Goodstein sequences for prominent ordinals up to the ordinal of $\Pi^1_1-\mathrm{CA}_0$

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Abstract

We introduce strong Goodstein principles which are true but unprovable in strong impredicative theories like ID_n.

Keywords: Goodstein sequence, proof-theoretic ordinal, unprovability.

1. Introduction

Goodstein sequences provide examples for strictly mathematical statements which are true (by Goodstein, see [Goo44]) but (according to Kirby and Paris, see [KP82]) not provable in PA. In the 80s several attempts have been made to define Goodstein principles capturing larger complexities using Π^1_2-logic. Unfortunately, even slight extensions of the original Goodstein principle led in some articles (see for instance [Abr89]) to somewhat messy expositions which were not completely transparent, at least from our point of view.

Quite recently an alternative and transparent method to generate Goodstein principles has been provided by De Smet and Weiermann in [DSW12]. Their Goodstein principles ranged in strength between Peano Arithmetic (PA) and the theory ID_1 of non-iterated monotone inductive definitions, and they asked whether an extension to the theories ID_n was possible. In this article we provide an affirmative answer by elementary calculations based on Buchholz style tree ordinals and a trick suggested by Cichon, see [Cic83].

There is some indication that Goodstein principles have no canonical extension to a strength beyond ID_ν and we expect having reached a canonical limit for strong Goodstein principles.

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[1] This author's research is supported in part by Fonds Wetenschappelijk Onderzoek (FWO) and the John Templeton Foundation. Parts of the research related to this article have been carried out during a visit of this author at the Isaac Newton Institute, Cambridge, UK in January 2012.

2. Tree Ordinals

We introduce tree ordinals, following lecture notes by Wilfried Buchholz. Minor technical modifications are motivated by our specific purposes.

Definition 2.1. Inductive definition of classes $\mathbb{T}_i$, $i < \omega$, of tree ordinals.

1. $\mathbf{0} := () \in \mathbb{T}_i$.
2. $\boldsymbol{\alpha} \in \mathbb{T}_i \;\Rightarrow\; \boldsymbol{\alpha} + \mathbf{1} := (\boldsymbol{\alpha}) \in \mathbb{T}_i$.
3. $\forall n \in \mathbb{N}(\boldsymbol{\alpha}_n \in \mathbb{T}_i) \;\Rightarrow\; (\boldsymbol{\alpha}_n)_{n\in\mathbb{N}} \in \mathbb{T}_i$.
4. $j < i \;\&\; \forall \boldsymbol{\xi} \in \mathbb{T}_j(\boldsymbol{\alpha}_{\boldsymbol{\xi}} \in \mathbb{T}_i) \;\Rightarrow\; (\boldsymbol{\alpha}_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in\mathbb{T}_j} \in \mathbb{T}_i$.

The set of tree ordinals, denoted by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, etc., is thus given by

$$\mathbb{T}_{<\omega} := \bigcup_{i<\omega} \mathbb{T}_i.$$

We also use the notation $\mathbf{1} := (()) = \mathbf{0} + \mathbf{1}$.

Note that every $\boldsymbol{\alpha} \in \mathbb{T}_i$ is of a form $(\boldsymbol{\alpha}_\iota)_{\boldsymbol{\iota}\in I}$ where I is one of the sets $\emptyset, \{0\}, \mathbb{N}$, or $\mathbb{T}_j$ for some $j < i$. We define

$$\|(\boldsymbol{\alpha}_\iota)_{\iota\in I}\| := \sup_{\iota\in I}(\|\boldsymbol{\alpha}_\iota\| + 1).$$

By transfinite induction on $\|\boldsymbol{\alpha}\|$ it is easy to show that $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\iota)_{\iota\in I} \in \mathbb{T}_i$ implies $\boldsymbol{\alpha}_\iota \in \mathbb{T}_i$ for all $\iota \in I$.

We introduce the following abbreviations:

$$\underline{0} := \mathbf{0}, \quad \underline{n+1} := \underline{n} + \mathbf{1}$$

and

$$\Omega_0 := (\underline{n})_{n\in\mathbb{N}}, \quad \Omega_{i+1} := (\boldsymbol{\xi})_{\boldsymbol{\xi}\in\mathbb{T}_i},$$

so that $\Omega_i \in \mathbb{T}_i - \bigcup_{j<i} \mathbb{T}_j$. We will sometimes write ω for both $\underline{\omega} := \Omega_0$ and $\mathbb{N}$, assuming that ambiguity is excluded by context. Likewise, we will sometimes identify Ω_{i+1} with $\mathbb{T}_i$.

Addition is defined by

$$\boldsymbol{\alpha} + \mathbf{0} := \boldsymbol{\alpha}, \quad \boldsymbol{\alpha} + (\boldsymbol{\beta}_\iota)_{\iota\in I} := (\boldsymbol{\alpha} + \boldsymbol{\beta}_\iota)_{\iota\in I} \;\text{ if } I \neq \emptyset,$$

consistent with the above definition of the special case $\boldsymbol{\alpha} + \mathbf{1}$, and multiples are defined by

$$\boldsymbol{\alpha} \cdot 0 := \mathbf{0}, \quad \boldsymbol{\alpha} \cdot (n+1) := (\boldsymbol{\alpha} \cdot n) + \boldsymbol{\alpha}.$$

Proposition 2.2. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{T}_{<\omega}$.

