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Active Current, Reactive Current, Kirchhoff's Laws and Tellegen's Theorem

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Summary: The relations between the basic laws of circuit theory (Kirchhoff's laws and the conservation of generalized power) are discussed with respect to the active and reactive current components. It is shown that the active and reactive currents do not satisfy Kirchoff's current law. It is proved that this is not in contradiction with the property that the active and reactive powers satisfy the conservation property.

1. INTRODUCTION

The fundamental laws of circuit theory and network analysis are [1] Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL) and the law of the conservation of power or more generally Tellegen's theorem [2]:

— KCL expresses the conservation of charge, or the property that the sum of the currents in a node is zero.
— KVL expresses that the voltage difference is independent of the path, or the property that the voltage in a loop or a closed circuit is zero.
— Tellegen's theorem expresses the law of conservation of power, which means that the total (generalized) power in a circuit is zero or that the same total power is generated as dissipated in a network. This property not only holds for the physical real power, but also for the active power, the real power and the complex power.

These three basic laws are not independent. It is a very important feature that the generalized law of conservation of power holds for all voltages and currents satisfying Kirchhoff's laws, even if the currents and the voltages do not satisfy equations of physically realisable network elements. Moreover it has been proved [1, 2] that any two of the three basic laws (KCL, KVL and Tellegen's theorem) imply the third one. The fact that KCL and KVL imply Tellegen's theorem is well known, as pointed out above. It is not so well known that KVL and Tellegen's theorem imply KCL, and similarly that KCL together with Tellegen's theorem implies KVL. More explicitly, if in a graph for a set of branch currents the total power is zero for all voltages satisfying KVL, then the currents satisfy KCL. Conversely, if in a graph for a set of branch voltages the total power is zero for all currents satisfying KCL, then the voltages satisfy KVL.

Let us now consider the sinusoidal steady state. The currents and voltages can be represented by complex numbers, the phasors. For every branch in an electrical network the current can be split up into two parts:

— the active current: this current is in phase with the voltage and corresponds to the same active power as the actual current, and to zero reactive power;
— the reactive current: this current is orthogonal to the voltage and corresponds to the same reactive power as the actual current, and to zero active power.

It is well known that the conservation of power also holds for the complex power, and hence also for its real and imaginary parts. Thus the conservation property also holds for the active power as well as for the reactive power. From this property and from the fact that the active power and the reactive power are respectively generated by the branch voltage and respectively the active current and the reactive current, one may expect that the active currents as well as the reactive currents satisfy KCL.

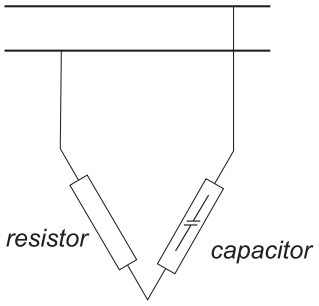

Fig. 1. A node where active and reactive currents do not satisfy Kirchoff's current law

However the following very simple example shows that this is not the case. Consider a node in a network connecting two branches with a resistive impedance and a reactive impedance respectively, as shown on Figure 1. The current in the resistive branch is obviously purely active and contains no reactive component. Similarly the current in the reactive branch is purely reactive and contains no active component. The sum of the currents in the two branches is zero; this corresponds to Kirchhoff's law in the node. But the sum of the active currents as well as the sum of the reactive currents is clearly non-zero.

In this paper this apparent contradiction is analysed. It is shown why the active currents and the reactive currents do not satisfy KCL. It is also explained why this does not contradict the property that Tellegen's theorem and KVL together imply KCL.

