The Hermitian variety $H(5,4)$ has no ovoid

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Abstract

In this paper the non-existence of ovoids of the polar space $H(5,4)$ is shown using a geometrical and combinatorial approach. We also give a new and unified proof for the non-existence of ovoids in the polar spaces $Q^-(2n+1,q)$, $H(2n,q^2)$ and $W(2n+1,q)$ for $n \geq 2$.

Keywords: ovoid, polar space, hermitian variety

1 Introduction

Let $\mathcal{P}$ be a finite polar space of rank $r \geq 2$. An *ovoid* $\mathcal{O}$ is a pointset of $\mathcal{P}$ which has exactly one point in common with each generator of $\mathcal{P}$.

For the finite classical polar spaces, existence or non-existence of ovoids is an important problem. A survey on this topic can be found in [4].

It is known that the polar space $H(2n,q^2)$, $n \geq 2$, has no ovoids [3]. Furthermore, in [2], it is proved that the polar space $H(2n+1,q^2)$, $q = p^h$, p prime, $n > 1$, has no ovoids if

$$p^{2n+1} > \binom{2n+p}{2n+1}^2 - \binom{2n+p-1}{2n+1}^2.$$

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This result is proved with algebraic techniques. Recently, A. Klein [1] obtained a comparable result, with a bound not as good as in [2], but using only geometrical and combinatorial arguments. If we consider again the bound of [2], then a computation shows that for all values of q the existence of ovoids of $H(5, q^2)$ is not excluded.

In this paper, the non-existence of ovoids of $H(5, 4)$ is proved using geometrical and combinatorial arguments. Up to now, no other comparable results for the polar space $H(5, q^2)$ are known.

In section 2 we present a new proof for known non-existence results of ovoids in the polar spaces $H(2n, q^2)$, $Q^-(2n+1, q)$, and $W(2n+1, q)$ for $q \geq 2$. This proof is very short and the same for these three types of polar spaces. The proof for the non-existence of ovoids in $H(5, 4)$ is given in the third section.

2 A new non-existence proof

We first show that $H(2n, q^2)$, $n \geq 2$, does not possess an ovoid. In fact, it is sufficient to show this for $n = 2$, because, for larger n, an ovoid $\mathcal{O}$ of $H(2n, q^2)$ induces ovoids in the polar spaces $H(2n-2, q^2)$ seen in the quotient geometries $P^\perp/P$ for points $P \in H(2n, q^2) \setminus \mathcal{O}$.

Suppose thus that $H(4, q^2)$ has an ovoid $\mathcal{O}$. Then $|\mathcal{O}| = q^5 + 1$. Embed $H(4, q^2)$ in $\mathrm{PG}(4, q^2)$ and consider a line l of $\mathrm{PG}(4, q^2)$ that does not belong to $H(4, q^2)$ and that meets $\mathcal{O}$ in $c \geq 1$ points. Every point of $\mathcal{O} \setminus l$ lies in $X^\perp$ for a unique point $X \in l \setminus \mathcal{O}$. If $X \in H(4, q^2)$, then X lies on $q^3 + 1$ generators (totally isotropic lines) that all meet $\mathcal{O}$ once and thus $|X^\perp \cap \mathcal{O}| = q^3 + 1$. If $X \notin H(4, q^2)$, then $X^\perp \cap H(4, q^2) = H(3, q^2)$ and $X^\perp \cap \mathcal{O}$ is an ovoid of this $H(3, q^2)$. Hence again $|X^\perp \cap \mathcal{O}| = q^3 + 1$. It follows that $|\mathcal{O}| - c = (q^2 + 1 - c)(q^3 + 1)$, that is $cq^3 = q^2 + q^3$, a contradiction.