1. $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{T}_i \;\Rightarrow\; \boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathbb{T}_i$.
2. $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma}) = (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma}$.

Definition 2.3. We define mappings $\mathbb{D}_i : \mathbb{T}_{<\omega} \to \mathbb{T}_i$ simultaneously for $i \in \mathbb{N}$. $\mathbb{D}_i(\boldsymbol{\alpha})$ is defined by transfinite recursion on $\|\boldsymbol{\alpha}\|$ as follows:

1. $\mathbb{D}_i(\mathbf{0}) := \Omega_i$.
2. $\mathbb{D}_i(\boldsymbol{\alpha}+\mathbf{1}) := \mathbb{D}_i(\boldsymbol{\alpha}) + \mathbb{D}_i(\boldsymbol{\alpha}) + \mathbf{1}$.
3. $\mathbb{D}_i((\boldsymbol{\alpha}_\iota)_{\iota\in I}) := (\mathbb{D}_i(\boldsymbol{\alpha}_\iota))_{\iota\in I}$ if $I \in \{\mathbb{N}\} \cup \{\mathbb{T}_j \mid j < i\}$.
4. $\mathbb{D}_i((\boldsymbol{\alpha}_\iota)_{\iota\in\mathbb{T}_j}) := \mathbb{D}_i(\boldsymbol{\alpha}_{\mathbb{D}_j(\boldsymbol{\alpha}_{\Omega_j})})$ if $j \geq i$.

Remark 2.4. We generally have

$$\mathrm{Im}(\mathbb{D}_{i+1}) \subseteq \mathbb{T}_{i+1} - \mathbb{T}_i.$$

Clause 2 in the above definition has been chosen in order to have direct access to exponentiation. This is crucial to approximate the appropriate epsilon numbers in applications. The following proof would go through also for the modified version $\mathbb{D}_i(\alpha+1) := \mathbb{D}_i\alpha + 1$ but then the underlying semantics in terms of order types would be affected. In the next definition Clause 4 will only be used for $j = i$.

With these preparations we may now define a set of terms for tree ordinals.

Definition 2.5. The set BT of terms for tree ordinals is defined inductively as follows. We simultaneously define the level $\mathrm{lv}(\alpha)$ of a term $\alpha \in \mathrm{BT}$.

1. $0, 1 \in \mathrm{BT}$, and
   $\mathrm{lv}(\alpha) := 0$ for $\alpha = 0, 1$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathrm{BT} - \{0\}$ where $n > 1$, then $\alpha := \sum_{i=1}^n \alpha_i \in \mathrm{BT}$, and
   $\mathrm{lv}(\alpha) := \max\{\mathrm{lv}(\alpha_i) \mid 1 \leq i \leq n\}$.
3. If $\alpha \in \mathrm{BT}$ with $\mathrm{lv}(\alpha) \leq i+1$, then $D_i\alpha \in \mathrm{BT}$, and
   $\mathrm{lv}(D_i\alpha) := i$.

We introduce the following notations.

$$\mathrm{BT}^{\leq i} := \{\alpha \in \mathrm{BT} \mid \mathrm{lv}(\alpha) \leq i\}, \quad \mathrm{BT}^i := \{\alpha \in \mathrm{BT} \mid \mathrm{lv}(\alpha) = i\}.$$

The *canonical interpretation* of $\alpha \in \mathrm{BT}$, sometimes written as $\underline{\alpha}$, is given in the obvious way, interpreting terms $D_i\alpha$ by $\mathbb{D}_i(\underline{\alpha})$ where $\underline{\alpha}$ is the canonical interpretation of α.

If ambiguity is unlikely, we will tacitly use canonical interpretations when dealing with terms in BT or writing BT-terms instead of tree ordinals.

Lemma 2.6. Let $\alpha \in \mathrm{BT}$. We have

$$\underline{\alpha} \in \mathbb{T}_i \;\Leftrightarrow\; \mathrm{lv}(\alpha) \leq i.$$

Proof. Trivial induction on the buildup of $\alpha \in \mathrm{BT}$. $\square$

Definition 2.7. Before defining $\mathrm{tp} : \mathrm{BT} \to \{0, 1, \omega\} \cup \{\mathbb{T}_i \mid i \in \mathbb{N}\}$ and $\alpha[\boldsymbol{\xi}] \in \mathbb{T}_{<\omega}$ for $\alpha \in \mathrm{BT}$ and $\boldsymbol{\xi} \in \mathrm{tp}(\alpha)$, we introduce a few standard conventions in order to ease notation. For the range of tp we declare the ordering

$$0 < 1 < \omega < \mathbb{T}_0 < \mathbb{T}_1 < \ldots,$$

for $\mathrm{tp}(\alpha) = \omega$ we sometimes write $\mathrm{tp}(\alpha) = \Omega_0$, and for $\mathrm{tp}(\alpha) = \mathbb{T}_i$ we also write $\mathrm{tp}(\alpha) = \Omega_{i+1}$. We then also declare the ordering