2. ACTIVE AND REACTIVE CURRENTS AND KIRCHHOFF'S CURRENT LAW

Consider a branch in a network with V the phasor of the branch voltage and I the phasor of the branch current. The active current I_a and the reactive current I_r are defined [3] as

the current components in phase and in quadrature with the branch voltage. They are explicitly given by the phasors:

$$I_a = \frac{\mathbf{Re}(VI^*)}{|V|^2}\ V \tag{1}$$

and:

$$I_r = \frac{\mathbf{Im}(VI^*)}{|V|^2}\ (-jV) \tag{2}$$

It is well known and also apparent from these expressions:

— that the active current corresponds to the active power in the network branch and to zero reactive power,
— that the reactive current corresponds to the reactive power in the network branch and to zero active power,
— that the branch current I equals the sum of the active current and the reactive current.

Let I_k for $k = 1, ..., m$, denote the currents in the m branches connected to a particular node of an electrical network. Assume that the positive direction of the currents is e.g. chosen towards the node. Each current is split up into its active and reactive current components I_{ak} and I_{rk} according to the expressions (1) and (2) above. Kirchhoff's current law implies:

$$\sum_{k=1}^{m} I_k = 0 \tag{3}$$

As already mentioned in the previous section, the active powers as well as the reactive powers in any electrical network satisfy the conservation property. This is not a direct consequence of the fundamental physical law of the conservation of power. Indeed neither the active power nor the reactive power are actual real powers. It is however a consequence of Tellegen's theorem, which is much more general that the fundamental physical law of power conservation. The question hence arises what can be said about the relationship between the active and reactive currents (corresponding to the active and reactive powers) and Kirchoff's current law.

It can be readily seen from expressions (1) and (2) that in general the active and the reactive current components in the branches connected to a node do not sum up to zero. The fundamental reason is that the proportionality factor between the active current (or the reactive current) and the branch current is not the same for all branches connected to a particular node, but clearly depends on the voltage of the branch under consideration. Hence in general the active currents in the branches connected to a node do not sum up to zero and do not satisfy KCL with respect to the considered node. The same is true for the reactive currents in these branches. If the ratios of the active current to the branch current are the same for all branches connected to the node, then the KCL is satisfied for the active currents. In the same way the KCL is satisfied for the reactive currents if the ratios of the reactive current to the branch current are the same for all branches connected to the node. Thus both the active currents and the reactive currents satisfy the KCL if the power factor of all branches connected to a node is the same, which also corresponds to the situation that the ratios of active to the reactive current are the same for all branches. Otherwise the active or reactive currents almost never satisfy the KCL.

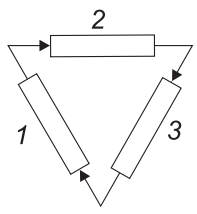

Fig. 2. Illustrative example of a single-loop electrical circuit

3. ILLUSTRATIVE EXAMPLE

As an illustrative example, we consider a circuit consisting of a single loop with three branches, shown on Figure 2. The branch reference directions are indicated in the figure. The references for voltages and currents are chosen as associated reference directions [1]. Thus the product of voltage and current in each branch corresponds to the power dissipated in the branch.

Let the current phasor be equal to 1 in the three branches and the voltage phasors respectively be equal to $1 + j$, $1 - j$ and -2. These voltages and currents clearly satisfy Kirchoff's laws. The phasors of the active and reactive currents and the active and reactive powers are shown in Table 1.

The complex powers in the three branches are respectively $1 + j$, $1 - j$ and -2, the phasors of the active currents $\frac{1}{2}(1 + j)$, $\frac{1}{2}(1 - j)$, and 1, the phasors of the reactive currents $\frac{1}{2}(1 - j)$, $\frac{1}{2}(1 + j)$, and 0, the active powers 1, 1, and –2, and the reactive powers 1, –1, and 0. Branch 1 is a resistive-inductive branch, branch 2 an resistive-capacitive branch and branch 3 delivers active power to branch 1 and to branch 2. The reactive power needed by branch 1 is delivered by branch 2. It is clear from Table 1 that the complex powers, the active powers and the reactive powers satisfy the conservation property, or equivalently Tellegen's theorem. It is however readily seen that only the total branch currents satisfy KCL in the three nodes, whereas the active currents as well as the reactive currents do not satisfy KCL.

