n-1}$

(ii) $\partial_{\overline{z}_n} g^\nu_\mu = b\, g^\nu_{(a,b-1)\underline{\mu}}$ *if* $b = b_n > b_{n-1}$*, while* $\partial_{\overline{z}_n} g^\nu_\mu = 0$ *if* $b = b_n = b_{n-1}$

(iii) $\partial_\mu g^\nu_\mu = \mu!\, \mathfrak{f}^\dagger_\nu$

Remark 4. *Note that* $\{\mathfrak{f}^\dagger_{r,\underline{\nu}}\}_{\underline{\nu}}$ *is a basis of* $\mathbb{S}^r_n$*, i.e. for each* $\lambda \in \mathbb{S}^r_n$ *there exist unique* $\lambda_{r,\underline{\nu}} \in \mathbb{C}$ *such that* $\lambda = \sum_{\underline{\nu}} \lambda_{r,\underline{\nu}}\, \mathfrak{f}^\dagger_{r,\underline{\nu}}$*. This implies that each function* $G(z, z^\dagger) : \mathbb{C}^n \longrightarrow \mathbb{S}^r_n$ *can be written as*

$$G(z, z^\dagger) = \sum_{\underline{\nu}} G_{r,\underline{\nu}}(z, \overline{z})\, \mathfrak{f}^\dagger_{r,\underline{\nu}}$$

where the functions $G_{r,\underline{\nu}}(z, \overline{z})$ *are* $\mathbb{C}$*–valued.*

By way of illustration we list the orthogonal basis polynomials of the spaces $\mathcal{M}^{2,1}_{2,1}$, $\mathcal{M}^{2,0}_{2,1}$ and $\mathcal{M}^{2,1}_{1,1}$ in complex dimension 2; the Appell properties of Proposition 8 are easily verified:

- 5 basis polynomials for the space $\mathcal{M}^{2,1}_{2,1}$
  $g^{1,1}_{2,1,2,0} = z_1^2\overline{z}_2\,\mathfrak{f}_1^\dagger I$
  $g^{1,1}_{2,1,1,0} = z_1^2\overline{z}_2\,\mathfrak{f}_2^\dagger I + (-z_1^2\overline{z}_1 + 2z_1z_2\overline{z}_2)\,\mathfrak{f}_1^\dagger I$
  $g^{1,1}_{2,1,0,0} = (-z_1^2\overline{z}_1 + 2z_1z_2\overline{z}_2)\,\mathfrak{f}_2^\dagger I + (-2z_1z_2\overline{z}_1 + z_2^2\overline{z}_2)\,\mathfrak{f}_1^\dagger I$
  $g^{1,0}_{2,1,0,1} = z_2^2\overline{z}_1\,\mathfrak{f}_2^\dagger I$
  $g^{1,0}_{2,1,0,0} = -z_2^2\overline{z}_1\,\mathfrak{f}_1^\dagger I + (-2z_1z_2\overline{z}_1 + z_2^2\overline{z}_2)\,\mathfrak{f}_2^\dagger I$
- 4 basis polynomials for the space $\mathcal{M}^{2,1}_{2,0}$
  $g^{1,1}_{2,0,2,0} = z_1^2\,\mathfrak{f}_1^\dagger I$
  $g^{1,1}_{2,0,1,0} = z_1^2\,\mathfrak{f}_2^\dagger I + 2z_1z_2\,\mathfrak{f}_1^\dagger I$
  $g^{1,1}_{2,0,0,0} = 2z_1z_2\,\mathfrak{f}_2^\dagger I + z_2^2\,\mathfrak{f}_1^\dagger I$
  $g^{1,0}_{2,0,0,0} = z_2^2\,\mathfrak{f}_2^\dagger I$
- 4 basis polynomials for the space $\mathcal{M}^{2,1}_{1,1}$
  $g^{1,1}_{1,0} = z_1\overline{z}_2\,\mathfrak{f}_1^\dagger I$
  $g^{1,1}_{0,0} = z_1\overline{z}_2\,\mathfrak{f}_2^\dagger I + (-z_1\overline{z}_1 + z_2\overline{z}_2)\,\mathfrak{f}_1^\dagger I$
  $g^{1,0}_{0,1} = z_2\overline{z}_1\,\mathfrak{f}_2^\dagger I$
  $g^{1,0}_{0,0} = -z_2\overline{z}_1\,\mathfrak{f}_1^\dagger I + (-z_1\overline{z}_1 + z_2\overline{z}_2)\,\mathfrak{f}_2^\dagger I$

