On Dependencies and Independencies of Fuzzy Implication Axioms

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Abstract

A fuzzy implication, commonly defined as a two-place operation on the unit interval, is an extension of the classical binary implication. It plays important roles in both mathematical and applied sides of fuzzy set theory. Besides the basic axioms, there are many potential fuzzy implication axioms, among which eight are widely used in the literature. Different fuzzy implications satisfying different subgroups of these eight axioms can be found. However, certain interrelationships exist between these eight axioms. But the results remain incomplete. This paper aims to lay bare the interrelationships between these eight axioms. The result is instrumental to propose a classification of fuzzy implications.

Key words: Fuzzy implication, fuzzy implication axioms, fuzzy logic operators, S-implication, R-implication

1 Introduction

One of the most important and interesting topics in fuzzy logic is to extend the classical binary logical operators *conjunction*, *disjunction*, *negation* and *implication* to fuzzy logic operators. The classical binary implication has the truth table

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

So the extension of the classical binary implication to the unit interval, a fuzzy implication, should be a $[0,1]^2 \to [0,1]$ mapping I that at least satisfies the boundary conditions:

$$I(0,0) = I(0,1) = I(1,1) = 1 \quad \text{and} \quad \mathrm{I}(1,0) = 0. \tag{1}$$

Fuzzy implications play significant roles both in fuzzy logic and fuzzy set theory. A fuzzy implication can be used to determine

1. the truth value of a conditional rule. The conditional rule 'If p then q' is defined by
$$Tr(p \Rightarrow q) = I(Tr(p), Tr(q)),$$
where p and q are two propositions and $Tr(p)$ denotes the truth value of the proposition p.
2. a conditional relation between two linguistic variables X and Y on the universes of discourse U and V, respectively. The rule 'If X is A then Y is B' is defined by
$$R_{X,Y}(u,v) = I(A(u), B(v)),$$
where $R_{X,Y}$ denotes the conditional relation on $U \times V$, A and B are fuzzy sets on U and V, respectively.

In the first aspect, a fuzzy implication is considered to be the extension of the implication in binary classical logic to the multivalued domain [8,12]. In the second aspect, a fuzzy implication is used in many applications as an operation between two fuzzy sets. For example, in the generalized modus ponens [12,15–17], in fuzzy subsethood measures [5], in fuzzy morphology operations [11,13], and in determining the association rules in data mining [21].
There is no standard definition for a fuzzy implication as for a fuzzy conjunction, a fuzzy disjunction and a fuzzy negation. As the author of [10] states 'One of the main difficulties I have met during the preparation of lecture notes on some basic material concerning fuzzy set theory, consisted of a lack of standard definitions for basic elementary notions'. By taking into account the extensive literature about fuzzy implications [3–5,8,9,20] we propose in this paper the following definition:

Definition 1.1 *A fuzzy implication I is a $[0,1]^2 \to [0,1]$ mapping that satisfies:*

FI1. the first place antitonicity FA:
$(\forall (x_1, x_2, y) \in [0,1]^3)(x_1 < x_2 \Rightarrow I(x_1, y) \geq I(x_2, y))$;
FI2. the second place isotonicity SI:
$(\forall (x, y_1, y_2) \in [0,1]^3)(y_1 < y_2 \Rightarrow I(x, y_1) \leq I(x, y_2))$;
FI3. dominance of falsity of antecedent DF: $(\forall x \in [0,1])(I(0,x) = 1)$;
FI4. dominance of truth of consequent DT: $(\forall x \in [0,1])(I(x,1) = 1)$;
FI5. boundary condition BC: $I(1,0) = 0$.

Because of DF, DT and BC, a fuzzy implication I satisfies the conditions in (1). There are several equivalent definitions for a fuzzy implication (as defined in Definition 1.1). For example, according to ([2], Lemma 1), the assumption that I satisfies (1), FA and SI is an equivalent definition.
Many other potential axioms have been proposed in the literature devoted to fuzzy set theory, in order to obtain fuzzy implications fulfilling different requirements [3–5,7,8,13,15,21,22], among which the most important ones are:

FI6. neutrality of truth NT: $(\forall x \in [0,1])(I(1,x) = x)$;
FI7. exchange principle EP: $(\forall (x,y,z) \in [0,1]^3)(I(x, I(y,z)) = I(y, I(x,z)))$;
FI8. ordering principle OP: $(\forall (x,y) \in [0,1]^2)(I(x,y) = 1 \Leftrightarrow x \leq y)$;
FI9. strong fuzzy negation principle SN: the mapping N_I defined as $(\forall x \in [0,1])(N_I(x) = I(x,0))$, is a strong fuzzy negation;
FI10. consequent boundary CB: $(\forall (x,y) \in [0,1]^2)(I(x,y) \geq y)$;
FI11. identity ID: $(\forall x \in [0,1])(I(x,x) = 1)$;
FI12. contrapositive principle (CP): $(\forall (x,y) \in [0,1]^2)(I(x,y) = I(N(y), N(x)))$, where N is a strong fuzzy negation;
FI13. continuity CO: I is a continuous mapping.

It is necessary to have a complete view of the interrelationships between these eight axioms NT-CO. On one hand, this helps to give a classification for all the fuzzy implications. On the other hand, this helps to solve some functional equations. Many works have studied the interrelationships between axioms NT-CO (e.g., [1,3,4,8]), but the complete interrelationship between these eight axioms remains missing. This paper aims to obtain the complete interrelationship between these axioms. Section 2 gives the necessary preliminaries. Section 3 gives the interrelationships between the axioms NT-CO. We provide each dependent case with a proof or citation, and each independent case with a counterexample. Section 4 summarizes the obtained results and illustrates the meanings.

2 Preliminaries

Definition 2.1 ([4], Definition 0) A mapping φ: $[a, b] \rightharpoonup [a, b]$ ($[a, b] \subset \mathbb{R}$) is *an order automorphism of the interval* $[a, b]$ if it is continuous, strictly increasing and satisfies the boundary conditions: $\varphi(a) = a$ and $\varphi(b) = b$.

Definition 2.2 ([1], Definition 2) Two mappings F, G: $[0, 1]^n \to [0, 1]$, $n \in \mathbb{N}$, are *conjugate*, if there exists an order automorphism φ of the unit interval such that $G = F_\varphi$, where

$$F_\varphi(x_1, x_2, \cdots, x_n) = \varphi^{-1}(F(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n))), \tag{2}$$

Definition 2.3 A mapping N: $[0, 1] \to [0, 1]$ is a *fuzzy negation* if it is decreasing and satisfies: $N(0) = 1$, $N(1) = 0$. Moreover, if N is involutive, i.e., $N(N(x)) = x$, for all $x \in [0, 1]$, then it is called a *strong fuzzy negation*.

