Hyperplanes of embeddable Grassmannians arise from projective embeddings: a short proof

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Abstract

In this note, we give an alternative and considerably shorter proof of a result of Shult [2] stating that all hyperplanes of embeddable Grassmannians arise from projective embeddings.

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1 Introduction

Let $n \geq 1$ and let $\mathbb{K}$ be a field. Let V denote an $(n+1)$-dimensional vector space over $\mathbb{K}$ and let $\mathrm{PG}(n,\mathbb{K}) = \mathrm{PG}(V)$ denote the projective space associated with V. Let $k \in \{0,\ldots,n-1\}$. Then the following point-line geometry $A_{n,k+1}$ can be defined:

- The points of $A_{n,k+1}$ are the k-dimensional subspaces of $\mathrm{PG}(n,\mathbb{K})$.
- The lines of $A_{n,k+1}$ are the sets $A(\pi_1,\pi_2)$ of k-dimensional subspaces of $\mathrm{PG}(n,\mathbb{K})$ which contain a given $(k-1)$-dimensional subspace π_1 and are contained in a given $(k+1)$-dimensional subspace π_2 ($\pi_1 \subset \pi_2$).
- Incidence is containment.

The geometry $A_{n,k+1}$ is called the *Grassmannian of the k-dimensional subspaces* of $\mathrm{PG}(n,\mathbb{K})$. We will denote the point-set of $A_{n,k+1}$ by $\mathcal{P}$. If x and y are two points of $A_{n,k+1}$, then $\mathrm{d}(x,y) := k - \dim(x \cap y)$ is the distance between x and y in the collinearity graph of $A_{n,k+1}$. A *hyperplane* of $A_{n,k+1}$ is a proper subspace of $A_{n,k+1}$ which meets every line of $A_{n,k+1}$.

Now, let $\bigwedge^{k+1} V$ denote the $(k+1)$-th exterior power of V. For every k-dimensional subspace $\alpha = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k+1} \rangle$ of $\mathrm{PG}(n,\mathbb{K})$, let $e(\alpha)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_{k+1} \rangle$ of $\mathrm{PG}(\bigwedge^{k+1} V)$. Notice that the point $e(\alpha)$ is independent of the generating set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k+1}\}$ of the subspace α. The map e defines a full projective embedding of $A_{n,k+1}$ into $\mathrm{PG}(\bigwedge^{k+1} V)$ which is called the *Grassmann-embedding* of $A_{n,k+1}$. If π is a hyperplane of $\mathrm{PG}(\bigwedge^{k+1} V)$, then $e^{-1}(\pi \cap e(\mathcal{P}))$ is a hyperplane of $A_{n,k+1}$. We say that the hyperplane $e^{-1}(\pi \cap e(\mathcal{P}))$ *arises* from the Grassmann-embedding of $A_{n,k+1}$.

The aim of this note is to give a short and elementary proof of the following result due to Shult [2].

Theorem 1.1 *All hyperplanes of $A_{n,k+1}$ arise from the Grassmann-embedding of $A_{n,k+1}$.*

Theorem 1.1 has an equivalent formulation in terms of alternating k-linear forms. This fact together with results of Ronan [1] and Wells [3] was exploited in [2] to prove Theorem 1.1. The alternative proof for Theorem 1.1 which we will now give is considerably shorter and only uses basic projective geometry.

2 Some useful results

We continue with the notation of Section 1.

Lemma 2.1 *Every hyperplane H of $A_{n,k+1}$ is a maximal subspace of $A_{n,k+1}$.*