Introducing the Bergman space of $\mathbb{S}^r_n$–valued hermitian monogenic square–integrable functions in the unit ball $\mathbb{B}_{2n}$ of $\mathbb{C}^n$

$$\mathcal{A}^2(\mathbb{B}_{2n};\mathbb{S}^r_n) = L^2(\mathbb{B}_{2n};\mathbb{S}^r_n) \cap \operatorname{Ker}\partial$$

we have found for this separable Hilbert space an orthogonal Appell basis $\left\{g^{r,\underline{\nu}}_\mu\right\}_{\mu,\underline{\nu}}$ and the following important result holds.

Theorem 2. *(Generalized Taylor series for spherical monogenics)*
Each function $G = \sum_{\underline{\nu}} G_{r,\underline{\nu}}\,\mathfrak{f}^\dagger_{r,\underline{\nu}} \in \mathcal{A}^2(\mathbb{B}_{2n};\mathbb{S}^r_n)$ may be expanded into an orthogonal series

$$G(z,z^\dagger) = \sum_{\mu,\underline{\nu}} \mathbf{t}^{r,\underline{\nu}}_\mu[G]\, g^{r,\underline{\nu}}_\mu$$

where the coefficients are given by

$$\mathbf{t}^{r,\underline{\nu}}_\mu[G] = \frac{1}{\mu!}\,\partial_\mu\, G_{r,\underline{\nu}}(z,\overline{z})\,|_{z=0}$$

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References

[1] P. Appell, Sur une classe de polynomes, *Ann. Sci. Ec. Norm. Sup.* **9**, 1880, 119–144.

[2] S. Bock, *Über funktionentheoretische Methoden in der räumlichen Elastizitätstheorie (German)*, Ph.D thesis, Bauhaus University, Weimar, 2009.

[3] S. Bock, K. Gürlebeck, On a polynomial basis generated from the generalized Kolosov–Muskhelishvili formulae, *Adv. Appl. Clifford Alg.* **19**, 2009, 191–209.

[4] S. Bock, K. Guerlebeck, R. Lávička, V. Souček, The Gel'fand-Tsetlin bases for spherical monogenics in dimension 3, *Rev. Mat. Iberoamericana* **28(4)**, 2012, 1165–1192.

[5] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitian Clifford analysis – Part I: Complex structure, *Compl. Anal. Oper. Theory* **1** (3), 2007, 341–365.

[6] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen, V. Souček, Fundaments of Hermitian Clifford analysis – Part II: Splitting of h–monogenic equations, *Complex Var. Elliptic Eq.* **52** (10-11), 2007, 1063–1079.

[7] F. Brackx, B. De Knock, H. De Schepper, F. Sommen, On Cauchy and Martinelli-Bochner Formulae in Hermitian Clifford Analysis, *Bull. Braz. Math. Soc.* **40** (3), 2009, 395–416.

[8] F. Brackx, B. De Knock, H. De Schepper, A matrix Hilbert transform in Hermitian Clifford Analysis, *J. Math. Anal. Appl.* **344** (2), 2008, 1068–1078.

[9] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Publishers (Boston-London-Melbourne, 1982).

[10] F. Brackx, H. De Schepper, N. De Schepper, F. Sommen, Hermitian Clifford-Hermite polynomials, *Adv. Appl. Clifford Alg.* **17** (3), 2007, 311–330.

[11] F. Brackx, H. De Schepper, D. Eelbode, V. Souček, The Howe Dual Pair in Hermitian Clifford Analysis, *Rev. Mat. Iberoamericana* **26** (2), 2010, 449–479.