$$0 < 1 < \Omega_0 < \Omega_1 < \ldots.$$

1. $\mathrm{tp}(0) := 0$.
2. $\mathrm{tp}(1) := 1$, and $1[0] := 0$.
3. $\mathrm{tp}(\alpha) := \mathrm{tp}(\alpha_n)$, if $\alpha = \sum_{i=1}^{n} \alpha_i$ with $n > 1$, and for $\boldsymbol{\xi} \in \mathrm{tp}(\alpha_n)$
$$\alpha[\boldsymbol{\xi}] := \alpha_1 + \ldots + \alpha_{n-1} + \alpha_n[\boldsymbol{\xi}].$$
4. $\mathrm{tp}(D_i 0) := \Omega_i$, and for $\boldsymbol{\xi} \in \Omega_i$
$$(D_i 0)[\boldsymbol{\xi}] := \boldsymbol{\xi}.$$
5. If $\mathrm{tp}(\alpha) = 1$ and $\mathrm{lv}(\alpha) \leq i + 1$, then $\mathrm{tp}(D_i \alpha) := 1$ and
$$(D_i \alpha)[0] := D_i(\alpha[0]) + D_i(\alpha[0]).$$
6. If $\mathrm{tp}(\alpha) \in \{\Omega_j \mid j \leq i\}$ and $\mathrm{lv}(\alpha) \leq i + 1$, then $\mathrm{tp}(D_i \alpha) := \mathrm{tp}(\alpha)$, and for $\boldsymbol{\xi} \in \mathrm{tp}(\alpha)$
$$(D_i \alpha)[\boldsymbol{\xi}] := D_i(\alpha[\boldsymbol{\xi}]).$$
7. If $\mathrm{tp}(\alpha) \in \{\Omega_{j+1} \mid j \geq i\}$ and $\mathrm{lv}(\alpha) \leq i + 1$, then $\mathrm{tp}(D_i \alpha) := \Omega_i$, and for $\boldsymbol{\xi} \in \Omega_i$
$$(D_i \alpha)[\boldsymbol{\xi}] := D_i(\alpha[D_j(\alpha[\boldsymbol{\xi}])]).$$

In the case $\mathrm{tp}(\alpha) = 1$, for convenience we set $\alpha[k] := \alpha[0]$, $k \in \omega$. In the case $\mathrm{tp}(\alpha) = 0$ we similarly set $\alpha[k] := 0$.

Tree ordinals are by definition identified with their fundamental sequences. Notice that $\mathrm{tp}(\alpha)$ can be understood as the domain of the canonical interpretation of α. The notation $\cdot[\cdot]$ makes the fundamental sequences of the canonical interpretations of BT-terms visible.

In general there does not always exist a term in BT whose canonical interpretation is $\alpha[\boldsymbol{\xi}]$. However, in case there is, the above definition clearly indicates which BT-term has to be taken. This will be made precise by the next lemma.

Lemma 2.8. Let $\alpha, \xi \in \mathrm{BT}$ and identify ξ with its canonical interpretation $\underline{\xi}$. Suppose $\xi \in \mathrm{tp}(\alpha)$. Then we have
$$\alpha[\xi] \in \mathrm{BT} \quad \text{and} \quad \mathrm{lv}(\alpha[\xi]) \leq \mathrm{lv}(\alpha)$$
where we have identified $\alpha[\xi] \in \mathbb{T}_{<\omega}$ with the term indicated in Definition 2.7.

Proof. Using Lemma 2.6, we proceed by trivial induction along the buildup of α. □

Lemma 2.9. Let $\alpha \in \mathrm{BT}$ be such that $\underline{\alpha} \in \mathbb{T}_i$ for some $i \in \mathbb{N}$. Then we have
$$\mathrm{tp}(\alpha) \in \mathbb{T}_i.$$
Since in general $\alpha \in \mathbb{T}_{\mathrm{lv}(\alpha)}$, it follows that
$$\mathrm{tp}(\alpha) < \mathbb{T}_{\mathrm{lv}(\alpha)}.$$

Proof. The lemma follows immediately from the definitions involved. $\square$

Remark 2.10. Notice that, due to the condition $\mathrm{lv}(\alpha) \leq i+1$ and the above lemma, Clause 3 of Definition 2.5 implies that terms $D_i\alpha \in \mathrm{BT}$ always satisfy $\mathrm{tp}(\alpha) \leq \Omega_{i+1}$. This shows that Clause 7 of Definition 2.7 can be simplified to

7.' If $\mathrm{tp}(\alpha) = \Omega_{i+1}$ and $\mathrm{lv}(\alpha) = i+1$, then $\mathrm{tp}(D_i\alpha) := \Omega_i$, and for $\boldsymbol{\xi} \in \Omega_i$

$$(D_i\alpha)[\boldsymbol{\xi}] := D_i(\alpha[D_i(\alpha[\boldsymbol{\xi}])]),$$

where the condition $\mathrm{lv}(\alpha) \leq i+1$ has cristallized to $\mathrm{lv}(\alpha) = i$ by Lemma 2.6.

