The pseudo-hyperplanes and homogeneous pseudo-embeddings of the generalized quadrangles of order $(3, t)$

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be

Abstract

In the paper [8], we introduced the notions of pseudo-hyperplane and pseudo-embedding of a point-line geometry and proved that every generalized quadrangle of order (s, t), $2 \leq s < \infty$, has faithful pseudo-embeddings. The present paper focuses on generalized quadrangles of order $(3, t)$. Using the computer algebra system GAP [12] and invoking some theoretical relationships between pseudo-hyperplanes and pseudo-embeddings obtained in [8], we are able to give a complete classification of all pseudo-hyperplanes of $\mathcal{Q}$. We hereby find several new examples of tight sets of generalized quadrangles, as well as a complete classification of all 2-ovoids of $\mathcal{Q}$. We use the classification of the pseudo-hyperplanes of $\mathcal{Q}$ to obtain a list of all homogeneous pseudo-embeddings of $\mathcal{Q}$.

Keywords: generalized quadrangle, pseudo-hyperplane, (universal, homogeneous) pseudo-embedding, pseudo-embedding rank, tight set, m-ovoid
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1 Introduction

1.1 Overview

Pseudo-hyperplanes and pseudo-embeddings are notions which were introduced in De Bruyn [8]. For partial linear spaces with three points per line, these concepts coincide with the well known notions of hyperplanes and projective embeddings. In [8], it was proved that every generalized quadrangle of order (s, t), $2 \leq s < \infty$, has faithful pseudo-embeddings. In the present paper, we study the pseudo-hyperplanes and pseudo-embeddings of the generalized quadrangles (GQ's) with four points per line. There are, up to isomorphism, five such GQ's, namely the (4×4)-grid, $W(3)$, $Q(4, 3)$, $Q(5, 3)$ and the unique generalized quadrangle $\mathrm{GQ}(3, 5)$ of order $(3, 5)$.

With the aid of the computer algebra system GAP [12] and some theoretical relationships between pseudo-hyperplanes and pseudo-embeddings, we are able to obtain a complete classification of all pseudo-hyperplanes for each of the five GQ's of order $(3, t)$. We also list several properties of these pseudo-hyperplanes.

Tight sets and m-ovoids of GQ's are certain nice sets of points. They are also called intriguing sets in the literature. We have examined each of the obtained pseudo-hyperplanes to determine whether it was an intriguing set or not. This investigation led to the discovery of several new examples of tight sets of GQ's. The classification of the pseudo-hyperplanes automatically gives rise to a complete classification of all 2-ovoids of each of the five GQ's of order $(3, t)$. Such a classification of the 2-ovoids was previously known for four of the five GQ's, but not for GQ$(3, 5)$. We will find that the GQ GQ$(3, 5)$ has up to isomorphism two 2-ovoids. One of these two 2-ovoids seems to be new.

A point-line geometry which has a pseudo-embedding must admit the so-called universal pseudo-embedding. Every other pseudo-embedding can be derived from this universal pseudo-embedding by taking so-called quotients. These quotients can usually be taken in many ways, leading to many nonisomorphic pseudo-embeddings. We will therefore narrow our point of view to a very nice class of pseudo-embeddings, namely the homogeneous pseudo-embeddings. These are pseudo-embeddings with the property that the full group of automorphisms of the geometry lifts to a group of automorphisms of the pseudo-embedding space. We will describe a method in terms of pseudo-hyperplanes to determine all homogeneous pseudo-embeddings and apply it to the GQ's of order $(3, t)$, leading to a complete list of homogeneous pseudo-embeddings. The determination of all these homogeneous pseudo-embeddings will also be realized with the aid of the computer algebra system GAP.

We will state our main results in Section 1.4. But before we can do that, we need to give the necessary technical background to understand these results. This will be done in Sections 1.2 and 1.3.

1.2 Pseudo-embeddings and pseudo-hyperplanes of point-line geometries

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three.

A *pseudo-hyperplane* of $\mathcal{S}$ is a proper subset H of $\mathcal{P}$ such that every line contains an even number of points of $\mathcal{P} \setminus H$. If H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{S}$, then the complement $H_1 * H_2 := \mathcal{P} \setminus (H_1 \Delta H_2)$ of the symmetric difference $H_1 \Delta H_2$ of H_1 and H_2 is again a pseudo-hyperplane of $\mathcal{S}$. In the case $\mathcal{S}$ is finite, the definition of pseudo-hyperplane can be rephrased in coding theoretical terms: a proper subset H of $\mathcal{P}$ is a pseudo-hyperplane if and only if the characteristic vector of its complement $\mathcal{P} \setminus H$ belongs to the dual code of $\mathcal{S}$. Here, the code of $\mathcal{S}$ is defined as the subspace of $\mathbb{F}_2^{\mathcal{P}}$ generated by the characteristic vectors of the lines.

Suppose V is a vector space over the field $\mathbb{F}_2$ of order 2. A *pseudo-embedding* of $\mathcal{S}$ into the projective space $\Sigma = \mathrm{PG}(V)$ is a mapping e from $\mathcal{P}$ to the point set of Σ satisfying:

(1) $< e(\mathcal{P}) >_\Sigma = \Sigma$; (2) if L is a line of $\mathcal{S}$ with points $x_1, x_2, \dots, x_k$, then the points $e(x_1), e(x_2), \dots, e(x_{k-1})$ of Σ are linearly independent and $e(x_k) =< \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{k-1} >$ where $\bar{v}_i$, $i \in \{1, 2, \dots, k-1\}$, is the unique vector of V for which $e(x_i) =< \bar{v}_i >_\Sigma$. If moreover e is an injective mapping, then the pseudo-embedding $e : \mathcal{S} \to \Sigma$ is called *faithful*. Two pseudo-embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $e_2 = \phi \circ e_1$.

Suppose $e : \mathcal{S} \to \mathrm{PG}(V)$ is a pseudo-embedding of $\mathcal{S}$ and G is a group of automorphisms of $\mathcal{S}$. We say that e is *G-homogeneous* if for every $\theta \in G$, there exists a (necessarily unique) projectivity η_θ of $\mathrm{PG}(V)$ such that $e(x^\theta) = e(x)^{\eta_\theta}$ for every point x of $\mathcal{S}$. If G is the full automorphism group of $\mathcal{S}$, then e is also called a *homogeneous pseudo-embedding.*

Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and α is a subspace of Σ satisfying the following two properties:

(Q1) if x is a point of $\mathcal{S}$, then $e(x) \notin \alpha$;

(Q2) if L is a line of $\mathcal{S}$ with points $x_1, x_2, \dots, x_k$, then $\alpha \cap < e(x_1), e(x_2), \dots, e(x_k) >_\Sigma = \emptyset$.

Then a new pseudo-embedding $e/\alpha : \mathcal{S} \to \Sigma/\alpha$ can be defined which maps each point x of $\mathcal{S}$ to the point $< \alpha, e(x) >$ of the quotient projective space Σ/α. This new pseudo-embedding e/α is called a *quotient* of e. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two pseudo-embeddings of $\mathcal{S}$, then we say that $e_1 \geq e_2$ if e_2 is isomorphic to a quotient of e_1. A pseudo-embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ is called *universal* if $\widetilde{e} \geq e$ for any pseudo-embedding e of $\mathcal{S}$. By [8, Theorem 1.2(1)], we know that if $\mathcal{S}$ has a pseudo-embedding, then $\mathcal{S}$ also has a universal pseudo-embedding. This universal pseudo-embedding is unique, up to isomorphism. If $\mathcal{S}$ has a faithful pseudo-embedding, then the universal pseudo-embedding of $\mathcal{S}$ is also faithful. If $\widetilde{e} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ is the universal pseudo-embedding of $\mathcal{S}$, where $\widetilde{V}$ is some vector space over $\mathbb{F}_2$, then the dimension of $\widetilde{V}$ is called the *pseudo-embedding rank* of $\mathcal{S}$.

Most of the notions defined in this subsection were introduced in De Bruyn [8]. Observe that for partial linear spaces with three points per line, the notions of pseudo-hyperplanes and pseudo-embeddings coincide with the well known notions of hyperplanes and projective embeddings. We hope that the theory of pseudo-hyperplanes and pseudo-embeddings will find some interesting applications in finite geometry. A few applications can already be found in [8] and the present paper. The concept of a pseudo-embedding has also implicitly played some role in a number of papers, like for instance in the papers [14, 15] where Li was able to determine the universal embedding dimensions of certain symplectic and Hermitian dual polar spaces. We hope that the study of pseudo-hyperplanes and pseudo-embeddings will offer some additional insight in these problems, as well as some related problems.

1.3 Generalized quadrangles and some of their special subsets

Let s and t be possibly infinite cardinal numbers. A point-line geometry $\mathcal{Q}$ is called a *generalized quadrangle* (GQ) of order (s, t) if the following properties are satisfied:

• every two distinct points are incident with at most 1 line;
• every line is incident with precisely $s+1 \geq 2$ points;
• every point is incident with precisely $t+1 \geq 2$ lines;
• for every line L and every point x not incident with L, there exists a unique point on L collinear with x.

By De Bruyn [8, Corollary 3.11(1)], every generalized quadrangle of order (s,t), $2 \leq s < \infty$, has faithful pseudo-embeddings.

In the present paper, we are interested in the GQ's of order $(3,t)$. By Brouwer [4], any such GQ must be finite. A classification of all finite GQ's of order $(3,t)$ was obtained by Dixmier and Zara [11], see also Payne and Thas [19, Section 6.2]. There are, up to isomorphism, five examples of such GQ's: (1) the (4×4)-grid; (2) $W(3)$; (3) $Q(4,3)$; (4) $Q(5,3)$; (5) the unique GQ of order $(3,5)$, which we will denote by $\mathrm{GQ}(3,5)$. The examples mentioned in (2), (3), (4) and (5) belong to infinite families of GQ's which we will now describe.

Suppose ζ is a symplectic polarity of the projective space $\mathrm{PG}(3,q)$. Then the points and lines of $\mathrm{PG}(3,q)$ which are totally isotropic with respect to ζ define a GQ of order (q,q) which we will denote by $W(q)$.

Consider in the projective space $\mathrm{PG}(4,q)$ a quadric Q whose equation with respect to some reference system is given by $X_0^2 + X_1X_2 + X_3X_4 = 0$. Then the points and lines of $\mathrm{PG}(4,q)$ contained in Q define a GQ of order (q,q) which we will denote by $Q(4,q)$. The GQ $Q(4,q)$ is isomorphic to $W(q)$ if and only if q is even.

Consider in the projective space $\mathrm{PG}(5,q)$ a quadric Q whose equation with respect to some reference system is given by $X_0^2 + \delta X_0X_1 + X_1^2 + X_2X_3 + X_4X_5 = 0$. Here, $\delta \in \mathbb{F}_q$ such that the quadratic polynomial $X^2 + \delta X + 1 \in \mathbb{F}_q[X]$ is irreducible. Then the points and lines of $\mathrm{PG}(5,q)$ contained in Q define a GQ of order (q,q^2) which we will denote by $Q(5,q)$.

Let $\mathrm{PG}(2,q)$, q even, be embedded as a hyperplane in $\mathrm{PG}(3,q)$, and let $\mathcal{H}$ be a hyperoval of $\mathrm{PG}(2,q)$. Then the points of $\mathrm{PG}(3,q) \setminus \mathrm{PG}(2,q)$ together with those lines of $\mathrm{PG}(3,q)$ which are not contained in $\mathrm{PG}(2,q)$ and which contain a unique point of $\mathcal{H}$ define a GQ of order $(q-1,q+1)$ which we will denote by $T_2^*(\mathcal{H})$. If $q=4$, then $T_2^*(\mathcal{H})$ is isomorphic to the unique generalized quadrangle of order $(3,5)$.

A set X of points of a generalized quadrangle $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s,t), $s < \infty$, is said to be *even* [resp., *odd*] if it intersects each line in an even [resp. odd] number of points. If s is odd, then the pseudo-hyperplanes of $\mathcal{Q}$ are precisely the even sets distinct from $\mathcal{P}$. If s is even, then the pseudo-hyperplanes of $\mathcal{Q}$ are precisely the odd sets distinct from $\mathcal{P}$.

Let $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized quadrangle of order (s,t), with s,t finite.

A set X of points of $\mathcal{Q}$ is called an *m-ovoid*, $m \in \{0,1,\ldots,s+1\}$, if every line of $\mathcal{Q}$ intersects X in precisely m points. The notion of m-ovoid was introduced by Thas [22].