[12] F. Brackx, H. De Schepper, R. Lávička, V. Souček, The Cauchy–Kovalevskaya Extension Theorem in Hermitian Clifford Analysis, *J. Math. Anal. Appl.* **381** (2), 2011, 649–660.

[13] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Gel'fand-Tsetlin procedure for the construction of orthogonal bases in Hermitian Clifford analysis. In: T.E. Simos, G. Psihoyios, Ch. Tsitouras, *Numerical Analysis and Applied Mathematics*, AIP Conference Proceedings 1281, Rhodos, Greece (2010), 1508–1511.

[14] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Fischer decompositions of kernels of Hermitian Dirac operators. In: T.E. Simos, G. Psihoyios, Ch. Tsitouras, *Numerical Analysis and Applied Mathematics*, AIP Conference Proceedings 1281, Rhodos, Greece (2010), 1484–1487.

[15] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Orthogonal bases of Hermitian monogenic polynomials: an explicit construction in complex dimension 2. In: T.E. Simos, G. Psihoyios, Ch. Tsitouras, *Numerical Analysis and Applied Mathematics, AIP Conference Proceedings 1281*, Rhodos, Greece (2010), 1451–1454.

[16] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Gel'fand-Tsetlin Bases of Orthogonal Polynomials in Hermitian Clifford Analysis, *Math. Meth. Appl. Sci.* **34**, 2011, 2167–2180.

[17] F. Brackx, H. De Schepper, R. Lávička, V. Souček, Embedding Factors for Branching in Hermitian Clifford Analysis, accepted for publication in *Complex Anal. Oper. Theory*, DOI 10.1007/s11785-014-0365-3.

[18] F. Brackx, H. De Schepper, F. Sommen, The Hermitian Clifford analysis toolbox, *Adv. Appl. Cliff. Alg.* **18** (3-4), 2008, 451–487.

[19] F. Brackx, H. De Schepper, F. Sommen, A Theoretical Framework for Wavelet Analysis in a Hermitian Clifford Setting, *Comm. Pure Appl. Anal.* **6** (3), 2007, 549–567.

[20] F. Brackx, H. De Schepper, V. Souček, Fischer Decompositions in Euclidean and Hermitian Clifford Analysis, *Archivum Mathematicum (Brno)* **46** (5), 2010, 301–321.

[21] I. Cação, *Constructive approximation by monogenic polynomials*, PhD thesis, University of Aveiro (2004).

[22] I. Cação, K. Gürlebeck, SḂock, Complete orthonormal systems of spherical monogenics – a constructive approach. In: L. H. Son. W. Tutschke, S. Jain (eds.), *Methods of Complex and Clifford Analysis*, Proceedings of ICAM, SAS Int. Pub. (Hanoi, 2004).

[23] I. Caçao, K. Gürlebeck, S. Bock, On derivatives of spherical monogenics, *Compl. Var. Elliptic Equ.* **51**, 2006, 847–869.

[24] I. Caçao, K. Gürlebeck, H.R. Malonek, Special monogenic polynomials and L_2-approximation, *Adv. Appl. Clifford Alg.* **11**, 2001 (S2), 47–60.

[25] F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Birkhäuser (Boston, 2004).

[26] A. Damiano and D. Eelbode, Invariant Operators Between Spaces of h-Monogenic Polynomials, *Adv. Appl. Cliff. Alg.* **19** (2), 2009, 237–251.

[27] R. Delanghe, F. Sommen, V. Souček, *Clifford algebra and spinor-valued functions – A function theory for the Dirac operator*, Kluwer Academic Publishers (Dordrecht, 1992).

[28] D. Eelbode, Stirling numbers and Spin–Euler polynomials, *Exp. Math.* **16** (1), 2007, 55–66.

[29] D. Eelbode, Irreducible $\mathfrak{sl}(m)$–modules of Hermitian monogenics, *Complex Var. Elliptic Equ.* **53** (10), 2008, 975–987.

[30] D. Eelbode, Fu Li He, Taylor series in Hermitian Clifford Analysis, *Compl. Anal. Oper. Theory*, **5(1)**, 2011, 97–111.