Lemma 2.11. Let $\alpha \in \mathrm{BT}$ be such that $\mathrm{tp}(\alpha) = \Omega_j$, and let $\beta \in \mathrm{BT}$ be such that $\mathrm{lv}(\beta) < j$ and $\mathrm{tp}(\beta) = \Omega_i$ where $i < j$. Then we have

$$\mathrm{tp}(\alpha[\beta]) = \Omega_i.$$

Proof. The proof proceeds again by induction along the buildup of α. The interesting case is where $\alpha = D_j(\gamma)$ for some γ such that $\mathrm{tp}(\gamma) = \Omega_{j+1}$ and $\mathrm{lv}(\gamma) = j+1$. We then have $\alpha[\beta] = D_j(\gamma[D_j(\gamma[\beta])])$, and by definition and i.h. we have $\mathrm{tp}(D_j(\gamma[\beta])) = \mathrm{tp}(\gamma[\beta]) = \Omega_i$. We apply the i.h. again to obtain $\mathrm{tp}(\gamma[D_j(\gamma[\beta])]) = \Omega_i$ which implies that $\mathrm{tp}(\alpha[\beta]) = \Omega_i$. $\square$

3. A Term Rewriting System to Base k

We now define a modified version of the Grzegorczyk hierarchy along countable tree ordinals represented as BT-terms. The argument k will serve as the base parameter in the generalized Goodstein process that we are going to define later. The approach to handle base-k representations via term rewriting is as in [DSW12].

Definition 3.1. For $k \in \mathbb{N} - \{0\}$, $\alpha, \lambda \in \mathrm{BT}^0$ with $\mathrm{tp}(\lambda) = \omega$, define

$$\begin{aligned} B_0(k) &:= k \\ B_{\alpha+1}(k) &:= B_\alpha(k) \cdot 2 + 1 \\ B_\lambda(k) &:= B_{\lambda[B_{\lambda[k]}(k)]}(k). \end{aligned}$$

Definition 3.2. Let $k > 0$ be fixed. We define the following four sets of terms: (principal) k-terms and (countable) ordinal k-terms. Any principal k-term is a k-term, any k-term is a countable ordinal k-term, which in turn is an ordinal k-term. The *level* $\mathrm{lv}(t)$ of an ordinal k-term t is defined simultaneously, compatibly with the corresponding definition for BT-terms, cf. Definition 2.5, by setting $\mathrm{lv}(t) := 0$ if t is a k-term. Countable ordinal k-terms will thus be characterized as the ordinal k-terms of level 0.

1. 0 is a k-term.
2. 1 is a principal k-term.
3. If α is a countable ordinal k-term, then $B_\alpha(k)$ is a principal k-term.

4. Suppose $\alpha_1, \dots, \alpha_n$ ($n > 0$) are ordinal k-terms such that each α_i is either a principal k-term or a term of the form $D_i\beta$ where β is an ordinal k-term such that $\mathrm{lv}(\beta) \le i + 1$, then

$$\alpha := \alpha_1 + \ldots + \alpha_n$$

is an ordinal k-term. If each term of a form $D_i\beta$ among $\alpha_1, \dots, \alpha_n$ satisfies $i = 0$, then α is a countable ordinal k-term. If all terms $\alpha_1, \dots, \alpha_n$ are k-terms, then α is a k-term.

Let T_k denote the set of k-terms and OT_k denote the set of ordinal k-terms.

Since any term t of the form $B_\alpha(k)$ is evaluated as a nonzero natural number n we may identify t with the corresponding tree ordinal denoting n and set $\mathrm{tp}(B_\alpha(k)) := 1$ and $t[0] := n - 1$. We also may identify any term t of a form $(t_1, \dots, t_n)$ with a natural number, namely the sum over the evaluations of the terms t_i. We thus obtain a set of terms for tree ordinals, OT_k, compatibly extending BT.

The letters s, t, u (with or without indices) range over k-terms and the letters α, β, γ (with or without indices) more generally range over ordinal k-terms.

Definition 3.3 ($\to$). We evaluate k-terms, where $k > 0$, partially, following a canonical and deterministic evaluation strategy. In each clause below suppose $n \in \mathbb{N}$ and $\alpha, \beta, \alpha_1, \dots, \alpha_n \in \mathrm{OT}_k$.

1. $B_0(k) \to k$.
2. $B_{\alpha+1}(k) \to B_\alpha(k) + B_\alpha(k) + 1$.
3. If α is $\to$-irreducible and countable and not of a form $\beta + 1$ or 0, then $B_\alpha(k) \to B_{\alpha[B_{\alpha[k]}(k)]}(k)$.
4. $\alpha_1 + \cdots + \alpha_n + D_i(\alpha + 1) \to \alpha_1 + \cdots + \alpha_n + D_i\alpha + D_i\alpha + 1$.
5. If $\alpha \to \beta$ then $B_\alpha(k) \to B_\beta(k)$.
6. If $\alpha \to \beta$ then $D_i\alpha \to D_i\beta$.
7. If $\alpha \to \beta$ then $\alpha_1 + \cdots + \alpha_n + \alpha \to \alpha_1 + \cdots + \alpha_n + \beta$.

Let $\to^*$ be the reflexive transitive closure of $\to$.

Lemma 3.4. If t is a nonzero k-term, then either $t \to^* 1$ or there is a unique term s such that $t \to^* s + 1$.

Proof. The relation $\to$ terminates under a monotone intepretation with ordinals below the ordinal of $\mathrm{ID}_{<\omega}$. Normal forms of terms are either of the form 0, 1, or $s + 1$. $\square$

Definition 3.5. Let $k, l \in \mathbb{N}$. We will define the change-of-base mapping

$$\cdot[k \leftarrow l] : \mathrm{OT}_k \to \mathrm{OT}_l$$

such that the change of base of k-terms results in l-terms. For convenience we use the abbreviation α' for $\alpha[k \leftarrow l]$.

