Intriguing sets of vertices of regular graphs

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Abstract

Intriguing and tight sets of vertices of point-line geometries have recently been studied in the literature. In this paper, we indicate a more general framework for dealing with these notions. Indeed, we show that some of the results obtained earlier can be generalized to larger classes of graphs. We also give some connections and relations with other notions and results from algebraic graph theory. One of the main tools in our study will be the Bose-Mesner algebra associated with the graph.

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1 Introduction

Suppose $\Gamma = (X, R)$ is a finite connected regular graph of valency k and diameter $d \geq 2$ and that $Y \subseteq X$ is a set of vertices of Γ.

The set Y is said to be *intriguing* if there exist constants h_1 and h_2 such that every vertex $y \in Y$ is adjacent to precisely h_1 vertices of Y and every vertex $z \notin Y$ is adjacent to precisely h_2 vertices of Y. Clearly, $\emptyset$ and X are examples of intriguing sets. An intriguing set is said to be *nontrivial* if it is a nonempty proper subset of X. The set Y is a nontrivial intriguing set if and only if $\{Y, X \setminus Y\}$ is a so-called regular or equitable partition of Γ. Intriguing sets are called *regular sets* in [18].

If $k = \theta_0 > \theta_1 > \cdots > \theta_s$ are the $s + 1 \geq 3$ distinct eigenvalues of Γ and if N denotes the total number of ordered pairs of adjacent vertices of Y, then it can be shown (see Proposition 3.8) that $\theta_s \cdot |Y| + \frac{k-\theta_s}{|X|} \cdot |Y|^2 \leq N \leq \theta_1 \cdot |Y| + \frac{k-\theta_1}{|X|} \cdot |Y|^2$. If the lower [resp. upper] bound is attained, then Y is called a *tight set of Type I* [resp. *tight set of Type II*]. A set Y of vertices is called *tight* if it is either a tight set of Type I or a tight set of Type II. Clearly, $\emptyset$ and X are examples of tight sets. Since $\theta_s \cdot |Y| + \frac{k-\theta_s}{|X|} \cdot |Y|^2 = \theta_1 \cdot |Y| + \frac{k-\theta_1}{|X|} \cdot |Y|^2$ if and only if $|Y| \in \{0, |X|\}$, $\emptyset$ and X are the only tight sets of Γ which are both of Type I and II. A tight set is said to be *nontrivial* if it is a nonempty proper subset of X.

Suppose $\mathcal{C}$ is a set of cliques of Γ such that any two adjacent vertices of Γ are contained in a unique clique of $\mathcal{C}$. Then the pair $(X, \mathcal{C})$ is also called a *point-line geometry*. In this context, the elements of X are also called *points*, those of $\mathcal{C}$ *lines*, and Γ is called the *collinearity graph* of $(X, \mathcal{C})$.

Intriguing and tight sets of vertices have been studied for (the collinearity graphs of) certain families of point-line geometries. These point-line geometries include the generalized quadrangles [5, 20, 21], the polar spaces [4, 11], certain half-spin geometries [10] and the partial quadrangles [2]. These papers mainly deal with the construction and classification (sometimes with the aid of a computer) of intriguing sets, as well as the derivation of some of their properties (in the style of Propositions 3.4, 3.7, 3.8 and Corollaries 3.6, 3.12 below). An interesting class of intriguing sets of generalized quadrangles are provided by the so-called hemisystems of generalized quadrangles of order (s^2, s), s odd. Several new classes of such hemisystems have recently been constructed, see [1, 3, 8, 9].

In this paper, we show that the theory of intriguing and tight sets can be developed for a rather large class of graphs, namely the regular graphs. Several of the propositions and corollaries which we will give are more general versions of earlier results. We will also give some connections and relations with other notions and results from algebraic graph theory. One of the main tools in our study will be the Bose-Mesner algebra associated with the graph.

Remark. We would like to warn the reader that the notion "tight set" as it occurs in [4, 5, 20, 21] is equivalent with the notion "tight set of Type II" as defined above. In [4, 5] it was shown that every intriguing set of a finite generalized quadrangle, or (more generally) of any finite nondegenerate polar space, is either a tight set (as defined in [4, 5]) or an m-ovoid (which is just a set of points intersecting each maximal singular subspace in precisely m points). In [11, Proposition 2.1], it was shown that the notion "m-ovoid" (of a polar space) is equivalent with the notion "tight set of Type I". We will see later (Corollary 3.9) that in any connected strongly regular graph, every intriguing set is indeed a tight set of Type I or a tight set of Type II.

2 Basic notions regarding graphs

We will now explain some of the terminology regarding graphs which we will use throughout this paper. All graphs considered in this paper are supposed to be finite.

Suppose $\Gamma = (X, R)$ be a connected regular graph of valency k and diameter d. The distance between two vertices x and y of Γ will be denoted by $\partial(x, y)$. If x is a vertex of Γ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of vertices at distance i from x. The adjacency matrix of Γ will be denoted by A. Let $\mathcal{M} = \mathbb{R}[A]$ (i.e., the set of polynomials in A with real coefficients) be the Bose-Mesner algebra of Γ. Clearly, every matrix of $\mathcal{M}$ is symmetric. Let $\theta_0, \theta_1, \ldots, \theta_s$ denote the distinct eigenvalues of Γ, where $\theta_0 > \theta_1 > \cdots > \theta_s$. Then the eigenvalue θ_0 is equal to k and has multiplicity 1. Since the matrices $I, A, \ldots, A^d$ are linearly independent elements of $\mathbb{R}[A]$, we necessarily have $s \geq d$. Let E_i, $i \in \{0, \ldots, s\}$, denote the minimal idempotent corresponding to the eigenvalue θ_i. Then

- $E_0 = \frac{1}{|X|} J$,

- $I = E_0 + E_1 + \cdots + E_s$,
- $A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_s E_s = \frac{k}{|X|} J + \theta_1 E_1 + \cdots + \theta_s E_s$,
- $E_i E_j = O$ for any two distinct $i, j \in \{0, \ldots, s\}$,
- $E_i \boldsymbol{j} = \mathbf{0}$ for every $i \in \{1, \ldots, s\}$,
- $\mathcal{M} = \mathrm{Span}(I, A, A^2, \ldots A^s) = \mathrm{Span}(E_0, E_1, \ldots, E_s)$.

Here, I, O and J are $|X| \times |X|$ matrices which are respectively equal to the identity matrix, the zero matrix and the matrix with all entries equal to 1. We denote by $\boldsymbol{j} \in \mathbb{R}^X$ [resp. $\mathbf{0} \in \mathbb{R}^X$] the $|X| \times 1$ matrix with all entries equal to 1 [resp. equal to 0]. For every set Y of vertices of Γ, we denote by $\boldsymbol{j}_Y \in \mathbb{R}^X$ the characteristic vector of Y. The number $r(Y) := |\{i \mid E_i \boldsymbol{j}_Y \neq \mathbf{0}\}| - 1$ is called the *dual degree* of Y. Since $E_i E_j = O$ for any two distinct $i, j \in \{0, \ldots, s\}$, the vectors $E_0 \boldsymbol{j}_Y, E_1 \boldsymbol{j}_Y, \ldots, E_s \boldsymbol{j}_Y$ are mutually orthogonal and hence the space $\mathcal{M}\boldsymbol{j}_Y = \mathrm{Span}(E_0 \boldsymbol{j}_Y, E_1 \boldsymbol{j}_Y, \ldots, E_s \boldsymbol{j}_Y)$ has dimension $r(Y) + 1$. In was Delsarte who introduced the notion of dual degree for association schemes (see note right above Theorem 11.1.1 of [6]). The above definition of dual degree is an obvious generalization to regular graphs.