Table 1.

	branch 1	branch 2	branch 3
branch current	1	1	1
branch voltage	1+j	1-j	-2
active current	½(1+j)	½(1-j)	1
reactive current	½(1-j)	½(1+j)	0
active power	1	1	-2
reactive power	1	-1	0
complex power	1+j	1-j	-2

4. ACTIVE AND REACTIVE VOLTAGES AND KIRCHHOFF'S VOLTAGE LAW

In a similar way as for the branch current, the branch voltage can be split up into an active and a reactive component. For a passive branch this corresponds to the voltage component across the active (or resistive) part of the branch impedance, and the voltage component across the reactive (inductive or capacitive) part of the branch impedance. Explicitly the active voltage V_a and the reactive voltage V_r are defined by:

$$V_a = \frac{\mathbf{Re}(VI^*)}{|I|^2}\ I \tag{4}$$

and:

$$V_r = \frac{\mathbf{Im}(VI^*)}{|I|^2}\ (jI) \tag{5}$$

The active and the reactive voltage components together with the branch current correspond respectively to the active and the reactive power of the branch.

Kirchhoff's voltage law states that the sum of the voltages in the branches (with positive direction in the same circulation sense) is zero. A similar discussion as in Section 2 shows that in general the active voltage components in the branches of a loop do not satisfy KVL. Similarly in general the reactive voltage components in the branches of a loop also do not satisfy KVL. Here also the question arises if this is not in contradiction with the fact that the currents satisfy KCL and the active and reactive powers satisfy the conservation property or, otherwise stated, Tellegen's theorem.

5. KIRCHHOFF'S LAWS AND TELLEGEN'S THEOREM

As already stated in Section 1, it is a very important general result of circuit theory [1] that any two properties expressed by the KVL, KCL and Tellegen's theorem, imply the third one. This leads to an apparent paradox. Indeed, considering the active and reactive currents in the branches of a circuit, the following properties hold:

— The branch voltages satisfy KVL.
— The active powers satisfy the conservation property (hence Tellegen's theorem).
— The reactive powers satisfy the conservation property (hence Tellegen's theorem).

This seems to imply that the active currents and the reactive currents should satisfy KCL. This is however not the case as discussed in the previous section.

To explain this apparent contradiction it should be clearly stated what is meant by „***KVL and Tellegen's theorem imply KCL***". As already stated in Section 1, this means that the currents satisfy KCL if the corresponding sum of the powers is zero for ***all*** voltages satisfying KVL, and not only for the voltages which are present in the network. It is indeed not sufficient that the sum of the powers is zero for the set of actual voltages (which satisfy KVL); from this it cannot be concluded that the currents satisfy KCL. For the active and the reactive currents we only know that their sums vanish (and hence Tellegen's theorem or the conservation property holds) for one particular set of voltages satisfying KVL, namely the actual branch voltages. This does not justify to conclude that the (active or reactive) currents satisfy KCL and explains the apparent contradiction. This is illustrated most clearly in the next section by the simple example considered in Section 3.

A similar discussion holds for the branch currents, the active voltage components and the active powers on the one hand, and the branch currents, the reactive voltage components and the reactive powers on the one hand. the following properties hold:

— The branch currents satisfy KCL.
— The active powers satisfy the conservation property (hence Tellegen's theorem).
— The reactive powers satisfy the conservation property (hence Tellegen's theorem).

This seems to imply that the active voltage components in the branches of a loop should satisfy KVL, and that also the reactive voltage components of the branches of a loop should satisfy KVL. This is however not the case. The explanation of this apparent contradiction is similar to the explanation given for the currents. Indeed the active and reactive powers only satisfy the conservation property for the given set of branch currents, not for all currents satisfying KCL.