The standard strong fuzzy negation N_0 is defined by

$$(\forall x \in [0, 1])(N_0(x) = 1 - x).$$

Any strong fuzzy negation N is conjugate with the standard strong fuzzy negation N_0 [18].

Definition 2.4 A mapping T: $[0, 1]^2 \to [0, 1]$ is a *triangular norm* (*t-norm* for short) if for all $x, y, z \in [0, 1]$ it satisfies:

T1. boundary condition: $T(x, 1) = x$,
T2. isotonicity: $y \le z$ implies $T(x, y) \le T(x, z)$,
T3. commutativity: $T(x, y) = T(y, x)$,
T4. associativity: $T(x, T(y, z)) = T(T(x, y), z)$.

Three important continuous t-norms are:

1. $T_{\mathbf{M}}(x, y) = \min(x, y)$, (minimum)
2. $T_{\mathbf{P}}(x, y) = xy$, (product)
3. $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$. (Łukasiewicz)

Definition 2.5 A mapping S: $[0, 1]^2 \to [0, 1]$ is a *triangular conorm* (*t-conorm* for short) if for all $x, y, z \in [0, 1]$ it satisfies:

S1. boundary condition: $S(x, 0) = x$,
S2. isotonicity: $y \le z$ implies $S(x, y) \le S(x, z)$,
S3. commutativity: $S(x, y) = S(y, x)$,
S4. associativity: $S(x, S(y, z)) = S(S(x, y), z)$.

Three classes of fuzzy implications generated by t-norms, t-conorms and fuzzy negations are:

- S-implication: $I(x, y) = S(N(x), y)$,
- R-implication: $I(x, y) = \sup\{t \in [0, 1] | T(x, t) \leq y\}$,
- QL-implication: $I(x, y) = S(N(x), T(x, y))$.

In addition to the above-mentioned three classes of fuzzy implications, we also consider in this paper other possible fuzzy implications that fulfill certain requirements.

3 Getting FI6(NT) from the Other Axioms

Theorem 3.1 *([3], Lemma 1.54(v), Corollary 1.57 (iii)) A fuzzy implication I satisfying SN and CP w.r.t. a strong fuzzy negation N satisfies NT iff $N_I = N$.*

In the rest of this section we consider the condition that $N_I \neq N$.

Proposition 3.2 *([1], Lemma 6) A fuzzy implication I satisfying EP and OP satisfies NT.*

Proposition 3.3 *([3], Lemma 1.56(ii)) A fuzzy implication I satisfying EP and SN satisfies NT.*

Proposition 3.4 *A fuzzy implication I satisfying EP and CO satisfies NT.*

PROOF. Because I satisfies EP, we have for all $x \in [0, 1]$,

$$I(1, N_I(x)) = I(1, I(x, 0)) = I(x, I(1, 0)) = I(x, 0) = N_I(x). \tag{3}$$

Because I is a continuous mapping, N_I is a continuous mapping. Thus expression (3) is equivalent to $I(1, a) = a$, for all $a \in [0, 1]$. Hence I satisfies NT. □

Remark 3.5 In Propositions 3.2, 3.3 and 3.4 we considered the following three cases:

EP $\wedge$ OP $\Rightarrow$ NT
EP $\wedge$ SN $\Rightarrow$ NT
EP $\wedge$ CO $\Rightarrow$ NT

So we still need to consider the following two cases:

$\mathrm{EP} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CP} \overset{?}{\Rightarrow} \mathrm{NT}$
$\mathrm{OP} \wedge \mathrm{SN} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CP} \wedge \mathrm{CO} \overset{?}{\Rightarrow} \mathrm{NT}$

Proposition 3.6 *There exists a fuzzy implication I satisfying EP, CB, ID, CP and not NT.*

PROOF. The fuzzy implication I_1 stated in [6] is defined by

$$I_1(x,y) = \begin{cases} 0 & \text{if } x = 1 \text{ and } y = 0 \\ 1 & \text{else} \end{cases}, \quad x, y \in [0,1].$$

Notice that I_1 is the greatest fuzzy implication. For all $x, y, z \in [0,1]$ we have

$$\begin{aligned} I_1(x, I_1(y,z)) &= \begin{cases} 1, & \text{if } x < 1 \text{ or } y < 1 \text{ or } z > 0 \\ 0, & \text{else} \end{cases} \\ &= I_1(y, I_1(x,z)). \end{aligned}$$

Therefore I_1 satisfies EP. Moreover, for all x, $y \in [0,1]$, we obtain:

$I_1(x,y) \geq y$,
$I_1(x,x) = 1$,
$I_1(N(y), N(x)) = I(x,y)$, for any strong fuzzy negation N.

Therefore I_1 satisfies CB, ID and CP w.r.t. any strong fuzzy negation N. However, in case that $x \neq 1$, $I_1(1,x) = 1 \neq x$. Therefore I_1 does not satisfy NT. $\square$

Proposition 3.7 *There exists a fuzzy implication I satisfying OP, SN, CB, ID, CP, CO and not NT.*

PROOF. Let a $[0,1]^2 \to [0,1]$ mapping I_2 be defined by

$$I_2(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \sqrt{1-(x-y)^2} & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

Checking that I_2 is a fuzzy implication is easy and therefore it is omitted. For all x, $y \in [0,1]$, we obtain:

$I_2(x,y) = 1$ iff $x \leq y$,
$N_{I_2}(x) = I_2(x,0) = \sqrt{1-x^2} = \varphi^{-1}(1-\varphi(x))$, where $\varphi(x) = x^2$ is an order automorphism of the unit interval,
$I_2(x,y) \geq y$,

$I_2(x, x) = 1$,
$I_2(1 - y, 1 - x) = I_2(x, y)$,
I_2 is a continuous mapping.

Therefore I_2 satisfies OP, SN, CB, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO. However, in case that $x \neq 1$ and $x \neq 0$, $I_2(1, x) = \sqrt{2x - x^2} \neq x$. Therefore I_2 does not satisfy NT. $\square$

So we considered all the possibilities that the fuzzy implication axiom NT can be implied from the other seven axioms. Moreover we stated for each independent case a counterexample.