A connected graph $\Gamma = (X, R)$ of diameter $d \geq 2$ is called *distance-regular* if there exist integers a_i, b_i, c_i ($i \in \{0, \ldots, d\}$) such that for any two vertices x and y at distance $i = \partial(x, y)$ from each other, there are precisely a_i neighbors of y in $\Gamma_i(x)$, b_i neighbors of y in $\Gamma_{i+1}(y)$ and c_i neighbors of y in $\Gamma_{i-1}(x)$. Clearly, $a_0 = c_0 = b_d = 0$ and $c_1 = 1$. If Γ is distance-regular, then Γ is regular with valency $k = a_0 + b_0 + c_0 = a_1 + b_1 + c_1 = \cdots = a_d + b_d + c_d$. Moreover, the total number of distinct eigenvalues of Γ is equal to $s + 1 = d + 1$.

The distance-regular graphs of diameter 2 are precisely the connected strongly regular graphs. In this case, we will denote a_1 also by λ and c_2 by μ and we will say that the distance-regular graph Γ is strongly regular with *parameters* (v, k, λ, μ). Here, v denotes the total number of vertices of Γ and (as usual) k denotes the valency of Γ. These parameters must satisfy the condition $\mu(v - 1 - k) = k(k - \lambda - 1)$. The connected strongly regular graph Γ has three distinct eigenvalues $\theta_0 = k$, $\theta_1 \geq 0$ and $\theta_2 < -1$, where θ_1 and θ_2 are the roots of the following quadratic polynomial

$$X^2 + (\mu - \lambda)X + (\mu - k).$$

Hence,

$$\theta_1 + \theta_2 = \lambda - \mu, \quad (1)$$
$$\theta_1 \cdot \theta_2 = \mu - k. \quad (2)$$

The connected strongly regular graphs are precisely the connected regular graphs having precisely three distinct eigenvalues.

More information on (strongly) regular graphs and distance-regular graphs can be found in the books [6], [13] and [14].

3 Intriguing and tight sets

Definition. Let $\Gamma = (X, R)$ be a connected graph of diameter d. A *code* in Γ is a nonempty subset Y of X. The distance between a vertex $x \in X$ to Y is defined as $\partial(x, Y) := \min\{\partial(x, y) \,|\, y \in Y\}$ and the number $t(Y) := \max\{\partial(x, Y) \,|\, x \in X\}$ is called the *covering radius* of Y. Y is called *completely regular* if there exist constants B_{ij}, $i \in \{0, \ldots, t(Y)\}$ and $j \in \{0, \ldots, d\}$, such that $|\Gamma_j(x) \cap Y| = B_{ij}$ for every $j \in \{0, \ldots, d\}$ and every vertex x at distance i from Y. More information on completely regular codes can be found in [6, Section 11.1], [12], [13, Section 11.7] and [19].

The following is a special case of a known result, see e.g. [13, Theorem 7.1].

Proposition 3.1 *Let $\Gamma = (X, R)$ be a distance regular graph of diameter $d \geq 2$. Then the nontrivial intriguing sets of Γ are precisely the completely regular codes of covering radius* 1.

We will meet completely regular codes again in Section 4 and in the remarks following Corollary 3.9. The examples given in Section 4 show that the conclusion of Proposition 3.1 is not valid for arbitrary regular graphs.

Proposition 3.2 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ and valency k. Let Y be a nonempty set of vertices of Γ and put $Z = X \setminus Y$. Then* $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) \subseteq \mathcal{M}\boldsymbol{j}_Y$ *with equality if and only if Y is an intriguing set of vertices of* Γ.

Proof. Since I and $J = |X| \cdot E_0$ belong to $\mathcal{M}$, the vectors $\boldsymbol{j}_Y = I\boldsymbol{j}_Y$ and $|Y| \cdot \boldsymbol{j} = J\boldsymbol{j}_Y$ belong to $\mathcal{M}\boldsymbol{j}_Y$. Hence, also the vector $\boldsymbol{j}_Z = \boldsymbol{j} - \boldsymbol{j}_Y$ belongs to $\mathcal{M}\boldsymbol{j}_Y$, proving that $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) \subseteq \mathcal{M}\boldsymbol{j}_Y$.

Suppose now that $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) = \mathcal{M}\boldsymbol{j}_Y$. Then since $A\boldsymbol{j}_Y$ belongs to $\mathcal{M}\boldsymbol{j}_Y$, there exists real numbers h_1 and h_2 such that $A\boldsymbol{j}_Y = h_1 \cdot \boldsymbol{j}_Y + h_2 \cdot \boldsymbol{j}_Z$. This implies that every vertex of Y is adjacent to precisely h_1 vertices of Y and that every vertex of Z is adjacent to precisely h_2 vertices of Y. So, Y is an intriguing set of vertices of Γ.

Conversely, suppose that Y is an intriguing set of vertices of Γ. Then there exist constants h_1 and h_2 such that every vertex of Y is adjacent to precisely h_1 vertices of Y and that every vertex of Z is adjacent to precisely h_2 vertices of Y. This implies that $A\boldsymbol{j}_Y = h_1 \cdot \boldsymbol{j}_Y + h_2 \cdot \boldsymbol{j}_Z$ and $A\boldsymbol{j}_Z = A(\boldsymbol{j} - \boldsymbol{j}_Y) = k \cdot \boldsymbol{j} - h_1 \cdot \boldsymbol{j}_Y - h_2 \cdot \boldsymbol{j}_Z = (k - h_1) \cdot \boldsymbol{j}_Y + (k - h_2) \cdot \boldsymbol{j}_Z$. Hence, $\mathcal{M}\boldsymbol{j}_Y \subseteq \mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z)$. As a consequence, $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) = \mathcal{M}\boldsymbol{j}_Y$. ■

Proposition 3.3 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ and valency k and let Y be a nonempty proper subset of X. Then Y is intriguing if and only if $r(Y) = 1$.*

Proof. Let $Z = X \setminus Y$. By Proposition 3.2, $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) \subseteq \mathcal{M}\boldsymbol{j}_Y$ with equality if and only if Y is intriguing. So, the dimension $r(Y) + 1$ of $\mathcal{M}\boldsymbol{j}_Y$ is equal to 2 if and only if $\mathrm{Span}(\boldsymbol{j}_Y, \boldsymbol{j}_Z) = \mathcal{M}\boldsymbol{j}_Y$, i.e. if and only if Y is intriguing. ■

Definition. Let $\Gamma = (X, R)$ be a connected regular graph, let $\theta_0 > \theta_1 > \cdots > \theta_s$ denote the distinct eigenvalues of Γ and let $E_0, E_1, \ldots, E_s$ denote the corresponding minimal idempotents. An intriguing set of vertices of Γ is said to be of *index* $i \in \{1, \ldots, s\}$ if $E_j \boldsymbol{j}_Y = \mathbf{0}$ for every $j \in \{1, \ldots, s\} \setminus \{i\}$. The sets $\emptyset$ and X are intriguing sets of index i for every $i \in \{1, \ldots, s\}$. By Proposition 3.3, the index of a nontrivial intriguing set is uniquely determined.

