Points and hyperplanes of the universal embedding space of the dual polar space $DW(5,q)$, q odd

B. N. Cooperstein and B. De Bruyn*

June 14, 2008

Abstract

In [10], one of the authors proved that there are 6 isomorphism classes of hyperplanes in the dual polar space $DW(5,q)$, q even, which arise from its Grassmann-embedding. In the present paper, we extend these results to the case that q is odd. Specifically, we determine the orbits of the full automorphism group of $DW(5,q)$, q odd, on the projective points (or equivalently, the hyperplanes) of the projective space $PG(13,q)$ which affords the universal embedding of $DW(5,q)$.

Keywords: (symplectic) dual polar space, hyperplane, Grassmann-embedding
MSC2000: 51A45, 51A50

1 Introduction

A *partial linear rank two incidence geometry*, also called a *point-line geometry*, is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ whose elements are called *points* and a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$ whose elements are called *lines*, such that any two distinct points are contained in at most 1 line. The *point-collinearity graph* of Γ is the graph with vertex set $\mathcal{P}$ where two points are adjacent if they are collinear, that is, lie on a common line. By a

*The research of the second author is supported by a grant of the Fund for Scientific Research - Flanders (Belgium)

subspace of Γ we mean a subset S of $\mathcal{P}$ such that if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subset S$. A subspace S is *singular* provided each pair of points in S is collinear, that is, S is a clique in the collinearity graph of Γ. $(\mathcal{P}, \mathcal{L})$ is said to be *Gamma space* (see [13]) if, for every $x \in \mathcal{P}$, $\{x\} \cup \Gamma(x)$ is a subspace. A subspace $S \neq \mathcal{P}$ is a *geometric hyperplane* if it meets every line.

Let e be a positive integer, p a prime and V a 6-dimensional vector space over the finite field $\mathbb{F}_q$, $q = p^e$, equipped with a non-degenerate alternating form f. Then every vector $\bar{u} \in V$ is *isotropic*, that is, satisfies $f(\bar{u}, \bar{u}) = 0$. A subspace U of V is called *totally isotropic* (*with respect to* f) if $f(\bar{u}_1, \bar{u}_2) = 0$ for all $\bar{u}_1, \bar{u}_2 \in U$.

Associated with (V, f) is a polar space which is denoted by $W(5, q)$. Here, by a *polar space* we shall mean a point-line geometry (P, L) which satisfies the following properties:

1. (P, L) is a Gamma space and for every point p and line l, p is collinear with some point of l (this means that p is collinear with 1 or all points of l).

2. No point p is collinear with every other point; and

3. There is an integer n called the *rank* of (P, L) such that if $S_0 \subset S_1 \subset \cdots \subset S_k$ is a properly ascending chain of singular subspaces then $k \leq n$. When $n = 2$ (P, L) is said to be a *generalized quadrangle*.

The points, respectively lines, of $W(5, q)$ are the 1-dimensional, respectively 2-dimensional, subspaces of V which are totally isotropic with respect to f and incidence is containment. In $W(5, q)$ two points $\langle \bar{u}_1 \rangle_V$ and $\langle \bar{u}_2 \rangle_V$ are collinear if and only if $f(\bar{u}_1, \bar{u}_2) = 0$, i.e. if and only if $\bar{u}_1$ and $\bar{u}_2$ are orthogonal.

Also associated with the alternating form f of V, there is a *dual polar space* $DW(5, q)$. The points, respectively lines, of $DW(5, q)$ are the 3-spaces, respectively 2-spaces, of V which are totally isotropic with respect to f and incidence is reverse containment. We denote the point-set and line-set of $DW(5, q)$ by $\mathcal{P}$ and $\mathcal{L}$, respectively. In the incidence system $(\mathcal{P}, \mathcal{L})$ two "points" U_1 and U_2 are collinear if and only if $\dim(U_1 \cap U_2) = 2$. More generally, one can say that the distance $\mathrm{d}(U_1, U_2)$ (in the collinearity graph of $(\mathcal{P}, \mathcal{L})$) between two points U_1 and U_2 of $DW(5, q)$ is equal to $3 - \dim(U_1 \cap U_2)$. The lines of the dual polar space $DW(5, q)$ are maximal singular subspaces and consequently, this geometry is also a Gamma space.

Alternatively, the geometries (P, L) and $(\mathcal{P}, \mathcal{L})$ can be defined as Lie incidence geometries (see [4]) making use of a construction of Gamma spaces from a symmetrical orbital (orbit of the Symplectic group on the Cartesian products P^2 or $\mathcal{P}^2$ (see [13]).

By Shult and Yanushka [21] or Cameron [1], the set of totally isotropic 3-spaces of V which contain a given 1-space of V is a convex subspace of diameter 2 of $DW(5, q)$. Such a convex subspace is called a *quad* of $DW(5, q)$. The points and lines contained in a quad define a generalized quadrangle which is isomorphic to the classical generalized quadrangle $Q(4, q)$ (Payne and Thas [16, Section 3.1]).

This paper is concerned with classifying all the geometric hyperplanes of $DW(5, q)$, q odd, which arise from an embedding (defined below). We will show (See Main Theorem) that there are always six isomorphism classes of such hyperplanes.

The notion of a geometric hyperplane was introduced by Veldkamp (see [23], [24]) in his characterization of polar geometries for the explicit purpose of proving that such a geometry is embeddable. Geometric hyperplanes have been studied in many other contexts as well: for example, they arise in the classification by Cohen and Shult of the affine polar spaces (see [3]), in Cuypers' characterization of the graph on 2300 vertices with automorphism group Co_2, the second Conway group ([8]). Often by removing a geometric hyperplane with certain properties from an incidence geometry one can create interesting affine geometries and this was the motivation of Pasini and Shpectorov [15] in studying uniform hyperplanes in dual polar spaces as well as Cooperstein and Pasini [7] in proving that ovoidal hyperplanes do not exist in $DW(5, q)$.

The research carried out in the present paper is part of the larger project of classifying all hyperplanes of finite dual polar spaces of small rank. A complete classification of all hyperplanes of the Hermitian dual polar space $DH(5, q^2)$ was obtained in De Bruyn and Pralle [11], [12]. All hyperplanes of the dual polar space $DQ^-(7, q)$ arising from an embedding were classified in De Bruyn [9]. The classification of all hyperplanes of the dual polar spaces $DQ(6, q)$ and $DQ(8, q)$ which arise from their spin-embeddings was obtained in Cardinali, De Bruyn & Pasini [2], De Bruyn [9], Shult [19] and Shult & Thas [20]. A complete list of all hyperplanes of $DW(5, q)$, q even, arising from an embedding was given in Pralle [17] (for $q = 2$) and De Bruyn [10]

(for arbitrary q even).

2 Technical description of the results

2.1 The Grassmann-embedding of $DW(5,q)$

We continue with the notation introduced in Section 1. Choose a basis $\mathcal{S} = \{\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\}$ in V such that $f(\bar{v}_i, \bar{w}_i) = 1$ and $f(\bar{v}_i, \bar{v}_j) = f(\bar{w}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j) = 0$ for all $i, j \in \{1,2,3\}$ with $i \neq j$. Let $W := \bigwedge^3 V$ be the third exterior product of V which is a vector space of dimension $\binom{6}{3} = 20$ over $\mathbb{F}_q$. Define now a bilinear form $g(\cdot,\cdot)$ from $W \times W$ to $\mathbb{F}_q$ by setting $\alpha \wedge \beta$ equal to $g(\alpha,\beta)$ $(\bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{v}_3 \wedge \bar{w}_3)$ for all $\alpha, \beta \in W$. Since $(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) \wedge (\bar{u}_4 \wedge \bar{u}_5 \wedge \bar{u}_6) = (-1)^9(\bar{u}_4 \wedge \bar{u}_5 \wedge \bar{u}_6) \wedge (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)$ for all vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_6 \in V$, the form $g(\cdot,\cdot)$ is alternative. Obviously, it is also non-degenerate.

For every point $x = \langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle_V$ of $DW(5,q)$, let $\epsilon(x)$ denote the 1-space $\langle \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \rangle_W$ of $W = \bigwedge^3 V$. This 1-space is independent from the generating set $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ of x. It is well-known that the subspace M of W generated by all 1-spaces $\epsilon(x)$, $x \in \mathcal{P}$, is 14-dimensional. One readily verifies that a basis of M is given by the set $\mathcal{S}_M := \{p_i \mid 1 \leq i \leq 14\}$, where

$$p_1 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3,\ p_2 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3,\ p_3 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3,\ p_4 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3,$$

$$p_5 = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3,\ p_6 = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3,\ p_7 = \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3,\ p_8 = \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3,$$

$$p_9 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 - \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{w}_3, \quad p_{10} = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 - \bar{w}_1 \wedge \bar{v}_3 \wedge \bar{w}_3,$$

$$p_{11} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_2 - \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_3, \quad p_{12} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{w}_2 - \bar{w}_2 \wedge \bar{v}_3 \wedge \bar{w}_3,$$

$$p_{13} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_3 - \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{v}_3, \quad p_{14} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{w}_3 - \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{w}_3.$$

For all $i, j \in \{1, \ldots, 14\}$, $g(p_i, p_j) = 0$, except when $\{i,j\}$ is equal to $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$, $\{9,10\}$, $\{11,12\}$ or $\{13,14\}$. Hence, the form $g(\cdot,\cdot)$ defines a non-degenerate alternating form in the 14-space M. For every subspace U of M, let $U^{\perp_g} = \{m \in M \mid g(u,m) = 0 \text{ for all } u \in U\}$.