Remark. The crucial point in the proof is that the non-tangent hyperplanes meet $H(4, q^2)$ in a polar space of the same rank, here $H(3, q^2)$, so that its intersection with the ovoid can be calculated. For this reason the same proof works for $Q^-(5, q)$ and hence shows the non-existence of ovoids of $Q^-(2n+1, q)$ for all $n \geq 2$. The proof also works for $W(5, q)$; here there are no non-tangent hyperplanes, and for a point $X \in l \setminus \mathcal{O}$, we have $|X^\perp \cap \mathcal{O}| = q^2 + 1$, since $\mathcal{O}$ induces in the polar space $W(3, q)$ of $X^\perp/X$ an ovoid. For $W(3, q)$, the final equation is $q^2 - 1 - c = (q + 1 - c)(q + 1)$, and does not give a contradiction but implies only that $c = 2$. Thus a non-symplectic line meets $\mathcal{O}$ either in 0 or 2 points. From this one deduces easily that every tangent hyperplane $P^\perp$ on a point $P \notin \mathcal{O}$ meets $\mathcal{O}$ in an arc with $q+1$ points. Also this set together with P forms a hyperoval in $P^\perp$, and thus q must be even. Hence $W(3, q)$, q odd, has no ovoid, and $W(2n+1, q)$ for $n \geq 2$ has no ovoid.

3 Non-existence of ovoids in $H(5, 4)$

Consider a hermitian variety $H(d, q^2)$ and its ambient projective space $\mathrm{PG}(d, q^2)$. Every plane of $\mathrm{PG}(d, q^2)$ is called a *hermitian plane*, if it meets $H(d, q^2)$ in a hermitian curve $H(2, q^2)$. A line of $\mathrm{PG}(d, q^2)$ is called a *hermitian line*, if it meets $H(d, q^2)$ in a $H(1, q^2)$ (that is a Baer-subline). A point of $\mathrm{PG}(d, q^2)$ is called a *hermitian point*, if it belongs to $H(d, q^2)$. If π is a plane meeting $H(d, q^2)$ in a $H(2, q^2)$, S_0 a point not belonging to π, then we denote with $S_0H(2, q^2)$ the cone with vertex S_0 and base $H(2, q^2)$, i.e. the points on the lines of $\mathrm{PG}(d, q^2)$ meeting π in a point of $H(2, q^2)$.

Lemma 3.1 *(a) $H(3, 4)$ has only two types of ovoids. One is $H(2, 4)$, and the other one has the following properties.*

(i) There exists a hermitian plane π and a hermitian line h of π, such that he ovoid consists of the six hermitian points of $\pi \setminus h$ and of the three hermitian points of the hermitian line $h^\perp$.

(ii) Every $H(2, 4)$ in $H(3, 4)$ meets the ovoid in $0, 1, 2, 3$ or 6 points.

(iii) Every $H(2, 4)$ that meets the ovoid in three points, meets it in three collinear points.

(b) Every partial ovoid of $H(3, 4)$ with 8 points can be extended to an ovoid.

Proof. (a) Let $\mathcal{O}$ be an ovoid of $H(3, 4)$. We consider a hermitian plane π and put $m := |\pi \cap \mathcal{O}|$. If $m = 9$, then the ovoid consists of the hermitian points in π and we are done. Suppose therefore that $m \neq 9$. Then there exists a point $P_1 \in \mathcal{O}$ with $P_1 \notin \pi$. Then $P_1^\perp \cap \pi$ is a hermitian line; its three hermitian points do not belong to $\mathcal{O}$, since they are perpendicular with P_1. Hence $m \leq 6$. If $m = 6$, then there are three points $P_1, P_2, P_3 \in \mathcal{O}$ and not in π and the previous argument shows that each P_i is perpendicular to the same hermitian line h of π. Then P_1, P_2, P_3 are the hermitian points of the hermitian line $h^\perp$. Thus, $\mathcal{O}$ is the ovoid described in (i). It is easy to see that (ii) and (iii) hold. Property (ii) also follows from the rest of the proof of part (a).