4 Getting FI7(EP) from the Other Axioms

Proposition 4.1 *There exists a fuzzy implication I satisfying NT, OP, SN, CB, ID, CP, CO and not EP.*

PROOF. Let a mapping I_3 be defined by

$$I_3(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (1 - y + xy)(x - y) & \text{if } x > y \end{cases}, \quad x, y \in [0, 1].$$

First we check that I_3 is a fuzzy implication. It is straightforward to check that $I_3(x, y) \in [0, 1]$ for all x, $y \in [0, 1]$. Moreover, for a fixed $y \in [0, 1]$, let $0 \leq x_1 < x_2 \leq 1$. If $x_1 \leq y$ then $I_3(x_1, y) = 1 \geq I_3(x_2, y)$. If $y < x_1 < x_2 \leq 1$ then we have

$$\begin{aligned} & x_1 y < x_2 y \\ \Leftrightarrow & 1 - y + x_1 y < 1 - y + x_2 y \\ \Leftrightarrow & (x_1 - y)(1 - y + x_1 y) < (x_2 - y)(1 - y + x_2 y) \\ \Leftrightarrow & I_3(x_1, y) > I_3(x_2, y). \end{aligned}$$

Therefore I_3 satisfies FA.
Furthermore, for a fixed $x \in [0, 1]$, let $0 \leq y_1 < y_2 \leq 1$. If $y_2 \geq x$ then $I_3(x, y_2) = 1 \geq I_3(x, y_1)$. If $0 \leq y_1 < y_2 < x$ then we have

$$\begin{aligned} & x y_2 - x y_1 < y_2 - y_1 \\ \Leftrightarrow & 1 - y_2 + x y_2 < 1 - y_1 + x y_1 \\ \Leftrightarrow & (x - y_2)(1 - y_2 + x y_2) < (x - y_1)(1 - y_1 + x y_1) \\ \Leftrightarrow & I_3(x, y_2) > I_3(x, y_1). \end{aligned}$$

Therefore I_3 also satisfies SI.
Checking that I_3 satisfies DF, DT and BC is easy and therefore it is omitted.
Hence I_3 is a fuzzy implication.
For all x, $y \in [0, 1]$, we obtain:

$I_3(1, x) = x,$
$I_3(x, y) = 1$ iff $x \le y,$
$N_{I_3}(x) = I_3(x, 0) = 1 - x,$
$I_3(x, y) \ge y,$
$I_3(x, x) = 1,$
$I_3(1 - y, 1 - x) = I_3(x, y),$
I_3 is a continuous mapping.

Therefore I_3 satisfies NT, OP, SN, CB, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO. However, take $x_0 = 0.3$, $y_0 = 0.9$ and $z_0 = 0.1$, we obtain $I(x_0, I(y_0, z_0)) \approx 0.9214$ and $I(y_0, I(x_0, z_0)) \approx 0.9210$. Therefore I_3 does not satisfy EP. □

Remark 4.2 The fuzzy implication I_{MM} presented in ([3], Table 1.5) is also an example that satisfies NT, OP, SN, CB, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO but not EP. It is interesting to note that the implications I_3 and I_{MM}, despite satisfying exactly the same properties among FA-CO, are not conjugated to each other. We omit the proof here because it is rather technical.

EP is thus independent of any of the other seven axioms.

5 Getting FI8(OP) from the Other Axioms

Proposition 5.1 *There exists a fuzzy implication I satisfying NT, EP, SN, CB, ID, CP, CO and not OP.*

PROOF. Given the strong fuzzy negation $N(x) = \sqrt{1 - x^2}$, for all $x \in [0, 1]$. The S-implication I_4 generated by the t-conorm $S_{\mathbf{L}}$ and the strong fuzzy negation N is defined by

$$I_4(x, y) = S_{\mathbf{L}}(N(x), y) = \min(\sqrt{1 - x^2} + y, 1), x, y \in [0, 1].$$

Because I_4 is an S-implication generated from a continuous t-conorm and a strong fuzzy negation, it satisfies NT, EP, SN, CB, CP w.r.t. the strong fuzzy negation N and CO [8]. Moreover, for all $x, y \in [0, 1]$, $I_4(x, x) = 1$. Therefore I_4 also satisfies ID. However, take $x_0 = 0.5$ and $y_0 = 0.4$, we obtain $I(x_0, y_0) = 1$ while $x_0 > y_0$. Therefore I_4 does not satisfy OP. □

Therefore OP is thus independent of any of the other seven axioms.

6 Getting FI9(SN) from the Other Axioms

Proposition 6.1 *([3], Lemma 1.5.4(v)) A fuzzy implication I satisfying NT and CP w.r.t. a strong fuzzy negation N satisfies SN. Moreover, $N_I = N$.*

Corollary 6.2 *A fuzzy implication I satisfying EP, OP and CP w.r.t. a strong fuzzy negation N satisfies SN. Moreover, $N_I = N$.*

PROOF. Straightforward from Propositions 3.2 and 6.1. □

Corollary 6.3 *A fuzzy implication I satisfying EP, CP w.r.t. a strong fuzzy negation N and CO satisfies SN. Moreover, $N_I = N$.*

PROOF. Straightforward from Propositions 3.4 and 6.1. □

Proposition 6.4 *([1], Lemma 14)([8], Corollary 1.1) A fuzzy implication I satisfying EP, OP and CO satisfies SN.*

Remark 6.5 In Proposition 6.1, Corollary 6.2, Corollary 6.3 and Proposition 6.4 we considered the following four cases:

$\mathrm{NT} \wedge \mathrm{CP} \Rightarrow \mathrm{SN}$
$\mathrm{EP} \wedge \mathrm{OP} \wedge \mathrm{CP} \Rightarrow \mathrm{SN}$
$\mathrm{EP} \wedge \mathrm{OP} \wedge \mathrm{CO} \Rightarrow \mathrm{SN}$
$\mathrm{EP} \wedge \mathrm{CP} \wedge \mathrm{CO} \Rightarrow \mathrm{SN}$

So we still need to consider the following five cases:

$\mathrm{NT} \wedge \mathrm{EP} \wedge \mathrm{OP} \wedge \mathrm{CB} \wedge \mathrm{ID} \overset{?}{\Rightarrow} \mathrm{SN}$
$\mathrm{NT} \wedge \mathrm{EP} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CO} \overset{?}{\Rightarrow} \mathrm{SN}$
$\mathrm{NT} \wedge \mathrm{OP} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CO} \overset{?}{\Rightarrow} \mathrm{SN}$
$\mathrm{EP} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CP} \overset{?}{\Rightarrow} \mathrm{SN}$
$\mathrm{OP} \wedge \mathrm{CB} \wedge \mathrm{ID} \wedge \mathrm{CP} \wedge \mathrm{CO} \overset{?}{\Rightarrow} \mathrm{SN}$