Proposition 3.4 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ with $s+1$ eigenvalues. Let Y_1 and Y_2 be two intriguing sets of vertices of Γ of the same index $i \in \{1, \ldots, s\}$.*
(i) If $Y_1 \subseteq Y_2$, then $Z = Y_2 \setminus Y_1$ is an intriguing set of index i.
(ii) If $Y_1 \cap Y_2 = \emptyset$, then $Z = Y_1 \cup Y_2$ is an intriguing set of index i.

Proof. We have $\boldsymbol{j}_Z = \boldsymbol{j}_{Y_2} - \boldsymbol{j}_{Y_1}$ (case (i)) or $\boldsymbol{j}_Z = \boldsymbol{j}_{Y_2} + \boldsymbol{j}_{Y_1}$ (case (ii)). In any case, the fact that $E_j \boldsymbol{j}_{Y_1} = E_j \boldsymbol{j}_{Y_2} = \mathbf{0}$ for every $j \in \{1, \ldots, s\} \setminus \{i\}$ implies that $E_j \boldsymbol{j}_Z = \mathbf{0}$ for every $j \in \{1, \ldots, s\} \setminus \{i\}$. So, Z is an intriguing set of index i. ■

Proposition 3.5 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$, let $\theta_0 > \theta_1 > \cdots > \theta_s$ denote the distinct eigenvalues of Γ and let $E_0, E_1, \ldots, E_s$ denote the corresponding idempotents. Suppose Y and Z are sets of vertices of Γ such that $E_i \boldsymbol{j}_Y = \mathbf{0}$ or $E_i \boldsymbol{j}_Z = \mathbf{0}$ for every $i \in \{1, \ldots, s\}$. Then $|Y \cap Z| = \frac{|Y| \cdot |Z|}{|X|}$.*

Proof. Since $I = E_0 + E_1 + \cdots + E_s$ and $E_0 = \frac{1}{|X|} J$, we have

$$|Y \cap Z| = \boldsymbol{j}_Y^T \cdot \boldsymbol{j}_Z = \boldsymbol{j}_Y^T (E_0 + E_1 + \cdots + E_s) \boldsymbol{j}_Z = \frac{1}{|X|} \boldsymbol{j}_Y^T J \boldsymbol{j}_Z = \frac{|Y| \cdot |Z|}{|X|}.$$

■

The following is an immediate corollary of Proposition 3.5.

Corollary 3.6 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ having $s+1$ distinct eigenvalues. Let $i_1, i_2 \in \{1, \ldots, s\}$ with $i_1 \neq i_2$. If Y_1 is an intriguing set of index i_1 and Y_2 is an intriguing set of index i_2, then $|Y_1 \cap Y_2| = \frac{|Y_1| \cdot |Y_2|}{|X|}$.*

Proposition 3.7 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ and valency k. Let $k = \theta_0 > \theta_1 > \cdots > \theta_s$ be the distinct eigenvalues of Γ. Let Y be a subset of X and let N be the total number of ordered pairs of adjacent vertices of Y. If Y is an intriguing set of index $i \in \{1, \ldots, s\}$, then $N = \theta_i \cdot |Y| + \frac{k - \theta_i}{|X|} \cdot |Y|^2$. Moreover, every vertex of Y is adjacent to precisely $\theta_i + \frac{k - \theta_i}{|X|} \cdot |Y|$ vertices of Y and every vertex not contained in Y is adjacent to precisely $\frac{k - \theta_i}{|X|} \cdot |Y|$ vertices of Y.*

Proof. We may suppose that $\emptyset \neq Y \neq X$. Then $E_i \boldsymbol{j}_Y \neq \mathbf{0}$ and $E_j \boldsymbol{j}_Y = \mathbf{0}$ for every $j \in \{1, \ldots, s\} \setminus \{i\}$. Put $Z := X \setminus Y$. By Proposition 3.2, $E_i \boldsymbol{j}_Y = h' \cdot \boldsymbol{j}_Y + h \cdot \boldsymbol{j}$ for some real numbers h' and h. Since $\mathbf{0} \neq E_i \boldsymbol{j}_Y = E_i^2 \boldsymbol{j}_Y = h' \cdot E_i \boldsymbol{j}_Y + h \cdot E_i \boldsymbol{j} = h' \cdot E_i \boldsymbol{j}_Y$, we necessarily have $h' = 1$. Now, since Y is an intriguing set $A\boldsymbol{j}_Y = h_1 \cdot \boldsymbol{j}_Y + h_2 \cdot \boldsymbol{j}_Z$, where h_1 and h_2 are integers such that every vertex of Y is adjacent to precisely h_1 vertices of Y and every vertex of Z is adjacent to precisely h_2 vertices of Y. Standard counting yields $|Y| \cdot (k - h_1) = (|X| - |Y|) \cdot h_2$. Since $A = \frac{k}{|X|}J + \theta_1 E_1 + \cdots + \theta_s E_s$, we have $(h_1 - h_2) \cdot \boldsymbol{j}_Y + h_2 \boldsymbol{j} = A\boldsymbol{j}_Y = \frac{k \cdot |Y|}{|X|}\boldsymbol{j} + \theta_i \cdot E_i \boldsymbol{j}_Y = \frac{k \cdot |Y|}{|X|}\boldsymbol{j} + \theta_i \cdot \boldsymbol{j}_Y + \theta_i h \cdot \boldsymbol{j}$. We find that $h_1 - h_2 = \theta_i$. Together with $|Y| \cdot (k - h_1) = (|X| - |Y|) \cdot h_2$, this implies that $h_1 = \theta_i + \frac{k - \theta_i}{|X|} \cdot |Y|$, $h_2 = \frac{k-\theta_i}{|X|} \cdot |Y|$ and $N = \theta_i \cdot |Y| + \frac{k-\theta_i}{|X|} \cdot |Y|^2$. ∎

Definition. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be two sequences of real numbers, with $n > m$. The second sequence is said to *interlace* the first one whenever $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for every $i \in \{1, \ldots, m\}$. The interlacing is called *tight* if there exists an integer $k \in \{0, \ldots, m\}$ such that $\lambda_i = \mu_i$ for every $i \in \{1, \ldots, k\}$ and $\lambda_{n-m+i} = \mu_i$ for every $i \in \{k+1, \ldots, m\}$.

The following result follows from [16, Theorem 3.5]. For reasons of completeness, we give a sketch of the proof.

Proposition 3.8 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ and valency k. Let $k = \theta_0 > \theta_1 > \cdots > \theta_s$ be the distinct eigenvalues of Γ. Let Y be a subset of X and let N be the total number of ordered pairs of adjacent vertices of Y. Then*

$$\theta_s \cdot |Y| + \frac{k - \theta_s}{|X|} \cdot |Y|^2 \leq N \leq \theta_1 \cdot |Y| + \frac{k - \theta_1}{|X|} \cdot |Y|^2.$$

Moreover, if equality holds in one of the inequalities above, then Y is intriguing.