1. $0' := 0$.

2. $1' := 1$.
3. $(B_\alpha(k))' := B_{\alpha'}(l)$.
4. $(\alpha_1 + \ldots + \alpha_n)' := \alpha_1' + \ldots + \alpha_n'$.

Thus the argument k of each B-(sub-)term is replaced by l.

The above change of base is the analogue of the original change of base in the Goodstein process, the subtraction of 1 in the Goodstein process has its analogue in the following definition which is based on Lemma 3.4.

Definition 3.6. The function $P : \mathrm{T}_k \to \mathrm{T}_k$ is defined as follows. Set $P0 := 0$. For nonzero $t \in \mathrm{T}_k$ let s be the unique k-term such that $t \to^* s + 1$, if that exists, and $s := 0$ otherwise. Then $Pt := s$.

4. Pointwise Collapsing and Change of Base

We now define the mapping which plays the central role in this article.

Definition 4.1. The pointwise collapsing functions $C_k : \mathrm{BT} \to \mathrm{OT}_k$ for $k > 0$, are defined as follows.

1. $C_k 0 := 0$.
2. $C_k 1 := 1$.
3. $C_k(\sum_{i=1}^n \alpha_i) := \sum_{i=1}^n C_k \alpha_i$.
4. $C_k D_0 \alpha := B_{C_k \alpha}(k)$.
5. $C_k D_{i+1} \alpha := D_i C_k \alpha$.

The following lemma addresses the relationship of C_k with tp and lv.

Lemma 4.2. Let $\alpha \in \mathrm{BT}$ and $k > 0$.

1. $C_k \alpha = 0 \;\Leftrightarrow\; \alpha = 0$.
2. $\mathrm{tp}(\alpha) = 1 \;\Rightarrow\; \mathrm{tp}(C_k \alpha) = 1$.
3. $\mathrm{lv}(\alpha) = 0 \;\Rightarrow\; \mathrm{lv}(C_k \alpha) = 0 \;\&\; C_k \alpha \in \omega$.
4. $\mathrm{lv}(\alpha) = i + 1 \;\Rightarrow\; \mathrm{lv}(C_k \alpha) = i$.
5. $\mathrm{tp}(\alpha) = \Omega_{i+1} \;\Rightarrow\; \mathrm{tp}(C_k \alpha) = \Omega_i$.

Proof. The proof is by induction on the buildup of α. Parts 1-4 are trivial. As for part 5, we consider the most interesting case $\alpha = D_{i+1}\gamma$ where $\mathrm{tp}(\gamma) = \Omega_{i+2}$ and $\mathrm{lv}(\gamma) = i+2$, cf. Clause 7.' of Remark 2.10. By Part 4 we have $\mathrm{lv}(C_k\gamma) = i+1$, and the i.h. yields $\mathrm{tp}(C_k\gamma) = \Omega_{i+1}$. We have $C_k\alpha = D_i C_k \gamma$, hence by 7.' $\mathrm{tp}(C_k\alpha) = \Omega_i$. □

We have the following crucial lemma regarding changes of base.

Lemma 4.3. Let $\alpha \in \mathrm{BT}$ and $k > 0$. Then we have

$$(C_k\alpha)[k \leftarrow k+1] = C_{k+1}\alpha.$$

Proof. Straightforward induction on the buildup of α. □

5. Collapsing and Fundamental Sequences

In this section we establish the key Lemma 5.2 that will allow us to apply "Cichon's trick", cf.[Cic83]. For Lemma 5.2 we will need the following

Lemma 5.1. Let $k > 0$ and suppose $\lambda \in \mathrm{BT}$ satisfies $\mathrm{tp}(\lambda) = \Omega_{j+1}$. We have

1. $C_k\lambda$ is $\to$-irreducible and neither 0 nor of a form $\alpha + 1$.
2. For any $\beta \in \mathrm{BT}$ such that $\mathrm{lv}(\beta) \leq j$ and $\mathrm{tp}(\beta) < \Omega_{j+1}$

$$C_k(\lambda[\beta]) = (C_k\lambda)[C_k\beta].$$

Proof. We first show that

$$C_k\beta \in \Omega_j.$$

In the case $j > 0$ parts 3 and 4 of Lemma 4.2 yield $\mathrm{lv}(C_k\beta) < j$, which by Lemma 2.6 implies that $C_k\beta \in \Omega_j$. If on the other hand $j = 0$, part 3 of Lemma 4.2 directly allows us to conclude that $C_k\beta \in \omega$. We now show the lemma by induction on the buildup of $\lambda \in \mathrm{BT}$.

Case $\lambda = D_{j+1}0$.
Then $C_k\lambda = D_j0$ and $\lambda[\beta] = \beta$, and both claims are immediate.