Assume that $m = 5$. Then there are four points in π that do not belong to $\mathcal{O}$ and four points P of $\mathcal{O}$ that are not in π. As before, each such point P gives rise to the hermitian line $h_P = P^\perp \cap \pi$, whose three hermitian points do not belong to $\mathcal{O}$. As there are four hermitian points in π that do not belong to $\mathcal{O}$, there is at most one such line h_P; hence all four choices of P lead to the same hermitian line $h = h_P$. But then $h^\perp$ contains four points P, which is not possible.

Assume that $m = 4$. Now there are five points P in $\mathcal{O} \setminus \pi$. Again consider the hermitian lines $h_p := P^\perp \cap \pi$, and keep in mind that their three hermitian points do not belong to $\mathcal{O}$. It is not possible that all five points P give the same line h_P. Then the are exactly two different such lines and they meet in a hermitian point (use $|\pi \setminus \mathcal{O}| = 5$). The intersection point of these two hermitian lines is thus in the perp of five points P, which is impossible.

Hence, if there exists an ovoid different from the two described in the lemma, then every hermitian plane meets it in at most three points. But then every plane meets it in at most three points, since the non-hermitian planes meet $H(3,4)$ in the union of three lines. But it is not possible to find nine points in $\mathrm{PG}(3,4)$ such that every plane contains at most three of these.

(b) As $H(3,4)$ has 27 lines, there are three lines l_1, l_2, l_3 of $H(3,4)$ not meeting the partial ovoid $\mathcal{O}$. The line l_i meets 10 other lines of $H(3,4)$, and eight of these meet $\mathcal{O}$. Thus the three lines l_i mutually meet. Hence, they pass through a common point P, and then $\mathcal{O} \cup \{P\}$ is an ovoid. □

Suppose that $\mathcal{O}$ is an ovoid of $H(5,q^2)$. Then $|\mathcal{O}| = q^5 + 1$. If P is a point of $H(5,q^2)$, then the $H(5,q^2)$ induces a $H(3,q^2)$ in the quotient geometry $P^\perp/P$. If $P \notin \mathcal{O}$, then the points of $\mathcal{O}$ in $P^\perp$ induce an ovoid of this $H(3,q^2)$. In particular, $P^\perp$ meets $\mathcal{O}$ in exactly $q^3 + 1$ points. Applying the results on ovoids in $H(3,4)$, we obtain the following.

Lemma 3.2 *Suppose that $\mathcal{O}$ is an ovoid of $H(5,4)$. Let P be a point of $H(5,4)$ not in $\mathcal{O}$. Then $|P^\perp \cap \mathcal{O}| = 9$. If π is a hermitian plane in $P^\perp$, then*

$|\langle P, \pi \rangle \cap \mathcal{O}| \in \{0, 1, 2, 3, 6, 9\}$.

Lemma 3.3 *Suppose $\mathcal{O}$ is an ovoid of $H(5, q^2)$. Consider a plane π that meets the variety in $H(2, q^2)$ and put $m := |\pi \cap \mathcal{O}|$. Suppose furthermore that $m \geq 1$. Let A resp. B be the set consisting of all points $X \in \mathcal{O}$ with $X \notin \pi$ such that $\langle \pi, X \rangle$ meets $H(5, q^2)$ in a $S_0H(2, q^2)$ resp. an $H(3, q^2)$.*

(a) We have $|A| = (q^2 - 1)(q^2 - 1 + m)$ and $|B| = q^2(q^3 - q^2 + 2 - m)$.

(b) If $q = 2$ and X is a hermitian point of $\pi \backslash \mathcal{O}$, then $|X^\perp \cap B| \in \{0, 3, 6, 7, 8, 9\}$.

Proof. Put $\alpha := |A|$ and $\beta := |B|$. We have $|\mathcal{O}| = q^5+1$, so $\alpha+\beta = q^5+1-m$. We count in two ways the number of pairs (U, W) with $U \in \pi \cap H(5, q^2)$ and $W \in A \cup B$ and such that UW is a totally isotropic line.