Proof. We may assume that $\emptyset \neq Y \neq X$. For subsets S and T of X, let $e(S,T) = |\{(s,t) \mid \{s,t\} \in R,\ s \in S,\ t \in T\}|$. Then $N = e(Y,Y)$. Put $X_0 := Y$, $X_1 := X \setminus Y$ and $\alpha := \frac{N}{|Y|}$. Let A be the adjacency matrix of Γ and let B be the following 2×2 matrix:

$$B = \begin{bmatrix} \frac{1}{|X_0|}e(X_0, X_0) & \frac{1}{|X_0|}e(X_0, X_1) \\ \frac{1}{|X_1|}e(X_1, X_0) & \frac{1}{|X_1|}e(X_1, X_1) \end{bmatrix} = \begin{bmatrix} \alpha & k - \alpha \\ \frac{|Y|(k-\alpha)}{|X|-|Y|} & k - \frac{|Y|(k-\alpha)}{|X|-|Y|} \end{bmatrix}.$$

Since k is an eigenvalue of B, the other eigenvalue λ is

$$\lambda = \mathrm{tr}(B) - k = \alpha - \frac{|Y|(k - \alpha)}{|X| - |Y|}.$$

By [6, Corollary 3.3.3] or [15, Theorem 1.2.3], the eigenvalues of B interlace those of A. So, we have

$$\theta_s \leq \alpha - \frac{|Y|(k - \alpha)}{|X| - |Y|} \leq \theta_1.$$

Therefore

$$\theta_s \cdot |Y| + \frac{k - \theta_s}{|X|} \cdot |Y|^2 \leq N \leq \theta_1 \cdot |Y| + \frac{k - \theta_1}{|X|} \cdot |Y|^2$$

as desired.

Moreover, if equality holds in one of the inequalities, then the interlacing is tight and $\{Y, Z\}$ is a regular partition of Γ by [6, Corollary 3.3.3] or [15, Theorem 1.2.3]. So, in this case Y is an intriguing set of vertices of Γ. ∎

We recall that if the lower [resp. upper] bound in Proposition 3.8 is attained then Y is called a *tight set of Type I* [resp. *tight set of Type II*].

Examples. We will now give two special cases of Proposition 3.8. As before, let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ and valency k. Let $k = \theta_0 > \theta_1 > \cdots > \theta_s$ be the distinct eigenvalues of Γ. Let Y be a subset of X.

(1) If the induced subgraph on Y is regular with valency α, then by Proposition 3.8, $(\alpha - \theta_1) \cdot \frac{|X|}{k-\theta_1} \leq |Y| \leq (\alpha - \theta_s) \cdot \frac{|X|}{k-\theta_s}$ (notice that $N = \alpha \cdot |Y|$). If the upper [resp. lower] bound is attained, then Y is a tight set of Type I [resp. Type II].

(2) If Y is a coclique of a connected graph Γ of valency k, then the lower bound in Proposition 3.8 implies that $|Y| \leq (1 + \frac{k}{-\theta_s})^{-1} \cdot |X|$. (Alternatively, one can put $\alpha = 0$ in the upper bound mentioned in (1).) This bound is known as the *Hoffman's coclique bound*, see [6, Propositions 1.3.2 and 3.7.2].

In the following corollary we collect some properties of tight sets which immediately follow from earlier results.

Corollary 3.9 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$ having precisely $s + 1$ eigenvalues. Then the following hold:*

(*i*) *A set of vertices of Γ is a tight set of Type I if and only if it is an intriguing set of index s.*

(*ii*) *A set of vertices of Γ is a tight set of Type II if and only if it is an intriguing set of index* 1.

(*iii*) *Suppose Γ is a strongly regular graph. Then a set of vertices of Γ is tight if and only if it is intriguing.*

(*iv*) *If Y_1 is a tight set of Type I and Y_2 is a tight set of Type II, then $|Y_1 \cap Y_2| = \frac{|Y_1| \cdot |Y_2|}{|X|}$.*

Proof. Claims (i) and (ii) are corollaries of Propositions 3.7 and 3.8. Claim (iii) is a corollary of Claims (i) and (ii) and the fact that every connected strongly regular graph has precisely 3 distinct eigenvalues. Claim (iv) follows from Corollary 3.6 and Claims (i) and (ii). ∎

Remarks. (1) The result mentioned in Corollary 3.9(iii) also follows from [18, Proposition 2]. The conclusion of Corollary 3.9(iii) may not be valid for connected regular graphs which are not strongly regular (i.e. for connected regular graphs with more than three eigenvalues), see Section 4.

(2) For the Johnson graph $J(v, d)$ and the Hamming graph $H(d, q)$, all completely regular codes of strength zero, i.e., all regular codes Y for which $E_1 \boldsymbol{j}_Y \neq \mathbf{0}$, are classified in [7, 17]. In particular all tight sets of Type II are classified.

(3) In [23], Tanaka classified completely regular codes of strength zero with extra conditions in various classical distance-regular graphs.

Definition. Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d \geq 2$. Let $k = \theta_0 > \theta_1 > \cdots > \theta_d$ denote the $d+1$ distinct eigenvalues of Γ and let $E_0, E_1, \ldots, E_d$ denote the corresponding idempotents. Let T be a subset of $\{1, 2, \ldots, d\}$. A *T-design* of Γ is defined to be a subset Y of X such that $E_i \boldsymbol{j}_Y = \mathbf{0}$ for all $i \in T$. An *anti-T-design* of Γ is a subset Z of X such that $E_i \boldsymbol{j}_Z = \mathbf{0}$ for all $j \in \{1, 2, \ldots, d\} \setminus T$. The notion of T-design and anti-T-design is due to Delsarte [12].

The following is an immediate corollary of Proposition 3.5.

Corollary 3.10 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d \geq 2$, let T be a subset of $\{1, 2, \ldots, d\}$, let Y be a T-design of Γ and let Z be an anti-T-design of Γ. Then $|Y \cap Z| = \frac{|Y| \cdot |Z|}{|X|}$.*

Proposition 3.11 *Let $\Gamma = (X, R)$ be a distance-regular graph of diameter $d \geq 2$, let $k = \theta_0 > \theta_1 > \cdots > \theta_d$ denote the $d+1$ distinct eigenvalues of Γ and let $E_0, E_1, \ldots, E_d$ denote the corresponding idempotents. Let T be a subset of $\{1, 2, \ldots, d\}$, let Y be a set of vertices of Γ and let $\mathcal{F}$ be a nonempty family of T-designs of Γ satisfying the following properties:*

- (i) *all elements of $\mathcal{F}$ have the same number of vertices;*
- (ii) *there exist constants α_i, $i \in \{0, 1, \ldots, d\}$, such that if x and y are two vertices of Γ at distance i from each other, then $\alpha_i = |\{Z \in \mathcal{F} \mid \{x, y\} \subset Z\}|$;*
- (iii) *for every $i \in T' := \{1, \ldots, d\} \setminus T$, there exists some $Z \in \mathcal{F}$ such that $E_i \boldsymbol{j}_Z \neq \mathbf{0}$.*

Then the following are equivalent:

- (a) *Y is an anti-T-design;*
- (b) *every element of $\mathcal{F}$ intersects Y in a constant number of vertices.*

Proof. That (a) implies (b) follows directly from Corollary 3.10. We will now also prove that (b) implies (a). So, suppose that every element of $\mathcal{F}$ intersects Y in a constant number of vertices, say m. For every $i \in \{0, 1, \ldots, d\}$, let A_i denote the $|X| \times |X|$ matrix whose rows and columns are indexed by X defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A_0 = I$ and $A_1 = A$. It is well-known that $\mathcal{M} = \text{Span}(I, A, A^2, \ldots, A^d) = \text{Span}(E_0, E_1, \ldots, E_d) = \text{Span}(A_0, A_1, \ldots, A_d)$.