If X is a set of points of $\mathcal{Q}$, then the total number of ordered pairs of distinct collinear points of X is bounded above by $|X| \cdot \left(\frac{|X|}{s+1} + (s-1)\right)$, see e.g. Payne and Thas [19, Theorem 1.10.1]. If equality occurs, then X is called *i-tight* where $i = \frac{|X|}{s+1}$. The number i is a nonnegative integer. The notion of a tight set was introduced by Payne [16].

A set X of points of $\mathcal{Q}$ is called *intriguing* if there exist constants h_1 and h_2 such that every point of X is collinear with precisely h_1 points of X and every point not contained in X is collinear with precisely h_2 points of X. The notion of an intriguing set was introduced in Bamberg, Law and Penttila [3], where it was shown ([3, Theorem 4.1]) that the intriguing sets of GQ's are precisely the tight sets and the m-ovoids for some $m \geq 0$. If X is a set of points of $\mathcal{Q}$ such that the setwise stabilizer of X (in the full automorphism group $Aut(\mathcal{Q})$ of $\mathcal{Q}$) has one orbit on X and one orbit on the complement of X, then X must be an intriguing set. We will discover several examples of such intriguing sets later.

A 1-ovoid of a generalized quadrangle is also called an *ovoid*. All ovoids of all generalized quadrangles of order $(3, t)$ are known. Obviously, the (4×4)-grid has up to isomorphism a unique ovoid. The generalized quadrangles $W(3)$ and $Q(5, 3)$ have no ovoids, see e.g. Payne and Thas [19, Theorems 1.8.3 and 1.8.4].

It is known that the generalized quadrangle $Q(4, 3)$ has up to isomorphism a unique ovoid. If Q is the underlying quadric of $Q(4, 3)$ and $\mathrm{PG}(4, 3)$ the ambient projective space of Q, then every ovoid of $Q(4, 3)$ is obtained by intersecting Q with a hyperplane Π of $\mathrm{PG}(4, 3)$ such that $\Pi \cap Q$ is a nonsingular elliptic quadric of Π, see e.g. Brouwer and Wilbrink [5, p, 160].

The generalized quadrangle $\mathrm{GQ}(3, 5)$ has up to isomorphism a unique ovoid. Suppose $\mathrm{GQ}(3, 5) = T_2^*(\mathcal{H})$, where $\mathcal{H}$ is a hyperoval of a hyperplane $\mathrm{PG}(2, 4)$ of the projective space $\mathrm{PG}(3, 4)$. If α is a plane of $\mathrm{PG}(3, 4)$ disjoint from $\mathcal{H}$, then $\alpha \setminus \mathrm{PG}(2, 4)$ clearly is an ovoid of $\mathrm{GQ}(3, 5)$. By Payne [17, VI1], every ovoid of $\mathrm{GQ}(3, 5)$ can be obtained in this way. Observe also that $\mathrm{GQ}(3, 5)$ has partitions in ovoids. Indeed, if L is a line of $\mathrm{PG}(2, 4)$ disjoint from $\mathcal{H}$ and if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the four planes of $\mathrm{PG}(3, 4)$ through L distinct from $\mathrm{PG}(2, 4)$, then $\{\alpha_i \setminus \mathrm{PG}(2, 4) \,|\, i \in \{1, 2, 3, 4\}\}$ is a partition in ovoids.

The generalized quadrangle $Q(4, 3)$ has many subquadrangles which are (4×4)-grids. If L_1 and L_2 are two disjoint lines of $Q(4, 3)$ and $Q \subseteq \mathrm{PG}(4, 3)$ is the underlying quadric of $Q(4, 3)$, then $< L_1, L_2 > \cap\, Q$ determines a (4×4)-subgrid of $Q(4, 3)$. Every (4×4)-subgrid of $Q(4, 3)$ is obtained in this way.

The generalized quadrangle $\mathrm{GQ}(3, 5)$ has many subquadrangles which are (4×4)-grids. Suppose $\mathrm{GQ}(3, 5) = T_2^*(\mathcal{H})$, where $\mathcal{H}$ is a hyperoval of a hyperplane $\mathrm{PG}(2, 4)$ of the projective space $\mathrm{PG}(3, 4)$. If $\alpha \neq \mathrm{PG}(2, 4)$ is a plane of $\mathrm{PG}(3, 4)$ containing precisely two points of $\mathcal{H}$, then $\alpha \setminus \mathrm{PG}(2, 4)$ determines a (4×4)-subgrid of $\mathrm{GQ}(3, 5)$. Every (4×4)-subgrid of $\mathrm{GQ}(3, 5)$ is obtained in this way.

The generalized quadrangle $Q(5, 3)$ has many subquadrangles isomorphic to $Q(4, 3)$ and many (4×4)-subgrids. Any $Q(4, 3)$-subquadrangle of $Q(5, 3)$ is obtained by intersecting the underlying quadric Q of $Q(5, 3)$ with a suitable hyperplane of the ambient projective space of Q. If L_1 and L_2 are two disjoint lines of $Q(5, 3)$, then $< L_1, L_2 > \cap\, Q$ determines a (4×4)-subgrid of $Q(5, 3)$, and every (4×4)-subgrid of $Q(5, 3)$ is obtained in this way.

1.4 The main results

In this paper, we realize the following goals for each of the five GQ's of order $(3, t)$.

(a) Determine the pseudo-embedding rank of the GQ.
(b) Enumerate all pseudo-hyperplanes of the GQ and list some of their properties.
(c) Determine which of these pseudo-hyperplanes are intriguing sets.
(d) Enumerate all homogeneous pseudo-embeddings of the GQ.

The above goals will be achieved with the aid of the computer algebra system GAP [12]. In order to realize goals (b) and (d), we will make use of some theoretical relationships between pseudo-hyperplanes and pseudo-embeddings which will be discussed in Section 2, see Proposition 2.3, Corollary 2.7 and the discussion after that corollary.

Theorem 1.1 (a) *The pseudo-embedding rank of the* (4×4)*-grid is equal to* 9*.*
(b) *The pseudo-embedding rank of* $W(3)$ *is equal to* 15*.*
(c) *The pseudo-embedding rank of* $Q(4,3)$ *is equal to* 15*.*
(d) *The pseudo-embedding rank of* $Q(5,3)$ *is equal to* 21*.*
(e) *The pseudo-embedding rank of* $\mathrm{GQ}(3,5)$ *is equal to* 18*.*

In Theorem 1.1, there are a number of results which are basically known. We have added these claims to Theorem 1.1 for reasons of completeness.

The result stated in Theorem 1.1(a) is known, see De Bruyn [8, Proposition 3.7].

The result stated in Theorem 1.1(b) is basically known. If $\mathcal{S}$ is a finite point-line geometry admitting a pseudo-embedding, then the pseudo-embedding rank of $\mathcal{S}$ is equal to $v - \mathrm{rank}_{\mathbb{F}_2}(M)$ where v is the total number of points of $\mathcal{S}$ and M is an incidence matrix of $\mathcal{S}$ (see Proposition 2.2). If $\mathcal{S}$ is isomorphic to the symplectic generalized quadrangle $W(q)$, then this number is equal to $\frac{1}{2}q(q^2+1)$ by Theorem 9.4 of Bagchi, Brouwer and Wilbrink [1].

Since $Q(4,3)$ is the point-line dual of $W(3)$ and both GQ's have the same number of points (namely 40), the pseudo-embedding ranks of these GQ's must coincide by De Bruyn [8, Proposition 3.1].

Theorem 1.2 (a) *Up to isomorphism, the* (4×4)*-grid has* 10 *even sets.*
(b) *Up to isomorphism,* $W(3)$ *has* 20 *even sets.*
(c) *Up to isomorphism,* $Q(4,3)$ *has* 21 *even sets.*
(d) *Up to isomorphism,* $Q(5,3)$ *has* 22 *even sets.*
(e) *Up to isomorphism,* $GQ(3,5)$ *has* 47 *even sets.*

It is straightforward to determine all even sets of the (4×4)-grid $\mathcal{G}$. All ten examples of such sets will be described in Section 4. For the other generalized quadrangles we will invoke GAP to classify all even sets. These even sets, together with some of their properties, are listed in the various tables of Section 5. In Sections 4 and 5, we will use the classification of the even sets to determine all homogeneous pseudo-embeddings of all GQ's of order $(3,t)$. All these homogeneous pseudo-embeddings turn out to be faithful.

Theorem 1.3 *(a) Up to isomorphism, the* (4×4)*-grid has two homogeneous pseudo-embeddings, the universal one in* $\mathrm{PG}(8,2)$ *and another one in* $\mathrm{PG}(4,2)$*.*

(b) Up to isomorphism, there exists a unique homogeneous pseudo-embedding of $W(3)$*, namely the universal pseudo-embedding in* $\mathrm{PG}(14,2)$*.*

(c) Up to isomorphism, there exist two homogeneous pseudo-embeddings of $Q(4,3)$*, the universal one in* $\mathrm{PG}(14,2)$ *and another one in* $\mathrm{PG}(8,2)$*.*

(d) Up to isomorphism, there exists a unique homogeneous pseudo-embedding of $Q(5,3)$*, namely the universal pseudo-embedding in* $\mathrm{PG}(20,2)$*.*

(e) Up to isomorphism, GQ$(3,5)$ *has five homogeneous pseudo-embeddings. The corresponding pseudo-embedding spaces are respectively* $\mathrm{PG}(17,2)$ *(universal pseudo-embedding),* $\mathrm{PG}(16,2)$*,* $\mathrm{PG}(12,2)$*,* $\mathrm{PG}(10,2)$ *and* $\mathrm{PG}(6,2)$*.*

The universal pseudo-embedding of the (4×4)-grid was explicitly described in De Bruyn [8, Proposition 3.7]. The homogeneous pseudo-embedding of the (4×4)-grid in $\mathrm{PG}(4,2)$ will be described in Theorem 4.1. In Section 5, we will show that the homogeneous pseudo-embeddings of $\mathrm{GQ}(3,5)$ into $\mathrm{PG}(6,2)$ and $\mathrm{PG}(12,2)$ are induced by homogeneous pseudo-embeddings of the affine space $\mathrm{AG}(3,4)$ into which $\mathrm{GQ}(3,5)$ is fully embeddable. These two pseudo-embeddings of $\mathrm{AG}(3,4)$ are in fact all homogeneous pseudo-embeddings of $\mathrm{AG}(3,4)$. Indeed, in De Bruyn [9, Theorem 1.5], we will show that every affine space $\mathrm{AG}(n,4)$, $n \geq 2$, has up to isomorphism two homogeneous pseudo-embeddings, the universal one in $\mathrm{PG}(n^2+n,2)$ and another one in $\mathrm{PG}(2n,2)$. In [9, Theorem 1.7], we will show that the two homogeneous pseudo-embeddings of $Q(4,3)$ are induced by the two homogeneous pseudo-embeddings of $\mathrm{AG}(4,4)$ into which $Q(4,3)$ is fully embeddable. In [9], also explicit constructions will be given for the two homogeneous pseudo-embeddings of $\mathrm{AG}(n,4)$, $n \geq 2$. Using this, it is thus possible to give explicit constructions for the two homogeneous pseudo-embeddings of $Q(4,3)$, as well as two of the five homogeneous pseudo-embeddings of $\mathrm{GQ}(3,5)$.

Now that we have a complete list of all pseudo-hyperplanes of all GQ's of order $(3,t)$, we can check whether they are also intriguing sets. This indeed turns out to be the case for several of the pseudo-hyperplanes we have found, see the Tables 1, 2, 3, 4, 6 and 7 of Sections 4 and 5. From these tables we also observe that every nonempty pseudo-hyperplane which is intriguing is very nice in the sense that the stabilizer of the pseudo-hyperplane has one orbit on the pseudo-hyperplane and also one orbit on the complement of the pseudo-hyperplane.

In the following corollary to our classification of the pseudo-hyperplanes, we collect the number of nonempty pseudo-hyperplanes which are also tight. Most of the tight sets of $Q(5,3)$ and $\mathrm{GQ}(3,5)$ alluded to in the corollary have not been described before in the literature. (Several constructions and classification results for tight sets of GQ's can be found in Payne [16, 18] and Bamberg, Law & Penttila [3].) For the tight sets of $W(3)$ occurring in Corollary 1.4, one can give rather easy constructions, see Section 5. The tight set of the (4×4)-grid is just the union of two disjoint lines.

