THE CONDITION NUMBER FOR CIRCULANT NETWORKS

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In this paper we prove that it is possible to use techniques specific to electromagnetic field synthesis in the study of some electrical circuits. The definition of a circulant network will be presented. The system matrix of such a network is a circular matrix, which allows an analytical evaluation of all the eigenvalues and all the singular values. Resonance frequencies can then be calculated exactly as will be demonstrated on passive and active circulant networks.

1. INTRODUCTION

The condition number is an essential tool in the field synthesis problem [1]. Using conditioning also in the study of electrical circuits [2] could prove beneficial. A condition number close to unity guarantees the system matrix is well conditioned. High values of condition number indicates ill conditioning which can be either due to numerical problems but also due to the fact the electric network is not stable.

Examples for this are the circulant networks. A circulant network is defined as a series connection of identical basic cells connected in a closed loop. It is known [3] that a closed loop can give rise to resonance or instability especially if active elements are involved.

As it will be shown further, the system matrix of such networks turns out to be a circular matrix [4], the reason the name circulant networks has been chosen. The major property is that all the eigenvalues of a circular matrix can be calculated analytically. This offers the unique possibility to evaluate condition numbers of the system matrix exactly, so that eventual ill conditioning or resonances can be evaluated exactly.

The importance of circulant networks derives from the fact that if in a chain of identical quadrupoles the matrix of the fundamental parameters at the n-th power is the identity matrix then a chain of n such quadrupoles can be studied as a

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circulant network. The applications can be extended to the study of filters or transmition lines.

2. **BASIC THEORY**

Let us consider the circulant network shown in Fig. 1. The unit cell is composed of an impedance $\underline{Z}$ and an admittance $\underline{Y}$. The current sources $\underline{I}_i$ can be diferent. A circulant network implies that the chain of Fig. 1 must form a closed loop or the node $n + 1$ = node 1.

The electromagnetic field quasi-stationary equations lead to the following equation:

$$\frac{\underline{V}_i - \underline{V}_{i-1}}{\underline{Z}} - \frac{\underline{V}_i - \underline{V}_{i+1}}{\underline{Z}} + \underline{Y}\underline{V}_i = \underline{I}_i \,, \tag{1}$$

for $i = 1, 2, \ldots n$. If $i = 1$ then $\underline{V}_{i-1}$ has to be replaced by $\underline{V}_n$ and if $i = n$ then $\underline{V}_{i+1}$ has to be replaced by $\underline{V}_1$ in (1).

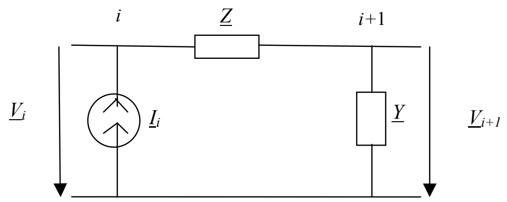

Fig. 1 – The unit cell of circulant network.

(1) can be rewritten as:

$$(2+\underline{Z}\,\underline{Y})\underline{V}_i - \underline{V}_{i-1} - \underline{V}_{i+1} = \underline{Z}\underline{I}_i. \tag{2}$$

Or in matrix form [5, 6]:

$$[A][V] = [Z][I]. \tag{3}$$

The system matrix $[A]$ is then:

$$\begin{bmatrix} 2+\underline{Z}\underline{Y} & -1 & & & & & & -1 \\ -1 & 2+\underline{Z}\underline{Y} & -1 & & & & & \\ & -1 & 2+\underline{Z}\underline{Y} & -1 & & & & \\ & & -1 & 2+\underline{Z}\underline{Y} & \ldots & \ldots & & \\ & & & -1 & \ldots & \ldots & & \\ & & & & \ldots & \ldots & & \\ & & & & & -1 & 2+\underline{Z}\underline{Y} & -1 \\ -1 & & & & & & -1 & 2+\underline{Z}\underline{Y} \end{bmatrix}, \tag{4}$$

[A] is obviously a circular n x n matrix, the eigenvalues of which can be calculated analytically.