Case $\lambda = D_{i+1}\gamma$ and $\mathrm{tp}(\gamma) = \Omega_{j+1}$, $\mathrm{lv}(\gamma) \leq i + 2$, where $j \leq i$.
Notice first, that $C_k(\lambda)$ is $\to$-irreducible since $C_k(\gamma)$ is $\to$-irreducible by the i.h. We proceed to show the second claim. We have $C_k\lambda = D_iC_k\gamma$ and $\lambda[\beta] = D_{i+1}(\gamma[\beta])$. Lemma 4.2 yields $\mathrm{tp}(C_k\gamma) = \Omega_j$ and $\mathrm{lv}(C_k\gamma) \leq i + 1$. Using the induction hypothesis we obtain

$$\begin{aligned} C_k(\lambda[\beta]) &= D_iC_k(\gamma[\beta]) \\ &= D_i((C_k\gamma)[C_k\beta]) \\ &= (D_iC_k\gamma)[C_k\beta] \\ &= (C_k\lambda)[C_k\beta]. \end{aligned}$$

Case $\lambda = D_{j+1}\gamma$ and $\mathrm{tp}(\gamma) = \Omega_{j+2}$, $\mathrm{lv}(\gamma) = \Omega_{j+2}$.
Again, $C_k\lambda$ is $\to$-irreducible since $C_k\gamma$ is $\to$-irreducible. We have $C_k\lambda = D_jC_k\gamma$ and $\lambda[\beta] = D_{j+1}(\gamma[D_{j+1}(\gamma[\beta])])$. Notice that we may apply the i.h. to γ, both with β and $D_{j+1}(\gamma[\beta])$ in the role of β in the original statement, since for the latter we clearly have $\mathrm{lv}(D_{j+1}(\gamma[\beta])) = j + 1$ and $\mathrm{tp}(D_{j+1}(\gamma[\beta])) < \Omega_{j+2}$. By Lemma 4.2, parts 4 and 5, we further have $\mathrm{tp}(C_k\gamma) = \Omega_{j+1}$ and $\mathrm{lv}(C_k\gamma) = j+1$, which together with the initially shown $C_k\beta \in \Omega_j$ allows us to apply Clause 7.' of Remark 2.10. We therefore have

$$\begin{aligned} (C_k\lambda)[C_k\beta] &= (D_jC_k\gamma)[C_k\beta] \\ &= D_j((C_k\gamma)[D_j((C_k\gamma)[C_k\beta])]) \quad \text{by 7.'} \\ &= D_j((C_k\gamma)[D_j(C_k(\gamma[\beta]))]) \quad \text{by i.h.} \\ &= D_j((C_k\gamma)[C_k(D_{j+1}(\gamma[\beta]))]) \\ &= D_j(C_k(\gamma[D_{j+1}(\gamma[\beta])])) \quad \text{by i.h.} \\ &= C_k(\lambda[\beta]). \end{aligned}$$

If λ is a sum then C_k distributes accordingly, and the claim follows. $\square$

Lemma 5.2. For all $k > 0$ and all $\lambda \in \mathrm{BT}$ such that $\mathrm{tp}(\lambda) = \omega$ we have

$$C_k\lambda \to C_k(\lambda[k]).$$

Proof. By induction on the buildup of $\lambda \in \mathrm{BT}$.

CASE $\lambda = D_0 0$. Then

$$C_k\lambda = B_0(k)$$

and

$$C_k(\lambda[k]) = C_k k = k$$

The definition of $\to$ yields the assertion.

CASE $\lambda = D_{i+1}\gamma$ and $\mathrm{tp}(\gamma) = \omega$. Then

$$C_k D_{i+1}\gamma = D_i C_k \gamma$$

and

$$C_k((D_{i+1}\gamma)[k]) = D_i C_k(\gamma[k])$$

By the induction hypothesis, $C_k\gamma \to C_k(\gamma[k])$, and so the assertion follows.

CASE $\lambda = D_0\gamma$ and $\mathrm{tp}(\gamma) = \omega$. Then

$$C_k D_0\gamma = B_{C_k\gamma}(k)$$

and

$$C_k((D_0\gamma)[k]) = B_{C_k(\gamma[k])}(k)$$

By the induction hypothesis, $C_k\gamma \to C_k(\gamma[k])$, and so the assertion follows.

CASE $\lambda = D_0\gamma$ and $\mathrm{tp}(\gamma) = \Omega_1$, $\mathrm{lv}(\gamma) = 1$. Then by Claim 1 of Lemma 5.1 $C_k\gamma$ is $\to$-irreducible and neither 0 nor of a form $\beta + 1$. Thus

$$\begin{aligned} C_k\lambda &= C_k D_0\gamma \\ &= B_{C_k\gamma}(k) \\ &\to B_{(C_k\gamma)[B_{(C_k\gamma)[k]}(k)]}(k). \end{aligned} \tag{1}$$

According to 7.' of Remark 2.10 we have

$$(D_0\gamma)[k] = D_0(\gamma[D_0(\gamma[k])]), \tag{2}$$

and by Lemma 5.1 we have

$$\begin{aligned} C_k(\gamma[D_0(\gamma[k])]) &= (C_k\gamma)[C_k D_0(\gamma[k])] \\ &= (C_k\gamma)[B_{C_k(\gamma[k])}(k)] \end{aligned} \tag{3}$$

as well as

$$C_k(\gamma[k]) = (C_k\gamma)[k] \tag{4}$$

since $C_k k = k$. We thus obtain

$$\begin{aligned}
C_k(\lambda[k]) &= C_k((D_0\gamma)[k]) \\
&= C_k(D_0(\gamma[D_0(\gamma[k])])) \quad \text{by (2)} \\
&= B_{C_k(\gamma[D_0(\gamma[k])])}(k) \\
&= B_{(C_k\gamma)[B_{C_k(\gamma[k])}(k)]}(k) \quad \text{by (3)} \\
&= B_{(C_k\gamma)[B_{(C_k\gamma)[k]}(k)]}(k) \quad \text{by (4), which is (1),}
\end{aligned}$$

hence

$$C_k\lambda \to C_k(\lambda[k]).$$

Finally, if λ is a sum then C_k and $\cdot[\cdot]$ distribute accordingly, and the claim follows. □

6. Goodstein-Sequences for $\mathbf{ID}_n$ and $\mathbf{\Pi^1_1{-}CA_0}$

The next definition will allow us to model subtraction by 1 in the Goodstein process in the context of ordinal terms in BT.