The map ϵ defines a full projective embedding of the dual polar space $DW(5,q)$ into the projective space $\mathrm{PG}(M) \cong \mathrm{PG}(13,q)$. This embedding is called the *Grassmann-embedding* of $DW(5,q)$. If $q \neq 2$, then by results

of Cooperstein [5] and Kasikova & Shult [14], we know that the Grassmann-embedding of $DW(5,q)$ is absolutely universal (Ronan [18]). This implies that all full embeddings of $DW(5,q)$, $q \neq 2$, can be obtained from its Grassmann-embedding by taking so-called quotients.

If π is a hyperplane of $\mathrm{PG}(M)$, then $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \pi)$ is a (geometric) hyperplane of $DW(5,q)$, namely a proper subset of $\mathcal{P}$ intersecting each line of $DW(5,q)$ in either a unique point or the whole line. We will say that the hyperplane $\epsilon^{-1}(\epsilon(\mathcal{P}) \cap \pi)$ *arises from the embedding* ϵ.

2.2 The automorphism groups of $W(5,q)$ and $DW(5,q)$

Before proceeding to our main theorem we describe the automorphism groups of $W(5,q)$ and $DW(5,q)$. Suppose θ is a permutation of the point-set of $W(5,q)$. Then θ will be an automorphism of $W(5,q)$ if and only if it induces a permutation on the set of all ordered pairs of distinct collinear points of $W(5,q)$. Similarly, a permutation of $\mathcal{P}$ will be an automorphism of $DW(5,q)$ if and only if it induces a permutation of the set of all ordered pairs of distinct collinear points of $DW(5,q)$. It is not difficult to see that automorphism groups of $DW(5,q)$ and $W(5,q)$ are isomorphic.

That automorphisms of $W(5,q)$ induce automorphisms of $DW(5,q)$ is fairly straightforward. That automorphisms of $DW(5,q)$ induce automorphisms of $W(5,q)$ follows from the fact that the quads of $DW(5,q)$ are characterized as the convex subspaces of diameter 2 and that these are in one-to-one correspondence with the points of $W(5,q)$. We proceed to describe the group $Aut(W(5,q)) \cong Aut(DW(5,q))$.

Recall that $\mathcal{S} = \{\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\}$ is a basis of V such that $f(\bar{v}_i, \bar{w}_i) = 1$ and $f(\bar{v}_i, \bar{v}_j) = f(\bar{w}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j) = 0$ for all $i, j \in \{1,2,3\}$ with $i \neq j$. A *similarity* of (V,f) is a linear transformation $\sigma \in GL(V)$ such that $f(\sigma(\bar{u}_1), \sigma(\bar{u}_2)) = \lambda_\sigma \cdot f(\bar{u}_1, \bar{u}_2)$ for all $\bar{u}_1, \bar{u}_2 \in V$. Here λ_σ is a non-zero scalar which depends on σ but is independent of $\bar{u}_1$ and $\bar{u}_2$. We denote by $G_f \leq GL(V)$ the group of all similarities. An *isometry* is a similarity σ with $\lambda_\sigma = 1$. We denote by S_f the group of all isometries. S_f is normal in G_f and isomorphic to $Sp(6, \mathbb{F}_q)$. Clearly similarities induce automorphisms of $W(5,q)$. The kernel of the action of G_f on P is the center of G_f and consists of all the scalar transformations $\lambda \cdot I_V$, where λ is a non-zero scalar. Denote by PG_f the quotient $G_f/Z(G_f)$ and by PS_f the quotient of S_f by

$S_f \cap Z(G_f)$. We note that $S_f \cap Z(G_f) = Z(S_f) = \langle -I_V \rangle$ and therefore $S_f Z(G_f)/Z(G_f) \cong S_f/Z(S_f)$. Thus, we may consider PS_f to be a subgroup of PG_f. The group PS_f is the simple group $PSp(6, \mathbb{F}_q)$. The index of PS_f in PG_f is two (see Steinberg [22]). If σ^* is the linear transformation of V which fixes $\bar{v}_i$, $i = 1, 2, 3$ and takes $\bar{w}_i$ to $d\bar{w}_i$, $i = 1, 2, 3$ with d a given non-square in $\mathbb{F}_q$ then $\sigma^* \in G_f \setminus S_f$ and consequently, $PG_f = PS_f\langle\sigma^*\rangle$. This describes the automorphisms of $W(5, q)$ which are induced by linear transformations of V. In addition, there are so-called "field automorphisms".

For $\bar{u} \in V$ denote by $[\bar{u}]_{\mathcal{S}}$ the coordinate vector of $\bar{u}$ with respect to the basis $\mathcal{S}$ of V. For every $\gamma \in Aut(\mathbb{F}_q)$, define a map $T_\gamma : V \to V$ by $[T_\gamma(\bar{v})]_{\mathcal{S}} = \gamma([\bar{v}]_{\mathcal{S}})$. Then T_γ induces a permutation of the point-set of $W(5, q)$ which preserves orthogonality and therefore induces an automorphism of $W(5, q)$. If $A = \{T_\gamma \,|\, \gamma \in Aut(\mathbb{F}_q)\}$, then $Aut(W(5, q)) = PG_f A = PS_f\langle\sigma^*\rangle A$.

2.3 The main results

Every element θ of G_f gives rise to a unique element $\theta' \in GL(\bigwedge^3 V)$ such that $\theta'(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. Obviously, θ' fixes M and hence gives rise to an element $\widehat{\theta} \in GL(M)$. For all $\alpha, \beta \in W = \bigwedge^3 V$, we have

$$g(\theta'(\alpha), \theta'(\beta)) = \det(\theta) \cdot g(\alpha, \beta). \tag{1}$$

Hence, $\widehat{\theta}$ is a similarity of (M, g). Now, define $\widehat{G_f} := \{\widehat{\theta} \,|\, \theta \in G_f\}$ and $\widehat{S_f} := \{\widehat{\theta} \,|\, \theta \in S_f\}$. By (1),

$$(\phi(U))^{\perp_g} = \phi(U^{\perp_g}) \tag{2}$$

for every $\phi \in \widehat{G_f}$ and every subspace U of M.

Suppose now that $\gamma \in Aut(\mathbb{F}_q)$. Let $\mathcal{B}$ be the basis $\{\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3\} \cup \{\bar{v}_i \wedge \bar{v}_j \wedge \bar{w}_k, \bar{v}_k \wedge \bar{w}_i \wedge \bar{w}_j \,|\, 1 \le i, j, k \le 3, i < j\}$ of W. Let T'_γ be the $\mathbb{F}_p$-linear map of W defined by $[T'_\gamma(\alpha)]_{\mathcal{B}} = \gamma([\alpha]_{\mathcal{B}})$. We have $T'_\gamma(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = T_\gamma(\bar{u}_1) \wedge T_\gamma(\bar{u}_2) \wedge T_\gamma(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. For all $\alpha, \beta \in W$, we have

$$g(T'_\gamma(\alpha), T'_\gamma(\beta)) = \gamma(g(\alpha, \beta)). \tag{3}$$

T'_γ fixes each of the vectors of the basis $\mathcal{S}_M$ of M and hence induces an $\mathbb{F}_p$-linear map $\widehat{T_\gamma} : M \to M$. By (3),

$$(\widehat{T_\gamma}(U))^{\perp_g} = \widehat{T_\gamma}(U^{\perp_g}) \tag{4}$$

for every subspace U of M.

Let $\overline{G_f}$ (respectively $\widehat{\overline{G_f}}$) denote the group of $\mathbb{F}_p$-linear maps of V (respectively W) generated by G_f and T_γ, $\gamma \in Aut(\mathbb{F}_q)$ (respectively $\widehat{G_f}$ and $\widehat{T_\gamma}$, $\gamma \in Aut(\mathbb{F}_q)$). By the discussion above, for every $\theta \in \overline{G_f}$, there exists a unique $\mathbb{F}_p$-linear map $\theta' : W \to W$ such that $\theta'(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. The map θ' stabilizes M and hence induces an $\mathbb{F}_p$-linear map $\widehat{\theta} : M \to M$. Obviously, $\widehat{\theta} \in \widehat{\overline{G_f}}$. Moreover, the map $\theta \mapsto \widehat{\theta}$ is an isomorphism between the groups $\overline{G_f}$ and $\widehat{\overline{G_f}}$.

It is the main purpose of this paper to determine the orbits of the group $Aut(DW(5,q))$ on the hyperplanes of $DW(5,q)$, q odd, which arise from its Grassmann-embedding. Since the Grassmann-embedding of $DW(5,q)$, q odd, is absolutely universal, the hyperplanes of $DW(5,q)$, q odd, arising from the Grassmann-embedding are all the hyperplanes of that dual polar space which arise from an embedding.

Determining the orbits of $Aut(DW(5,q))$ on the hyperplanes of $DW(5,q)$ is equivalent to the enumeration of all $\widehat{\overline{G_f}}$-orbits on the hyperplanes of M. By equations (2) and (4), this is equivalent to enumerating the orbits of $\widehat{\overline{G_f}}$ on the 1-spaces of M, i.e. the points of $\mathrm{PG}(M)$. We will achieve our objective by first enumerating the orbits of $\widehat{S_f}$ on the 1-spaces of M and then determining when these $\widehat{S_f}$-orbits fuse when we extend the group to all of $Aut(DW(5,q))$.