3. **EIGENVALUES ANALYSIS**

The eigenvalues of a circular matrix [A] can be evaluated as follows:

$$\lambda_0 = \sum\nolimits_{k=1}^{n} A_{1,k} = ZY\,, \tag{5}$$

$$\lambda_m = \sum\nolimits_{k=0}^{n-1} A_{1,k+1} \exp\left(2\pi \mathrm{j}\frac{mk}{n}\right) = ZY + 4\sin^2\frac{\pi m}{n}\,. \tag{6}$$

If $\underline{Z}\ \underline{Y}$ is real and positive and n even the maximum eigenvalue occurs for $m = n/2$ or $\lambda_{n/2} = \underline{Z}\,\underline{Y} + 4$.

The condition number is then (for circular matrix):

$$\kappa = |\lambda_{n/2}|/|\lambda_0| = |\underline{Z}\ \underline{Y}+4|/|\underline{Z}\ \underline{Y}|, \tag{7}$$

or the largest eigenvalue (in absolute value) divided by the smallest one.

Generally the condition number is defined with the so-called singular values σ_i. These are defined by [2]:

$$\sigma_i = \sqrt{\mu_i}\,, \tag{8}$$

where μ_i are the eigenvalues of the matrix $[A]^{\mathrm{H}}[A]$, $[A]^{\mathrm{H}}$ being the Hermitean (the transpose conjugate) [4] of the system matrix [A].

If $\sigma_0 \le \sigma_1 \le \sigma_2 \le \ldots \le \sigma_{n-1}$ the condition number κ is then defined by:

$$k = \frac{\sigma_{n-1}}{\sigma_0}\,. \tag{9}$$

High values of κ indicates ill conditioning which can be either due to numerical problems but also due to the fact the electric network is not stable (oscillator *e.g.*).

An important difference between eigenvalues and singular values is that the latter ones are always real and positive numbers.

$[A]^{\mathrm{H}}[A]$ is a circular matrix too and all the elements are real numbers. The eigenvalues μ_i of $[A]^{\mathrm{H}}[A]$ are found similarly by using (5) or (6):

$$\mu_0 = \sum\nolimits_{k=1}^{n} A^H A_{1,k} = \underline{Z}\underline{Y}\underline{Z}^*\underline{Y}^* \tag{10}$$

$$\mu_m = \sum\nolimits_{k=0}^{n-1} A^H A_{1,k+1} \exp\left(2\pi \mathrm{j}\frac{mk}{n}\right) = \tag{11}$$

$$= \underline{Z}\underline{Y}\underline{Z}^*\underline{Y}^* + 4\left(\underline{Z}\underline{Y}\underline{Z}^*\underline{Y}^*\right)\sin^2\frac{\pi m}{n} + 16\sin^2\frac{\pi m}{n} =$$

$$= (\underline{Z}\underline{Y} + 4\sin^2\frac{\pi m}{n})\left(\underline{Z}^*\underline{Y}^* + \sin^2\frac{\pi m}{n}\right).$$

It can be easily verified that

$$\mu_m = \left|\lambda_m\right|^2. \tag{12}$$

We note $\underline{Z} = r + jx$, $\underline{Y} = g + jy$ so:

$$\underline{Z}\,\underline{Y} = rg - xy + j\,(ry + gx) = a + j\,b. \tag{13}$$

Let p and q corresponding to:

$$\mu_{\max} = \mu_p = \left(a + 4\sin^2\frac{p\pi}{n}\right)^2 + b^2;$$

$$\mu_{\min} = \mu_q = \left(a + 4\sin^2\frac{q\pi}{n}\right)^2 + b^2.$$

The condition number is:

$$K = \sqrt{\frac{\mu_p}{\mu_q}}. \tag{14}$$

If $a > 0$ the condition number indicates a well conditioned system matrix. In this case $p = \frac{n}{2}$ and $q = 0$ (the (7) formula is found once again for even n and $b = 0$.

The ill-conditioned system is obtained only [7] for $b = 0$ and $a \in [-4, 0]$ so $\underline{Z} = j\omega L$ and $\underline{Y} = j\omega C$, $p = 0$ and $a + 4\sin^2\frac{\pi q}{n} = 0$ so the resonance condition is:

$$\omega^2 LC = 4\sin^2\frac{\pi q}{n}. \tag{15}$$

For example if $n = 3$ the condition is:

$$\omega^2 LC = 3. \tag{16}$$

After the triangle-star transformation in the circulant network of Fig. 1, the circuit is presented in Fig. 2, where $E_i = \frac{1}{j\omega C} I_i$.