Corollary 1.4 *(a) Up to isomorphism, the* (4×4)*-grid has a unique nonempty pseudo-hyperplane which is also a tight set.*

(b) Up to isomorphism, the generalized quadrangle $W(3)$ *has four nonempty pseudo-hyperplanes which are also tight sets.*

(c) The generalized quadrangle $Q(4,3)$ *has no nonempty pseudo-hyperplanes which are also tight sets.*

(d) Up to isomorphism, the generalized quadrangle $Q(5,3)$ *has seven nonempty pseudo-hyperplanes which are also tight sets.*

(e) Up to isomorphism, the generalized quadrangle $GQ(3,5)$ *has six nonempty pseudo-hyperplanes which are also tight sets.*

A complete classification of all m-ovoids of all GQ's of order $(3,t)$ exists, except for the 2-ovoids of GQ$(3,5)$. Since 2-ovoids of GQ's of order $(3,t)$ are examples of pseudo-hyperplanes, we must find them back in our classification of the pseudo-hyperplanes. Consulting Tables 1, 2, 3, 4, 6 and 7 of Sections 4 and 5, we easily find:

Corollary 1.5 *(a) Up to isomorphism, the* (4×4)*-grid has two* 2*-ovoids.*

(b) Up to isomorphism, the generalized quadrangle $W(3)$ *has a unique* 2*-ovoid.*

(c) Up to isomorphism, the generalized quadrangle $Q(4,3)$ *has a unique* 2*-ovoid.*

(d) Up to isomorphism, the generalized quadrangle $Q(5,3)$ *has a unique* 2*-ovoid.*

(e) Up to isomorphism, the generalized quadrangle $GQ(3,5)$ *has two* 2*-ovoids.*

It is straightforward to verify that the (4×4)-grid has up to isomorphism two 2-ovoids. These two 2-ovoids will be described in Section 4 (Sets of Type 6 and 10, respectively).

The generalized quadrangle $W(3)$ is known to have 2-ovoids, see Cossidente, Culbert, Ebert & Marino [7, Theorem 2.1] and Bamberg, Law & Penttila [3, Theorem 5.1]. By a computer result of Bamberg, Kelly, Law and Penttila [2, Section 7.1], $W(3)$ has up to isomorphism a unique 2-ovoid.

By Segre [21], the generalized quadrangle $Q(5,3)$ has up to isomorphism a unique 2-ovoid. The uniqueness of the 2-ovoid of $Q(5,3)$ is an immediate consequence of the uniqueness of the hemi-system on the Hermitian surface $H(3,9)$.

Since $Q(4,3)$ occurs as a subquadrangle of $Q(5,3)$ and the latter has 2-ovoids, also $Q(4,3)$ must have 2-ovoids. By a computer result of Bamberg, Kelly, Law and Penttila [2, Section 7.3], the generalized quadrangle $Q(4,3)$ has up to isomorphism a unique 2-ovoid.

It is not so difficult to construct a 2-ovoid in the generalized quadrangle GQ$(3,5)$. Indeed, the union of any two disjoint ovoids of GQ$(3,5)$ is a 2-ovoid. The other 2-ovoid of GQ$(3,5)$ seems to be unknown.

The close relationship between pseudo-hyperplanes and pseudo-embeddings (to be described in Proposition 2.3) together with the fact that all 2-ovoids are pseudo-hyperplanes will allow us to find all 2-ovoids using a computer algorithm that does not need to invoke backtracking.

2 Homogeneous pseudo-embeddings

In this section, we realize the following goals.

- We prove that if a point-line geometry $\mathcal{S}$ has a pseudo-embedding, then its universal pseudo-embedding is homogeneous.
- We give a method in terms of pseudo-hyperplanes to determine whether a given pseudo-embedding is G-homogeneous, where G is some group of automorphisms of the point-line geometry.
- We describe a method for constructing and classifying G-homogeneous pseudo-embeddings.

We start with recalling three results from De Bruyn [8]. These results generalize some earlier results of Ronan [20] regarding hyperplanes and projective embeddings of point-line geometries with three points per line.

Proposition 2.1 ([8, Theorem 1.1]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and Π is a hyperplane of Σ. Then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a pseudo-hyperplane of $\mathcal{S}$.*

If a pseudo-hyperplane H is obtained from a pseudo-embedding e as described in Proposition 2.1, then H is said to *arise from* e. Recall that if a point-line geometry has a pseudo-embedding, then it also has a universal pseudo-embedding which is unique, up to isomorphism. In the following proposition a construction is given which yields this universal pseudo-embedding.

Proposition 2.2 ([8, Theorem 1.2(2)]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Let V be a vector space over the field $\mathbb{F}_2$ with a basis B whose vectors are indexed by the elements of $\mathcal{P}$, say $B = \{\bar{v}_x \,|\, x \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors of the form $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_k}$ where $\{x_1, x_2, \ldots, x_k\}$ is the point set of some line of $\mathcal{S}$. If $\mathcal{S}$ has at least one pseudo-embedding, then $\bar{v}_x \notin W$ for every point x of $\mathcal{S}$. Moreover, the map $\widetilde{e}$ which associates with each point $x \in \mathcal{P}$ the 1-space $\{\bar{v}_x + W, W\}$ of V/W defines a pseudo-embedding of $\mathcal{S}$ into $\mathrm{PG}(V/W)$ which is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.*

Proposition 2.3 ([8, Theorem 1.3]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. If $\mathcal{S}$ has at least one pseudo-embedding, then every pseudo-hyperplane of $\mathcal{S}$ arises from the universal pseudo-embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ of $\mathcal{S}$. Moreover, the formula $H = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$ determines a one-to-one correspondence between the pseudo-hyperplanes H of $\mathcal{S}$ and the hyperplanes Π of $\widetilde{\Sigma}$.*

Theorem 2.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. If $\mathcal{S}$ has a pseudo-embedding, then its universal pseudo-embedding is homogeneous.*

Proof. Let $\widetilde{e} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ denote the universal pseudo-embedding of $\mathcal{S}$ and let $\alpha \in Aut(\mathcal{S})$. Since $\widetilde{e} \circ \alpha$ and $\widetilde{e} \circ \alpha^{-1}$ are pseudo-embeddings of $\mathcal{S}$ into $\mathrm{PG}(\widetilde{V})$, there exist by the universality of $\widetilde{e}$ linear mappings $f_\alpha : \widetilde{V} \to \widetilde{V}$ and $f'_\alpha : \widetilde{V} \to \widetilde{V}$ such that $f_\alpha \circ \widetilde{e} = \widetilde{e} \circ \alpha$ and $f'_\alpha \circ \widetilde{e} = \widetilde{e} \circ \alpha^{-1}$. Then $f'_\alpha \circ f_\alpha \circ \widetilde{e} = f'_\alpha \circ \widetilde{e} \circ \alpha = \widetilde{e} \circ \alpha^{-1} \circ \alpha = \widetilde{e}$. As $\widetilde{V}$ is defined over $\mathbb{F}_2$ and $\widetilde{e}(\mathcal{S})$ spans $\mathrm{PG}(\widetilde{V})$, this forces $f'_\alpha \circ f_\alpha = id_{\widetilde{V}}$. Similarly, $f_\alpha \circ f'_\alpha = id_{\widetilde{V}}$. So, $f_\alpha \in GL(V)$. Since $f_\alpha \circ \widetilde{e} = \widetilde{e} \circ \alpha$ for every $\alpha \in Aut(\mathcal{S})$, the universal pseudo-embedding $\widetilde{e}$ of $\mathcal{S}$ is homogeneous. ■

The following theorem allows us to determine whether a given pseudo-embedding is G-homogeneous, where G is some group of automorphisms of the point-line geometry.

Theorem 2.5 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three, and let G be a group of automorphisms of $\mathcal{S}$. A pseudo-embedding $e : \mathcal{S} \to \Sigma$ of $\mathcal{S}$ is G-homogeneous if and only if G stabilizes the set of pseudo-hyperplanes of $\mathcal{S}$ arising from e.*

Proof. (1) Suppose $e : \mathcal{S} \to \Sigma$ is a G-homogeneous pseudo-embedding. Let H be a pseudo-hyperplane of $\mathcal{S}$ arising from e, let $\theta \in G$ and let η_θ be the unique projectivity of Σ such that $e(x)^{\eta_\theta} = e(x^\theta)$ for every point x of $\mathcal{S}$. If Π denotes the unique hyperplane of Σ for which $H = e^{-1}(e(\mathcal{P}) \cap \Pi)$, then we have $e(H) = e(\mathcal{P}) \cap \Pi$, $e(H)^{\eta_\theta} = e(\mathcal{P})^{\eta_\theta} \cap \Pi^{\eta_\theta}$, $e(H^\theta) = e(\mathcal{P}^\theta) \cap \Pi^{\eta_\theta}$ and hence $H^\theta = e^{-1}(e(\mathcal{P}) \cap \Pi^{\eta_\theta})$. So, also the pseudo-hyperplane H^θ arises from the pseudo-embedding e.

(2) Conversely, suppose that $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ with the property that for every $\theta \in G$ and every pseudo-hyperplane H of $\mathcal{S}$ arising from e, also the pseudo-hyperplane H^θ arises from e. Let $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ denote the universal pseudo-embedding of $\mathcal{S}$ and let α denote the subspace of $\widetilde{\Sigma}$ for which $\widetilde{e}/\alpha \cong e$. Without loss of generality, we may suppose that $e = \widetilde{e}/\alpha$.

For every $\theta \in G$, let η_θ be the unique projectivity of $\widetilde{\Sigma}$ such that $\widetilde{e}(x)^{\eta_\theta} = \widetilde{e}(x^\theta)$ for every point x of $\mathcal{S}$. For every pseudo-hyperplane H of $\mathcal{S}$, let Π_H denote the unique hyperplane of $\widetilde{\Sigma}$ for which $H = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi_H)$. With a similar reasoning as in part (1), we know that $H^\theta = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi_H^{\eta_\theta})$ for every $\theta \in G$. Hence, $\Pi_{H^\theta} = \Pi_H^{\eta_\theta}$ for every $\theta \in G$.

Let $\theta \in G$ and suppose α contains a point p not belonging to α^{η_θ}. Then there exists a hyperplane Π of $\widetilde{\Sigma}$ through α such that Π^{η_θ} does not contain p. Put $H := \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$. Then $H^\theta = \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi^{\eta_\theta})$. But this is impossible since this would imply that H arises from e but H^θ not.

Let $\theta \in G$ and suppose that α^{η_θ} contains a point not belonging to α. Then α contains a point not belonging to $\alpha^{\eta_\theta^{-1}} = \alpha^{\eta_{\theta^{-1}}}$. Applying the previous paragraph with θ^{-1} instead of θ, we obtain a contradiction.

So, we have that $\alpha^{\eta_\theta} = \alpha$ for every $\theta \in G$. This implies that η_θ induces a projectivity η'_θ of the quotient projective space Σ/α. For every point p of $\mathcal{S}$, we have that $e(p)^{\eta'_\theta} = e(p^\theta)$. So, e is indeed a G-homogeneous pseudo-embedding of $\mathcal{S}$. ■

Our next aim is to describe a method that can be used to construct G-homogeneous pseudo-embeddings. This method will be based on a modification of the following result from [8].

Proposition 2.6 ([8, Theorem 1.4(3)]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three.*

- *If $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$, then the set $\mathcal{H}$ of all pseudo-hyperplanes of $\mathcal{S}$ arising from e satisfies the following properties:*
  (a) if H_1 and H_2 are two distinct elements of $\mathcal{H}$, then also $H_1 * H_2$ belongs to $\mathcal{H}$;
  (b) if L is a line of $\mathcal{S}$ containing an odd number of points, then for every point x of L there exists a pseudo-hyperplane of $\mathcal{H}$ which has only the point x in common with L;
  (c) if L is a line of $\mathcal{S}$ containing an even number of points, then for any two distinct points x_1 and x_2 of L, there exists a pseudo-hyperplane of $\mathcal{H}$ having only the points x_1 and x_2 in common with L;
  (d) for every point x of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ not containing x.