Proof. Let X_1 and X_2 be two points of $A_{n,k+1}$ not contained in H. Recall that $\mathrm{d}(X_1, X_2) = k - \dim(X_1 \cap X_2)$. We prove by induction on $\mathrm{d}(X_1, X_2)$ that X_1 and X_2 are contained in a path which entirely consists of points of $\mathcal{P} \setminus H$. Obviously, this holds if $\mathrm{d}(X_1, X_2) \leq 1$. So, suppose that $\delta := \mathrm{d}(X_1, X_2) \geq 2$. For every $i \in \{1, 2\}$, let (y_i, α_i) be a non-incident point-hyperplane pair of X_i such that $X_1 \cap X_2 \subseteq \alpha_i$. Put $\beta_1 := \langle X_1, y_2 \rangle$ and $\beta_2 := \langle X_2, y_1 \rangle$. Then $A(\alpha_1, \beta_1)$ and $A(\alpha_2, \beta_2)$ are two lines of $A_{n,k+1}$. Moreover, for every point Z_1 of $A(\alpha_1, \beta_1)$, there exists a unique point $Z_2 \in A(\alpha_2, \beta_2)$ at distance $\mathrm{d}(X_1, X_2) - 1$ from Z_1, namely $Z_2 = \langle \alpha_2, z \rangle$, where z is the unique point in $Z_1 \cap y_1 y_2$. Since $X_i \notin H$ and $X_i \in A(\alpha_i, \beta_i)$, $|A(\alpha_i, \beta_i) \cap H| = 1$. So, it is possible to choose a $Z_1 \in A(\alpha_1, \beta_1)$ and a $Z_2 \in A(\alpha_2, \beta_2)$ such that $Z_1 \notin H$, $Z_2 \notin H$ and $\mathrm{d}(Z_1, Z_2) = \mathrm{d}(X_1, X_2) - 1$. By the induction hypothesis, Z_1 and Z_2 are connected by a path entirely consisting of points of $\mathcal{P} \setminus H$. Hence, also X_1 and X_2 are connected by such a path. $\square$

Suppose now that H_1 and H_2 are two distinct hyperplanes of $A_{n,k+1}$. Let Γ be the graph with vertex set $\mathcal{P} \setminus (H_1 \cup H_2)$, with two vertices x and y adjacent if and only if $\mathrm{d}(x, y) = 1$ and the line xy meets $H_1 \cap H_2$. Let $\mathcal{C}$ denote the set of all connected components of Γ and put $\mathcal{H} := \{H_1, H_2\} \cup \{C \cup (H_1 \cap H_2) \,|\, C \in \mathcal{C}\}$.

Lemma 2.2 *If H is a hyperplane of $A_{n,k+1}$ such that $H \cap H_1 = H_1 \cap H_2 = H \cap H_2$, then $H \in \mathcal{H}$.*

Proof. Since H_1 and H_2 are distinct maximal subspaces, $H_1 \cap H_2$ is not a maximal subspace. Since $H_1 \cap H_2 \subseteq H$ and H is a maximal subspace, there exists an $x^* \in H \setminus (H_1 \cap H_2)$. Clearly, $x^* \notin H_1 \cup H_2$. So, x^* is a vertex of Γ

and there exists a unique element $W \in \mathcal{H}$ containing x^*. We will prove that $H = W$.

We first show that $W \subseteq H$. In view of the fact that $x^* \in W \cap H$, we need to show that if $x \in H \setminus (H_1 \cap H_2)$ and y is a vertex of Γ adjacent to x, then also $y \in H \setminus (H_1 \cap H_2)$. Now, since (i) $\mathrm{d}(x, y) = 1$, (ii) xy meets $H_1 \cap H_2 = H \cap H_1$, and (iii) H is a subspace, it follows that $xy \subseteq H$. In particular, $y \in H$.

We next show that $H \subseteq W$. It suffices to prove the following (by induction on i): if $x, y \in H \setminus (H_1 \cap H_2)$ with $\mathrm{d}(x, y) = i$ and $x \in W$, then also $y \in W$. The claim then immediately follows from the fact that $x^* \in H \cap W$. If $\mathrm{d}(x, y) = 1$, then the line xy meets $H \cap H_1 = H_1 \cap H_2$. Hence, x and y are adjacent points of Γ and $y \in W$. So, we will suppose that $\delta = \mathrm{d}(x, y) \geq 2$. We show that there exists a point u_x of x, a point u_y of y, a hyperplane α_x of x and a hyperplane α_y of y such that $x \cap y \subseteq \alpha_x$, $x \cap y \subseteq \alpha_y$, $u_x \notin \alpha_x$, $u_y \notin \alpha_y$, $\langle \alpha_x, u_y \rangle \in H$ and $\langle \alpha_y, u_x \rangle \in H$.