Definition 6.1. For $k > 0$ we define $P_k : \mathrm{BT}^0 \to \mathrm{BT}^0$ according to the following clauses.

1. $P_k 0 := 0$.
2. $P_k\alpha := \beta$ if $\alpha = \beta + 1$ for some $\beta \in \mathrm{BT}$, i.e. $\mathrm{tp}(\alpha) = 1$.
3. $P_k\alpha := P_k(\alpha[k])$ if $\mathrm{tp}(\alpha) = \omega$.

Notice that $\mathrm{tp}(\alpha) \le \omega$ follows from $\mathrm{lv}(\alpha) = 0$ by Lemma 2.9. Note further that if α is of the form $D_0(\gamma + 1)$ we have $P_k\alpha = D_0\gamma + D_0\gamma$.

Lemma 6.2. Let $\alpha \in \mathrm{BT}^0$ and $k > 0$. Then we have

$$C_k P_k\alpha = PC_k\alpha.$$

Proof. We argue by induction on $\|\alpha\|$, identifying α with its canonical interpretation in $\mathbb{T}_{<\omega}$.

Case $\alpha = 0$. Trivial.

Case $\alpha = \beta + 1$. Then we have $P_k\alpha = \beta$, hence $C_k\alpha = C_k\beta + 1$, and thus $PC_k\alpha = C_k\beta$. In particular, for α of the form $D_0(\gamma + 1)$ we obtain $PC_k\alpha = PB_{C_k\gamma+1}(k) = B_{C_k\gamma}(k) \cdot 2$.

Case $\alpha \in \mathrm{Lim}$, i.e. $\mathrm{tp}(\alpha) = \omega$. We then have $P_k\alpha = P_k(\alpha[k])$. By Lemma 2.8 we have $\mathrm{lv}(\alpha[k]) = 0$, and since $\|\alpha[k]\| < \|\alpha\|$, using the i.h. we obtain

$$\begin{aligned}
C_k P_k\alpha &= C_k P_k(\alpha[k]) \\
&= PC_k(\alpha[k]) \\
&= PC_k\alpha
\end{aligned}$$

by the definition of P, since $C_k\alpha \to C_k(\alpha[k])$ according to Lemma 5.2. □

By the work of Buchholz in [B81, B87] we know that

$$\sup_{m\in\mathbb{N}} \|D_0 \dots D_m 0\| = |\mathrm{ID}_{<\omega}| = |\Pi^1_1\text{−CA}_0| =: \tau_0$$

and that, setting

$$\sigma_n := \sup_{m\in\mathbb{N}} \|D_0 \dots D_n D_n^{(m)} 0\|$$

for $n \in \mathbb{N}$, where $D_n^{(m)}0$ is the m-fold application of D_n to 0, we have

$$\sigma_0 = |\mathrm{PA}| = \varepsilon_0$$

and

$$\sigma_{n+1} = |\mathrm{ID}_{n+1}|.$$

It is well-established that defining the fast-growing hierarchy $h_\alpha : \mathbb{N} \to \mathbb{N}$ for $\alpha \in \mathrm{BT}^0 \cup \{\tau_0\}$ by

1. $h_0(x) := x$,
2. $h_{\alpha+1}(x) := h_\alpha(x) + 1$,
3. $h_\lambda(x) := h_{\lambda[x]}(x)$ for $\lambda \in \mathrm{Lim}$,

where $\tau_0[m] := D_0 \dots D_m 0$, the function h_{τ_0} is not provably recursive in the theory $\mathrm{PA} + \mathrm{TI}({<}\tau_0)$ and hence not provably recursive in $\Pi^1_1\text{−CA}_0$. Furthermore, h_{σ_n} is not provably recursive in the theory $\mathrm{PA} + \mathrm{TI}({<}\sigma_n)$, cf. [FS95].

Lemma 6.3. Let $\alpha \in \mathrm{BT}^0$ be nonzero, and let $k \geq 2$. Setting

$$s(\alpha, k) := \min\{l > k \mid P_l P_{l-1} \dots P_{k+1}\alpha = 0\}$$

we have

$$s(\alpha, k) = h_\alpha(k + 1) - 1.$$

Proof. We proceed by induction on $\|\alpha\|$, as in [DSW12].

Case $\alpha = 1$. Then we have $P_{k+1}1 = 0$ and $s(1, k) = k + 1 = h_1(k + 1) - 1$.