With (16) we have serie resonance in Fig. 2 and if the sources are different the network is in great difficulty.

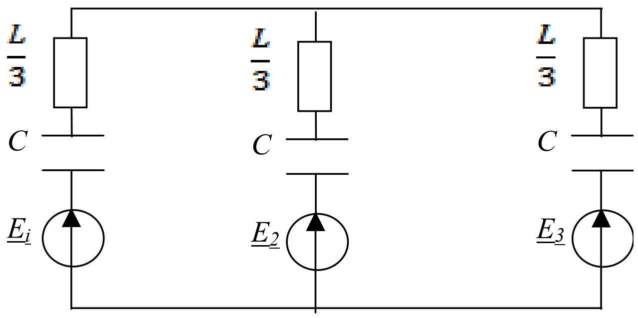

Fig. 2 – Resonance in circulant network with 3 cells.

For $q = 0$ one gets then a resonance at "DC" which means physically that all the capacitors in the lumped LC are being charged to the same constant but arbitrary voltage. The singular values are now:

$$\sigma_o = \omega^2 LC\,,$$
$$\sigma_m = \left|\,\omega^2 LC - 4\sin^2\frac{\pi m}{n}\right|. \tag{17}$$

4. APPLICATION TO AN ACTIVE NETWORK

Let us consider now the circulant network shown in Fig. 3. Each cell consist of an operational amplifier [3] and two impedances $\underline{Z_1}$ and $\underline{Z_2}$. The operational amplifier is considerer as an ideal one: infinite gain, infinite input resistance and zero output resistance.

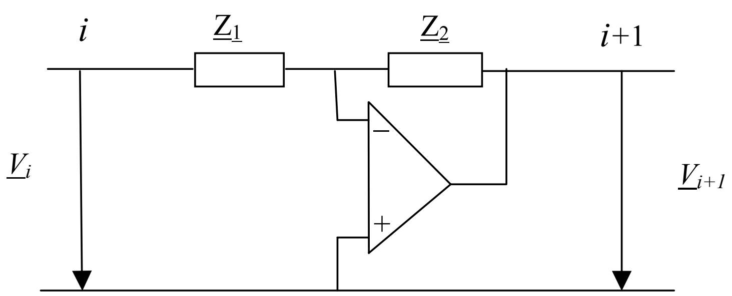

Fig. 3 – The unit cell of active network.

As can be found in several textbooks one has:

$$\frac{\underline{V_{i+1}}}{\underline{V_i}} = -\frac{\underline{Z_2}}{\underline{Z_1}} = -\alpha\,, \tag{18}$$

or

$$\alpha \underline{V}_{\mathrm{i}} + \underline{V}_{\mathrm{i}+1} = 0. \tag{19}$$

The system matrix [A] is now:

$$[A] = \begin{bmatrix} \alpha & 1 & & & & & & \\ & \alpha & 1 & & & & & \\ & & \alpha & 1 & & & & \\ & & & \alpha & \dots & \dots & & \\ & & & & \dots & \dots & & \\ & & & & \dots & \dots & & \\ & & & & & & \alpha & 1 \\ 1 & & & & & & & \alpha \end{bmatrix}. \tag{20}$$

Obviously, (20) is a circular matrix, the eigenvalues [4] of which are easily found be using (5) and (6):

$$\lambda_0 = \alpha + 1, \tag{21}$$

$$\lambda_m = \alpha + \exp\left(2\pi \,\mathrm{j} \frac{m}{n}\right). \tag{22}$$

Example 1. As a first practical example we consider a circulant network composed of inverters $\underline{Z_1} = \underline{Z_2} = R$ or $\alpha = 1$. We have then:

$$\lambda_0 = 2, \tag{23}$$

$$\lambda_m = 1 + \exp\left(2\pi j \frac{m}{n}\right), \tag{24}$$

$$\left|\lambda_m\right| = \sqrt{2\left(1 + \cos\frac{2\pi m}{n}\right)} = 2\cos\frac{\pi m}{n}. \tag{25}$$

λ_m can be zero for n even and $m = n/2$. The network has then an infinite number of solutions:

$$\underline{V_i} = \underline{V_0} \text{ (i even) and } \underline{V_i} = -\underline{V_0} \text{ (i odd)}, \tag{26}$$

where $\underline{V_0}$ can be any value.