Before stating our Main Theorem, we need to define some extra vectors in M. Unless otherwise stated we will always assume in the sequel that q is an odd prime power. Let $d \in \mathbb{F}_q$ such that d is a non-square and if -1 is a non-square, then we take d equal to -1. Define the following additional vectors of M:

$$p_{15} = p_1 + p_4,\ p_{16} = p_1 + dp_4,\ p_{17} = p_1 + p_4 + p_6,\ p_{18} = p_1 + p_8,\ p_{19} = p_1 + dp_8,$$

$$p_{20} = dp_1 + p_4 + p_6 + p_7,\ p_{21} = dp_2 + dp_3 + dp_5 + p_8.$$

Also, set $P_i = \langle p_i \rangle_W$ and $H_i = \epsilon^{-1}(P_i^{\perp_g} \cap \epsilon(\mathcal{P}))$ for every $i \in \{1, \ldots, 21\}$. We can now state our main theorem:

Type	Representative	Orbit size
I	H_1	$(q^3+1)(q^2+1)(q+1)$
II	H_{15}	$\frac{q^6-1}{q-1}\frac{q^2(q^2+1)}{2}$
III	H_{16}	$\frac{q^6-1}{q-1}\frac{q^2(q^2-1)}{2}$
IV	H_{17}	$q^3(q^6-1)(q^2+1)(q+1)$
V	H_{18}	$\frac{q^6(q^4-1)(q^3+1)}{2}$
VI	H_{20}	$\frac{q^6(q^4-1)(q^3-1)}{2}$

Table 1: The orbits of $Aut(DW(5,q))$, q odd, on the geometric hyperplanes of $DW(5,q)$.

Main Theorem. *Let q be an odd prime power. Then the group $Aut(DW(5,q))$ has six orbits on the geometric hyperplanes of $DW(5,q)$ which arise from an embedding with representatives H_1, H_{15}, H_{16}, H_{17}, H_{18} and H_{20}. The sizes of the orbits are given in Table* 1.

The Main Theorem is a consequence of the following two results, which we will prove in Sections 3 and 4.

Point Enumeration Theorem (*i*) *If -1 is a non-square in $\mathbb{F}_q$, q odd, then the group $\widehat{S_f}$ has six orbits on the point-set of* $\mathrm{PG}(M)$ *with representatives P_1, P_{15}, P_{16}, P_{17}, P_{18}, and P_{20}. The orbit sizes and the stabilizers of a representative are given in Table* 2.

(*ii*) *If -1 is a square in $\mathbb{F}_q$, q odd, then the group $\widehat{S_f}$ has eight orbits on the point-set of* $\mathrm{PG}(M)$ *with representatives P_1, P_{15}, P_{16}, P_{17}, P_{18}, P_{19}, P_{20} and P_{21}. The orbit sizes and stabilizers are given in Table* 3.

To prove the Point Enumeration Theorem we will show in both cases that the conjectured representatives given in the tables are all in different orbits, compute their stabilizers and hence their orbit sizes. Since in both cases the sum of the orbit sizes is $\frac{q^{14}-1}{q-1}$ it will follow that we have enumerated all the $\widehat{S_f}$-orbits on the points of M.

Fusion Theorem (*i*) *Assume that -1 is a non-square in $\mathbb{F}_q$ with q odd. Then the automorphisms of $DW(5,q)$ induced by σ^* and T_γ, $\gamma \in Aut(\mathbb{F}_q)$, fix each of the $\widehat{S_f}$-orbits of the hyperplanes H_1, H_{15}, H_{16}, H_{17}, H_{18}, and H_{20}.*

Type	Representative	Orbit size	Stabilizer
I	P_1	$(q^3+1)(q^2+1)(q+1)$	$q^6.GL(3,q)$
II	P_{15}	$\frac{q^6-1}{q-1}\frac{q^2(q^2+1)}{2}$	$q^5.SL(2,q)\times SL(2,q)\times \mathbb{Z}_{q-1}.2$
III	P_{16}	$\frac{q^6-1}{q-1}\frac{q^2(q^2-1)}{2}$	$q^5.SL(2,q^2)\times \mathbb{Z}_{q-1}.2$
IV	P_{17}	$q^3(q^6-1)(q^2+1)(q+1)$	$q^5.SL(2,q)*\mathbb{Z}_{q-1}.2$
V	P_{18}	$\frac{q^6(q^4-1)(q^3+1)}{2}$	$\mathbb{Z}_2\times SL(3,q)$
VI	P_{20}	$\frac{q^6(q^4-1)(q^3-1)}{2}$	$\mathbb{Z}_2\times SU(3,q)$

Table 2: The $\widehat{S_f}$-orbits on the points of PG(M): the case that -1 is a non-square in $\mathbb{F}_q$.

Type	Representative	Orbit size	Stabilizer
I	P_1	$(q^3+1)(q^2+1)(q+1)$	$q^6.GL(3,q)$
II	P_{15}	$\frac{q^6-1}{q-1}\frac{q^2(q^2+1)}{2}$	$q^5.SL(2,q)\times SL(2,q)\times \mathbb{Z}_{q-1}.2$
III	P_{16}	$\frac{q^6-1}{q-1}\frac{q^2(q^2-1)}{2}$	$q^5.SL(2,q^2)\times \mathbb{Z}_{q-1}.2$
IV	P_{17}	$q^3(q^6-1)(q^2+1)(q+1)$	$q^5.SL(2,q)*\mathbb{Z}_{q-1}.2$
Va	P_{18}	$\frac{q^6(q^4-1)(q^3+1)}{4}$	$\mathbb{Z}_2\times SL(3,q).2$
Vb	P_{19}	$\frac{q^6(q^4-1)(q^3+1)}{4}$	$\mathbb{Z}_2\times SL(3,q).2$
VIa	P_{20}	$\frac{q^6(q^4-1)(q^3-1)}{4}$	$\mathbb{Z}_2\times SU(3,q).2$
VIb	P_{21}	$\frac{q^6(q^4-1)(q^3-1)}{4}$	$\mathbb{Z}_2\times SU(3,q).2$

Table 3: The $\widehat{S_f}$-orbits on the points of PG(M): the case that -1 is a square in $\mathbb{F}_q$.

(*ii*) *Assume that* -1 *is a square in* $\mathbb{F}_q$ *with* q *odd. Then the automorphisms of* $DW(5,q)$ *induced by* σ^* *and* T_γ, $\gamma \in Aut(\mathbb{F}_q)$, *fix each of the* $\widehat{S_f}$*-orbits of the hyperplanes* H_1, H_{15}, H_{16} *and* H_{17}. *On the other hand, the* $\widehat{S_f}$*-orbits of* H_{18} *and* H_{19} *become a single orbit as do the* $\widehat{S_f}$*-orbits of* H_{20} *and* H_{21}.

The Main Theorem classifies all hyperplanes of $DW(5,q)$, q odd, arising from an embedding. As previously mentioned, all hyperplanes of $DW(5,q)$, q even, arising from an embedding were already classified in Pralle [17] (for $q = 2$ with the aid of the computer) and De Bruyn [10] (for arbitrary $q = 2^m$ without the use of the computer).

Several combinatorial properties of the hyperplanes of $DW(5,q)$, q odd, arising from an embedding were already obtained by the authors in [6]. For each hyperplane H of $DW(5,q)$, q odd, they determined by purely combinatorial and geometrical techniques the total number of quads Q for which $Q \cap H$ is a certain configuration of points in Q and the total number of points x for which $\Delta(x) \cap H$ is a certain configuration of points in $\Delta(x)$. Here, $\Delta(x)$ denotes the set of points equal to or collinear with x. On basis of these combinatorial properties, the authors were able to divide the set of hyperplanes of $DW(5,q)$, q odd, into 6 classes: Type I-hyperplanes, Type II-hyperplanes, ..., Type VI-hyperplanes. This terminology is consistent with the one used in the present paper. By our Main Theorem, each of the 6 classes defined in [6] is actually an isomorphism class, except when -1 is a non-square in $\mathbb{F}_q$. Then the Type VI hyperplanes (which all have the same combinatorial properties mentioned above) split into 2 isomorphism classes: the Type VIa hyperplanes and the Type VIb hyperplanes.

3 Proof of the Point Enumeration Theorem

3.1 Notations and a few lemmas

We will continue with the notations introduced in Sections 1 and 2.

Let $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\delta^2 = d$. We may suppose that (i) $\bar{w}_i = \delta \bar{v}_i$ for every $i \in \{1,2,3\}$ and (ii) V is a 3-dimensional vector space over $\mathbb{F}_{q^2}$ with basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and a 6-dimensional vector space over $\mathbb{F}_q$ with basis

$\mathcal{S} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3\}$. Recall that $\bigwedge^3 V$ must be regarded as the third exterior power of V as a vector space over the field $\mathbb{F}_q$.