Let H be an $|X| \times |\mathcal{F}|$ matrix whose rows are indexed by X and columns are indexed by $\mathcal{F}$ defined by

$$H_{x,Z} = \begin{cases} 1 & \text{if } x \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M := HH^T = \alpha_0 A_0 + \alpha_1 A_1 + \cdots + \alpha_d A_d.$$

Since $\mathrm{Span}(E_0, E_1, \ldots, E_d) = \mathrm{Span}(A_0, A_1, \ldots, A_d)$, there exist real numbers $\beta_0, \beta_1, \ldots, \beta_d$ such that

$$M = HH^T = \beta_0 E_0 + \beta_1 E_1 + \cdots + \beta_d E_d.$$

Suppose $j \in T$. Then for every $Z \in \mathcal{F}$, $E_j \boldsymbol{j}_Z = \mathbf{0}$. So, $E_j H = O$ since the Z-th column of H is $\boldsymbol{j}_Z$. Since $\beta_j E_j = E_j M = E_j HH^T = O$, we have $\beta_j = 0$.

Suppose $j \in T'$. Then there is some $Z \in \mathcal{F}$ such that $E_j \boldsymbol{j}_Z \neq \mathbf{0}$. Hence, $E_j H \neq O$ and $E_j M E_j = (E_j H)(E_j H)^T \neq O$. So, $E_j M = \beta_j E_j \neq O$ and $\beta_j \neq 0$.

Since every element of $\mathcal{F}$ intersects Y in precisely m vertices, we have

$$\beta_0 E_0 \boldsymbol{j}_Y + \sum_{j \in T'} \beta_j E_j \boldsymbol{j}_Y = M \boldsymbol{j}_Y = HH^T \boldsymbol{j}_Y = \alpha_0 \cdot m \boldsymbol{j}.$$

Recall that $E_0 \boldsymbol{j}_Y = \frac{|Y|}{|X|} \boldsymbol{j}$ and $\beta_j \neq 0$ for every $j \in T'$. The fact that the column matrices $E_j \boldsymbol{j}_Y$, $j \in T' \cup \{0\}$, are mutually orthogonal then implies that $E_j \boldsymbol{j}_Y = \mathbf{0}$ for every $j \in T'$. So, Y is an anti-T-design, proving that (b) also implies (a). ■

Remark. We continue with the notation introduced in the statement of Proposition 3.11. Suppose G is a group of automorphisms of Γ such that for any vertices x_1, y_1, x_2, y_2 satisfying $\partial(x_1, y_1) = \partial(x_2, y_2)$, there exists an element of G mapping $\{x_1, y_1\}$ to $\{x_2, y_2\}$. (Such a group exists if Γ is distance-transitive.) Suppose also that Z^* is a T-design such that $E_i \boldsymbol{j}_{Z^*} \neq \mathbf{0}$ for every $i \in T'$. If $\mathcal{F}$ denotes the orbit of Z^* under the action of the group G, then the conditions (i), (ii) and (iii) of Proposition 3.11 are satisfied. So, in this case, the anti-T-designs of Γ can be characterized as those sets of vertices of Γ which intersect each element of $\mathcal{F}$ in a constant number of vertices.

The following is the special case $d = 2$ of Proposition 3.11.

Corollary 3.12 *Let $\Gamma = (X, R)$ be a connected strongly regular graph. Put $\{A, B\} = \{I, II\}$. Let Y be a set of vertices of Γ and let $\mathcal{F}$ be a nonempty family of nontrivial tight sets of Type A of Γ satisfying the following properties:*

- *(i) all elements of $\mathcal{F}$ have the same number of vertices;*
- *(ii) there exist constants α_i, $i \in \{0, 1, 2\}$, such that if x and y are two vertices of Γ at distance i from each other, then $\alpha_i = |\{Z \in \mathcal{F} \mid \{x, y\} \subset Z\}|$.*

Then the following are equivalent:

- *(a) Y is a tight set of Type B;*

(*b*) *every element of $\mathcal{F}$ intersects Y in a constant number of vertices.*

Unlike in many of the other results of this section, we suppose in Corollary 3.12 that the regular graph is strongly regular. In Appendix A, we will prove that if a connected regular graph Γ of diameter 2 has a nonempty family $\mathcal{F}$ of nontrivial intriguing sets of a given index satisfying conditions (i) and (ii) of Corollary 3.12, then Γ necessarily is a strongly regular graph.

Corollary 3.12 is especially interesting in the case the strongly regular graph Γ has some natural family of nontrivial tight sets of Type A which satisfies the conditions (i) and (ii) of the corollary. In this case, Corollary 3.12 provides a natural characterization for tight sets of Type B of Γ. Strongly regular graphs which have such a natural family of tight sets (of a certain type) include the polar spaces and the half dual polar graphs of diameter 2, see [10] and [11]. In fact, these papers already contain a proof of Corollary 3.12 for these particular graphs. The methods of proof used in these papers are different from the ones we used above, but they work as well for arbitrary connected strongly regular graphs. In Appendix B we will use these methods to give a combinatorial proof for Corollary 3.12.

We conclude this section by giving a result, taken from [11, Corollary 1.4], which can be proved by means of Corollary 3.12. This result is stated in terms of quadrics and Hermitian varieties of projective spaces, but with each of these sets of points there naturally corresponds a polar space. Proposition 3.13 characterizes tight sets of these polar spaces in terms of their intersections with certain hyperplanes of the projective spaces (namely those hyperplanes whose intersection with the quadric or Hermitian variety is of a certain type). Observe here also that the notions "ovoidal set" and "tight set" which occur in the statement of Corollary 1.4 of [11] are respectively equivalent with the notions "tight sets of Type I" and "tight sets of Type II" of the present paper (see [11, Proposition 2.1]).