- *Conversely, suppose that $\mathcal{H}$ is a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions (a), (b), (c) and (d) above. Then there exists a pseudo-embedding e of $\mathcal{S}$ such that the pseudo-hyperplanes of $\mathcal{S}$ arising from e are precisely the elements of $\mathcal{H}$. This pseudo-embedding e is uniquely determined, up to isomorphism. This pseudo-embedding e is faithful if and only if for any two distinct points x_1 and x_2 of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ containing x_1 but not x_2.*

Observe that condition (d) in Proposition 2.6 follows from conditions (b) and (c) if there is at least one line incident with x.

The following result is an immediate consequence of Theorem 2.5 and Proposition 2.6.

Corollary 2.7 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with the property that the number of points on each line is finite and at least three. Let G be a group of automorphisms of $\mathcal{S}$. Let $\mathcal{H}$ be a finite set pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions (a)-(d) of Proposition 2.6 and let e be the unique pseudo-embedding of $\mathcal{S}$ such that $\mathcal{H}$ is the set of pseudo-hyperplanes of $\mathcal{S}$ arising from e. Then e is G-homogeneous if and only if G stabilizes $\mathcal{H}$.*

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite point-line geometry with the property that the number of points on each line is finite and at least three, and let G be a group of automorphisms of $\mathcal{S}$. The G-homogeneous pseudo-embeddings of $\mathcal{S}$ can possibly be classified in the following way.

(1) Determine all G-orbits of pseudo-hyperplanes of $\mathcal{S}$. We denote these orbits by $\mathcal{H}_i$, $i \in I$, where I is some suitable index set.

(2) Find all subsets $J \subseteq I$ such that $\bigcup_{j\in J} \mathcal{H}_j$ satisfies condition (a) of Proposition 2.6. For each such subset J of I, the number $|\bigcup_{j\in J} \mathcal{H}_j| + 1$ must be a power of 2.

(3) For every subset $J \subseteq I$ for which $\bigcup_{j\in J} \mathcal{H}_j$ satisfies condition (a) of Proposition 2.6, verify the conditions (b), (c) and (d) of Proposition 2.6 to find out whether there exists a pseudo-embedding e of $\mathcal{S}$ for which $\bigcup_{j\in J} \mathcal{H}_j$ is the set of all pseudo-hyperplanes arising from e.

If we have successfully been able to perform the steps (1), (2) and (3), then we have found all G-homogeneous pseudo-embeddings of $\mathcal{S}$.

We will successfully use this method in Sections 4 and 5 to find all homogeneous pseudo-embeddings of all generalized quadrangles of order $(3, t)$.

3 Some constructions of pseudo-hyperplanes

As we have mentioned in Section 1, every GQ of order $(3, t)$ has 2-ovoids. Each such 2-ovoid is an example of a pseudo-hyperplane. We also mentioned in Section 1 that if H_1 and H_2 are two distinct pseudo-hyperplanes, then also $H_1 * H_2$ is a pseudo-hyperplane. We now give a number of other constructions for pseudo-hyperplanes of GQ of order $(3, t)$.

(1) Let $\mathcal{Q}$ be a generalized quadrangle of order $(3, t)$. If X is a nonempty pseudo-hyperplane of $\mathcal{Q}$, then the complement of X is again a pseudo-hyperplane of $\mathcal{Q}$.

(2) Let $\mathcal{Q}$ be a generalized quadrangle of order $(3, t)$, let L be a line of $\mathcal{Q}$ and let x_1, x_2 be two distinct points of L. Let Y denote the set of all points of $\mathcal{Q}$ not contained in L which are collinear with either x_1 or x_2. Then $Y \cup \{x_1, x_2\}$ is a pseudo-hyperplane of $\mathcal{Q}$.

(3) Let $\mathcal{Q}$ be a generalized quadrangle of order $(3, t)$ and let x_1, x_2 be two noncollinear points of $\mathcal{Q}$, then $(x_1^\perp \cup x_2^\perp) \setminus (\{x_1, x_2\} \cup (x_1^\perp \cap x_2^\perp))$ is a pseudo-hyperplane of $\mathcal{Q}$. (For a point x of $\mathcal{Q}$, $x^\perp$ denotes the set of points of $\mathcal{Q}$ collinear with or equal to x.)

(4) The union of two orthogonal hyperbolic lines of $W(3)$ is a pseudo-hyperplane of $W(3)$.

(5) Let $\mathcal{Q}$ be either $Q(4, 3)$, GQ$(3, 5)$ or the (4×4)-grid. Let O be an ovoid of $\mathcal{Q}$ and $x \in O$. Then $x^\perp \cup O$ is a pseudo-hyperplane of $\mathcal{Q}$.

(6) Let $\mathcal{Q}$ be either $Q(4, 3)$, GQ$(3, 5)$ or the (4×4)-grid. Let O be an ovoid of $\mathcal{Q}$ and x a point of $\mathcal{Q}$ not contained in O. Then $(x^\perp \cup O) \setminus (\{x\} \cup (x^\perp \cap O))$ is a pseudo-hyperplane of $\mathcal{Q}$.

(7) Let $\mathcal{Q}$ be either $Q(4, 3)$ or GQ$(3, 5)$. Let O_1 and O_2 be two distinct ovoids which intersect in four points. Then $(O_1 \cup O_2) \setminus (O_1 \cap O_2)$ is a pseudo-hyperplane of $\mathcal{Q}$.

(8) Let Q be a nonsingular parabolic quadric of PG$(4, 3)$ and let ζ be the orthogonal polarity of PG$(4, 3)$ corresponding to Q. The points and lines of PG$(4, 3)$ contained in Q define a generalized quadrangle $Q(4, 3)$. Let α be a plane of PG$(4, 3)$ intersecting Q in a nonsingular conic of α such that the line α^ζ of PG$(4, 3)$ is disjoint from Q. By De Soete and Thas [10, Section 2], $\alpha \cap Q$ is a so-called $\{0, 2\}$-*set of points* of $Q(4, 3)$. This means

that every point of $Q(4,3)$ not contained in $\alpha \cap Q$ is collinear with either 0 or 2 points of $\alpha \cap Q$. The GQ $W(3)$ is the dual GQ of $Q(4,3)$. Let L_1, L_2, L_3 and L_4 denote those lines of $W(3)$ which correspond to the four points of $\alpha \cap Q$. Then every line of $W(3)$ distinct from L_1, L_2, L_3 and L_4 meets either 0 or 2 lines of the set $\{L_1, L_2, L_3, L_4\}$. So, $L_1 \cup L_2 \cup L_3 \cup L_4$ is a pseudo-hyperplane of $W(3)$.

(9) By Knarr [13, Section 5], the generalized quadrangle $Q(5,3)$ has a set $\mathcal{L}$ of 10 mutually disjoint lines forming a $\{0,2\}$-set of lines of $Q(5,3)$. The union of the 10 lines of $\mathcal{L}$ is a pseudo-hyperplane of $Q(5,3)$.

(10) The union of two disjoint (4×4)-subgrids of the generalized quadrangle GQ$(3,5)$ is a pseudo-hyperplane of GQ$(3,5)$.

Lemma 3.1 *Let $\mathcal{Q}_1$ be a generalized quadrangle of order $(3, t_1)$ which is a subgeometry of a generalized quadrangle $\mathcal{Q}_2$ of order $(3, t_2)$. Let X be an even set of $\mathcal{Q}_1$ and let Y be the set of all points y of $\mathcal{Q}_2 \setminus \mathcal{Q}_1$ such that $y^\perp \cap \mathcal{Q}_1$ intersects X in an even number of points. Then $H = \mathcal{Q}_1 \cup Y$ is an even set of $\mathcal{Q}_2$.*

Proof. Observe that t_1 and t_2 are odd.

We prove that X contains an even number of points. Let L be an arbitrary line of $\mathcal{Q}_1$. Then $|L \cap X|$ is even. Each point of $X \setminus L$ is collinear with a unique point of L. Now, there are an even number of points in $L \cap X$ (respectively, $L \setminus X$), through each such point there are t_1 lines of $\mathcal{Q}_1$ distinct from L and on each such line there are an odd (respectively, even) number of points of $X \setminus L$. From these facts, one readily sees that there are an even number of points in $X \setminus L$. Hence, also $|X| = |L \cap X| + |X \setminus L|$ is even.

We now prove that every line L of $\mathcal{Q}_2$ intersects H in an even number of points. We distinguish four cases.

• Suppose L is a line of $\mathcal{Q}_1$. Then $|L \cap H| = 4$.

• Suppose L is a line of $\mathcal{Q}_2$ which intersects $\mathcal{Q}_1$ in a point u belonging to X. Then $(\mathcal{Q}_1 \setminus u^\perp) \cap X$ contains an odd number of points. This follows from the fact that $|X|$ is even, $t_1 + 1$ is even and that each of the $t_1 + 1$ lines of $\mathcal{Q}_1$ through u contains an even number of points of X. If $L = \{u, y_1, y_2, y_3\}$, then $\{(y_1^\perp \cap \mathcal{Q}_1) \setminus \{u\}, (y_2^\perp \cap \mathcal{Q}_1) \setminus \{u\}, (y_3^\perp \cap \mathcal{Q}_1) \setminus \{u\}\}$ is a partition of $\mathcal{Q}_1 \setminus u^\perp$. Since $(\mathcal{Q}_1 \setminus u^\perp) \cap X$ contains an odd number of points, there are an odd number of $i \in \{1,2,3\}$ for which $(y_i^\perp \cap \mathcal{Q}_1 \cap X) \setminus \{u\}$ contains an odd number of points, or equivalently, for which $(y_i^\perp \cap \mathcal{Q}_1 \cap X)$ contains an even number of points. So, $|L \cap H|$ is even.

• Suppose L is a line of $\mathcal{Q}_2$ which intersects $\mathcal{Q}_1$ in a point u not belonging to X. Then $(\mathcal{Q}_1 \setminus u^\perp) \cap X$ contains an even number of points. This follows from the fact that $|X|$ is even and that each of the $t_1 + 1$ lines of $\mathcal{Q}_1$ through u contains an even number of points of X. If $L = \{u, y_1, y_2, y_3\}$, then $\{(y_1^\perp \cap \mathcal{Q}_1) \setminus \{u\}, (y_2^\perp \cap \mathcal{Q}_1) \setminus \{u\}, (y_3^\perp \cap \mathcal{Q}_1) \setminus \{u\}\}$ is a partition of $\mathcal{Q}_1 \setminus u^\perp$. Since $(\mathcal{Q}_1 \setminus u^\perp) \cap X$ contains an even number of points, there are an even number of $i \in \{1,2,3\}$ for which $(y_i^\perp \cap \mathcal{Q}_1 \cap X) \setminus \{u\} = y_i^\perp \cap \mathcal{Q}_1 \cap X$ contains an odd number of points. This implies that there are an even number of points of L which are not contained in H.

• Suppose $L = \{y_1, y_2, y_3, y_4\}$ is a line of $\mathcal{Q}_2$ which is disjoint from $\mathcal{Q}_1$. Then $\{y_1^\perp \cap \mathcal{Q}_1, y_2^\perp \cap \mathcal{Q}_1, y_3^\perp \cap \mathcal{Q}_1, y_4^\perp \cap \mathcal{Q}_1\}$ is a partition of $\mathcal{Q}_1$ into ovoids. Since $|X|$ is even, there are an even number of $i \in \{1,2,3,4\}$ for which $y_i^\perp \cap \mathcal{Q}_1 \cap X$ contains an even number of points. So, also in this case we have that $|L \cap H|$ is even. ∎

Observe that, with the notations of Lemma 3.1, we have:

> If X' is the even set of $\mathcal{Q}_1$ which is the complement of X (in $\mathcal{Q}_1$) and H' is the even set of $\mathcal{Q}_2$ which arises from X' (in the sense of Lemma 3.1), then $H = H'$.

4 The pseudo-hyperplanes and homogeneous pseudo-embeddings of the (4×4)-grid

Let $k_1, k_2 \in \mathbb{N} \setminus \{0,1,2\}$. By De Bruyn [8, Proposition 3.7], the pseudo-embedding rank of the $(k_1 \times k_2)$-grid is equal to $(k_1-1)(k_2-1)$. Actually the proof of Proposition 3.7 of [8] contains an explicit construction of the universal pseudo-embedding of this grid. In the following theorem, we construct another homogeneous pseudo-embedding of the $(k_1 \times k_2)$-grid in case k_1 and k_2 are even.

Theorem 4.1 *Let $k_1, k_2 \geq 4$ be even integers. Then the $(k_1 \times k_2)$-grid has a homogeneous faithful pseudo-embedding in* $\mathrm{PG}(k_1+k_2-4, 2)$.