Case $\alpha = \beta + 1$ for some nonzero $\beta \in \mathrm{BT}$. Then $P_{k+1}\alpha = \beta$, hence by i.h.

$$s(\alpha, k) = s(\beta, k + 1) = h_\beta(k + 2) - 1 = h_\alpha(k + 1) - 1.$$

Case $\alpha \in \mathrm{Lim}$, i.e. $\mathrm{tp}(\alpha) = \omega$. We then have

$$P_{k+1}\alpha = P_{k+1}(\alpha[k + 1]) = h_{\alpha[k+1]}(k + 1) - 1 = h_\alpha(k + 1) - 1$$

by the i.h. □

Definition 6.4. We define canonical Goodstein processes for the theories ID_n, where $\mathrm{ID}_0 := \mathrm{PA}$, and Π^1_1-CA_0 which are parametrized in $m \in \mathbb{N}$, by the sequences $(t^n_k(m))_{k\in\mathbb{N}}$ and $(t_k(m))_{k\in\mathbb{N}}$, respectively, as follows:

$$\begin{aligned} t^n_0(m) &:= C_2(D_0 \dots D_n D_{n+1}^{(m+1)} 0), \\ t^n_{k+1}(m) &:= P(t^n_k(m)[k + 2 \leftarrow k + 3]) \quad \text{and} \\ t_0(m) &:= C_2(D_0 \dots D_m 0), \\ t_{k+1}(m) &:= P\left(t_k(m)[2 + k \leftarrow 2 + k + 1]\right). \end{aligned}$$

Remark 6.5. By an application of Lemma 4.3 we obtain

$$t_0(m)[2 \leftarrow 3] = C_3(D_0 \dots D_m 0),$$

and Lemma 6.2 shows that $t_1(m) = C_3 P_3(D_0 \dots D_m 0)$. Applying Lemma 4.3 again we obtain

$$t_1(m)[3 \leftarrow 4] = C_4 P_3(D_0 \dots D_m 0),$$

whence by Lemma 6.2 $t_2(m) = C_4 P_4 P_3(D_0 \dots D_m 0)$, etc. In general we therefore have

$$t_k(m) = C_{k+2} P_{k+2} \dots P_3(D_0 \dots D_m 0). \tag{5}$$

If we define BT-terms $\alpha_m := D_0 \dots D_m 0$, whose interpretations are the ordinals $\tau_0[m]$, Equation (5) yields, according to part 1 of Lemma 4.2, that the existence of (the minimal) k such that $t_k(m) = 0$ is equivalent to the existence of $s(\alpha_m, 2)$, which then is equal to $k + 2$.

Similarly, we obtain

$$t_k^n(m) = C_{k+2} P_{k+2} \dots P_3(D_0 \dots D_n D_{n+1}^{(m+1)} 0). \tag{6}$$

Theorem 6.6 (Unprovability Results).

1. The number-theoretic assertion

$$\forall m \exists k (t_k^n(m) = 0)$$

is unprovable in $\mathrm{PA} + \mathrm{TI}(<\sigma_n)$, hence unprovable in ID_n.

2. The number-theoretic assertion

$$\forall m \exists k (t_k(m) = 0)$$

is unprovable in $\mathrm{PA} + \mathrm{TI}(<\tau_0)$, hence unprovable in $\mathrm{ID}_{<\omega}$ and $\Pi_1^1{-}\mathrm{CA}_0$.

Proof. We argue as in [DSW12], starting from Definition 6.4 and using Remark 6.5, Equations 6 and 5, in order to see that the assertions are equivalent to the statement

$$\forall \alpha \forall x \exists y ((\alpha < \rho \;\&\; x > 0) \to (y > x \;\&\; P_y P_{y-1} \dots P_{x+1} \alpha = 0)), \tag{7}$$

where $\rho = \sigma_n$ for Claim 1 and $\rho = \tau_0$ for Claim 2. We use Lemma 6.3 in order to see that the provability of (7) would imply the provable totality and hence recursiveness of h_ρ in $\mathrm{PA} + \mathrm{TI}(<\rho)$, contradicting the results in [FS95]. □

7. Final Remarks

1. It seems to be straightforward to extend the results of this paper to ordinal segments given by the proof-theoretic ordinals of the theories ID_ν for, say, $\nu \leq \Gamma_0$ or more general for those ν for which the nature of the slow growing hierarchy has been classified in terms of the Hardy hierarchy.

2. A natural problem for future research would be to classify the phase transition threshold for the Goodstein principles considered in this paper or from [DSW12]. To this end one replaces the transition $t[k \leftarrow k+1]$ by $t[f(k) \leftarrow f(k+1)]$ for some definable sublinear unbounded function f (cf., e.g., [MW08]) and one is interested in classifying those functions f for which the underlying Goodstein principle remains unprovable in the theory in question. Corresponding results for Buchholz-style Hydra games have already been obtained by Frederik Meskens in [M09]. We conjecture that similar thresholds will also apply in the context of strong Goodstein principles and that these results will provide intrinsic information on the first subrecursively inaccessible ordinal, i.e. the first ordinal where the slow and fast growing hierarchies match up.
3. A technical stumbling block for extending Goodstein principles to larger ordinals comes from the largely unknown behaviour of the slow growing hierarchy for indices above $\psi_0\Omega_\omega$ (assuming Buchholz style notations). We expect that progress will be possible in case that one would be able to show that $\psi_0\Omega_{\Omega_\omega}$ is the second subrecursively inaccessible ordinal, i.e. the second ordinal where the slow and fast growing hierarchy match up (under the standard Buchhholz-style assignment of fundamental sequences). But this problem seems to be rather difficult to us although some initial progress is reported in [Ar91] and [Wei95].

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