Lemma 3.1 *If τ is an $\mathbb{F}_{q^2}$-linear transformation of V with $\det(\tau) = 1$, then $\widehat{\tau}$ centralizes the vectors p_{20} and p_{21}.*

Proof. Let E_{ij} denote the $(3 \times 3)-$matrix with a "1" in the $(i,j)-$entry and 0's elsewhere and set $\chi_{ij} = \{I_3 + \alpha E_{ij} \,|\, \alpha \in \mathbb{F}_{q^2}\}$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. Also, set $w_1 = E_{12} - E_{21} + E_{33}$ and $w_2 = E_{11} + E_{23} - E_{32}$. Then the group $SL(3, q^2)$ is generated by χ_{13}, w_1 and w_2. So, it suffices to prove that the induced action of each of these centralizes p_{20} and p_{21}.

Let $\alpha = a + b\delta$ where $a, b \in \mathbb{F}_q$ and suppose τ is the $\mathbb{F}_{q^2}$-linear transformation of V whose associated matrix with respect to the basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is equal to $I_3 + \alpha E_{13}$. Then the matrix of τ with respect to $\mathcal{S}$ is $\begin{pmatrix} A & dB \\ B & A \end{pmatrix}$, where $A = I_3 + aE_{13}$ and $B = bE_{13}$. It is now quite straightforward to compute the induced action of τ on p_{20} and p_{21}. $\widehat{\tau}(p_{20})$ is equal to

$$\begin{aligned}
& \widehat{\tau}(d\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3) \\
= \;& d[\bar{v}_1 \wedge \bar{v}_2 \wedge (a\ \bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] + \bar{v}_1 \wedge \bar{w}_2 \wedge ((db)\ \bar{v}_1 + a\bar{w}_1 + \bar{w}_3) \\
& + \bar{w}_1 \wedge \bar{v}_2 \wedge ((db)\ \bar{v}_1 + a\bar{w}_1 + \bar{w}_3) + \bar{w}_1 \wedge \bar{w}_2 \wedge (a\ \bar{v}_1 + \bar{v}_3 + b\bar{w}_1) \\
= \;& d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (db)\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + a\ \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1 \\
& + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + (db)\ \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 + a\ \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 \\
= \;& p_{20},
\end{aligned}$$

since $\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 = -\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1$ and $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 = -\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1$.

Similarly, $\widehat{\tau}(p_{21})$ is equal to

$$\begin{aligned}
& \widehat{\tau}(d\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + d\ \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 + d\ \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3) \\
= \;& d\ [\bar{v}_1 \wedge \bar{v}_2 \wedge ((db)\ \bar{v}_1 + a\bar{w}_1 + \bar{w}_3)] + d\ [\bar{v}_1 \wedge \bar{w}_2 \wedge (a\ \bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] \\
& + d\ [\bar{w}_1 \wedge \bar{v}_2 \wedge (a\ \bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] + \bar{w}_1 \wedge \bar{w}_2 \wedge ((db)\ \bar{v}_1 + a\bar{w}_1 + \bar{w}_3) \\
= \;& (da)\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1 + d\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + (db)\ \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1 + d\ \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 \\
& + (da)\ \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 + d\ \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (db)\ \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 \\
= \;& p_{21}.
\end{aligned}$$

The matrix of w_1 with respect to $\mathcal{S}$ is $\begin{pmatrix} A & O \\ O & A \end{pmatrix}$ where $A = E_{12} - E_{21} + E_{33}$ and O is the (3×3)-matrix with all entries equal to 0. $\widehat{w_1}(p_{20})$ is equal to

$$\begin{aligned} &= \widehat{w_1}(d\ \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3) \\ &= d\ (-\bar{v}_2) \wedge \bar{v}_1 \wedge \bar{v}_3 + (-\bar{v}_2) \wedge \bar{w}_1 \wedge \bar{w}_3 + (-\bar{w}_2) \wedge \bar{v}_1 \wedge \bar{w}_3 + (-\bar{w}_2) \wedge \bar{w}_1 \wedge \bar{v}_3 \\ &= p_{20}, \end{aligned}$$

since $(-\bar{v}_2) \wedge \bar{v}_1 = \bar{v}_1 \wedge \bar{v}_2$, $(-\bar{w}_2) \wedge \bar{w}_1 = \bar{w}_1 \wedge \bar{w}_2$, $(-\bar{v}_2) \wedge \bar{w}_1 = \bar{w}_1 \wedge \bar{v}_2$ and $(-\bar{w}_2) \wedge \bar{v}_1 = \bar{v}_1 \wedge \bar{w}_2$.

In an entirely similar way, one shows that $\widehat{w_1}(p_{21}) = p_{21}$, $\widehat{w_2}(p_{20}) = p_{20}$ and $\widehat{w_2}(p_{21}) = p_{21}$. □

Recall that every point $x \in \mathcal{P}$ gives rise to a 1-space $\epsilon(x)$ of M, i.e. a point $\epsilon(x)$ of $\mathrm{PG}(M)$. For a line $l \in \mathcal{L}$, we define $\epsilon(l) := \{\epsilon(x) \,|\, x \in l\}$. We denote by $\tilde{l}$ the 2-space of M generated by the 1-spaces $\epsilon(x)$, $x \in l$. We put $\widetilde{\mathcal{P}} = \widehat{\mathcal{P}} = \{\epsilon(x) \,|\, x \in \mathcal{P}\}$, $\widehat{\mathcal{L}} = \{\epsilon(l) \,|\, l \in \mathcal{L}\}$ and $\widetilde{\mathcal{L}} = \{\tilde{l} \,|\, l \in \mathcal{L}\}$.

Let X be a point of $\mathrm{PG}(V)$. By abuse of notation, we will also write $X \in \mathrm{PG}(V)$. The set $Q(X) = \{x \in \mathcal{P} \,|\, X \subset x \subset X^\perp\}$ is a convex subspace of $DW(5,q)$ which defines a generalized quadrangle isomorphic to $Q(4,q)$. We set $\mathcal{Q} := \{Q(X) \,|\, X \in \mathrm{PG}(V)\}$ and refer to the elements of $\mathcal{Q}$ as *quads* of $DW(5,q)$. For $Q \in \mathcal{Q}$, we will denote by $\widehat{Q}$ the collection $\{\epsilon(x) \,|\, x \in Q\}$ and by $\widetilde{Q}$ the subspace of M spanned by the elements of $\widehat{Q}$. We refer to both $\widehat{Q}$ and $\widetilde{Q}$ as the *quads* of M. Set $\widetilde{\mathcal{Q}} = \{\widetilde{Q} \,|\, Q \in \mathcal{Q}\}$.

For every point u of $DW(5,q)$, $\Delta(u)$ denotes the set of points of $DW(5,q)$ collinear or equal to u. If $P = \epsilon(u) \in \widetilde{\mathcal{P}}$, then we define $\Delta(P) := \epsilon(\Delta(u))$ and $M(P)$ is the subspace of M spanned by the elements of $\Delta(P)$. We call $M(P)$ the *hemisphere* of P.

Lemma 3.2 *Let $X, Y \in \mathrm{PG}(V)$. Then the following holds:*
(i) If $X \perp_f Y$, then $\widetilde{Q}(X) \cap \widetilde{Q}(Y) \in \widetilde{\mathcal{L}}$.
(ii) If X and Y are not orthogonal, then $\widetilde{Q}(X) \cap \widetilde{Q}(Y) = 0$.

Proof. (i) The group $\overline{G_f}$ is transitive on pairs $\{X, Y\}$ of 1-spaces of V such that $X \perp_f Y$. Therefore we can take $X = \langle \bar{v}_1 \rangle$ and $Y = \langle \bar{v}_2 \rangle$. Then $\widetilde{Q}(X) = \langle p_1, p_2, p_3, p_4, p_9 \rangle$, $\widetilde{Q}(Y) = \langle p_1, p_2, p_5, p_6, p_{11} \rangle$ and $\widetilde{Q}(X) \cap \widetilde{Q}(Y) = \langle p_1, p_2 \rangle \in \widetilde{\mathcal{L}}$.

(ii) The group $\overline{G_f}$ is also transitive on pairs $\{X, Y\}$ of 1-spaces of V such that X and Y are non-orthogonal with respect to f. We can take $X = \langle \bar{v}_1 \rangle, Y = \langle \bar{w}_1 \rangle$. Now, $\widetilde{Q}(X) = \langle p_1, p_2, p_3, p_4, p_9 \rangle$ and $\widetilde{Q}(Y) = \langle p_5, p_6, p_7, p_8, p_{10} \rangle$. Hence, $\widetilde{Q}(X) \cap \widetilde{Q}(Y) = 0$ as claimed. □

This result implies the next corollary which is fundamental:

Corollary 3.3 *Let $\widetilde{Q} \in \widetilde{\mathcal{Q}}$ and $P \in \mathrm{PG}(\widetilde{Q}) \setminus \widetilde{\mathcal{P}}$. Then $\widetilde{Q}$ is the unique quad of M which contains P.*

Lemma 3.4 *Let $P \in \widetilde{\mathcal{P}}$ and $Q \in \widetilde{\mathcal{Q}}$ such that $P \notin Q$. Let R denote the unique point of $Q \cap \widetilde{\mathcal{P}}$ at distance 1 from P. Then $M(P) \cap Q = R$.*