Proof. We suppose that the point set of the $(k_1 \times k_2)$-grid $\mathcal{G}$ is equal to $\{1,2,\ldots,k_1\} \times \{1,2,\ldots,k_2\}$ where two distinct points (i_1,i_2) and (j_1,j_2) are collinear if and only if either $i_1 = j_1$ or $i_2 = j_2$. Let V be a (k_1+k_2-3)-dimensional vector space over $\mathbb{F}_2$ having a basis consisting of the following k_1+k_2-3 vectors: $\bar{e}[1,1]$, $\bar{e}[i_1,1]$, $\bar{e}[1,i_2]$ where $2 \leq i_1 \leq k_1-1$ and $2 \leq i_2 \leq k_2-1$. Define

$$\bar{e}[k_1,1] := \sum_{i=1}^{k_1-1} \bar{e}[i,1], \qquad \bar{e}[1,k_2] := \sum_{i=1}^{k_2-1} \bar{e}[1,i].$$

For every point (i_1,i_2) of $\mathcal{G}$ with $i_1, i_2 \geq 2$, we define

$$\bar{e}[i_1,i_2] := \bar{e}[1,1] + \bar{e}[i_1,1] + \bar{e}[1,i_2].$$

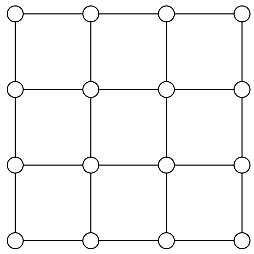
Observe that the last formula remains valid if at least one of i_1, i_2 is equal to 1. For every point $p = (i_1,i_2)$ of $\mathcal{G}$, we define

$$e(p) := \langle \bar{e}[i_1,i_2] \rangle \in \mathrm{PG}(V).$$

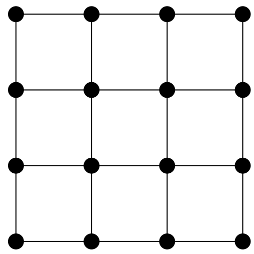
It is straightforward to verify that e defines a homogeneous faithful pseudo-embedding of $\mathcal{G}$ into $\mathrm{PG}(V)$. ∎

Now, consider the (4×4)-grid $\mathcal{G}$. The fact that the pseudo-embedding rank of $\mathcal{G}$ is equal to 9 implies by Proposition 2.3 that $\mathcal{G}$ has $2^9 = 512$ even sets. These even sets can easily be classified by hand. We list them below and call them *even sets of Type 1*, *Type 2*, ..., *Type 10*. In Table 1, we list a few of the basic properties of these sets.

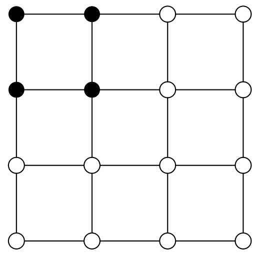
Type 1



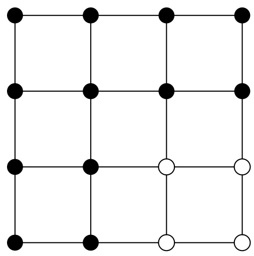
Type 2



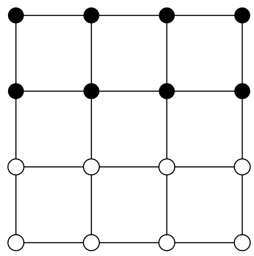
Type 3



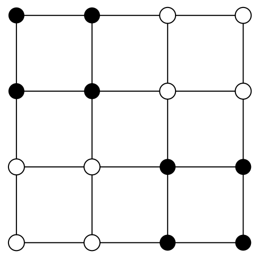
Type 4



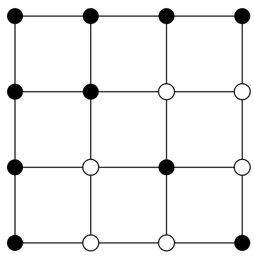
Type 5



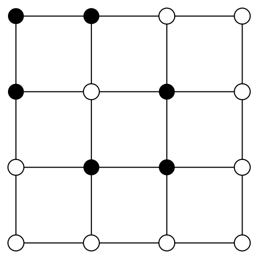
Type 6



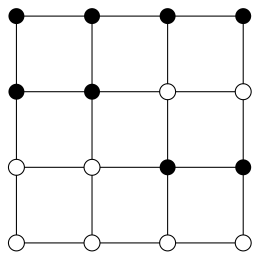
Type 7



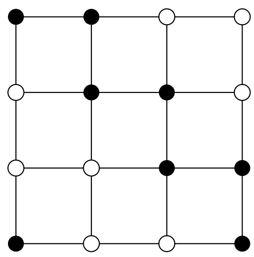
Type 8



Type 9



Type 10



Type	# even sets	# points	Type of complement	Remark
1	1	0	2	$\emptyset$
2	1	16	1	$\mathcal{P}$
3	36	4	4	–
4	36	12	3	–
5	12	8	5	2-tight
6	18	8	6	2-ovoid
7	96	10	8	–
8	96	6	7	–
9	144	8	9	–
10	72	8	10	2-ovoid

Table 1: The even sets of the (4×4)-grid

Now, let R be the following relation on the set $I := \{1, 2, \ldots, 10\}$. If $i_1, i_2 \in I$, then $(i_1, i_2) \in R$ if $i_1 = i_2$ or if there exist two even sets H_1 and H_2 of Type i_1 of $\mathcal{G}$ such that $H_1 * H_2$ is an even set of Type i_2. Let $\overline{R}$ be the transitive closure of R, i.e. the smallest transitive relation of the set I containing R. We determined R with the aid of a computer. In the following two tables, we describe the relations R and $\overline{R}$. An "X" in row i and column j means that the couple (i, j) belongs to the relation.

R	1	2	3	4	5	6	7	8	9	10
1	X	X	–	–	–	–	–	–	–	–
2	–	X	–	–	–	–	–	–	–	–
3	–	X	X	X	X	X	X	–	X	–
4	–	X	–	X	X	X	X	–	X	–
5	X	X	–	–	X	X	–	–	–	–
6	X	X	–	–	X	X	–	–	–	–
7	–	X	X	X	X	X	X	X	X	X
8	–	X	X	X	X	X	X	X	X	X
9	X	X	X	X	X	X	X	X	X	X
10	X	X	X	X	X	X	X	X	X	X

$\overline{R}$	1	2	3	4	5	6	7	8	9	10
1	X	X	–	–	–	–	–	–	–	–
2	–	X	–	–	–	–	–	–	–	–
3	X	X	X	X	X	X	X	X	X	X
4	X	X	X	X	X	X	X	X	X	X
5	X	X	–	–	X	X	–	–	–	–
6	X	X	–	–	X	X	–	–	–	–
7	X	X	X	X	X	X	X	X	X	X
8	X	X	X	X	X	X	X	X	X	X
9	X	X	X	X	X	X	X	X	X	X
10	X	X	X	X	X	X	X	X	X	X

Now, we make the following convention. If H is an even set of $\mathcal{G}$ and e is a pseudo-embedding of $\mathcal{G}$, then we say that H *arises from* e if either $H = \mathcal{P}$ or H is a pseudo-hyperplane arising from e.

For every $i \in I$, we define $T(i) := \{j \in I \,|\, (i,j) \in \overline{R}\}$. By Theorem 2.5 and Proposition 2.6, if e is a homogeneous pseudo-embedding of $\mathcal{G}$ and if H is an even set of Type i arising from e, then every even set whose type belongs to $T(i)$ must also arise from e. So, if we put $\Omega := \{T(i) \,|\, i \in I\}$, then the following holds.

> If e is a homogeneous pseudo-embedding of $\mathcal{G}$ and if I_e is the set of all $i \in I$ such that all even sets of Type i arise from e, then I_e is the union of a number of elements of Ω.

Now, we can easily see that Ω consists of the following four elements:
- $I_1 = \{2\}$;
- $I_2 = \{1, 2\}$;
- $I_3 = \{1, 2, 5, 6\}$;
- $I_4 = I$.

Since $I_1 \subset I_2 \subset I_3 \subset I_4$, there are only four possibilities for a nonempty subset of I that can be written as the union of a number of elements of Ω, namely the sets I_1, I_2, I_3 and I_4 themselves. Now, for every $j \in \{1, 2, 3, 4\}$, let $\mathcal{H}_j$ denote the set of those pseudo-hyperplanes of $\mathcal{G}$ whose type belong to I_j. We have verified that $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ satisfy condition (a) of Proposition 2.6, but that only the sets $\mathcal{H}_3$ and $\mathcal{H}_4$ also satisfy conditions (c) and (d).

We conclude that there are up to isomorphism two homogeneous pseudo-embeddings of $\mathcal{G}$, namely the homogeneous pseudo-embedding e_3 corresponding to $\mathcal{H}_3$ and the homogeneous pseudo-embedding e_4 corresponding to $\mathcal{H}_4$. The pseudo-embedding e_4 must be universal, while the pseudo-embedding e_3 must be the one which was already described in Theorem 4.1. Observe also that since $|\mathcal{H}_4| = 2^9 - 1$ and $|\mathcal{H}_3| = 2^5 - 1$, the corresponding pseudo-embedding spaces are isomorphic to $\mathrm{PG}(8, 2)$ and $\mathrm{PG}(4, 2)$, respectively.

5 The pseudo-hyperplanes and homogeneous pseudo-embeddings of the generalized quadrangles $W(3)$, $Q(4,3)$, $Q(5,3)$ and $\mathrm{GQ}(3,5)$

Let $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be one of the GQ's $W(3)$, $Q(4,3)$, $Q(5,3)$, $\mathrm{GQ}(3,5)$, and let v denote the total number of points of $\mathcal{Q}$. We have $v = 40$, $v = 40$, $v = 112$ and $v = 64$ in the respective cases.

Into the computer algebra system GAP [12], there are build many models of permutation groups, including a model of the permutation representation of $\mathrm{Aut}(\mathcal{Q})$ on the set $\{1, 2, \ldots, v\}$ which is equivalent with the permutation representation of $\mathrm{Aut}(\mathcal{Q})$ on the point set $\mathcal{P}$ of $\mathcal{Q}$. One can easily identify those subsets of size 4 of $\{1, 2, \ldots, v\}$ which correspond to the lines of $\mathcal{Q}$. In this way, we obtain a computer model of the GQ $\mathcal{Q}$ and a computer model for the permutation representation of $\mathrm{Aut}(\mathcal{Q})$ on the point set $\mathcal{P}$ of $\mathcal{Q}$.

Now, that we have identified the lines of $\mathcal{Q}$ in our computer model, we can implement computer models for the vector space V and the subspace $W \subseteq V$ which occur in Proposition 2.2. The pseudo-embedding rank of $\mathcal{Q}$ is then equal to $d := v - \dim(W)$. We find that $d = 15$, $d = 15$, $d = 21$ and $d = 18$ in the respective cases. By Propositions 2.2 and 2.3, we then know that the total number of even sets of $\mathcal{Q}$ is equal to 2^d.

By Propositions 2.2 and 2.3, we also know that there exists a one-to-one correspondence between the hyperplanes Π of V through W and the pseudo-hyperplanes H_Π of $\mathcal{Q}$. We have implemented a procedure in GAP which allows us to turn each hyperplane Π of V through W into the subset of $\{1, 2, \ldots, v\}$ which corresponds with $H_\Pi \subseteq \mathcal{P}$.

We have subsequently used the following procedure to find all even sets of $\mathcal{Q}$.

- **Step 1:** The empty set $\emptyset$ and the point set $\mathcal{P}$ of $\mathcal{Q}$ are examples of even sets. Put $N := 2$.

- **Step 2:** Let GAP choose a random hyperplane Π of V through W and let GAP calculate the corresponding pseudo-hyperplane H. Calculate the stabilizer S_H of H (in $\mathrm{Aut}(\mathcal{Q})$). Then the total number of pseudo-hyperplanes isomorphic to H is equal to $N_H := \frac{|\mathrm{Aut}(\mathcal{Q})|}{|S_H|}$. Verify whether H is isomorphic to one of the previous obtained even sets. If this is not the case, then put $N := N + N_H$.

- **Step 3:** If $N < 2^d$, then go again to Step 2. If $N = 2^d$, then we have found all even sets and we are done.