Proposition 3.13 ([11]) (1) *A set of points of a nonsingular quadric $\mathcal{Q}(2r,q)$ of Witt index $r \geq 2$ of the projective space $\mathrm{PG}(2r,q)$ is a tight set of Type II if and only if it intersects every $\mathcal{Q}^-(2r-1,q) \subseteq \mathcal{Q}(2r,q)$ in a constant number of points.*

(2) *A set of points of a nonsingular quadric $\mathcal{Q}(2r,q)$ of Witt index $r \geq 2$ of the projective space $\mathrm{PG}(2r,q)$ is a tight set of Type I if and only if it intersects every $\mathcal{Q}^+(2r-1,q) \subseteq \mathcal{Q}(2r,q)$ in a constant number of points.*

(3) *A set of points of a nonsingular quadric $\mathcal{Q}^+(2r-1,q)$ of Witt index $r \geq 2$ of the projective space $\mathrm{PG}(2r-1,q)$ is a tight set of Type II if and only if it intersects every $\mathcal{Q}(2r-2,q) \subseteq \mathcal{Q}^+(2r-1,q)$ in a constant number of points.*

(4) *A set of points of a nonsingular quadric $\mathcal{Q}^-(2r+1,q)$ of Witt index $r \geq 2$ of the projective space $\mathrm{PG}(2r+1,q)$ is a tight set of Type I if and only if it intersects every $\mathcal{Q}(2r,q) \subseteq \mathcal{Q}^-(2r+1,q)$ in a constant number of points.*

(5) *A set of points of a nonsingular Hermitian variety $\mathcal{H}(2r-1,q)$ of Witt index $r \geq 2$ of the projective space $\mathrm{PG}(2r-1,q)$ is a tight set of Type II if and only if it intersects every $\mathcal{H}(2r-2,q) \subseteq \mathcal{H}(2r-1,q)$ in a constant number of points.*

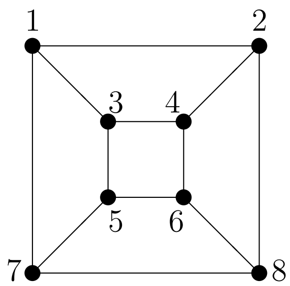
(6) *A set of points of a nonsingular Hermitian variety* $\mathcal{H}(2r, q)$ *of Witt index* $r \geq 2$ *of the projective space* $\mathrm{PG}(2r, q)$ *is a tight set of Type I if and only if it intersects every* $\mathcal{H}(2r-1, q) \subseteq \mathcal{H}(2r, q)$ *in a constant number of points.*

4 Some examples of intriguing sets in small graphs

As mentioned before, the references [2], [4], [5], [10], [20] and [21] contain plenty of examples of nontrivial intriguing sets of vertices. The graphs considered in these papers are all strongly regular. So, each of these intriguing sets is also tight (Corollary 3.9(iii)) and a completely regular code (Proposition 3.1). The aim of this section is to give some examples of intriguing sets in connected regular graphs which are not strongly regular. These examples show that not all intriguing sets are tight and that not all of them are completely regular codes. We consider a particular class of graphs, namely the regular graphs of valency 3 on 8 vertices.

By Read and Wilson [22, p. 127], there are 5 isomorphism classes of connected regular graphs of valency 3 on 8 vertices. For each such graph $\Gamma = (X, R)$, we give up to isomorphism all nontrivial intriguing sets of vertices. These sets are easily determined with the aid of a computer or even by hand. For each intriguing set Y, we also mention whether Y is a tight set and whether Y is a completely regular code. As usual we denote by $3 = \theta_0 > \theta_1 > \cdots > \theta_s$ the distinct eigenvalues of Γ. If Y is an intriguing set of vertices of index $i \in \{1, \ldots, s\}$ of Γ, then by Proposition 3.7, $\theta_i = h_1 - h_2$ where h_1 and h_2 are constants such that every vertex of Y is adjacent to precisely h_1 vertices of Y and every vertex outside Y is adjacent to precisely h_2 vertices of Y.

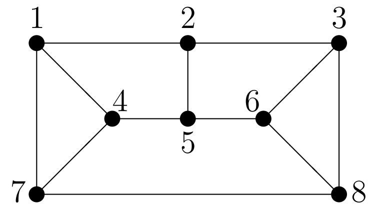
(I) Suppose $\Gamma = (X, R)$ is the cube whose vertices are labeled in the following way.



The eigenvalues of Γ are $\theta_0 = 3$, $\theta_1 = 1$, $\theta_2 = -1$ and $\theta_3 = -3$. Up to isomorphism, the cube Γ has the following nontrivial intriguing sets of vertices. Notice that by Proposition 3.1 all these intriguing sets are also completely regular codes.

Intriguing set	h_1	h_2	index	tight	compl. reg. code
$Y_1 = \{1, 6\}$	0	1	2	NO	YES
$Y_2 = \{1, 4, 5, 8\}$	0	3	3	Type I	YES
$Y_3 = \{1, 2, 5, 6\}$	1	2	2	NO	YES
$Y_4 = \{1, 2, 3, 4\}$	2	1	1	Type II	YES
$Y_5 = X \setminus Y_1 = \{2, 3, 4, 5, 7, 8\}$	2	3	2	NO	YES

(II) Let $\Gamma = (X, R)$ be the following regular graph of valency 3.

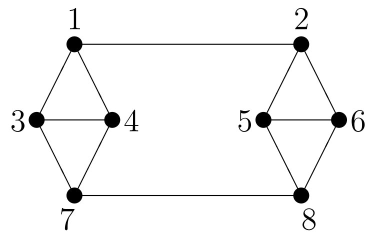


The eigenvalues of Γ are $\theta_0 = 3$, $\theta_1 = \sqrt{3}$, $\theta_2 = 1$, $\theta_3 = -1 + \sqrt{2}$, $\theta_4 = -1$, $\theta_5 = -\sqrt{3}$ and $\theta_6 = -1 - \sqrt{2}$. Up to isomorphism, Γ has the following nontrivial intriguing sets of vertices.

Intriguing set	h_1	h_2	index	tight	compl. reg. code
$Y_1 = \{1, 6\}$	0	1	4	NO	YES
$Y_2 = \{1, 3, 4, 6\}$	1	2	4	NO	YES
$Y_3 = X \setminus Y_2 = \{2, 5, 7, 8\}$	1	2	4	NO	YES
$Y_4 = X \setminus Y_1 = \{2, 3, 4, 5, 7, 8\}$	2	3	4	NO	NO*

(∗) Notice that $\Gamma_3(2) \cap Y_4 = \emptyset$ and $\Gamma_3(3) \cap Y_4 = \{4\}$. So, the complement of a completely regular code is not necessarily completely regular.

(III) Let $\Gamma = (X, R)$ be the following regular graph of valency 3.

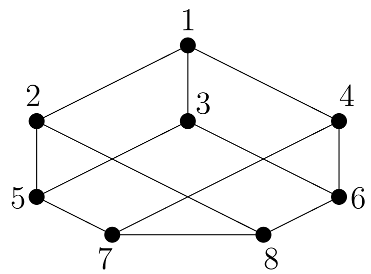


The eigenvalues of Γ are $\theta_0 = 3$, $\theta_1 = \sqrt{5}$, $\theta_2 = 1$, $\theta_3 = -1$ and $\theta_4 = -\sqrt{5}$. Up to isomorphism, Γ has the following nontrivial intriguing sets of vertices.