If n is odd, the only possible solution for the network is $\underline{V_i} = 0$ for all values of i. For n odd the smallest eigenvalue is evaluated as:

$$\left|\lambda_{\min}\right| = \sqrt{2\left(1 - \cos\frac{\pi}{n}\right)} = 2\sin\frac{\pi}{2n}. \tag{27}$$

The condition number [2] is then:

$$k = \left|\frac{\lambda_0}{\lambda_{\min}}\right| = \frac{1}{\sin\dfrac{\pi}{2n}}\,, \tag{28}$$

which can be quite hight for large values of the number of cells n.

Example 2. A second example involves an RC network with $\underline{Z_1} = R$ and $\underline{Z_2} = 1/\mathrm{j}\omega C$. This gives rise to $\alpha = 1/\mathrm{j}\omega RC$. The eigenvalue are then:

$$|\lambda_0| = \left|1 + \frac{1}{\mathrm{j}\omega RC}\right| = \sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}\,, \tag{30}$$

$$|\lambda_m| = \sqrt{1 + \frac{1}{\omega^2 R^2 C^2} - \frac{2}{\omega RC}\sin\frac{2\pi m}{n}}\,.$$

λ_m can be zero if $\omega RC = 1$ and $\sin(2\pi m/n) = 1$ which is only possible if n is a multiple of 4. The circuit is then oscillating [3] at an angular frequency $1/RC$.

Example 3. As a third example we consider a parallel connection of a resistor R and a capacitor C for $\underline{Z_2}$ and a single resistor for $\underline{Z_1}$:

$$\underline{Z_1} = \frac{R}{\sqrt{2}}\,, \tag{31}$$

$$\underline{Z_2} = \frac{R}{1 + \mathrm{j}\omega RC}\,. \tag{32}$$

We have now:

$$\alpha = \frac{\sqrt{2}}{1 + \mathrm{j}\omega RC}\,. \tag{33}$$

The eigenvalues are then found to be:

$$|\lambda_0| = \sqrt{\frac{2\left(1+\sqrt{2}\right)}{1 + \omega^2 R^2 C^2} + 1}\,, \tag{34}$$

$$|\lambda_m| = \sqrt{\frac{2}{1+\omega^2R^2C^2} + 1 + \frac{2\sqrt{2}}{1+\omega^2R^2C^2}\cos\frac{2\pi m}{n} - \frac{2\sqrt{2}\,\omega RC}{1+\omega^2R^2C^2}\sin\frac{2\pi m}{n}}\,. \tag{35}$$

One can easily verify that for $\omega RC = 1$ the eigenvalue $\lambda_m = 0$ on condition $2\pi m/n = 2\pi\ 3/8$ or:

$$\frac{m}{n} = \frac{3}{8}\,. \tag{36}$$

If the circulant network contains 8 cells, the eigenvalue, $|\lambda_3| = 0$ and the network will be oscillating at a frequency $\omega = 1/RC$. The physical explanation is quite obvious. If $\omega RC = 1$ each cell has a gain equal to unity and a phase shift of exactly $2\pi/8$. Hence the 8 cell will provide a total phase shift 2π so that oscillation can start. The same phenomenon will occur if n is any multiple of 8.

5. CONCLUSIONS

It has been proved in this paper that the system matrix of a so called circulant network turns out to be a circular matrix. These matrices have the major advantage that all the eigenvalues can be calculated analytically. It becomes then possible to analyse phenomena like resonance exactly, without any disturbance due to ill conditioned matrices. It was also observed that the results obtained from the eigenvalues or from the singular values are all in perfect agreement. This is a consequence of the fact that the singular values are equal to the absolute values of the eigenvalues for the circular matrix. The results and methods used can be applied in the study of quadrupoles, filters and transmition lines.

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