In this way, we found that $\mathcal{Q}$ has M even sets, where $M = 20$, $M = 21$, $M = 22$ and $M = 47$ in the respective cases. We have also written various procedures in GAP to find various properties of the even sets. These properties can be found in Tables 2, 3, 4, 5, 6 and 7, where we have ordered the even sets according to the size of their orbits and the number of points they contain[1]. Several of these even sets were already constructed in

[1]The structure of the stabilizers of the even sets of Type 6, 9 and 10 of $Q(5,3)$ (see Table 4) were computed by Tim Penttila using the computer algebra system MAGMA [6]. (GAP remained irresponsive when asked to compute these stabilizers.)

Section 3.

• The even sets of Type (5) of $W(3)$ were constructed in Section 3(2).

• The even sets of Type (9) of $W(3)$ were constructed in Section 3(3).

• The even sets of Type (3) of $W(3)$ were constructed in Section 3(4).

• The even sets of Type (6) of $W(3)$ were constructed in Section 3(8).

• The even sets of Type (7) of $Q(4,3)$ were constructed in Section 3(2).

• The even sets of Type (3) of $Q(4,3)$ were constructed in Section 3(3).

• The even sets of Type (12) of $Q(4,3)$ were constructed in Section 3(5).

• The even sets of Type (13) of $Q(4,3)$ were constructed in Section 3(6).

• The even sets of Type (8) of $Q(4,3)$ were constructed in Section 3(7).

• The even sets of Type (4) of $Q(4,3)$ can be constructed as in Lemma 3.1 of Section 3 if one starts from an even set of Type $i \in \{3,4,9,10\}$ of a (4×4)-subgrid $\mathcal{G}$ of $Q(4,3)$.

• The even sets of Type (6) of $Q(5,3)$ were constructed in Section 3(2).

• The even sets of type (7) of $Q(5,3)$ were constructed in Section 3(3).

• The even sets of Type (11) of $Q(5,3)$ were constructed in Section 3(9).

• The even sets of Type (5) of $Q(5,3)$ can be constructed as in Lemma 3.1 of Section 3 if one starts from an even set of Type $i \in \{7,8\}$ of a (4×4)-subgrid or an even set of Type $j \in \{8,9,10,15,18,21\}$ of a $Q(4,3)$-subquadrangle.

• The even sets of Type (8) of $Q(5,3)$ can be constructed as in Lemma 3.1 of Section 3 if one starts from an even set of Type $i \in \{5,6,11,12,13,14,16,17,19,20\}$ of a $Q(4,3)$-subquadrangle.

• The even sets of Type (9) of $Q(5,3)$ can be constructed as in Lemma 3.1 of Section 3 if one starts from an even set of Type $i \in \{3,4,9,10\}$ of a (4×4)-subgrid.

• The even sets of Type (14) of GQ$(3,5)$ were constructed in Section 3(2).

• The even sets of Type (17) of GQ$(3,5)$ were constructed in Section 3(3).

• The even sets of Type (13) of GQ$(3,5)$ were constructed in Section 3(5).

• The even sets of Type (21) of GQ$(3,5)$ were constructed in Section 3(6).

• The even sets of Type (7) of GQ$(3,5)$ were constructed in Section 3(7).

• The even sets of Type (4) of GQ$(3,5)$ were constructed in Section 3(10).

• Every even set of Type (3) of GQ$(3,5)$ is the union of two disjoint ovoids.

Remarks. (1) Every even set of GQ$(3,5)$ which can be obtained as described in Lemma 3.1 is either the whole set of points or the union of two disjoint (4×4)-subgrids.

(2) Let $\mathcal{Q}$ be a generalized quadrangle of order $(3,t)$. Recall that if H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{Q}$, then $H_1 * H_2$ is again a pseudo-hyperplane of $\mathcal{Q}$. Starting from the list of pseudo-hyperplanes of $\mathcal{Q}$ described above, one can find many other pseudo-hyperplanes of $\mathcal{Q}$ in this way. In fact, every pseudo-hyperplane of $\mathcal{Q}$ can be obtained from this list of pseudo-hyperplanes by successive application of this construction. We give one example. Let L_1 and L_2 be two orthogonal hyperbolic lines of $W(3)$, let $x_1 \in L_1$ and $x_2 \in L_2$. Then $H_1 = L_1 \cup L_2$ and $H_2 = \{x_1, x_2\} \cup ((x_1^\perp \cup x_2^\perp) \setminus x_1x_2)$ are two pseudo-hyperplanes of $W(3)$ (of respective Types (3) and (5)). The pseudo-hyperplane $H_1 * H_2$ of $W(3)$ contains 28 points and is of Type (12).

T	N	v	I_l	I_h	Stabilizer	O_1	O_2	C	Remark
1	1	0	[40,0,0,0,0]	[90,0,0,0,0]	$O(5,3):C_2$	0	1	2	$\emptyset$
2	1	40	[0,0,0,0,40]	[0,0,0,0,90]	$O(5,3):C_2$	1	0	1	$\mathcal{P}$
3	45	8	[24,0,16,0,0]	[24,64,0,0,2]	$((((SL(2,3):C_2):C_2):C_3):C_2):C_2$	1	1	4	2-tight
4	45	32	[0,0,16,0,24]	[2,0,0,64,24]	$(((((C_2\times D_8):C_2):C_3):C_3):C_2):C_2$	1	1	3	8-tight
5	240	20	[6,0,28,0,6]	[6,12,54,12,6]	$S_3\times S_3\times S_3$	2	2	5	–
6	270	16	[12,0,24,0,4]	[6,32,48,0,4]	$S_4\times D_8$	1	1	7	4-tight
7	270	24	[4,0,24,0,12]	[4,0,48,32,6]	$S_4\times D_8$	1	1	6	6-tight
8	432	20	[0,0,40,0,0]	[10,20,30,20,10]	$C_2\times A_5$	1	1	8	2-ovoid
9	540	16	[8,0,32,0,0]	[18,16,40,16,0]	$GL(2,3):C_2$	1	4	10	–
10	540	24	[0,0,32,0,8]	[0,16,40,16,18]	$GL(2,3):C_2$	4	1	9	–
11	720	12	[16,0,24,0,0]	[20,36,30,4,0]	$C_2\times S_3\times S_3$	1	4	12	–
12	720	28	[0,0,24,0,16]	[0,4,30,36,20]	$C_2\times S_3\times S_3$	4	1	11	–
13	2160	20	[6,0,28,0,6]	[2,28,30,28,2]	$C_2\times C_2\times S_3$	3	3	13	–
14	2880	16	[9,0,30,0,1]	[9,36,30,12,3]	D_{18}	4	4	16	–
15	2880	20	[3,0,34,0,3]	[5,28,24,28,5]	$(C_3\times C_3):C_2$	5	5	15	–
16	2880	24	[1,0,30,0,9]	[3,12,30,36,9]	D_{18}	4	4	14	–
17	3240	16	[10,0,28,0,2]	[10,32,32,16,0]	$C_2\times D_8$	2	4	18	–
18	3240	24	[2,0,28,0,10]	[0,16,32,32,10]	$C_2\times D_8$	4	2	17	–
19	5184	20	[5,0,30,0,5]	[5,20,40,20,5]	D_{10}	4	4	19	–
20	6480	20	[4,0,32,0,4]	[6,20,38,20,6]	$C_2\times C_2\times C_2$	4	4	20	–

T denotes the type of the even set H, N denotes the total number of even sets of Type T, $v := |H|$, the i-th component of I_l [resp. I_h] denotes the total number of lines [resp. hyperbolic lines] of $W(3)$ meeting H in precisely $i-1$ points, O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H], and C denotes the type of the complement of H.

Table 2: The even sets of the generalized quadrangle $W(3)$

T	N	v	I_l	I_g	Stabilizer	O_1	O_2	C	Remark
1	1	0	[40,0,0,0,0]	[45,0,0,0,0,0,0,0,0,0]	$O(5,3):C_2$	0	1	2	$\emptyset$
2	1	40	[0,0,0,0,40]	[0,45,0,0,0,0,0,0,0,0]	$O(5,3):C_2$	1	0	1	$\mathcal{P}$
3	135	16	[8,0,32,0,0]	[1,0,0,0,0,12,0,32,0,0]	$((((C_2 \times D_8):C_2):C_3):C_2):C_2$	1	2	4	–
4	135	24	[0,0,32,0,8]	[0,1,0,0,0,12,32,0,0,0]	$((((C_2 \times D_8):C_2):C_3):C_2):C_2$	2	1	3	–
5	216	10	[20,0,20,0,0]	[5,0,30,0,0,0,0,10,0,0]	$C_2 \times S_5$	1	2	6	–
6	216	30	[0,0,20,0,20]	[0,5,0,30,0,0,10,0,0,0]	$C_2 \times S_5$	2	1	5	–
7	240	20	[6,0,28,0,6]	[0,0,0,0,9,0,18,18,0,0]	$(((C_3 \times C_3):C_3):C_4):C_2$	2	2	7	–
8	270	12	[16,0,24,0,0]	[2,0,24,0,0,3,0,16,0,0]	$D_8 \times S_4$	1	2	9	–
9	270	28	[0,0,24,0,16]	[0,2,0,24,0,3,16,0,0,0]	$D_8 \times S_4$	2	1	8	–
10	324	20	[0,0,40,0,0]	[0,0,0,0,0,5,0,0,0,40]	$((C_2 \times C_2 \times C_2 \times C_2):C_5):C_2$	1	1	10	2-ovoid
11	360	18	[4,0,36,0,0]	[0,0,0,0,0,9,0,18,0,18]	$C_2 \times ((S_3 \times S_3):C_2)$	1	3	12	–
12	360	22	[0,0,36,0,4]	[0,0,0,0,0,9,18,0,0,18]	$C_2 \times ((S_3 \times S_3):C_2)$	3	1	11	–
13	1080	14	[12,0,28,0,0]	[1,0,12,0,0,0,0,26,0,6]	$C_2 \times S_4$	2	5	14	–
14	1080	26	[0,0,28,0,12]	[0,1,0,12,0,0,26,0,0,6]	$C_2 \times S_4$	5	2	13	–
15	3240	16	[10,0,28,0,2]	[0,0,12,0,1,0,0,12,16,4]	D_{16}	2	4	18	–
16	3240	18	[8,0,28,0,4]	[0,0,6,0,4,1,6,12,16,0]	$C_2 \times D_8$	4	4	17	–
17	3240	22	[4,0,28,0,8]	[0,0,0,6,4,1,12,6,16,0]	$C_2 \times D_8$	4	4	16	–
18	3240	24	[2,0,28,0,10]	[0,0,0,12,1,0,12,0,16,4]	D_{16}	4	2	15	–
19	4320	18	[6,0,32,0,2]	[0,0,3,0,0,3,3,15,12,9]	D_{12}	5	6	20	–
20	4320	22	[2,0,32,0,6]	[0,0,0,3,0,3,15,3,12,9]	D_{12}	6	5	19	–
21	6480	20	[4,0,32,0,4]	[0,0,1,1,0,1,8,8,16,10]	$C_2 \times C_2 \times C_2$	6	6	21	–

T denotes the type of the even set H, N denotes the total number of even sets of Type T, $v := |H|$, the i-th component of I_l denotes the total number of lines meeting H in precisely $i-1$ points, the i-th component of I_g denotes the total number of (4×4)-subgrids $\mathcal{G}$ of $Q(4,3)$ which intersect H in an even set of Type i of $\mathcal{G}$, O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H], and C denotes the type of the complement of H.