Proof. Since $\widehat{\overline{G_f}}$ is transitive on the pairs (P, Q) with $P \in \widetilde{\mathcal{P}}$, $Q \in \widetilde{\mathcal{Q}}$ and $P \notin Q$, we may without loss of generality suppose that $Q = \widetilde{Q}(\langle \bar{v}_1 \rangle)$ and $P = \langle p_8 \rangle$. Then $R = \langle p_4 \rangle$. Now, $Q = \langle p_1, p_2, p_3, p_4, p_9 \rangle$ and $M(P) = \langle p_4, p_6, p_7, p_8, p_{10}, p_{12}, p_{14} \rangle$ and hence $M(P) \cap Q = \langle p_4 \rangle = R$. □

Corollary 3.5 *Let $P \in \widetilde{\mathcal{P}}$ and $R \in \mathrm{PG}(M(P)) \setminus \widetilde{\mathcal{P}}$. If R is contained in a quad, then this quad necessarily contains P.*

Lemma 3.6 *Let $\widetilde{Q} \in \widetilde{\mathcal{Q}}$ and $R \in \mathrm{PG}(\widetilde{Q}) \setminus \widetilde{P}$. Then there exists a $P \in \widetilde{\mathcal{P}}$ such that $R \in \mathrm{PG}(M(P))$.*

Proof. Let $\widetilde{L}$ be contained in $\widetilde{Q}$ where $L \in \mathcal{L}$. Then $\widetilde{Q} = \bigcup_{P \in \widetilde{L}} \langle \widetilde{Q} \cap \Delta(P) \rangle \subset \bigcup_{P \in \widetilde{Q}} M(P)$. □

In our next lemma we will show that if a point is contained in two distinct hemispheres, in fact, it is contained in a quad.

Lemma 3.7 *Let P and P' be distinct points of $\widetilde{\mathcal{P}}$ and $X \in \mathrm{PG}(M(P) \cap M(P'))$. Then there is a quad $\widetilde{Q}$ containing P such that $X \subset \widetilde{Q}$.*

Proof. For every $t \in \{1, 2, 3\}$, $\widehat{\overline{G_f}}$ is transitive on the pairs (P, P') of points of $\widetilde{\mathcal{P}}$ with $d(P, P') = t$. Therefore we can take (P, P') to be one of $(P_1, P_2), (P_1, P_4), (P_1, P_8)$. For every $i \in \{1, 2, 4, 8\}$, set $M_i = M(P_i)$. Then

$$M_1 = \langle p_1, p_2, p_3, p_5, p_9, p_{11}, p_{13} \rangle, M_2 = \langle p_1, p_2, p_4, p_6, p_9, p_{11}, p_{14} \rangle,$$

$$M_4 = \langle p_2, p_3, p_4, p_8, p_9, p_{12}, p_{14}\rangle, M_8 = \langle p_4, p_6, p_7, p_8, p_{10}, p_{12}, p_{14}\rangle.$$

Now $M_1 \cap M_2 = \langle p_1, p_2, p_9, p_{11}\rangle$. This space is covered by $\bigcup \widetilde{Q}(\langle \bar{v}\rangle)$ where $\bar{v} \in \langle \bar{v}_1, \bar{v}_2\rangle$. $M_1 \cap M_4 = \langle p_2, p_3, p_9\rangle$ and this is contained in $\widetilde{Q}(\langle \bar{v}_1\rangle)$. Finally, $M_1 \cap M_8 = 0$. □

This also has an important corollary:

Corollary 3.8 *Assume $X \in \mathrm{PG}(M(P))$ for $P \in \widetilde{\mathcal{P}}$ and X is not contained in a quad which contains P. Then P is the unique point of $\widetilde{\mathcal{P}}$ for which $X \in \mathrm{PG}(M(P))$.*

3.2 Points contained in at least one hemisphere

We now show that the points P_1, P_{15}, P_{16} and P_{17} are in distinct orbits of $\widehat{S_f}$, with orbit sizes and stabilizers as shown in Tables 2 and 3. We also show that the union of these orbits constitute all points of $\mathrm{PG}(M)$ which are contained in at least one hemisphere.

The orbit of P_1 is just $\widetilde{\mathcal{P}}$. There are $(q^3+1)(q^2+1)(q+1)$ such points and the stabilizer $S_{P_1} := (\widehat{S_f})_{P_1}$ of P_1 is isomorphic to the subgroup of S_f which fixes a maximal totally isotropic subspace of V. The group S_{P_1} has a normal elementary Abelian subgroup $E(P_1)$ of order q^6. This subgroup has a complement $L(P_1) \cong GL(3,q)$. This justifies the entries of line I of the Tables 3 and 4.

For a point X of $\mathrm{PG}(V)$ the stabilizer in $\widehat{S_f}$ of $\widetilde{Q}(X)$ is isomorphic to $S_X := (S_f)_X$. The group S_X has a normal subgroup $E(X)$ of order q^5 which is a special group. This subgroup has a complement $L(X)$ which is isomorphic to $L(X)' \times Z(X)$, where $L(X)' \cong Sp(4,q)$ is the commutator subgroup of $L(X)$ and $Z(X) \cong \mathbb{Z}_{q-1}$. Note that $L(X)'/Z(L(X)') \cong \Omega(5,q)$. In fact, the group $L(X)$ preserves a quadratic form on $\widetilde{Q}(X)$ which we describe now.

Let $X = \langle \bar{v}_1\rangle$. Set $V(X) = \langle \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\rangle$. Note that $X \wedge \bigwedge^2(X^\perp) = X \wedge \bigwedge^2(V(X))$ has dimension six. We denote this space by $D(X)$. Any vector $\bar{v} \in D(X)$ and can be written as $\bar{v}_1 \wedge \alpha$ for $\alpha \in \bigwedge^2(V(X))$. Also, for $\alpha, \beta \in \bigwedge^2(V(X)), \alpha \wedge \beta$ is a multiple of $\bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3$. Thus, define $b : \bigwedge^2(V(X)) \times \bigwedge^2(V(X)) \to \mathbb{F}_q$ by $\alpha \wedge \beta = b(\alpha,\beta)\,(\bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3)$. This defines a non-degenerate symmetric bilinear form of Witt index 3. Now

define $\hat{b} : D(X) \times D(X) \to \mathbb{F}_q$ by $\hat{b}(\bar{v}_1 \wedge \alpha, \bar{v}_1 \wedge \beta) = b(\alpha, \beta)$. This also is a non-degenerate symmetric bilinear form of Witt index 3. The space $\widetilde{Q}(X)$ is the subspace of $D(X)$ which is orthogonal to $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 + \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{w}_3$ with respect to $\hat{b}$. The group $L(X)$ has three orbits on the projective points of $\widetilde{Q}(X)$: the singular points of the quadratic form $\hat{b}$, which are the points of $\widehat{Q}(X)$ and the two classes of non-singular points with respect to $\hat{b}$. Note that $\hat{b}(p_9, p_9) = \hat{b}(p_{15}, p_{15}) = 2$. Also, $p_9^{\perp_{\hat{b}}} = \langle p_1, p_2, p_3, p_4 \rangle$ which is a non-degenerate hyperbolic subspace of $(\widetilde{Q}(X), \hat{b})$. On the other hand, $\hat{b}(p_{16}, p_{16}) = 2d$ and since $\hat{b}(p_{15}, p_{15}) \cdot \hat{b}(p_{16}, p_{16}) = 4d$ a non-square, it follows that P_{15} and P_{16} are in different classes of non-singular points of $(\widetilde{Q}(X), \hat{b})$ and therefore representatives of the two classes. Since there are $\frac{q^6-1}{q-1}$ quads $Q(X)$ for $X \in \mathrm{PG}(V)$ and for each X there are $\frac{q^2(q^2+1)}{2}$ points in the class of P_{15} contained in $\widetilde{Q}(X)$ and $\frac{q^2(q^2-1)}{2}$ points in the class of P_{16} contained in $\widetilde{Q}(X)$ the entries of lines II and III of Tables 3 and 4 have now been justified.

We now make use of Corollary 3.3 and simple counting to show that for $P \in \widetilde{\mathcal{P}}$ there are points in $M(P)$ which are not from classes I, II and III.

Lemma 3.9 *The following holds for a point $P \in \widetilde{\mathcal{P}}$:*

(i) The number of points of type I in $\mathrm{PG}(M(P))$ *is* $1 + q(q^2 + q + 1)$.

(ii) The number of points of type II in $\mathrm{PG}(M(P))$ *is* $\frac{q^2(q^2+q+1)(q+1)}{2}$.

(iii) The number of points of type III in $\mathrm{PG}(M(P))$ *is* $\frac{q^2(q^2+q+1)(q-1)}{2}$.

(iv) There are $q^3(q^3-1)$ points in $\mathrm{PG}(M(P))$ *which do not belong to a quad.*

Proof. (i): The points of type I in $M(P)$ are precisely $\Delta(P)$. There are $q^2 + q + 1$ lines on P each with q points of $\Delta(P)$ apart from P.

(ii) and (iii): The point P belongs to q^2+q+1 quads. For a quad $\widetilde{Q}$ containing $P, M(P) \cap \widetilde{Q}$ is the hyperplane of $\widetilde{Q}$ spanned by $\Delta(P) \cap \widehat{Q}$. A simple count yields that $M(P) \cap \widetilde{Q}$ contains $\frac{q^2(q+1)}{2}$ points of type II and $\frac{q^2(q-1)}{2}$ points of type III. The second and third parts follow from this.