[31] M.I. Falcão, J.F. Cruz, H.R. Malonek, Remarks on the generation of monogenic functions. In: K. Gürlebeck, C. Könke (eds.), *Proceedings of the 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering*, ISSN 1611-4086, Bauhaus University Weimar, 2006.

[32] M.I. Falcão, H.R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. In: T.E. Simos, G. Psihoyios, Ch. Tsitouras (eds.), *Numerical Analysis and Applied Mathematics*, AIP Conference Proceedings **vol. 936**, American Institute of Physics, Melville, NY, 2007, 750–753.

[33] I.M. Gelfand, M.L. Tsetlin, Finite-dimensional representations of groups of orthogonal matrices (Russian), *Dokl. Akad. Nauk SSSR* **71**, 1950, 1017–1020. English transl. in: I.M. Gelfand, *Collected papers* **Vol II**, Berlin, Springer-Verlag, 1988, 657–661.

[34] J. Gilbert, M. Murray, *Clifford Algebra and Dirac Operators in Harmonic Analysis*, Cambridge University Press (Cambridge, 1991).

[35] K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic functions in the plane and n-dimensional space. Translated from the 2006 German original. With 1 CD-ROM (Windows and UNIX)*, Birkhäuser Verlag (Basel, 2008).

[36] K. Gürlebeck, W. Sprößig, *Quaternionic and Clifford Calculus for Physicists and Engineers*, J. Wiley & Sons (Chichester, 1997).

[37] R. Lávička, Canonical bases for $\mathfrak{sl}(2,\mathbb{C})$-modules of spherical monogenics in dimension 3, *Archivum Mathematicum* **46** (5) 2010, 339–349.

[38] R. Lávička, Complete orthogonal Appell systems for spherical monogenics, *Compl. Anal. Oper. Theory* **6** (2) 2012, 477–489.

[39] R. Lávička, Orthogonal Appell bases for Hodge-de Rham systems in Euclidean spaces, *Adv. Appl. Clifford Alg.* **23(1)**, 2013, 113–124.

[40] R. Lávička, V. Souček, P. Van Lancker, Orthogonal basis for spherical monogenics by step two branching, *Ann. Glob. Anal. Geom.* **41** (2), 2012, 161-186.

[41] H.R. Malonek, M.I. Falcão, Special monogenic polynomials - properties and applications. In: T.E. Simos, G. Psihoyios, Ch. Tsitouras (eds.), *Numerical Analysis and Applied Mathematics*, AIP Conference Proceedings **vol. 936**, American Institute of Physics, Melville, NY, 2007, 764–767.

[42] A. I. Molev, Gel'fand–Tsetlin bases for classical Lie algebras. In: M. Hazewinkel (ed.), *Handbook of Algebra, vol.4*, Elsevier, (2006), 109–170.

[43] I. Porteous, *Clifford Algebras and the Classical groups*, Cambridge University Press (Cambridge, 1995).

[44] R. Rocha-Chavez, M. Shapiro, F. Sommen, *Integral theorems for functions and differential forms in $\mathbb{C}_m$*, Research Notes in Math. 428, Chapman&Hall / CRC (New York, 2002).

[45] I. Sabadini, F. Sommen, Hermitian Clifford analysis and resolutions, *Math. Meth. Appl. Sci.* **25**(16-18), 2002, 1395–1414.

[46] F. Sommen, D. Peña Peña, A Martinelli-Bochner formula for the Hermitian Dirac equation, *Math. Meth. Appl. Sci.* **30** (9), 2007, 1049–1055.

[47] P. Van Lancker, Spherical Monogenics: An Algebraic Approach, *Adv. Appl. Clifford Alg.* **19**, 2009, 467–496.

[48] N. Ja. Vilenkin, A. U. Klimyk, *Representation of Lie Groups and Special Functions, Volume 2: Class I Representations, Special Functions, and Integral Transforms*, Kluwer Academic Publishers (Dordrecht – Boston – London, 1993).