Table 3: The even sets of the generalized quadrangle $Q(4,3)$

T	N	v	Stabilizer	O_1	O_2	C	Remark
1	1	0	$PSU(4,3).D_8$	0	1	2	$\emptyset$
2	1	112	$PSU(4,3).D_8$	1	0	1	$\mathcal{P}$
3	648	56	$PSL(3,4):C_2$	1	1	3	2-ovoid
4	1134	32	$(((C_2 \times C_2 \times C_2 \times C_2):A_6):C_2):C_2$	1	1	5	8-tight
5	1134	80	$(((C_2 \times C_2 \times C_2 \times C_2):A_6):C_2):C_2$	1	1	4	20-tight
6	1680	56	$((((((C_3.C_3^4).C_2).C_2).C_2).C_2).C_2).C_2$	2	2	6	–
7	4536	40	$(C_2 \times C_2 \times S_6):C_2$	1	3	8	–
8	4536	72	$(C_2 \times C_2 \times S_6):C_2$	3	1	7	–
9	8505	48	$((((((C_2^2.C_2^2).C_2^2).C_3).C_2).C_2).C_2).C_2$	1	1	10	12-tight
10	8505	64	$((((((C_2^2.C_2^2).C_2^2).C_3).C_2).C_2).C_2).C_2$	1	1	9	16-tight
11	9072	40	$C_2 \times ((A_6.C_2):C_2)$	1	1	12	10-tight
12	9072	72	$C_2 \times ((A_6.C_2):C_2)$	1	1	11	18-tight
13	38880	56	$C_2 \times (PSL(3,2):C_2)$	1	1	13	14-tight
14	45360	56	$S_4 \times S_4$	3	3	14	–
15	181440	48	$C_2 \times ((S_3 \times S_3):C_2)$	2	5	16	–
16	181440	64	$C_2 \times ((S_3 \times S_3):C_2)$	5	2	15	–
17	204120	48	$(D_8 \times D_8):C_2$	2	4	19	–
18	204120	56	$(D_8 \times D_8):C_2$	4	4	18	–
19	204120	64	$(D_8 \times D_8):C_2$	4	2	17	–
20	217728	56	S_5	3	3	20	–
21	362880	56	$(S_3 \times S_3):C_2$	3	3	21	–
22	408240	56	$(C_2 \times D_{16}):C_2$	4	4	22	–

T denotes the type of the even set H, N denotes the total number of even sets of Type T, $v := |H|$, O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H], and C denotes the type of the complement of H.

Table 4: The even sets of the generalized quadrangle $Q(5,3)$

T	I_l	I_g	I_q
1	[280,0,0,0,0]	[2835,0,0,0,0,0,0,0,0,0]	[252,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]
2	[0,0,0,0,280]	[0,2835,0,0,0,0,0,0,0,0]	[0,252,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]
3	[0,0,280,0,0]	[0,0,0,0,0,315,0,0,0,2520]	[0,0,0,0,0,0,0,0,0,252,0,0,0,0,0,0,0,0,0,0,0]
4	[120,0,160,0,0]	[255,0,1440,0,0,180,0,960,0,0]	[6,0,30,0,96,0,0,120,0,0,0,0,0,0,0,0,0,0,0,0,0]
5	[0,0,160,0,120]	[0,255,0,1440,0,180,960,0,0,0]	[0,6,0,30,0,96,0,0,120,0,0,0,0,0,0,0,0,0,0,0,0]
6	[18,0,244,0,18]	[0,0,0,0,81,486,648,648,0,972]	[0,0,0,0,0,0,36,0,0,0,108,108,0,0,0,0,0,0,0,0,0]
7	[80,0,200,0,0]	[90,0,720,0,0,225,0,1440,0,360]	[2,0,30,0,0,0,0,60,0,0,40,0,120,0,0,0,0,0,0,0,0]
8	[0,0,200,0,80]	[0,90,0,720,0,225,1440,0,0,360]	[0,2,0,30,0,0,0,0,60,0,0,40,0,120,0,0,0,0,0,0,0]
9	[64,0,192,0,24]	[24,3,576,0,192,120,384,768,768,0]	[0,0,0,12,0,0,0,48,0,0,0,0,0,0,0,192,0,0,0,0,0]
10	[24,0,192,0,64]	[3,24,0,576,192,120,768,384,768,0]	[0,0,12,0,0,0,0,0,48,0,0,0,0,0,0,0,192,0,0,0,0]
11	[90,0,180,0,10]	[90,0,1080,0,45,0,0,720,720,180]	[0,0,0,0,72,0,0,0,0,0,0,0,0,0,180,0,0,0,0,0,0]
12	[10,0,180,0,90]	[0,90,0,1080,45,0,720,0,720,180]	[0,0,0,0,0,72,0,0,0,0,0,0,0,0,0,0,0,180,0,0,0]
13	[42,0,196,0,42]	[0,0,252,252,189,42,504,504,1008,84]	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,42,84,84,42,0,0,0]
14	[24,0,232,0,24]	[3,3,72,72,0,261,528,528,576,792]	[0,0,6,6,0,0,0,3,3,18,24,24,0,0,0,0,0,0,48,48,72]
15	[58,0,204,0,18]	[21,0,486,18,81,81,252,852,720,324]	[0,0,0,0,6,0,6,12,0,0,6,0,30,0,54,54,0,0,36,12,36]
16	[18,0,204,0,58]	[0,21,18,486,81,81,852,252,720,324]	[0,0,0,0,0,6,6,0,12,0,0,6,0,30,0,0,54,54,12,36,36]
17	[56,0,208,0,16]	[21,0,456,24,40,86,192,864,736,416]	[0,0,4,0,8,0,0,4,0,4,0,0,32,0,64,16,8,0,64,0,48]
18	[32,0,216,0,32]	[4,4,128,128,64,147,576,576,768,440]	[0,0,4,4,0,0,0,2,2,0,8,8,8,8,0,32,32,0,32,32,80]
19	[16,0,208,0,56]	[0,21,24,456,40,86,864,192,736,416]	[0,0,0,4,0,8,0,0,4,4,0,0,0,32,0,8,16,64,0,64,48]
20	[30,0,220,0,30]	[0,0,150,150,15,120,480,480,840,600]	[0,0,0,0,0,0,0,0,0,12,0,0,0,0,30,0,0,30,60,60,60]
21	[36,0,208,0,36]	[0,0,198,198,90,81,504,504,936,324]	[0,0,0,0,0,0,0,0,0,0,0,0,0,0,36,36,36,36,36,36,36]
22	[34,0,212,0,34]	[2,2,160,160,81,98,552,552,832,396]	[0,0,0,0,0,0,4,0,0,0,4,4,8,8,16,32,32,16,32,32,64]

T denotes the type of the even set H, the i-th component of I_l denotes the total number of lines meeting H in precisely $i-1$ points, the i-th component of I_g [resp. I_q] denotes the total number of (4×4)-subgrids $\mathcal{G}$ [resp. $Q(4,3)$-subquadrangles $\mathcal{Q}$] of $Q(5,3)$ which intersect H in an even set of Type i of $\mathcal{G}$ [resp. $\mathcal{Q}$].

Table 5: The even sets of the generalized quadrangle $Q(5,3)$

T	N	v	I_l	I_g	Stabilizer	O_1	O_2	C	Remark
1	1	0	[96,0,0,0,0]	[60,0,0,0,0,0,0,0,0,0]	$((C_2)^6 : (C_3.A_6)) : C_2$	0	1	2	$\emptyset$
2	1	64	[0,0,0,0,96]	[0,60,0,0,0,0,0,0,0,0]	$((C_2)^6 : (C_3.A_6)) : C_2$	1	0	1	$\mathcal{P}$
3	36	32	[0,0,96,0,0]	[0,0,0,0,0,60,0,0,0,0]	$(((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2) : C_2$	1	1	3	2-ovoid
4	90	32	[16,0,64,0,16]	[2,2,0,0,32,24,0,0,0,0]	$((((C_2)^6 : C_3) : C_2) : C_2) : C_2$	1	1	4	8-tight
5	180	16	[48,0,48,0,0]	[6,0,48,0,0,6,0,0,0,0]	$(C_2 \times ((((C_2 \times D_8) : C_2) : C_3) : C_2)) : C_2$	1	1	6	4-tight
6	180	48	[0,0,48,0,48]	[0,6,0,48,0,6,0,0,0,0]	$(C_2 \times ((((C_2 \times D_8) : C_2) : C_3) : C_2)) : C_2$	1	1	5	12-tight
7	240	24	[24,0,72,0,0]	[3,0,0,0,0,9,0,48,0,0]	$S_4 \times S_4$	1	2	8	–
8	240	40	[0,0,72,0,24]	[0,3,0,0,0,9,48,0,0,0]	$S_4 \times S_4$	2	1	7	–
9	288	24	[30,0,60,0,6]	[5,0,0,0,15,0,0,40,0,0]	$(C_2 \times C_2 \times A_5) : C_2$	1	1	10	6-tight
10	288	40	[6,0,60,0,30]	[0,5,0,0,15,0,40,0,0,0]	$(C_2 \times C_2 \times A_5) : C_2$	1	1	9	10-tight
11	360	32	[0,0,96,0,0]	[0,0,0,0,0,12,0,0,0,48]	$((((C_2 \times D_8) : C_2) : C_3) : C_2) : C_2$	1	1	11	2-ovoid
12	384	30	[6,0,90,0,0]	[0,0,0,0,0,0,0,15,0,45]	$GL(2,4) : C_2$	1	3	13	–
13	384	34	[0,0,90,0,6]	[0,0,0,0,0,0,15,0,0,45]	$GL(2,4) : C_2$	3	1	12	–
14	576	32	[10,0,76,0,10]	[0,0,0,0,5,15,20,20,0,0]	$C_2 \times S_5$	2	2	14	–
15	640	18	[42,0,54,0,0]	[6,0,27,0,0,0,0,27,0,0]	$(((C_3 \times C_3) : C_3) : C_4) : C_2$	1	3	16	–
16	640	46	[0,0,54,0,42]	[0,6,0,27,0,0,27,0,0,0]	$(((C_3 \times C_3) : C_3) : C_4) : C_2$	3	1	15	–
17	720	24	[24,0,72,0,0]	[3,0,0,0,0,9,0,48,0,0]	$S_4 \times D_8$	1	3	18	–
18	720	40	[0,0,72,0,24]	[0,3,0,0,0,9,48,0,0,0]	$S_4 \times D_8$	3	1	17	–
19	1080	32	[16,0,64,0,16]	[0,0,8,8,8,4,0,0,32,0]	$(D_8 \times D_8) : C_2$	1	1	19	8-tight
20	1080	32	[8,0,80,0,8]	[1,1,0,0,0,10,0,0,32,16]	$(C_2 \times (((C_4 \times C_2) : C_2) : C_2)) : C_2$	2	2	20	–
21	1152	22	[30,0,66,0,0]	[0,0,30,0,0,0,0,15,0,15]	S_5	2	4	23	–
22	1152	32	[10,0,76,0,10]	[0,0,0,0,5,15,20,20,0,0]	$C_2 \times A_5$	2	2	22	–
23	1152	42	[0,0,66,0,30]	[0,0,0,30,0,0,15,0,0,15]	S_5	4	2	21	–

T denotes the type of the even set H, N denotes the total number of even sets of Type T, $v := |H|$, the i-th component of I_l denotes the total number of lines meeting H in precisely $i-1$ points, the i-th component of I_g denotes the total number of (4×4)-subgrids $\mathcal{G}$ of GQ$(3,5)$ which intersect H in an even set of Type i of $\mathcal{G}$, O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H], and C denotes the type of the complement of H.