Proposition 6.6 *([8], Table 1.1) There exists a fuzzy implication satisfying NT, EP, OP, CB, ID and not SN.*

PROOF. The Gödel implication

$$I_G(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}, \quad x, y \in [0,1]. \tag{4}$$

is an R-implication generated by the continuous t-norm $T_{\mathbf{M}}$. Therefore I_G satisfies NT, EP, OP, CB and ID [8]. However we have for all $x \in [0,1]$,

$$N_{I_G}(x) = I_G(x,0) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$$

Therefore I_G does not satisfy SN. □

Proposition 6.7 *There exists a fuzzy implication I satisfying NT, EP, CB, ID, CO and not SN.*

PROOF. Given the fuzzy negation $N(x) = 1 - x^2$, for all $x \in [0,1]$. The S-implication generated from the t-conorm $S_{\mathbf{L}}$ and the fuzzy negation N is defined by

$$I_5(x,y) = \min(1 - x^2 + y, 1) \quad x, y \in [0,1].$$

For all x, $y \in [0,1]$, we obtain:

$I_5(1,x) = x$,
$I_5(x,y) \ge y$,
$I_5(x,x) = 1$,
I_5 is a continuous mapping.

Therefore I_5 satisfies NT, CB, ID and CO. Moreover, because I_5 is an (S,N)-implication generated from the Łukasiewicz t-conorm and the strict fuzzy negation $N(x) = 1 - x^2$, it then also satisfies EP ([3], Proposition 2.4.3(i)). However, we have for all $x \in [0,1]$

$$N_{I_5}(x) = I_5(x,0) = 1 - x^2$$

which is not a strong fuzzy negation. Therefore I_5 does not satisfy SN. □

Proposition 6.8 *There exists a fuzzy implication I satisfying NT, OP, CB, ID, CO and not SN.*

PROOF. Let a mapping I_6 be defined by

$$I_6(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{1+\sqrt{1-x}} + \sqrt{1-x}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

First we show that I_6 is a fuzzy implication. It is straightforward to check that $I_6(x,y) \in [0,1]$ for all x, $y \in [0,1]$. Moreover, I_6 satisfying SI-BC is trivial. We prove that I_6 also satisfies FA. Indeed, for all $0 \le y < x \le 1$ we obtain

$$\frac{\partial I_6(x,y)}{\partial x} = \frac{-1}{2}(1 - \frac{y}{(1+\sqrt{1-x})^2})\frac{1}{\sqrt{1-x}},$$

and

$$\begin{aligned} & y < x < 2 - x + 2\sqrt{1-x} \\ \Rightarrow & y < (1+\sqrt{1-x})^2 \\ \Rightarrow & (1 - \frac{y}{(1+\sqrt{1-x})^2})\frac{1}{\sqrt{1-x}} > 0 \\ \Rightarrow & \frac{\partial I_6(x,y)}{\partial x} < 0. \end{aligned}$$

Furthermore, for all $0 \le x \le y \le 1$, $\frac{\partial I_6(x,y)}{\partial x} = 0$. Thus $\frac{\partial I_6(x,y)}{\partial x} \le 0$, for all x, $y \in [0,1]$. Therefore I_6 satisfies FA.
Next we show that I_6 satisfies NT, OP, CB, ID and CO but not SN.
Indeed, for all x, $y \in [0,1]$

$I_6(1,x) = x$,
$I_6(x,y) = 1$ iff $x \le y$,
$I_6(x,y) \ge y$,
I_6 is a continuous mapping.

Therefore I_6 satisfies NT, OP, CB and CO. However, we have for all $x \in [0,1]$

$$N_{I_6}(x) = I_6(x,0) = \sqrt{1-x}$$

which is not a strong fuzzy negation. Therefore I_6 does not satisfy SN $\square$

Proposition 6.9 *There exists a fuzzy implication I satisfying EP, CB, ID, CP and not SN.*

The fuzzy implication I_1 stated in the proof of Proposition 3.6 satisfies EP, CB, ID and CP w.r.t. any strong fuzzy negation N. However, we have

$$N_{I_1}(x) = I_1(x,0) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}, \quad x \in [0,1],$$

which is not a strong fuzzy negation. Therefore I_1 does not satisfy SN.

Proposition 6.10 *There exists a fuzzy implication I satisfying OP, CB, ID, CP, CO and not SN.*

PROOF. Let a $[0,1]^2 \to [0,1]$ mapping I_7 be defined by

$$I_7(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \sqrt{1-(x-y)}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

Checking that I_7 is a fuzzy implication is easy and therefore it is omitted. For all x, $y \in [0,1]$, we obtain:

$I_7(x,y) = 1$ iff $x \le y$,
$I_7(x,y) \ge y$,
$I_7(x,x) = 1$,
$I_7(1-y, 1-x) = I_7(x,y)$,
I_7 is a continuous mapping.

Therefore I_7 satisfies OP, CB, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO. However, we have for all $x \in [0,1]$

$$N_{I_7}(x) = I_7(x,0) = \sqrt{1-x},$$

which is not a strong fuzzy negation. Therefore I_7 does not satisfy SN. $\square$

Remark 6.11 The fuzzy implication I_{BZ} presented in ([3], Example 1.5.10(iv)) also satisfies OP, CB, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO but not SN. It is interesting to note that the two implications I_7 and I_{BZ}, despite satisfying exactly the same properties among FA-CO, are not conjugated to each other. Indeed, $I_7(x,0)$ is strictly decreasing while $I_{BZ}(x,0)$ is not.

So we considered all the possibilities that the fuzzy implication axiom SN can be implied from the other seven axioms. Moreover we stated for each independent case a counterexample.