Let α_x be an arbitrary hyperplane of x through $x \cap y$ and let S denote the set of all k-dimensional subspaces of $\mathrm{PG}(n, \mathbb{K})$ through α_x which intersect y in a subspace of dimension $k - \delta + 1$. Then S is a subspace of $A_{n,k+1}$ which carries the structure of a projective space isomorphic to $\mathrm{PG}(\delta - 1, \mathbb{K})$. The set $S \cap H$ is equal to either S or a hyperplane of S (if we regard S as a projective space).

Suppose $\delta \geq 3$ or $S \cap H = S$. Let u_x be an arbitrary point of $x \setminus \alpha_x$. Let S' denote the set of all k-dimensional subspaces through u_x which intersect y in a hyperplane of y containing $x \cap y$. Then S' is a subspace of $A_{n,k+1}$ which carries the structure of a projective space $\mathrm{PG}(\delta - 1, \mathbb{K})$. The set $S' \cap H$ is equal to either S' or a hyperplane of S'. If $\delta \geq 3$, then we see that there exist elements $\beta \in S \cap H$ and $\gamma \in S' \cap H$ such that $(\beta \cap y) \cap (\gamma \cap y) = x \cap y$. In this case, put $\alpha_y := \gamma \cap y$ and let u_y be an arbitrary point of $(\beta \cap y) \setminus (x \cap y)$. Suppose $\delta = 2$ and $S \cap H = S$. Let β be an arbitrary element of $S' \cap H$. Put $\beta \cap y = \alpha_y$ and let u_y be an arbitrary point of $y \setminus \alpha_y$. In both cases, one readily verifies that $(u_x, u_y, \alpha_x, \alpha_y)$ satisfies all required properties.

Suppose $\delta = 2$ and that $S \cap H$ is a singleton $\{\beta\}$. Let u_y be an arbitrary point of $(\beta \cap y) \setminus (x \cap y)$ and let α_y be an arbitrary hyperplane of y through $x \cap y$ not containing u_y. Let S' denote the set of all k-dimensional subspaces of $\mathrm{PG}(n, \mathbb{K})$ through α_y which intersect x in a hyperplane of x. Then S' is a line. Since $\langle \alpha_y, \alpha_x \rangle \notin H$, there exists a unique element $\gamma \in S'$ belonging to H. Let u_x be an arbitrary point of $(\gamma \cap x) \setminus (x \cap y)$. Then $(u_x, u_y, \alpha_x, \alpha_y)$ satisfies all required properties.

Now, let u_x, u_y, α_x and α_y as above. Then $L_x := A(\alpha_x, \langle x, u_y \rangle)$ and $L_y := A(\alpha_y, \langle y, u_x \rangle)$ are lines of $A_{n,k+1}$. Since L_x contains the points x and $\langle \alpha_x, u_y \rangle$ of H, all the points of L_x are contained in H. Similarly, since L_y contains the points y and $\langle \alpha_y, u_x \rangle$ of H, all the points of L_y are contained in H. Clearly, every point z_1 of L_x has distance $\mathrm{d}(x, y) - 1$ from a unique point z_2 of L_y, namely z_2 is the unique k-dimensional subspace containing α_y and the singleton $z_1 \cap u_x u_y$. Since $x \notin H_1$ and $y \notin H_1$, $|L_x \cap H_1| = 1 = |L_y \cap H_1|$. Hence, there exists a $z_1 \in L_x$ and a $z_2 \in L_y$ such that $z_1 \notin H_1$, $z_2 \notin H_1$ and $\mathrm{d}(z_1, z_2) = \mathrm{d}(x, y) - 1$. Now, applying the induction hypothesis 3 times, we find $z_1 \in W$, $z_2 \in W$ and $y \in W$. □

Corollary 2.3 *If H_1 and H_2 arise from the Grassmann-embedding of $A_{n,k+1}$, then also every hyperplane H of $A_{n,k+1}$ satisfying $H \cap H_1 = H_1 \cap H_2 = H \cap H_2$ arises from the Grassmann-embedding of $A_{n,k+1}$.*