(iv): The number of points that have been accounted for is

$$1+q+q^2+q^3+(q^2+q+1)[\frac{q^2(q+1)}{2}+\frac{q^2(q-1)}{2}] = 1+q+q^2+2q^3+q^4+q^5.$$

Since $|\mathrm{PG}(M(P))| = \frac{q^7-1}{q-1}$ there are $q^6 - q^3 = q^3(q^3-1)$ remaining points. □

Lemma 3.10 *The stabilizer S_P of a point $P \in \widetilde{\mathcal{P}}$ is transitive on the points of* $\mathrm{PG}(M(P))$ *which do not belong to quads.*

Proof. Since $\widehat{S_f}$ is transitive on $\widetilde{\mathcal{P}}$ we can take $P = P_2$ and $M(P) = M_2$. Recall that $S_P = E(P) \cdot L(P)$ where $E(P)$ is elementary Abelian of order q^6, and $L(P) \cong GL(3,q)$. The subgroup $E(P)$ fixes every projective line of the form $P + P', P' \in \Delta(P) \setminus \{P\}$ and for such a line, is transitive on $\mathrm{PG}(P + P') \setminus \{P\}$. This implies that $E(P)$ acts trivially on the six dimensional quotient space $M(P)/P$. The action of the complement, $L(P)$, on $M(P)/P$ is equivalent to the action of $GL(3,q)$ on the space $Sym(3,q)$ of $(3 \times 3)-$symmetric matrices where the action is given by $g \circ m = g^T m g$ (where g^T is the transpose of the matrix g). Under this action, every matrix is equivalent to a diagonal matrix and there are six orbits on non-zero vectors, two each for rank 1, 2 and 3. Representatives for the orbits on vectors are as follows:

$$1)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, 2)\begin{pmatrix}d&0&0\\0&0&0\\0&0&0\end{pmatrix}, 3)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, 4)\begin{pmatrix}1&0&0\\0&d&0\\0&0&0\end{pmatrix},$$

$$5)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, 6)\begin{pmatrix}d&0&0\\0&d&0\\0&0&d\end{pmatrix}.$$

Note that the vectors in 1) and 2) give rise to the same point of $\mathrm{PG}(Sym(3,q))$ as do the vectors in 5) and 6) but the vectors in 3) and 4) do not.

Consequently, $L(P)$ has four orbits on the points of $M(P)/P$. However, for any 2-space U of $M(P)$ containing P the group $E(P)$ is transitive on $\mathrm{PG}(U) \setminus \{P\}$ and therefore S_P has four orbits on the points of $\mathrm{PG}(M(P)) \setminus \{P\}$. The point P_1 is a representative of one orbit, and the points P_{15} and P_{16} are the representatives of two other orbits. Thus, there is one other orbit consisting of all those points of $\mathrm{PG}(M(P))$ which do not belong to quads. □

The point P_{17} is a point of $M(P_2)$ which does not belong to a quad. In view of Corollary 3.8 and Lemmas 3.9 and 3.10 it now follows that the orbit of P_{17} has $|\widetilde{\mathcal{P}}| \times (q^6 - q^3) = q^3(q^6-1)(q^2+1)(q+1)$.

3.3 Points not belonging to a hemisphere

We now turn our attention to points which do not belong to $M(P)$ for any point $P \in \widetilde{\mathcal{P}}$.

Since the group $\widehat{S_f}$ is transitive on $\widetilde{\mathcal{P}}$ and for a point $P \in \widetilde{\mathcal{P}}$ the normal Abelian group $E(P)$ acts regularly on the points P' with $d(P, P') = 3$, it follows that $\widehat{S_f}$ is transitive on ordered pairs (P, P') of points from $\widetilde{\mathcal{P}}$ at distance three. One such pair is (P_1, P_8). By Cooperstein and De Bruyn [6, Corollary 5.3], an element of $\widehat{S_f}$ which stabilizes a given point of $\langle P_1, P_8\rangle \setminus \{P_1, P_8\}$ must either stabilize the ordered pair (P_1, P_8) or interchange P_1 and P_8.

The stabilizer $S_{(P_1,P_8)}$ of the ordered pair (P_1, P_8) is isomorphic to $GL(3, q)$. The normal subgroup $SL(3, q)$ acts trivially on both the points P_1 and P_8 while an element of $Z(S_{(P_1,P_8)})$ will multiply p_8 by a scalar a and p_1 by $\frac{1}{a}$. Such an element takes the point $\langle p_1 + p_8\rangle$ to $\langle p_1 + a^2 p_8\rangle$.

There is also a group element which interchanges the points P_1 and P_8 and, specifically, takes p_1 to p_8 and p_8 to $-p_1$. This transformation takes the point $\langle p_1 + p_8\rangle$ to $\langle p_1 - p_8\rangle$. If -1 is a non-square in $\mathbb{F}_q$ then all the points of $\langle P_1, P_8\rangle \setminus \{P_1, P_8\}$ are in the same orbit. On the other hand, if -1 is a square in $\mathbb{F}_q$ then $\langle p_1 + p_8\rangle$ and $\langle p_1 + dp_8\rangle$ are in different orbits. In the former case we get a single orbit with representative P_{18} and orbit size $\frac{q^6(q^4-1)(q^3+1)}{2}$ and in the latter case two orbits, with representatives P_{18} and P_{19} each with orbit size $\frac{q^6(q^4-1)(q^3+1)}{4}$.

We next show that the group S_f contains a subgroup $G \cong GU(3, q^2)$. Recall that δ is an element of $\mathbb{F}_{q^2}$ such that $\delta^2 = d$ and $\bar{w}_i = \delta \bar{v}_i$ for every $i \in \{1, 2, 3\}$. For any $\alpha \in \mathbb{F}_{q^2}$, put $\bar{\alpha} := \alpha^q$. Note that for $\alpha = a + b\delta, \bar{\alpha} = a - b\delta$. Now, define a map $h : V \times V \to \mathbb{F}_{q^2}$ as follows ($\alpha_i, \beta_i \in \mathbb{F}_{q^2}$):

$$h(\sum_{i=1}^{3} \alpha_i \bar{v}_i, \sum_{i=1}^{3} \beta_i \bar{v}_i) = \frac{1}{2\bar{\delta}} \sum_{i=1}^{3} \alpha_i \bar{\beta}_i.$$

Since $tr(\delta) = 0$ this defines a skew Hermitian form on V. It then follows that the map $f' : V \times V \to \mathbb{F}_q$ given by $f'(\bar{v}, \bar{w}) = tr(h(\bar{v}, \bar{w}))$ is an alternating

form. We claim that $f' = f$. We compute $f'(\bar{v}_i, \bar{v}_j), f'(\bar{w}_i, \bar{w}_j), f'(\bar{v}_i, \bar{w}_j)$ for $i \neq j$ and $f'(\bar{v}_i, \bar{w}_i)$ for $i = 1, 2, 3$.

By the definition, $h(\bar{v}_i, \bar{v}_j) = h(\bar{w}_i, \bar{w}_j) = h(\bar{v}_i, \bar{w}_j) = 0$ for $i \neq j$ and consequently we only have to compute $f'(\bar{v}_i, \bar{w}_i)$. By definition this is $tr(h(\bar{v}_i, \delta\bar{v}_i)) = tr(\frac{\bar{\delta}}{2\delta}) = tr(\frac{1}{2}) = 1$. So, our claim holds.

It now follows that if σ is an isometry of (V, h), that is, a unitary transformation, then σ is an isometry of the symplectic space (V, f). So if $G = \{\sigma \in GL_{\mathbb{F}_{q^2}}(V) \mid h(\sigma(\bar{u}_1), \sigma(\bar{u}_2)) = h(\bar{u}_1, \bar{u}_2), \forall \bar{u}_1, \bar{u}_2 \in V\}$, then $G \cong GU(3, q^2)$ and $G < S_f$. Let G' be the derived subgroup of G, then G' is isomorphic to $SU(3, q^2)$. By Lemma 3.1 it follows that $\widehat{G'}$ centralizes $\langle p_{20}, p_{21} \rangle$.

We next determine the stabilizer of the point P_{20}. We will first show in a series of lemmas that if $\bar{v}, \bar{w} \in \langle p_{20}, p_{21} \rangle$ and $\theta \in S_f$ satisfies $\theta(\bar{v}) = \bar{w}$ then $\theta(\langle p_{20}, p_{21} \rangle = \langle p_{20}, p_{21} \rangle$.

Let V' denote the six dimensional vector space over $\mathbb{F}_{q^2}$ with basis $\mathcal{S}$. For a vector $\bar{x} = a_1\bar{v}_1 + a_2\bar{v}_2 + a_3\bar{v}_3 + b_1\bar{w}_1 + b_2\bar{w}_2 + b_3\bar{w}_3 \in V'$ we define $\bar{x}^q = a_1^q\bar{v}_1 + a_2^q\bar{v}_2 + a_3^q\bar{v}_3 + b_1^q\bar{w}_1 + b_2^q\bar{w}_2 + b_3^q\bar{w}_3$. For $\theta \in GL(V)$ we denote by $\overline{\theta}$ the element induced by θ in $GL(V')$ and $\overline{\theta'}$ the corresponding element of $GL(\bigwedge^3 V')$.