7 Getting FI10(CB) from the Other Axioms

Proposition 7.1 *([4],Lemma 1 (viii)) A fuzzy implication I satisfying NT satisfies CB.*

Corollary 7.2 *A fuzzy implication I satisfying EP and SN satisfies CB.*

PROOF. Straightforward from Propositions 3.3 and 7.1. □

Corollary 7.3 *A fuzzy implication I satisfying EP and CO satisfies CB.*

PROOF. Straightforward from Propositions 3.4 and 7.1. □

Proposition 7.4 *([1],Lemma 6) A fuzzy implication I satisfying EP and OP satisfies CB.*

Remark 7.5 In Proposition 7.1, Corollary 7.2, Corollary 7.3 and Proposition 7.4 we considered the following four cases:

NT $\Rightarrow$ CB
EP $\wedge$ OP $\Rightarrow$ CB
EP $\wedge$ SN $\Rightarrow$ CB
EP $\wedge$ CO $\Rightarrow$ CB

So we still need to consider the following two cases:

EP $\wedge$ ID $\wedge$ CP $\stackrel{?}{\Rightarrow}$ CB
OP $\wedge$ SN $\wedge$ ID $\wedge$ CP $\wedge$ CO $\stackrel{?}{\Rightarrow}$ CB

Proposition 7.6 *There exists a fuzzy implication I satisfying EP, ID, CP and not CB.*

PROOF. Let a fuzzy implication I_8 be defined by

$$I_8(x,y) = \begin{cases} 1, & \text{if } x \leq 0.5 \text{ or } y \geq 0.5 \\ 0, & \text{else} \end{cases}, \quad x, y \in [0,1].$$

For all $x, y, z \in [0,1]$ we obtain

$$\begin{aligned} I_8(x, I_8(y,z)) &= \begin{cases} 1, & \text{if } x \leq 0.5 \text{ or } y \leq 0.5 \text{ or } z \geq 0.5 \\ 0, & \text{else} \end{cases} \\ &= I_8(y, I_8(x,z)) \end{aligned}$$

Therefore I_8 satisfies EP. Moreover, for all x, $y \in [0,1]$, we obtain:

$I_8(x,x) = 1,$
$I_8(1-y, 1-x) = I_8(x,y).$

Therefore I_8 satisfies ID and CP w.r.t. the standard strong fuzzy negation N_0. However, take $x_0 = 1$ and $y_0 = 0.1$, we obtain $I_8(x_0, y_0) = 0 < y_0$. Therefore I_8 does not satisfy CB. $\square$

Proposition 7.7 *There exists a fuzzy implication I satisfying OP, SN, ID, CP, CO and not CB.*

PROOF. Let a $[0,1]^2 \to [0,1]$ mapping I_9 be represented as

$$I_9(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ (1-\sqrt{x-y})^2, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

Checking that I_9 is a fuzzy implication is easy and therefore it is omitted. For all x, $y \in [0,1]$, we obtain:

$I_9(x,y) = 1$ iff $x \leq y$,
$N_{I_9}(x) = I_9(x,0) = (1-\sqrt{x})^2 = \varphi^{-1}(1-\varphi(x))$, where $\varphi(x) = \sqrt{x}$ is an order automorphism of the unit interval,
$I_9(x,x) = 1$,
$I_9(1-y, 1-x) = I_9(x,y)$,
I_9 is a continuous mapping.

Therefore I_9 satisfies OP, SN, ID, CP w.r.t. the standard strong fuzzy negation N_0, and CO. However, take $x_0 = 1$ and $y_0 = 0.64$, we obtain $I_9(x_0, y_0) = 0.16 < y_0$. Therefore I_9 does not satisfy CB. $\square$

So we considered all the possibilities that the fuzzy implication axiom CB can be implied from the other seven axioms. Moreover we stated for each independent case a counterexample.

8 Getting FI11(ID) from the Other Axioms

Proposition 8.1 *A fuzzy implication I satisfying OP satisfies ID.*

PROOF. Straightforward. $\square$

Remark 8.2 In Proposition 8.1 we considered the following case:

OP $\Rightarrow$ ID.

So we still need to consider the following case:

NT $\wedge$ EP $\wedge$ SN $\wedge$ CB $\wedge$ CP $\wedge$ CO $\stackrel{?}{\Rightarrow}$ ID

Proposition 8.3 *There exists a fuzzy implication I satisfying NT, EP, SN, CB, CP, CO and not ID.*

PROOF. The Kleene-Dienes implication $I_{KD}(x,y) = \max(1-x,y)$, for all $(x,y) \in [0,1]^2$ is an S-implication generated from the t-conorm $S_{\mathbf{M}}$ and the standard strong fuzzy negation N_0. Therefore I_{KD} satisfies NT, EP, SN, CB, CP w.r.t. the standard strong fuzzy negation N_0, and CO. However, for $x_0 = 0.1$, we obtain $I_{KD}(x_0, x_0) = 0.9 \neq 1$. Therefore I_{KD} does not satisfy ID. $\Box$

So we considered all the possibilities that the fuzzy implication axiom ID can be implied from the other seven axioms, and stated for the independent case a counterexample.

9 Getting FI12(CP) from the Other Axioms

Proposition 9.1 *([4],Lemma 1(ix)) A fuzzy implication I satisfying EP and SN satisfies CP w.r.t. the strong fuzzy negation N_I.*

Proposition 9.2 *([1]) A fuzzy implication I satisfying EP, OP and CO satisfies CP w.r.t. the strong fuzzy negation N_I.*

Remark 9.3 In Propositions 9.1 and 9.2 we considered the following two cases:

EP $\wedge$ SN $\Rightarrow$ CP
EP $\wedge$ OP $\wedge$ CO $\Rightarrow$ CP

So we still need to consider the following three cases:

NT $\wedge$ EP $\wedge$ OP $\wedge$ CB $\wedge$ ID $\stackrel{?}{\Rightarrow}$ CP
NT $\wedge$ EP $\wedge$ CB $\wedge$ ID $\wedge$ CO $\stackrel{?}{\Rightarrow}$ CP
NT $\wedge$ OP $\wedge$ SN $\wedge$ CB $\wedge$ ID $\wedge$ CO $\stackrel{?}{\Rightarrow}$ CP

Proposition 9.4 *There exists a fuzzy implication I satisfying NT, EP, OP, CB, ID and not CP.*

According to the proof of Proposition 6.6, the Gödel implication I_G satisfies NT, EP, OP, CB and ID. However, for any strong fuzzy negation N we obtain

$$I_G(N(y), N(x)) = \begin{cases} 1, & \text{if } x \leq y \\ N(x), & \text{if } x > y \end{cases}$$

In case that $x > y$ and $N(x) \neq y$, $I_G(N(y), N(x)) \neq I_G(x, y)$. Therefore I_G does not satisfy CP w.r.t. any strong fuzzy negation.