Proof. Take a point $x^* \in H \setminus (H_1 \cap H_2)$. Since H_i, $i \in \{1,2\}$, is a maximal subspace, $\Sigma_i := \langle e(H_i) \rangle$ is a hyperplane of $\mathrm{PG}(\bigwedge^{k+1} V)$. Moreover, $\Sigma_i \cap e(\mathcal{P}) = e(H_i)$. So, $\Sigma_1 \neq \Sigma_2$ and $e(x^*) \notin \Sigma_1 \cap \Sigma_2$ since $\Sigma_1 \cap \Sigma_2 \cap e(\mathcal{P}) = e(H_1) \cap e(H_2) = e(H_1 \cap H_2)$. Put $\Sigma := \langle e(x^*), \Sigma_1 \cap \Sigma_2 \rangle$ and $H' := e^{-1}(\Sigma \cap e(\mathcal{P}))$. Then $x^* \in H'$ and $H' \cap H_1 = H_1 \cap H_2 = H' \cap H_2$. By the proof of Lemma 2.2, $H = W = H'$, where W is the unique element of $\mathcal{H}$ containing x^*. $\square$

3 Alternative proof of Theorem 1.1

We will prove Theorem 1.1 by induction on n. If $k \in \{0, n-1\}$, then $A_{n,k+1}$ is a projective space and the theorem trivially holds in this case. So, Theorem 1.1 holds if $n \leq 2$. In the sequel, we will suppose that $n \geq 3$ and $k \in \{1, \dots, n-2\}$.

Let (x, π) be a non-incident point-hyperplane pair of $\mathrm{PG}(n, \mathbb{K})$. Let S_x, respectively S_π, be the subspace of $A_{n,k+1}$ consisting of all k-dimensional subspaces of $\mathrm{PG}(n, \mathbb{K})$ which are incident with x, respectively π. The point-line geometry $\widetilde{S_x}$ (respectively $\widetilde{S_\pi}$) induced on S_x (respectively S_π) is isomorphic to $A_{n-1,k}$ (respectively $A_{n-1,k+1}$). The Grassmann-embedding e of $A_{n,k+1}$ induces an embedding e_x of $\widetilde{S_x}$ into a subspace Σ_x of $\mathrm{PG}(\bigwedge^{k+1} V)$ and an embedding e_π of $\widetilde{S_\pi}$ into a subspace Σ_π of $\mathrm{PG}(\bigwedge^{k+1} V)$. Choosing a basis $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{n+1}\}$ in V such that $\langle \bar{e}_1 \rangle = x$ and $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle = \pi$, we see that: (i) Σ_x is the subspace of $\mathrm{PG}(\bigwedge^{k+1} V)$ generated by all points of the form $\langle \bar{e}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_{k+1} \rangle$, where $\bar{f}_2, \dots, \bar{f}_{k+1}$ are vectors of $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle$; (ii) Σ_π is the subspace of $\mathrm{PG}(\bigwedge^{k+1} V)$ generated by all points of the form $\langle \bar{f}_1 \wedge \bar{f}_2 \wedge \cdots \wedge \bar{f}_{k+1} \rangle$, where $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_{k+1}$ are vectors of $\langle \bar{e}_2, \dots, \bar{e}_{n+1} \rangle$. Hence, Σ_x and Σ_π are complementary subspaces of $\mathrm{PG}(\bigwedge^{k+1} V)$. It is also clear that e_x and e_π are isomorphic to the Grassmann-embeddings of respectively $A_{n-1,k}$ and $A_{n-1,k+1}$.