A point U of $\pi \cap H(5, q^2)$ does not occur in such a pair, if $U \in \mathcal{O}$. If however, $U \notin \mathcal{O}$, then U occurs in q^3+1 such pairs, since $\mathcal{O}$ induces an ovoid in $U^\perp/U$ as seen in section 2. Hence the number of pairs is $(q^3 + 1 - m)(q^3 + 1)$.

Now consider a point $V \in \mathcal{O}$ with $V \notin \pi$. If $V \in B$, that is $\langle V, \pi \rangle$ meets $H(5, q^2)$ in a $H(3, q^2)$, then $V^\perp$ meets π in a $H(1, q^2)$, so V occurs in $q + 1$ pairs (U, V). If $V \in A$, that is $\langle V, \pi \rangle$ meets $H(5, q^2)$ in a cone with a point vertex over the $H(2, q^2)$ in π, then $V^\perp \cap \pi$ is a tangent line, containing exactly one point U (notice that V cannot be the vertex, since π contains at least one point of $\mathcal{O}$). In this case V occurs in exactly one pair (U, V). Hence the number of pairs is $\alpha + \beta(q + 1)$. Thus

$$\alpha + \beta(q + 1) = (q^3 + 1)(q^3 + 1 - m).$$

Using $\alpha + \beta = |\mathcal{O}| - m$, the assertion of (a) follows.

For the proof of (b) suppose that $q = 2$ and consider a hermitian point X of π that is not in $\mathcal{O}$. Then the 9 points of $X^\perp \cap \mathcal{O}$ lie in $A \cup B$. By the definition of A, the points of $X^\perp \cap A$ lie in the solid $\langle X, \pi^\perp \rangle$. By the definition of B, no point of B lies in $\langle X, \pi^\perp \rangle$. If consider the hermitian variety $H(3, q^2)$ induced in $X^\perp/X$ and the ovoid induced in $H(3, q^2)$ by $\mathcal{O}$, we see that this ovoid has a hermitian plane with $|X^\perp \cap A|$ points of the ovoid. Apply Lemma 3.2 to deduce that $|X^\perp \cap A| \in \{0, 1, 2, 3, 6, 9\}$. Using $|X^\perp \cap (A \cup B)| = 9$, the assertion of (b) follows. □

Lemma 3.4 *Suppose $\mathcal{O}$ is an ovoid of $H(5, 4)$. Consider a plane π that meets the variety in $H(2, 4)$ and put $m := |\pi \cap \mathcal{O}|$. Then $m \leq 6$ and $m \neq 4, 5$.*

Proof. Use the notation of the preceding lemma. As $|B| = 4(6 - m)$, we have $m \leq 6$.

Case 1. $m = 5$. Then $|B| = 4$ and $|A| = 24$. Consider $V \in B$. As $\langle \pi, V \rangle$ meets $H(5,4)$ in a $H(3,4)$ then $V^{\perp}$ meets π in a hermitian line. The three points of the hermitian line do not belong to $\mathcal{O}$ since V lies in $\mathcal{O}$. Now π has nine hermitian points and five of these belong to $\mathcal{O}$. Hence at most one hermitian line of π can miss $\mathcal{O}$. This shows that all 4 points $V \in B$ are perpendicular to the same hermitian line h of π. Thus, the points X of this hermitian line satisfy $|X^{\perp} \cap B| = 4$, contradicting part (b) of the previous lemma.

Case 2. $m = 4$. Then $|B| = 8$. As in the case $m = 5$ we see that each $V \in B$ gives rise to a hermitian line $h_V := V^{\perp} \cap \pi$ of π such that the three hermitian points of h_V do no lie in $\mathcal{O}$. This time five of the hermitian points of π do not belong to $\mathcal{O}$, so there is space for one or two such hermitian lines.