6. EXAMPLE (continued)

We again consider the illustrative example of the network shown in Figure 2 and discussed in Section 3. The explanation of the apparent contradiction discussed in Section 5 is that the complex powers should satisfy the conservation property or equivalently Tellegen's theorem for all situations corresponding to the currents and any set of voltages such that the sum of voltages over the loop is zero, and not only the voltages present in the circuit. Therefore the total branch currents satisfy KCL. The same is not true for the powers corresponding to the active currents. The sum of these powers is only zero for the actual voltages and the actual active currents, not for all other voltages satisfying the KVL and the active currents. Similarly for the reactive currents: the sum of the reactive powers is only zero for the actual voltages and the actual reactive currents, not for all voltages satisfying the KVL and the reactive currents. This is why the active currents and the reactive currents do not satisfy KCL.

This can readily be seen by considering e.g. the fictitious voltage phasors 1, 1 and –2 in the three branches. These are not the actual branch voltages, but nevertheless they satisfy KVL. As summarized in Table 2, the corresponding (complex) powers satisfy Tellegen's theorem, but not the powers corresponding to the given active and reactive currents and this set of voltages. Indeed with that set of voltages the complex powers corresponding to the total currents are 1, 1, –2, and hence satisfy Tellegen's theorem. On the other hand the powers corresponding to the (original) active currents

Table 2.

	branch 1	branch 2	branch 3
branch current	1	1	1
active current	½(1+j)	½(1-j)	1
reactive current	½(1-j)	½(1+j)	0
fictitious voltage	1	1	-2
(fictitious) complex power			
for branch current	1	1	-2
for active current	½(1+j)	½(1-j)	-2
for reactive current	½(1-j)	½(1+j)	0

and these voltages are ½(1 + j), ½(1 – j), –2, and the powers corresponding to the (original) reactive currents and these voltages are ½(1 + j),½(1 + j), 0. We see that the sums of these powers do not vanish.

7. FURTHER OBSERVATIONS

The concepts of active and reactive currents have not only been defined for sinusoidal currents and voltages, but also for other situations, such as the case of instantaneous currents and the case of periodic nonsinusoidal currents [4–6]. The same discussion as above with respect to the corresponding powers, Kirchhoff's laws and Tellegen's theorem, can be carried out for these currents. A similar apparent contradiction is obtained concerning the fact that the powers corresponding to the active (or reactive) currents satisfy the conservation property and that the voltages satisfy Kirchhoff's voltage law, but that nevertheless the active (or reactive) currents do not satisfy KCL. This contradiction can be explained in a similar way as in the sinusoidal case.

8. CONCLUSIONS

In this paper the relation between the basic laws of circuit theory (Kirchoff's voltage and current laws and Tellegen's theorem or the principle of the conservation of generalized power) is discussed in relation with the active and reactive current components and the corresponding active and reactive powers. It is emphasized that the active and reactive currents do not satisfy KCL, although the active and reactive powers satisfy the conservation property. Hence on the one hand it is correct to say that if an element in a network takes (or generates) active (or reactive) power, then another element in the same network should generate (or take) that active (or reactive) power. On the other hand it is not correct to state that if an element in a network takes (or generates) active (or reactive) current, then another element in the same network should generate (or take) that active (or reactive) current. It is shown why this is no contradiction with the fundamental property that KVL and Tellegen's theorem imply KCL.

The discussion of this paper obviously does not intend to throw any doubt on the validity of Kirchhoff's laws or Tellegen's theorem. It is also emphasized that one should not conclude that the active and reactive currents do not have an interesting physical meaning, even though they do not satisfy KCL. The only valid conclusion is that in general the active and reactive currents cannot exist ***separately*** in the network, because they do not satisfy KCL. Another conclusion is that one should be very careful in the interpretation of the (correct) statement that ***„any two of the three properties expressed by the fundamental circuit laws (KVL, KCL and Tellegen's theorem) imply the third one"***.

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