Intriguing set	h_1	h_2	index	tight	compl. reg. code
$Y_1 = \{1, 8\}$	0	1	3	NO	YES
$Y_2 = \{3, 5\}$	0	1	3	NO	NO*
$Y_3 = \{1, 2, 7, 8\}$	1	2	3	NO	YES
$Y_4 = \{1, 3, 5, 8\}$	1	2	3	NO	NO*
$Y_5 = X \setminus Y_3 = \{3, 4, 5, 6\}$	1	2	3	NO	YES
$Y_6 = X \setminus Y_1 = \{2, 3, 4, 5, 6, 7\}$	2	3	3	NO	NO*
$Y_7 = X \setminus Y_2 = \{1, 2, 4, 6, 7, 8\}$	2	3	3	NO	YES

(∗) Notice that $\Gamma_3(1) \cap Y_2 = \emptyset$, $\Gamma_3(4) \cap Y_2 = \{5\}$, $\Gamma_3(2) \cap Y_4 = \emptyset$, $\Gamma_3(4) \cap Y_4 = \{5\}$, $\Gamma_3(2) \cap Y_6 = \{7\}$ and $\Gamma_3(3) \cap Y_6 = \{5, 6\}$.

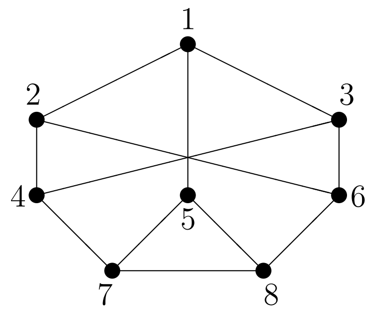
(IV) Let $\Gamma = (X, R)$ be the following regular graph of valency 3.



The eigenvalues of Γ are $\theta_0 = 3$, $\theta_1 = 1$, $\theta_2 = -1 + \sqrt{2}$, $\theta_3 = -1$ and $\theta_4 = -1 - \sqrt{2}$. Up to isomorphism, Γ has the following nontrivial intriguing sets of vertices.

Intriguing set	h_1	h_2	index	tight	compl. reg. code
$Y_1 = \{1, 3, 7, 8\}$	1	2	3	NO	YES
$Y_2 = \{1, 2, 3, 5\}$	2	1	1	Type II	YES

(V) Let $\Gamma = (X, R)$ be the following regular graph of valency 3.



The eigenvalues of Γ are $\theta_0 = 3$, $\theta_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{17}$, $\theta_2 = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $\theta_3 = 0$, $\theta_4 = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $\theta_5 = -\frac{1}{2} - \frac{1}{2}\sqrt{17}$. There are no nontrivial intriguing sets.

A Regular graphs of diameter 2 with a nice family of intriguing sets

We continue with the notations introduced in Section 2.

Lemma A.1 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter $d \geq 2$. Let E, E' be two nonzero mutually orthogonal idempotents in $\mathcal{M}$. If the adjacency matrix A can be written as a linear combination of E, E' and the identity matrix I, then Γ is strongly regular.*

Proof. Let $E'' = I - E - E'$. Then E, E', E'' are mutually orthogonal idempotents in $\mathcal{M}$. By assumption there exist $c, c', c'' \in \mathbb{R}$ such that $A = cE + c'E' + c''E''$. Since

$A^\ell = c^\ell E + c'^\ell E' + c''^\ell E''$, every power of A can be written as a linear combination of these idempotents and hence $\mathcal{M} = \mathbb{R}[A] = \mathrm{Span}(E, E', E'')$. Since

$$3 \le d+1 \le s+1 = \dim \mathcal{M} = \dim \mathrm{Span}(E, E', E'') \le 3,$$

Γ has precisely $s+1 = 3$ distinct eigenvalues. The lemma then follows from the fact that every connected regular graph with three distinct eigenvalues is strongly regular (see e.g. [14, Lemma 10.2.1]). ∎

Proposition A.2 *Let $\Gamma = (X, R)$ be a connected regular graph of diameter 2 and let $\theta_0 > \theta_1 > \cdots > \theta_s$ denote the $s+1$ distinct eigenvalues of Γ. Let $\mathcal{F}$ be a nonempty family of nontrivial intriguing sets of fixed index j of Γ satisfying the following properties:*

(i) all elements of $\mathcal{F}$ have the same number of vertices;

(ii) there exist constants α_i, $i \in \{0, 1, 2\}$, such that if x and y are two vertices of Γ at distance i from each other, then $\alpha_i = |\{Z \in \mathcal{F} \mid \{x, y\} \subset Z\}|$.

Then Γ is strongly regular.

Proof. We recycle some of the arguments of the proof of Proposition 3.11.

For every $i \in \{0, 1, 2\}$, let A_i denote the $|X| \times |X|$ matrix whose rows and columns are indexed by X defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M} = \mathbb{R}[A]$ be the Bose-Mesner algebra of Γ and let $E_0, E_1, \ldots, E_s$ be the primitive idempotents of $\mathcal{M}$ corresponding to the eigenvalues $\theta_0, \theta_1, \ldots, \theta_s$. Recall that $E_0 = \frac{1}{|X|}J$. Clearly, $A_0 = I$, $A_1 = A$ and $A_2 = J - I - A$ are all elements of $\mathcal{M}$.

Let H be an $|X| \times |\mathcal{F}|$ matrix whose rows are indexed by X and columns are indexed by $\mathcal{F}$ defined by

$$H_{x,Z} = \begin{cases} 1 & \text{if } x \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M = HH^T = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 \in \mathcal{M}.$$

Hence there exist real numbers $\beta_0, \beta_1, \ldots, \beta_s$ such that

$$M = HH^T = \beta_0 E_0 + \beta_1 E_1 + \cdots + \beta_s E_s.$$

Suppose $i \in \{1, 2, \ldots s\} \setminus \{j\}$. Then for every $Z \in \mathcal{F}$, $E_i \boldsymbol{j}_Z = \mathbf{0}$. So, $E_i H = O$ since the Z-th column of H is $\boldsymbol{j}_Z$. Since $\beta_i E_i = E_i M = E_i H H^T = O$, we have $\beta_i = 0$.

Moreover since every $Z \in \mathcal{F}$ is a nontrivial intriguing set of index j, $E_j \boldsymbol{j}_Z \neq \mathbf{0}$ and $E_j H \neq O$. Note that since $E_j H$ is a real matrix, $E_j H \neq O$ implies $O \neq E_j H H^T E_j^T$ and hence $\beta_j E_j = E_j M \neq O$. Therefore $\beta_j \neq 0$ and

$$M = \alpha_0 I + \alpha_1 A + \alpha_2 (J - I - A) = \beta_0 \frac{1}{|X|} J + \beta_j E_j.$$

Hence $E_j \in \mathrm{Span}(I, A, J)$. Since $E_j \notin \mathrm{Span}(I, J)$,

$$A \in \mathrm{Span}(I, E_j, J) = \mathrm{Span}(E_0, E_j, I).$$

Since E_0, E_j are nonzero mutually orthogonal idempotents, Γ is strongly regular by Lemma A.1. ∎

B A combinatorial proof for Corollary 3.12

The aim of this appendix is to give a combinatorial proof for Corollary 3.12. We continue with the notation introduced in the statement of that corollary. Suppose Γ is a connected strongly regular graph with parameters (v, k, λ, μ) and distinct eigenvalues $k = \theta_0 > \theta_1 > \theta_2$.

Put $(h, h') = (2, 1)$ if (A = I and B = II) and $(h, h') = (1, 2)$ if (A = II and B = I). Let N_1 denote the total number of ordered pairs of adjacent vertices of Y.