Lemma 3.11 *Let $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_6\}$ and $\{\bar{e}'_1, \bar{e}'_2, \ldots, \bar{e}'_6\}$ be two bases of V' such that $\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_6 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \bar{e}'_4 \wedge \bar{e}'_5 \wedge \bar{e}'_6$. Then $\{\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle, \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle\} = \{\langle \bar{e}'_1, \bar{e}'_2, \bar{e}'_3 \rangle, \langle \bar{e}'_4, \bar{e}'_5, \bar{e}'_6 \rangle\}$.*

Proof. Put $\alpha := \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \bar{e}_4 \wedge \bar{e}_5 \wedge \bar{e}_6 = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + \bar{e}'_4 \wedge \bar{e}'_5 \wedge \bar{e}'_6$. For every vector $\bar{x}$ of V', let $A_{\bar{x}}$ denote the subspace of V' consisting of all vectors $\bar{y}$ satisfying $\alpha \wedge \bar{x} \wedge \bar{y} = 0$. Let B be the subset of V' which consists of all vectors $\bar{x}$ of V' such that $\dim(A_{\bar{x}}) \geq 4$. We will now prove that $B = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle \cup \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle$. In a completely similar way, one can then also prove that $B = \langle \bar{e}'_1, \bar{e}'_2, \bar{e}'_3 \rangle \cup \langle \bar{e}'_4, \bar{e}'_5, \bar{e}'_6 \rangle$. This then implies that $\{\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle, \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle\} = \{\langle \bar{e}'_1, \bar{e}'_2, \bar{e}'_3 \rangle, \langle \bar{e}'_4, \bar{e}'_5, \bar{e}'_6 \rangle\}$.

Put $\bar{x} = \delta_1\bar{e}_1 + \delta_2\bar{e}_2 + \cdots + \delta_6\bar{e}_6$ and $\bar{y} = a_1\bar{e}_1 + a_2\bar{e}_2 + \cdots + a_6\bar{e}_6$. Then

the fact that $\alpha \wedge \bar{x} \wedge \bar{y} = 0$ implies that

$$\begin{bmatrix} -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_3 & 0 & \delta_1 & 0 & 0 & 0 \\ 0 & -\delta_3 & \delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \\ 0 & 0 & 0 & -\delta_6 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\dim(V_{\bar{x}}) \geq 4$ if and only if the rank of

$$\begin{bmatrix} -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_3 & 0 & \delta_1 & 0 & 0 & 0 \\ 0 & -\delta_3 & \delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \\ 0 & 0 & 0 & -\delta_6 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \end{bmatrix}$$

is at most 2. This happens precisely when $(\delta_1, \delta_2, \delta_3) = (0,0,0)$ or $(\delta_4, \delta_5, \delta_6) = (0,0,0)$, i.e. when $\bar{x} \in \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle \cup \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle$. □

The proof of the following lemma is straightforward.

Lemma 3.12 *For all $a, b \in \mathbb{F}_q$, $(a + b\delta) \cdot (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) + (a - b\delta) \cdot (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3) = 2a \cdot p_{21} + 2bd \cdot p_{20}$.*

Corollary 3.13 *The vectors of the 2-space $\langle p_{20}, p_{21} \rangle$ of $\bigwedge^3 V$ are precisely the vectors of the form $(a + b\delta) \cdot (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) + (a - b\delta) \cdot (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3)$, where $a, b \in \mathbb{F}_q$.*

By Lemma 3.11 and Corollary 3.13, we have

Corollary 3.14 *If $\theta \in S_f$ such that $\widehat{\theta}$ maps a nonzero vector of $\langle p_{20}, p_{21} \rangle$ to a nonzero vector of $\langle p_{20}, p_{21} \rangle$, then $\widehat{\theta}$ stabilizes $\langle p_{20}, p_{21} \rangle$. Moreover, one of the following holds:*

(1) *$\overline{\theta'}$ stabilizes the 1-spaces $\langle (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) \rangle$ and $\langle (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3) \rangle$ of $\bigwedge^3 V'$.*

(2) *$\overline{\theta'}$ interchanges the 1-spaces $\langle (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) \rangle$ and $\langle (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3) \rangle$ of $\bigwedge^3 V'$.*

Let W_f denote the subgroup of S_f consisting of all $\theta \in S_f$ for which $\widehat{\theta}$ stabilizes $\langle p_{20}, p_{21}\rangle$. Let U_f denote the normal subgroup of W_f consisting of all $\theta \in W_f$ for which case (1) of Corollary 3.14 occurs. Put $\widehat{W_f} := \{\widehat{\theta} \,|\, \theta \in W_f\}$ and $\widehat{U_f} := \{\widehat{\theta} \,|\, \theta \in U_f\}$.

Remark 3.15 Let θ be an element of U_f, let μ_1 be the restriction of $\overline{\theta}$ to the 3-space $\langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$ of V' and let μ_2 be the restriction of $\overline{\theta}$ to the 3-space $\langle \bar{w}_1 - \delta\bar{v}_1, \bar{w}_2 - \delta\bar{v}_2, \bar{w}_3 - \delta\bar{v}_3\rangle$ of V'. Then $1 = \det(\overline{\theta}) = \det(\mu_1) \cdot \det(\mu_2)$.

Now, let $\bar{x}$ be an arbitrary vector of $\langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$. Since $\bar{x} + \bar{x}^q \in V$, we have $\bar{y} := \overline{\theta}(\bar{x} + \bar{x}^q) = \overline{\theta}(\bar{x}) + \overline{\theta}(\bar{x}^q) \in V$. Also, $\bar{y} = \bar{y}^q = [\overline{\theta}(\bar{x}^q)]^q + [\overline{\theta}(\bar{x})]^q$. Since there exist unique $\bar{y}_1 \in \langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$ and $\bar{y}_2 \in \langle \bar{w}_1 - \delta\bar{v}_1, \bar{w}_2 - \delta\bar{v}_2, \bar{w}_3 - \delta\bar{v}_3\rangle$ such that $\bar{y} = \bar{y}_1 + \bar{y}_2$, we necessarily have $\overline{\theta}(\bar{x}^q) = [\overline{\theta}(\bar{x})]^q$. Hence, $\mu_2(\bar{x}^q) = \overline{\theta}(\bar{x}^q) = [\overline{\theta}(\bar{x})]^q = [\mu_1(\bar{x})]^q$.

By the previous paragraph, $\det(\mu_2) = [\det(\mu_1)]^q$. If $\det(\mu_1) = a + b\delta$, then $\det(\mu_2) = a - b\delta$ and since $\det(\mu_1) \cdot \det(\mu_2) = 1$, we have $a^2 - b^2 d = 1$.

Conversely, let $a, b \in \mathbb{F}_q$ such that $a^2 - b^2 d = 1$. Then the element of $GL(V)$ determined by $\bar{v}_1 \mapsto a \cdot \bar{v}_1 + b \cdot \bar{w}_1$, $\bar{w}_1 \mapsto bd \cdot \bar{v}_1 + a \cdot \bar{w}_1$, $\bar{v}_2 \mapsto \bar{v}_2$, $\bar{w}_2 \mapsto \bar{w}_2$, $\bar{v}_3 \mapsto \bar{v}_3$, $\bar{w}_3 \mapsto \bar{w}_3$ determines an element of U_f for which the corresponding value of $\det(\mu_1)$ is equal to $a + b\delta$.

Lemma 3.16 *Let $a_1, a_2, b_1, b_2 \in \mathbb{F}_q$ such that $(a_1, a_2) \neq (0,0) \neq (b_1, b_2)$. Then the 1-spaces $\langle a_1 p_{21} + a_2 p_{20}\rangle$ and $\langle b_1 p_{21} + b_2 p_{20}\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if $(a_1^2 - \frac{a_2^2}{d})(b_1^2 - \frac{b_2^2}{d})$ is a square.*

Proof. By Lemma 3.12, $a_1 p_{21} + a_2 p_{20} = (\frac{a_1}{2} + \frac{a_2}{2d}\delta) \cdot (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) + (\frac{a_1}{2} - \frac{a_2}{2d}\delta) \cdot (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3)$ and $b_1 p_{21} + b_2 p_{20} = (\frac{b_1}{2} + \frac{b_2}{2d}\delta) \cdot (\bar{w}_1 + \delta\bar{v}_1) \wedge (\bar{w}_2 + \delta\bar{v}_2) \wedge (\bar{w}_3 + \delta\bar{v}_3) + (\frac{b_1}{2} - \frac{b_2}{2d}\delta) \cdot (\bar{w}_1 - \delta\bar{v}_1) \wedge (\bar{w}_2 - \delta\bar{v}_2) \wedge (\bar{w}_3 - \delta\bar{v}_3)$. By Remark 3.15, the 1-spaces $\langle a_1 p_{21} + a_2 p_{20}\rangle$ and $\langle b_1 p_{21} + b_2 p_{20}\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if there exists a $\lambda \in \mathbb{F}_q^*$ and $c_1, c_2 \in \mathbb{F}_q$ satisfying $c_1^2 - c_2^2 d = 1$ such that $(\frac{a_1}{2} + \frac{a_2}{2d}\delta) \cdot (c_1 + c_2\delta) \cdot \lambda = \frac{b_1}{2} + \frac{b_2}{2d}\delta$. If c_1' and c_2' are the unique elements of $\mathbb{F}_q$ such that $(\frac{a_1}{2} + \frac{a_2}{2d}\delta)(c_1' + c_2'\delta) = \frac{b_1}{2} + \frac{b_2}{2d}\delta$, then one readily verifies that $b_1^2 - \frac{b_2^2}{d} = (a_1^2 - \frac{a_2^2}{d})({c_1'}^2 - {c_2'}^2 d)$. It now follows that $\langle a_1 p_{21} + a_2 p_{20}\rangle$ and $\langle b_1 p_{21} + b_2 p_{20}\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if $(a_1^2 - \frac{a_2^2}{d})(b_1^2 - \frac{b_2^2}{d})$ as a square. □

Lemma 3.17 *There are two $\widehat{U_f}$-orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

Proof. If $a_1 = 1$ and $a_2 = 0$, then $a_1^2 - \frac{a_2^2}{d} = 1$ is a square.