Lemma 3.1 *Let y be a point of $A_{n,k+1}$ not contained in $S_x \cup S_\pi$. Then there exists a unique line L_y through y meeting S_x and S_π.*

Proof. Regarding y as a k-dimensional subspace of $\mathrm{PG}(n, \mathbb{K})$, we have $x \notin y$ and $y \cap \pi$ is a $(k-1)$-dimensional subspace of π. The line $L_y := A(y \cap \pi, \langle x, y \rangle)$ contains y, intersects S_x in the point $\langle x, y \cap \pi \rangle$ and S_π in the point $\langle x, y \rangle \cap \pi$. The uniqueness of L_y is also obvious. $\square$

Now, let H be an arbitrary hyperplane of $A_{n,k+1}$. Then $H \cap S_x$ is either S_x or a hyperplane of $\widetilde{S_x}$. Similarly, $H \cap S_\pi$ is either S_π or a hyperplane of $\widetilde{S_\pi}$. By Lemma 3.1, it is impossible that $H \cap S_x = S_x$ and $H \cap S_\pi = S_\pi$.

Suppose $H \cap S_\pi = S_\pi$. Then $H \cap S_x$ is a hyperplane of $\widetilde{S_x}$. By the induction hypothesis, there exists a hyperplane β of Σ_x such that $H \cap S_x = e^{-1}(e(S_x) \cap \beta)$. Now, the hyperplane H is uniquely determined by $H \cap S_x$: a point $y \notin S_x \cup S_\pi$

is contained in H if and only if $L_y \cap S_x \subseteq H$. This implies that H is the hyperplane of $A_{n,k+1}$ arising from the hyperplane $\langle \beta, \Sigma_\pi \rangle$ of $\mathrm{PG}(\bigwedge^{k+1} V)$. In a completely similar way, one shows that if $H \cap S_x = S_x$, then H arises from the Grassmann-embedding of $A_{n,k+1}$.

Suppose $H \cap S_x$ is a hyperplane of $\widetilde{S_x}$ and $H \cap S_\pi$ is a hyperplane of $\widetilde{S_\pi}$. By the induction hypothesis, there exists a hyperplane β_1 of Σ_x and a hyperplane β_2 of Σ_y such that $H \cap S_x = e^{-1}(e(S_x) \cap \beta_1)$ and $H \cap S_\pi = e^{-1}(e(S_\pi) \cap \beta_2)$. Now, put $H_1 := e^{-1}(e(\mathcal{P}) \cap \langle \beta_1, \Sigma_\pi \rangle)$ and $H_2 := e^{-1}(e(\mathcal{P}) \cap \langle \beta_2, \Sigma_x \rangle)$. Then H_1 and H_2 are distinct hyperplanes of $A_{n,k+1}$. We show that

$$(H \cap L) \cap (H_1 \cap L) = (H_1 \cap L) \cap (H_2 \cap L) = (H \cap L) \cap (H_2 \cap L) \qquad (1)$$

for every line L meeting S_x and S_π.

If $L \cap S_x \subseteq H$ and $L \cap S_\pi \subseteq H$, then $H \cap L = H_1 \cap L = H_2 \cap L = L$ and (1) holds. If $L \cap S_x \subseteq H$ and $L \cap S_\pi \cap H = \emptyset$, then $H \cap L = L \cap S_x$, $H_1 \cap L = L$, $H_2 \cap L = L \cap S_x$ and (1) holds again. A similar reasoning applies to the case $L \cap S_\pi \subseteq H$ and $L \cap S_x \cap H = \emptyset$. Finally, suppose $L \cap S_x \cap H = L \cap S_\pi \cap H = \emptyset$. Then $H_1 \cap L = L \cap S_\pi$, $H_2 \cap L = L \cap S_x$ and $H \cap L$ is a singleton different from $H_1 \cap L$ and $H_2 \cap L$. So, (1) holds again.

By Lemma 3.1 and (1), $H_1 \cap H = H_1 \cap H_2 = H_2 \cap H$. By Corollary 2.3, H arises from the Grassmann-embedding of $A_{n,k+1}$.

References

[1] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179-185.

[2] E. E. Shult. Geometric hyperplanes of embeddable Grassmannians. *J. Algebra* 145 (1992), 55–82.

[3] A. L. Wells. Universal projective embeddings of the Grassmannian, half spinor, and dual orthogonal geometries. *Quart. J. Math. Oxford Ser.* 34 (1983), 375–386.