Proposition 9.5 *There exists a fuzzy implication I satisfying NT, EP, CB, ID, CO and not CP.*

The fuzzy implication I_5 stated in the proof of Proposition 6.7 satisfies NT, EP, CB, ID and CO. However, because for all $x \in [0,1]$, $N_{I_5}(x) = 1 - x^2$, which is not a strong fuzzy negation, according to Corollary 1.5.5 in [3], I_5 does not satisfy CP w.r.t. any strong fuzzy negation.

Proposition 9.6 *There exists a fuzzy implication I satisfying NT, OP, SN, CB, ID, CO and not CP.*

PROOF. Let a mapping I_{10} be defined by

$$I_{10}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y+(x-y)\sqrt{1-x^2}}{x}, & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

First we check that I_{10} is a fuzzy implication. It is straightforward to check that $I_{10}(x, y) \in [0,1]$ for all x, $y \in [0,1]$. Moreover, for a fixed $y \in [0,1]$, let $0 \leq x_1 < x_2 \leq 1$. If $x_1 \leq y$ then $I_{10}(x_1, y) = 1 \geq I_{10}(x_2, y)$. If $y < x_1 < x_2 \leq 1$ then define a mapping $f_y :]y, 1] \to [0,1]$ as $f_y(x) = \frac{y+(x-y)\sqrt{1-x^2}}{x}$, for all $x \in]y, 1]$. We have

$$\frac{df_y}{dx} = \frac{y - x}{\sqrt{1-x^2}} - \frac{y(1-\sqrt{1-x^2})}{x^2} < 0.$$

Thus f_y is a decreasing mapping. Therefore $I_{10}(x_1, y) > I_{10}(x_2, y)$. Hence I_{10} satisfies FA.
Furthermore, for a fixed $x \in [0,1]$, let $0 \leq y_1 < y_2 \leq 1$. If $y_2 \geq x$ then $I_{10}(x, y_2) = 1 \geq I_{10}(x, y_1)$. If $0 \leq y_1 < y_2 < x$ then we have

$$\begin{aligned} &I_{10}(x, y_2) - I_{10}(x, y_1) \\ =&\frac{(y_2 - y_1)(1 - \sqrt{1-x^2})}{x} > 0 \end{aligned}$$

Therefore I_{10} also satisfies SI.
Checking that I_{10} satisfies DF, DT and BC is easy and therefore it is omitted.

Hence I_{10} is a fuzzy fuzzy imlication.
For all x, $y \in [0,1]$, we obtain:

$I_{10}(1,x) = x$,
$I_{10}(x,y) = 1$ iff $x \leq y$,
$N_{I_{10}}(x) = I_{10}(x,0) = \sqrt{1-x^2} = \varphi^{-1}(1-\varphi(x))$, where $\varphi(x) = x^2$ is an order automorphism of the unit interval,
$I_{10}(x,x) = 1$,
I_{10} is a continuous mapping.

Therefore I_{10} satisfies NT, OP, SN, ID and CO. If I_{10} satisfies CP w.r.t. a strong fuzzy negation N, then for all $x \in [0,1]$, we obtain

$$N(x) = I_{10}(1, N(x)) = I_{10}(x,0) = N_{I_{10}}(x) = \sqrt{1-x^2}.$$

However, take $x_0 = 0.8$ and $y_0 = 0.1$, we obtain $I_{10}(x_0, y_0) = 0.65$ and $I_{10}(N(y_0), N(x_0)) \approx 0.643$. Therefore I_{10} does not satisfy CP w.r.t. any strong fuzzy negation N. $\square$

So we considered all the possibilities that the fuzzy implication axiom CP can be implied from the other seven axioms. Moreover we stated for each independent case a counterexample.

10 Getting FI13(CO) from the Other Axioms

Proposition 10.1 *There exists a fuzzy implication I satisfying NT, EP, OP, SN, CB, ID, CP and not CO.*

PROOF. Let N be a strong fuzzy negation. Recall the R_0-implication stated in [14] which is defined by

$$(I_{\min_0})_N(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(N(x), y), & \text{if } x > y \end{cases}, \quad x, y \in [0,1].$$

$(I_{\min_0})_N$ is the R-implication generated by the left-continuous t-norm, nilpotent minimum [7]:

$$(T_{\min_0})_N(x,y) = \begin{cases} \min(x,y), & \text{if } y > N(x) \\ 0, & \text{if } y \leq N(x) \end{cases}, \quad x, y \in [0,1].$$

$(I_{\min_0})_N$ satisfies NT, EP, OP, SN, CB, ID and CP w.r.t. N, and is right-continuous in the second place [14] but it is not continuous. $\square$

Therefore CO is independent of any of the other seven axioms.

11 Summary

From Sections 4, 5 and 10 the three axioms EP, OP and CO are essential because they are totally independent of the other axioms. On the other hand, they are really important because the combination of them can imply all the other five axioms. From Section 8, the axiom ID is relatively essential because only OP can imply it. The combination of the other six axioms cannot imply ID. However, none of the other axioms is dependent on ID.
Table 1 summarizes the results we obtained in Sections 3-10.
Let S_1 denote a subset of A defined by

$$A = \{\mathrm{NT}, \mathrm{EP}, \mathrm{OP}, \mathrm{SN}, \mathrm{CB}, \mathrm{ID}, \mathrm{CP}, \mathrm{CO}\},$$

and

$$S_2 = A - S_1.$$

Then from Table 1 we can judge if a fuzzy implication satisfies all the axioms in S_1 then it also satisfies the axioms of S_2. For example, let

$$S_1 = \{\mathrm{EP}, \mathrm{SN}, \mathrm{ID}\}.$$

Then

$$S_2 = \{\mathrm{NT}, \mathrm{OP}, \mathrm{CB}, \mathrm{CP}, \mathrm{CO}\}.$$

According to rows 2, 10 and 14 of Table 1, the fuzzy implication I_4 and $I_{\min_0}$ of Table 2, we obtain:

$$S_1 \Rightarrow \{\mathrm{NT}, \mathrm{CB}, \mathrm{CP}\}.$$

Finally we summarize all the examples in Table 2, where ‘Y’ denotes ‘yes’ and ‘N’ denotes ‘no’.