Table 6: The even sets of GQ$(3,5)$

T	N	v	I_l	I_g	Stabilizer	O_1	O_2	C	Remark
24	1440	32	[12,0,72,0,12]	[0,0,0,0,6,6,24,24,0,0]	$C_2 \times C_2 \times S_4$	2	2	24	–
25	1440	32	[12,0,72,0,12]	[0,0,0,0,6,6,24,24,0,0]	$C_2 \times C_2 \times S_4$	2	2	25	–
26	2880	32	[12,0,72,0,12]	[0,0,0,0,6,6,24,24,0,0]	$C_2 \times S_4$	2	2	26	–
27	2880	32	[12,0,72,0,12]	[0,0,6,6,6,6,0,0,24,12]	$C_2 \times S_4$	2	2	27	–
28	4320	24	[26,0,68,0,2]	[1,0,16,0,1,2,0,24,8,8]	$(C_2 \times C_2 \times C_2 \times C_2) : C_2$	2	4	30	–
29	4320	24	[28,0,64,0,4]	[1,0,20,0,2,1,0,16,16,4]	$C_2 \times C_2 \times D_8$	2	4	31	–
30	4320	40	[2,0,68,0,26]	[0,1,0,16,1,2,24,0,8,8]	$(C_2 \times C_2 \times C_2 \times C_2) : C_2$	4	2	28	–
31	4320	40	[4,0,64,0,28]	[0,1,0,20,2,1,16,0,16,4]	$C_2 \times C_2 \times D_8$	4	2	29	–
32	5760	26	[24,0,66,0,6]	[0,0,18,0,0,0,3,12,24,3]	S_4	5	4	33	–
33	5760	38	[6,0,66,0,24]	[0,0,0,18,0,0,12,3,24,3]	S_4	4	5	32	–
34	8640	32	[8,0,80,0,8]	[0,0,2,2,0,4,8,8,16,20]	$C_2 \times C_2 \times C_2 \times C_2$	6	6	34	–
35	8640	32	[14,0,68,0,14]	[0,0,4,4,3,1,12,12,24,0]	$C_2 \times D_8$	4	4	35	–
36	11520	26	[22,0,70,0,4]	[1,0,9,0,2,3,2,25,12,6]	D_{12}	4	9	39	–
37	11520	30	[12,0,78,0,6]	[0,0,3,0,0,6,9,18,12,12]	D_{12}	7	6	38	–
38	11520	34	[6,0,78,0,12]	[0,0,0,3,0,6,18,9,12,12]	D_{12}	6	7	37	–
39	11520	38	[4,0,70,0,22]	[0,1,0,9,2,3,25,2,12,6]	D_{12}	9	4	36	–
40	17280	30	[14,0,74,0,8]	[0,0,6,2,0,0,4,11,24,13]	D_8	7	10	44	–
41	17280	30	[16,0,70,0,10]	[0,0,6,1,4,2,11,16,16,4]	D_8	9	6	45	–
42	17280	32	[12,0,72,0,12]	[0,0,4,4,2,2,8,8,24,8]	$C_2 \times C_2 \times C_2$	6	6	42	–
43	17280	32	[10,0,76,0,10]	[0,0,2,2,1,3,12,12,16,12]	$C_2 \times C_2 \times C_2$	7	7	43	–
44	17280	34	[8,0,74,0,14]	[0,0,2,6,0,0,11,4,24,13]	D_8	10	7	40	–
45	17280	34	[10,0,70,0,16]	[0,0,1,6,4,2,16,11,16,4]	D_8	6	9	41	–
46	23040	28	[18,0,72,0,6]	[0,0,9,0,0,0,6,18,18,9]	S_3	7	9	47	–
47	23040	36	[6,0,72,0,18]	[0,0,0,9,0,0,18,6,18,9]	S_3	9	7	46	–

T denotes the type of the even set H, N denotes the total number of even sets of Type T, $v := |H|$, the i-th component of I_l denotes the total number of lines meeting H in precisely $i-1$ points, the i-th component of I_g denotes the total number of (4×4)-subgrids $\mathcal{G}$ of $\mathrm{GQ}(3,5)$ which intersect H in an even set of Type i of $\mathcal{G}$, O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H], and C denotes the type of the complement of H.

Table 7: The even sets of $\mathrm{GQ}(3,5)$

The decision whether a given pseudo-hyperplane is an intriguing set can easily be done with the aid of the following lemma.

Lemma 5.1 *Let X be a pseudo-hyperplane of $\mathcal{Q}$ and let $[a_0, 0, a_2, 0, a_4]$ denote the value of I_l corresponding to X (as occurring in the tables). Then:*

- *X is a 2-ovoid if and only if $a_0 = a_4 = 0$;*
- *X is a tight set if and only if $|X| \cdot (\frac{|X|}{4} + 2) = 2a_2 + 12a_4$.*

Moreover, if X is a tight set, then X is $\frac{|X|}{4}$-tight.

For the generalized quadrangle $W(3)$, we find that there exists up to isomorphism a unique 2-ovoid and a unique i-tight pseudo-hyperplane for every $i \in \{2, 4, 6, 8\}$. The uniqueness of the 2-ovoid was already proved in Bamberg, Kelly, Law and Penttila [2, Section 7.1]. The 2-tight and 4-tight pseudo-hyperplanes have been described above. The 8-tight pseudo-hyperplanes are precisely the complements of the 2-tight pseudo-hyperplanes and the 6-tight pseudo-hyperplanes are precisely the complements of the 4-tight pseudo-hyperplanes.

For the generalized quadrangle $Q(5, 3)$, we find that there exists up to isomorphism a unique 2-ovoid and a unique i-tight pseudo-hyperplane for every $i \in \{8, 10, 12, 14, 16, 18, 20\}$. As already told in Section 1, the uniqueness of the 2-ovoid is a consequence of the uniqueness of the hemi-system on the Hermitian surface $H(3, 9)$ which is due to Segre [21]. As mentioned above, the 10-tight pseudo-hyperplane of $Q(5, 3)$ is the union of ten mutually disjoint lines of $Q(5, 3)$ forming a $\{0, 2\}$-set of lines. As far as the author knows, the other tight sets have not been described before in the literature.

For the generalized quadrangle $Q(4, 3)$, we find that there exists up to isomorphism a unique 2-ovoid. The uniqueness of this 2-ovoid was already proved in Bamberg, Kelly, Law and Penttila [2, Section 7.3].

For the generalized quadrangle $\mathrm{GQ}(3, 5)$, we find that there are up to isomorphism two 2-ovoids. One of these 2-ovoids (Type 3) is the union of two disjoint ovoids. The other 2-ovoid seems to be unknown. Up to isomorphism, $\mathrm{GQ}(3, 5)$ has a unique i-tight pseudo-hyperplane for every $i \in \{4, 6, 10, 12\}$. Up to isomorphism, there are two 8-tight pseudo-hyperplanes. As told before, one of these 8-tight sets (Type 4) is the union of two disjoint (4×4)-grids. As far as the author knows, the other tight sets have not been described before in the literature.

We now turn our attention to the classification of the homogeneous pseudo-embeddings of $\mathcal{Q}$. We will follow the same procedure with which we were able to determine all homogeneous pseudo-embeddings of the (4×4)-grid. Recall that $\mathcal{Q}$ has up to isomorphism M even sets which we call the *even sets of Type 1, Type 2, ..., Type M*. Similarly as in the case of the (4×4)-grid, relations R and $\overline{R}$ can be defined on the set $I := \{1, 2, \ldots, M\}$, as well as a set Ω of subsets of I. For every subset J of I, we denote by $\mathcal{H}(J)$ the set of all pseudo-hyperplanes of $\mathcal{Q}$ whose type belongs to J. Let Ω' be the set of all nonempty $J' \subseteq I$ which can be written as the union of a number of elements of Ω (at least one) and for which $|\mathcal{H}(J')| + 1$ is a power of 2. Recall that if $\mathcal{H}(J)$ coincides with the set of pseudo-hyperplanes of $\mathcal{Q}$ arising from a particular homogeneous pseudo-embedding of $\mathcal{Q}$, then necessarily $J \cup \{2\} \in \Omega'$. With the aid of GAP, we have computed Ω' for each of the four possibilities of $\mathcal{Q}$.

If $\mathcal{Q} = W(3)$, then $M = 20$. We find that Ω' contains three elements, namely the sets $J_1 = \{2\}$, $J_2 = \{1, 2\}$ and $J_3 = I$. Among the sets $\mathcal{H}(J_1)$, $\mathcal{H}(J_2)$ and $\mathcal{H}(J_3)$, only $\mathcal{H}(J_3)$ satisfies the conditions (a), (c) and (d) of Proposition 2.6, implying that the universal pseudo-embedding of $W(3)$ is the up to isomorphism unique homogeneous pseudo-embedding of $W(3)$.

If $\mathcal{Q} = Q(4, 3)$, then $M = 21$. We find that $\Omega' = \{J_1, J_2, J_3, J_4\}$, where $J_1 = \{2\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 2, 3, 4, 7\}$ and $J_4 = I$. Among the sets $\mathcal{H}(J_1)$, $\mathcal{H}(J_2)$, $\mathcal{H}(J_3)$ and $\mathcal{H}(J_4)$, only $\mathcal{H}(J_3)$ and $\mathcal{H}(J_4)$ satisfy the conditions (a), (c) and (d) of Proposition 2.6, implying that $Q(4, 3)$ has up to isomorphism two homogeneous pseudo-embeddings, the universal one corresponding to $\mathcal{H}(J_4)$ and another one corresponding to $\mathcal{H}(J_3)$. Since $|\mathcal{H}(J_3)| = 2^9 - 1$, the pseudo-embedding space of the latter homogeneous pseudo-embedding is $\mathrm{PG}(8, 2)$. The two homogeneous pseudo-embeddings are faithful by Proposition 2.6.

If $\mathcal{Q} = Q(5, 3)$, then $M = 22$. We find that Ω' contains three elements, namely the sets $J_1 = \{2\}$, $J_2 = \{1, 2\}$ and $J_3 = I$. Among the sets $\mathcal{H}(J_1)$, $\mathcal{H}(J_2)$ and $\mathcal{H}(J_3)$, only $\mathcal{H}(J_3)$ satisfies the conditions (a), (c) and (d) of Proposition 2.6. This implies that the universal pseudo-embedding of $Q(5, 3)$ is the up to isomorphism unique homogeneous pseudo-embedding of $Q(5, 3)$.

If $\mathcal{Q} = \mathrm{GQ}(3, 5)$, then $M = 47$. We find that $\Omega' = \{J_1, J_2, \ldots, J_7\}$, where $J_1 = \{2\}$, $J_2 = \{1, 2\}$, $J_3 = \{1, 2, 3, 4\}$, $J_4 = \{1, 2, 3, 4, 7, 8, 24\}$, $J_5 = \{1, 2, 3, 4, 9, 10, 14, 17, 18, 22, 25, 26\}$, $J_6 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 17, 18, 19, 20, 22, 24, 25, 26, 27, 28, 29, 30, 31, 34, 35, 42, 43, 46, 47\}$ and $J_7 = I$. Among the sets $\mathcal{H}(J_1), \mathcal{H}(J_2), \ldots, \mathcal{H}(J_7)$, all but $\mathcal{H}(J_1)$ and $\mathcal{H}(J_2)$ satisfy the conditions (a), (c) and (d) of Proposition 2.6, implying that the generalized quadrangle $\mathrm{GQ}(3, 5)$ has up to isomorphism five homogeneous pseudo-embeddings. All these pseudo-embeddings are faithful by Proposition 2.6. The homogeneous pseudo-embedding corresponding to $\mathcal{H}(J_7)$ is the universal one. As above, the dimensions of the homogeneous pseudo-embeddings can be calculated by means of the number of pseudo-hyperplanes contained in $\mathcal{H}(J_i)$, $i \in \{3, 4, 5, 6, 7\}$. These dimensions are as stated in Theorem 1.3.

Observe also that $\mathrm{GQ}(3, 5)$ is a subgeometry of the affine 3-space $\mathrm{AG}(3, 4)$ (with the same points, but fewer lines) and that every automorphism of $\mathrm{GQ}(3, 5)$ is induced by an automorphism of $\mathrm{AG}(3, 4)$ (see e.g. Payne [17, V5]). So, every homogeneous pseudo-embedding of $\mathrm{AG}(3, 4)$ will induce a homogeneous pseudo-embedding of $\mathrm{GQ}(3, 5)$. Now, by De Bruyn [8, Corollary 4.4], the pseudo-embedding rank of $\mathrm{AG}(3, 4)$ is 13. Hence, the universal pseudo-embedding of $\mathrm{AG}(3, 4)$ will induce a homogeneous pseudo-embedding of $\mathrm{GQ}(3, 5)$ which is isomorphic to the homogeneous pseudo-embedding of $\mathrm{GQ}(3, 5)$ in $\mathrm{PG}(12, 2)$ corresponding to $\mathcal{H}(J_5)$. This implies the following.

Lemma 5.2 *The pseudo-hyperplanes of* AG$(3,4)$ *are precisely the pseudo-hyperplanes of* GQ$(3,5)$ *belonging to the set* $\mathcal{H}(J_5)$.

Those pseudo-hyperplanes of AG$(3,4)$ which are empty or can be written as the union of two disjoint planes of AG$(3,4)$ are precisely the pseudo-hyperplanes of GQ$(3,5)$ belonging to the set $\mathcal{H}(J_3)$. By Corollary 2.7, these pseudo-hyperplanes determine a homogeneous pseudo-embedding of AG$(3,4)$. So, also the homogeneous pseudo-embedding of GQ$(3,5)$ in PG$(6,2)$ corresponding to $\mathcal{H}(J_3)$ is induced by a homogeneous pseudo-embedding of AG$(3,4)$. By Lemma 5.2, the homogeneous pseudo-embedding of GQ$(3,5)$ corresponding to $\mathcal{H}(J_4)$ cannot be induced by a pseudo-embedding of AG$(3,4)$ since $\mathcal{H}(J_4)$ contains pseudo-hyperplanes which are not contained in $\mathcal{H}(J_5)$.

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