Suppose that every element of $\mathcal{F}$ contains precisely δ vertices and put $N_1' := \delta\Big(\theta_h + \frac{k-\theta_h}{v}\delta\Big)$. Then for every $Z \in \mathcal{F}$, the total number of ordered pairs of adjacent vertices of Z is equal to N_1' (recall Proposition 3.8). Also, put $N_2' := \delta(\delta-1) - N_1' = \delta(\delta-1) - \delta\Big(\theta_h + \frac{k-\theta_h}{v}\delta\Big)$. Then for every $Z \in \mathcal{F}$, the total number of ordered pairs of nonadjacent vertices of Z is equal to N_2'. An obvious counting argument yields

$$\alpha_0 = |\mathcal{F}| \cdot \frac{\delta}{v}, \quad \alpha_1 = |\mathcal{F}| \cdot \frac{N_1'}{v \cdot k}, \quad \alpha_2 = |\mathcal{F}| \cdot \frac{N_2'}{v(v-k-1)}.$$

For every $Z \in \mathcal{F}$, put $k_Z := |Y \cap Z|$. Summing over all $Z \in \mathcal{F}$, we obtain

$$\begin{aligned}
\sum 1 &= |\mathcal{F}|,\\
\sum k_Z &= |Y| \cdot \alpha_0,\\
\sum k_Z(k_Z - 1) &= N_1 \cdot \alpha_1 + \Big(|Y| \cdot (|Y| - 1) - N_1\Big) \cdot \alpha_2,\\
\sum k_Z(k_Z - 1) &= N_1(\alpha_1 - \alpha_2) + |Y|^2 \cdot \alpha_2 - |Y| \cdot \alpha_2,\\
\sum k_Z^2 &= N_1(\alpha_1 - \alpha_2) + |Y|^2 \cdot \alpha_2 + |Y| \cdot (\alpha_0 - \alpha_2).
\end{aligned}$$

From the Cauchy-Schwartz inequality $(\sum 1) \cdot (\sum k_Z^2) \geq (\sum k_Z)^2$, we find

$$N_1 \cdot (\alpha_1 \cdot |\mathcal{F}| - \alpha_2 \cdot |\mathcal{F}|) + |Y|^2 \cdot (\alpha_2 \cdot |\mathcal{F}| - \alpha_0^2) + |Y| \cdot (\alpha_0 \cdot |\mathcal{F}| - \alpha_2 \cdot |\mathcal{F}|) \geq 0. \qquad (3)$$

One calculates that

$$\alpha_1 \cdot |\mathcal{F}| - \alpha_2 \cdot |\mathcal{F}| = \frac{|\mathcal{F}|^2}{vk(v-k-1)}\Big(N_1'(v-k-1) - N_2'k\Big)$$

$$\begin{aligned}
&= \frac{|\mathcal{F}|^2}{vk(v-k-1)}\Big(N_1'(v-1)-(N_1'+N_2')k\Big)\\
&= \frac{|\mathcal{F}|^2\cdot\delta}{vk(v-k-1)}\Big(\theta_h(v-1)+\frac{(v-1)(k-\theta_h)}{v}\delta-(\delta-1)k\Big)\\
&= \frac{|\mathcal{F}|^2\cdot\delta(v-\delta)}{v^2k(v-k-1)}\Big(\theta_h(v-1)+k\Big).
\end{aligned}$$

In a similar way, one shows that

$$\begin{aligned}
\alpha_2\cdot|\mathcal{F}|-\alpha_0^2 &= \frac{|\mathcal{F}|^2}{v^2(v-k-1)}(vN_2'-(v-k-1)\delta^2)\\
&= -\frac{|\mathcal{F}|^2\cdot\delta(v-\delta)}{v^2(v-k-1)}(1+\theta_h),
\end{aligned}$$

and that

$$\begin{aligned}
\alpha_0\cdot|\mathcal{F}|-\alpha_2\cdot|\mathcal{F}| &= \frac{|\mathcal{F}|^2}{v(v-k-1)}\Big(\delta(v-k-1)-N_2'\Big)\\
&= \frac{|\mathcal{F}|^2\cdot\delta(v-\delta)}{v^2(v-k-1)}(v-k+\theta_h).
\end{aligned}$$

Consider now the two expressions

$$N_1\cdot(\alpha_1\cdot|\mathcal{F}|-\alpha_2\cdot|\mathcal{F}|)+|Y|^2\cdot(\alpha_2\cdot|\mathcal{F}|-\alpha_0^2)+|Y|\cdot(\alpha_0\cdot|\mathcal{F}|-\alpha_2\cdot|\mathcal{F}|), \tag{4}$$

$$N_1-\frac{k-\theta_{h'}}{v}|Y|^2-|Y|\cdot\theta_{h'}. \tag{5}$$

In each of these expressions, there are nonzero coefficients accompanying at least one of N_1, $|Y|^2$, $|Y|$. In the latter expression, the coefficient of N_1 is equal to 1. Suppose that in the former expression $\alpha_2\cdot|\mathcal{F}|-\alpha_0^2=\alpha_0\cdot|\mathcal{F}|-\alpha_2\cdot|\mathcal{F}|=0$. Then since $\delta\neq 0\neq v-\delta$, $1+\theta_h=0=v-k+\theta_h$, implying that $v=k+1$, which is impossible.

We now show that any of the expressions (4), (5) is a multiple of the other. In order to achieve this goal, we must show that 3 determinants are equal to 0. The determinant

$$\begin{vmatrix} \theta_h(v-1)+k & -k(1+\theta_h)\\ v & \theta_{h'}-k \end{vmatrix}$$

is equal to $\theta_h\theta_{h'}(v-1)+k\theta_{h'}-k(v-1)\theta_h-k^2+vk+vk\theta_h=\theta_h\theta_{h'}(v-1)+k(\theta_h+\theta_{h'})-k^2+vk$, the determinant

$$\begin{vmatrix} \theta_h(v-1)+k & k(v-k+\theta_h)\\ 1 & -\theta_{h'} \end{vmatrix}$$

is equal to $-\theta_h\theta_{h'}(v-1)-k\theta_{h'}-k(v-k)-k\theta_h=-\theta_h\theta_{h'}(v-1)-k(\theta_h+\theta_{h'})-kv+k^2$ and the determinant

$$\begin{vmatrix} -(1+\theta_h) & v-k+\theta_h\\ \theta_{h'}-k & -v\theta_{h'} \end{vmatrix}$$

is equal to $v\theta_{h'}+v\theta_h\theta_{h'}-(v-k)\theta'_h+k(v-k)-\theta_h\theta_{h'}+k\theta_h = (v-1)\theta_h\theta_{h'}+k(\theta_h+\theta_{h'})+kv-k^2$. By equations (1) and (2), $(v-1)\theta_h\theta_{h'} + k(\theta_h+\theta_{h'}) + kv - k^2 = (v-1)(\mu-k) + k(\lambda-\mu) - k^2 + vk = \mu(v-1-k) - k(k-\lambda-1) = 0$. So, each of these three determinants is indeed equal to 0.

Now, every element of $\mathcal{F}$ intersects Y in a constant number of vertices (i.e. k_Z is constant) if and only if the inequality (3) is an equality. By the above calculations, we know that this happens if and only if the expression (5) is equal to 0. But expression (5) is equal to 0 if and only if Y is a tight set of Type B (recall Proposition 3.8). ∎

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