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Figure

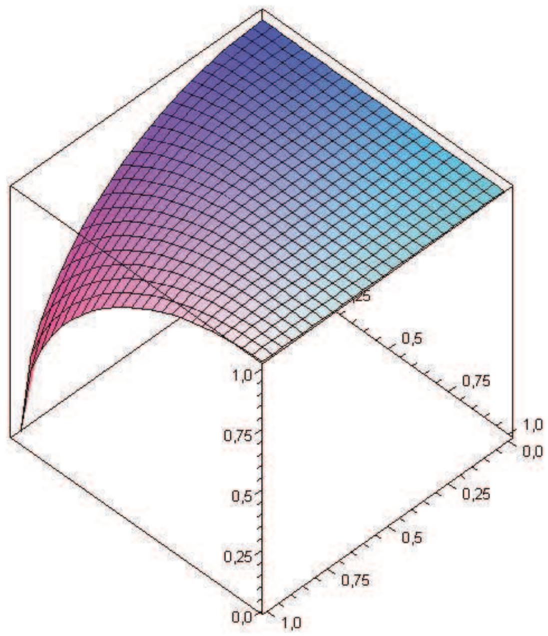

Fig. 1. The fuzzy implication I_2 in the proof of Proposition 3.7

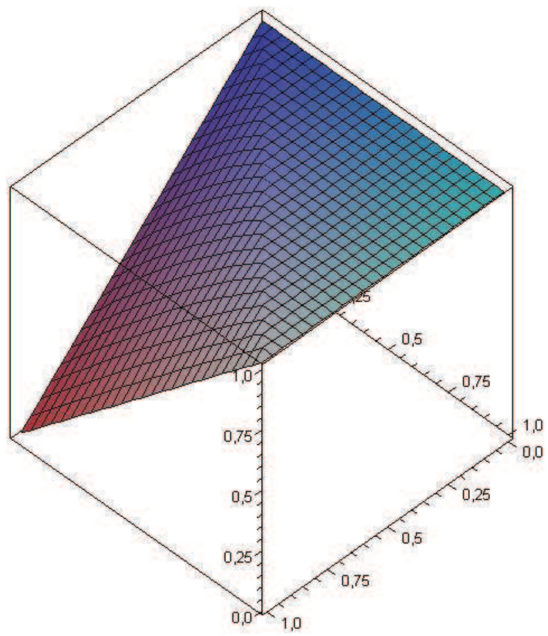

Fig. 2. The fuzzy implication I_3 in the proof of Proposition 4.1

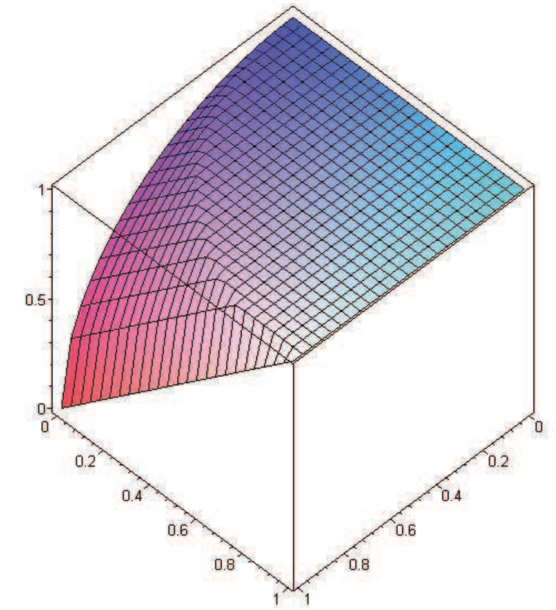

Fig. 3. The fuzzy implication I_4 in the proof of Proposition 5.1

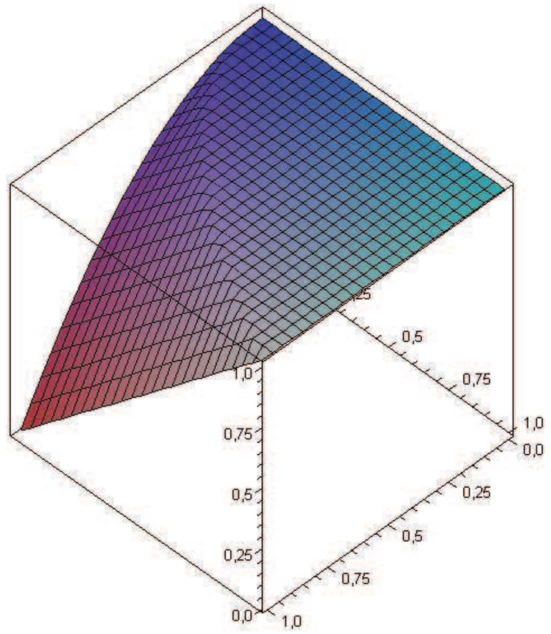

Fig. 4. The fuzzy implication I_5 in the proofs of Propositions 6.7 and 9.5

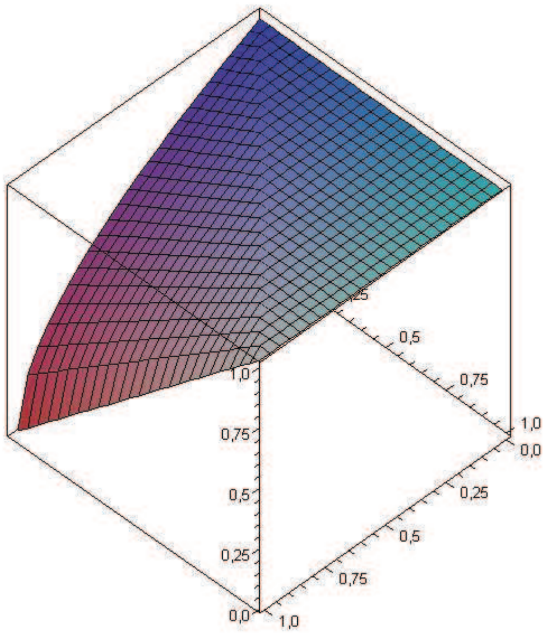

Fig. 5. The fuzzy implication I_6 in the proof of Proposition 6.8

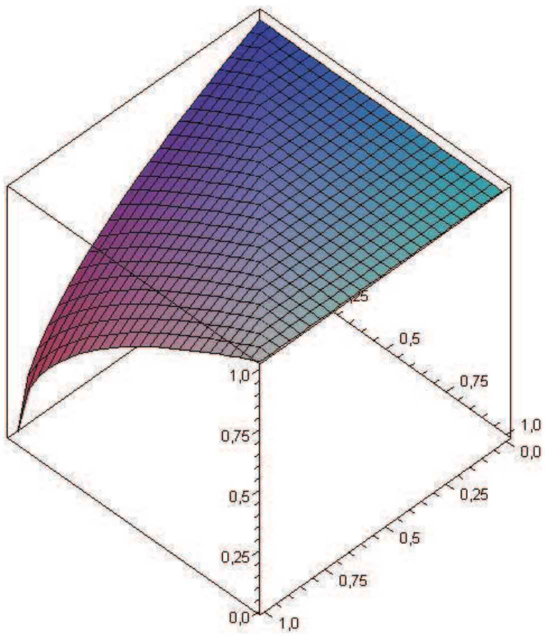

Fig. 6. The fuzzy implication I_7 in the proof of Proposition 6.10

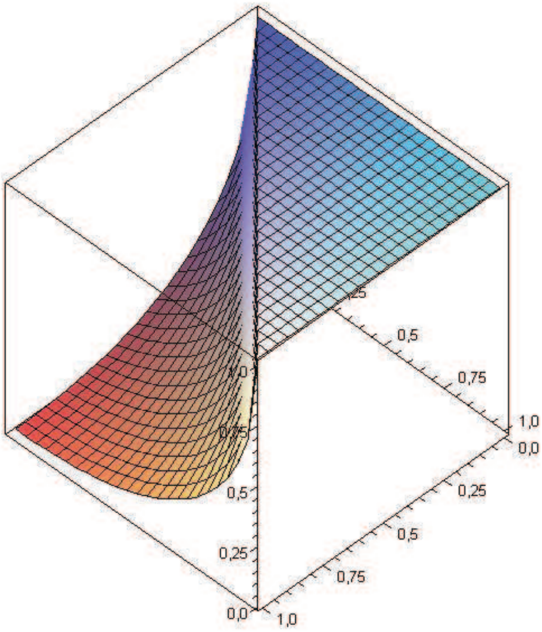

Fig. 7. The fuzzy implication I_9 in the proof of Proposition 7.7

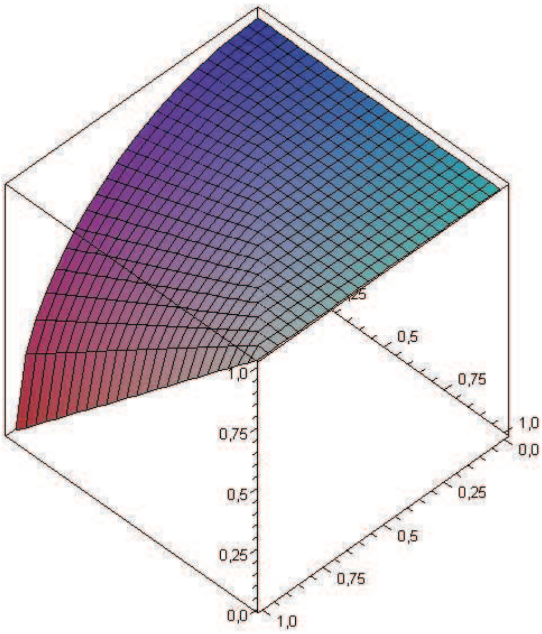

Fig. 8. The fuzzy implication I_{10} in the proof of Proposition 9.6

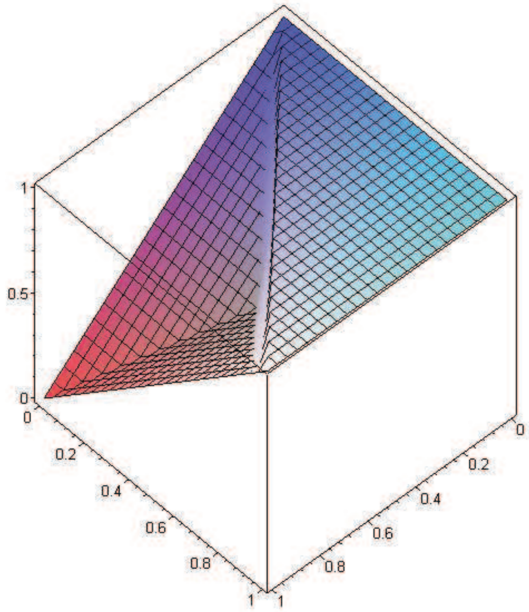

Fig. 9. The fuzzy implication $(I_{\min_0})_{N_0}$ in the proof of Proposition 10.1