Now, choose $a_1 \in \mathbb{F}_q^*$. Then there exist $a_2, a_3 \in \mathbb{F}_q^*$ such that $da_1^2 = a_2^2 + a_3^2$. Then $a_1^2 - \frac{a_2^2}{d} = \frac{a_3^2}{d}$ is a non-square.

The claim now follows from Lemma 3.16. □

We will now construct a particular element $\widehat{\theta^*}$ of $\widehat{W_f} \setminus \widehat{U_f}$. Let $A, B \in \mathbb{F}_q^*$ such that $(\frac{A}{B})^2 + (\frac{1}{B})^2 = d$ (Hence, $A^2 - B^2 d = -1$) and consider the following map θ^* of S_f:

$$\begin{cases} \bar{v}_1 & \mapsto & A \cdot \bar{v}_1 + B \cdot \bar{w}_1, \\ \bar{w}_1 & \mapsto & -Bd \cdot \bar{v}_1 - A \cdot \bar{w}_1, \\ \bar{v}_2 & \mapsto & A \cdot \bar{v}_2 - B \cdot \bar{w}_2, \\ \bar{w}_2 & \mapsto & Bd \cdot \bar{v}_2 - A \cdot \bar{w}_2, \\ \bar{v}_3 & \mapsto & A \cdot \bar{v}_3 + B \cdot \bar{w}_3, \\ \bar{w}_3 & \mapsto & -Bd \cdot \bar{v}_3 - A \cdot \bar{w}_3. \end{cases}$$

Then one readily verifies that $\theta^* \in W_f \setminus U_f$. Moreover, $\widehat{\theta^*}(p_{21}) = Ap_{21} + Bdp_{20}$.

Proposition 3.18 (1) *If -1 is a non-square, then $\widehat{W_f}$ has one orbit on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

(2) *If -1 is a square, then $\widehat{W_f}$ has two orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

Proof. Since $\widehat{U_f}$ is a normal subgroup of index 2 of $\widehat{W_f}$, we can conclude the following:

(1) If $\langle p_{21}\rangle$ and $\langle \widehat{\theta^*}(p_{21})\rangle$ belong to the same $\widehat{U_f}$-orbit, then $\widehat{\theta^*}$ stabilizes the two $\widehat{U_f}$-orbits. In this case $\widehat{W_f}$ has two orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.

(2) If $\langle p_{21}\rangle$ and $\langle \widehat{\theta^*}(p_{21})\rangle$ belong to different $\widehat{U_f}$-orbits, then $\widehat{\theta^*}$ interchanges the two $\widehat{U_f}$-orbits. In this case $\widehat{W_f}$ has one orbit on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.

Now, $\langle p_{21}\rangle$ and $\langle \widehat{\theta^*}(p_{21})\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if $(1^2 - \frac{0^2}{d})(A^2 - \frac{(Bd)^2}{d}) = A^2 - B^2 d = -1$ is a square. The proposition follows. □

4 Proof of the Fusion Theorem

Since $\widehat{S_f}$ is normal in $\widehat{G_f}$, if two $\widehat{S_f}$-orbits were to fuse via $\widehat{\sigma^*}$ or $\widehat{T_\gamma}$, $\gamma \in Aut(\mathbb{F}_q)$, then they must have the same size. When -1 is a non-square there are no such possibilities. When -1 is a square it could be the case that the orbits with representatives P_{18} and P_{19} fuse and the orbits with representatives P_{20} and P_{21} fuse. We show that this is indeed the case.

Suppose then that -1 is a square. Now $\widehat{\sigma^*}(p_1+p_8) = p_1+d^3p_8$ and d^3 is a non-square. The points $P_{19} = \langle p_1 + dp_8\rangle$ and $\langle p_1 + d^3p_8\rangle$ are in the same $\widehat{S_f}$-orbit. So, in this case we get the fusion of the $\widehat{S}_f$-orbits with representatives P_{18} and P_{19}. We also show that the orbits with representatives P_{20} and P_{21} fuse. Before doing so, we note that the points $P_{21} = \langle p_8+dp_2+dp_3+dp_5\rangle$ and $\langle p_1+dp_4+dp_6+dp_7\rangle$ are in the same $\widehat{S_f}$-orbit: let $\sigma(\bar{v}_i) = \bar{w}_i, \sigma(\bar{w}_i) = -\bar{v}_i, i = 1,2,3$. Then $\widehat{\sigma}(p_1+dp_4+dp_6+dp_7) = p_8+dp_2+dp_3+dp_5$ from which the claim follows. Now $\widehat{\sigma^*}(p_{20}) = \widehat{\sigma^*}(dp_1 + p_4 + p_6 + p_7) = dp_1 + d^2p_4 + d^2p_6 + d^2p_7 = d(p_1 + dp_4 + dp_6 + dp_7)$ and therefore $\widehat{\sigma^*}(P_{20}) = \langle p_1 + dp_4 + dp_6 + dp_7\rangle$ is in the $\widehat{S_f}$-orbit of P_{21}. This completes the proof of the Fusion Theorem.

References

[1] P. J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[2] I. Cardinali, B. De Bruyn and A. Pasini. Locally singular hyperplanes in thick dual polar spaces of rank 4. *J. Combin. Theory Ser. A* 113 (2006), 636–646.

[3] A. M. Cohen and E. E. Shult. Affine polar spaces. *Geom. Dedicata* 35 (1990), 43-76.

[4] B. N. Cooperstein. Some geometries associated with parabolic representations of groups of Lie type. *Canad. J. Math.* 28 (1976), 1021-1031.

[5] B. N. Cooperstein. On the generation of dual polar spaces of symplectic type over finite fields. *J. Combin. Theory Ser. A* 83 (1998), 221–232.

[6] B. N. Cooperstein and B. De Bruyn. The combinatorial properties of the hyperplanes of $DW(5,q)$ arising from embedding. *Des. Codes Cryptogr.* 47 (2008), 35–51.

[7] B. N. Cooperstein and A. Pasini. The non-existence of ovoids in the dual polar space $DW(5,q)$. *J. Combin. Theory Ser. A* 104 (2003), 351-364.

[8] H. Cuypers. Extended near hexagons and line system. *Adv. Geom.* 4 (2004), 181-214.

[9] B. De Bruyn. The hyperplanes of $DQ(2n,\mathbb{K})$ and $DQ^-(2n+1,q)$ which arise from their spin-embeddings. *J. Combin. Theory Ser. A* 114 (2007), 681–691.

[10] B. De Bruyn. The hyperplanes of $DW(5,2^h)$ arising from embedding. *Discrete Math.*, to appear.

[11] B. De Bruyn and H. Pralle. The hyperplanes of $DH(5,q^2)$. *Forum Math.* 20 (2008), 239–264.

[12] B. De Bruyn and H. Pralle. The exceptional hyperplanes of $DH(5,4)$. *European J. Combin.* 28 (2007), 1412–1418.

[13] D. G. Higman. Finite permutation groups of rank 3. *Math. Z.* 86 (1964), 145–156.

[14] A. Kasikova and E. E. Shult. Absolute embeddings of point-line geometries. *J. Algebra* 238 (2001), 265-291.

[15] A. Pasini and S. Shpectorov. Uniform hyperplanes of finite dual polar spaces of rank 3. *J. Combin. Theory Ser. A* 94 (2001), 276-288.

[16] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles.* Research Notes in Mathematics 110. Pitman, 1984.

[17] H. Pralle. The hyperplanes of $DW(5,2)$. *Experiment. Math.* 14 (2005), 373–384.

[18] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179-185.

[19] E. E. Shult. *Generalized hexagons as geometric hyperplanes of near hexagons.* In "Groups, Combinatorics and Geometry" (eds. M. Liebeck and J. Saxl), Cambridge Univ. Press (1992), 229-239.

[20] E. E. Shult and J. A. Thas. Hyperplanes of dual polar spaces and the spin module. *Arch. Math.* 59 (1992), 610-623.

[21] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.

[22] R. Steinberg. *Lectures on Chevalley Groups*. Yale University Lecture Notes, 1967.

[23] F. D. Veldkamp. Polar geometry. I, II, III, IV. *Indag. Math.* 21 (1959), 512-551.

[24] F. D. Veldkamp. Polar geometry V. *Indag. Math.* 22 (1959), 207-212.

Addresses:

Bruce Cooperstein
Mathematics Department
University of California at Santa Cruz
194 Baskin Engineering
Santa Cruz, CA 95064, USA
coop@ucsc.edu

Bart De Bruyn
Department of Pure Mathematics and Computer Algebra
Ghent University
Krijgslaan 281 (S22)
B-9000 Gent, Belgium
bdb@cage.ugent.be