Table 1
Summary of the interrelationships between the eight axioms

1.	$EP \wedge OP \Rightarrow NT$
2.	$EP \wedge SN \Rightarrow NT$
3.	$EP \wedge CO \Rightarrow NT$
4.	$NT \wedge CP \Rightarrow SN$
5.	$EP \wedge OP \wedge CP \Rightarrow SN$
6.	$EP \wedge OP \wedge CO \Rightarrow SN$
7.	$EP \wedge CP \wedge CO \Rightarrow SN$
8.	$NT \Rightarrow CB$
9.	$EP \wedge OP \Rightarrow CB$
10.	$EP \wedge SN \Rightarrow CB$
11.	$EP \wedge CO \Rightarrow CB$
12.	$OP \Rightarrow ID$
13.	$EP \wedge OP \wedge CO \Rightarrow CP$
14.	$EP \wedge SN \Rightarrow CP$

Table 2
Summary of the examples of fuzzy implications satisfying the indicated axioms

FI6 NT	FI7 EP	FI8 OP	FI9 SN	FI10 CB	FI11 ID	FI12 CP	FI13 CO	examples
N	Y	N	N	Y	Y	Y	N	I_1
N	N	Y	Y	Y	Y	Y	Y	I_2
Y	N	Y	Y	Y	Y	Y	Y	I_3
Y	Y	N	Y	Y	Y	Y	Y	I_4
Y	Y	N	N	Y	Y	N	Y	I_5
Y	N	Y	N	Y	Y	N	Y	I_6
N	N	Y	N	Y	Y	Y	Y	I_7
N	Y	N	N	N	Y	Y	N	I_8
N	N	Y	Y	N	Y	Y	Y	I_9
Y	N	Y	Y	Y	Y	N	Y	I_{10}
Y	Y	Y	N	Y	Y	N	N	I_G
Y	Y	N	Y	Y	N	Y	Y	I_{KD}
Y	Y	Y	Y	Y	Y	Y	N	$I_{\min_0}$
Y	Y	Y	Y	Y	Y	Y	Y	$I_{\mathbf{L}}$