First suppose that the points V produce two different lines h_V. Call them h_1 and h_2, and let x_i be the number of points $V \in B$ satisfying $h_V = h_i$. Then $1 \leq x_1, x_2$ and $x_1 + x_2 = |B| = 8$. A hermitian point X that lies in h_1 but not in h_2 satisfies $|X^{\perp} \cap B| = x_1$. Thus, the previous lemma shows that $x_1 \in \{0, 3, 6, 7, 8, 9\}$. The same holds for x_2. As $x_i > 0$ and $x_1 + x_2 = 8$, this is impossible.

Hence all points $V \in B$ produce the same hermitian line $h = h_V$. Thus $B \subseteq h^{\perp}$. Here $h^{\perp} \cap H(5,4)$ is a $H(3,q)$. Since $\pi^{\perp} \subseteq h^{\perp}$, and $h^{\perp} \cap \pi$ is a non-hermitian point, then no point of $\pi \cap \mathcal{O}$ lies in $h^{\perp}$. Each point of A lies in a 3-space $\langle \pi, U \rangle$ with a hermitian point $U \in \pi^{\perp}$, and thus lies in a 3-space $\langle \pi^{\perp}, U' \rangle$ with a hermitian point $U' \in \pi$. Hence $h^{\perp} \cap A = \emptyset$.

Thus the hermitian variety $H' := h^{\perp} \cap H(5,4)$, which is a $H(3,q)$ meets $\mathcal{O}$ only in the eight points of B. These eight points form a partial ovoid $\mathcal{O}'$ of H'. From Lemma 3.1, we see that $\mathcal{O}'$ can be extended to an ovoid $\mathcal{O}''$ of the $H(3,q^2)$. The ovoid $\mathcal{O}''$ can not be a hermitian curve, because otherwise the eight points of $\mathcal{O}'$ would lie in a hermitian plane; but we have already seen earlier in the proof that every hermitian plane has at most six points. Thus, $\mathcal{O}''$ is an ovoid of the second type. As $\mathcal{O}'$ is obtained by removing one point, it is easy to see that some hermitian plane meets $\mathcal{O}'$ in exactly five points. But we have seen earlier in this proof that hermitian planes can not meet $\mathcal{O}$ in precisely five points. This final contradiction shows that $m = 4$ is also not possible □

Lemma 3.5 *Suppose $\mathcal{O}$ is an ovoid of $H(5,4)$. Then $|\pi \cap \mathcal{O}| \leq 3$ for every hermitian plane π.*

Proof. From the preceding lemma we know that each hermitian plane meets $\mathcal{O}$ in six or at most three points. Assume that there exists a hermitian plane π with six points in $\mathcal{O}$. Let X_1, X_2 and X_3 be the three hermitian points of π that are not in $\mathcal{O}$. From Lemma 3.3 we see the following. For each of the 27 points P of $\mathcal{O}$ that are not in π, the solid $\langle \pi, P \rangle$ meets $H(5,4)$ in a cone over the hermitian curve in π; hence P lies in a solid $\langle \pi, U \rangle$ for a unique hermitian point $U \in \pi^{\perp}$.

Case 1. X_1, X_2, X_3 are non-collinear. Then there are three hermitian lines h_1, h_2, h_3 in π missing $\{X_1, X_2, X_3\}$. These three lines are not concurrent. Also the three hermitian points of h_i belong to $\mathcal{O}$. Consider a point $P \in A$. Then $\langle h_i, P \rangle$ is a plane with at least four points in $\mathcal{O}$. Hence, $\langle h_i, P \rangle$ is a hermitian plane and, by the previous lemma, it meets $\mathcal{O}$ in six points. Thus in the solid $\langle \pi, P \rangle$ we have six points of $\mathcal{O}$ in π and in each plane $\langle h_i, P \rangle$, $i = 1, 2, 3$, we see three more points of $\mathcal{O}$. As h_1, h_2, h_3 are not concurrent, this implies that $\langle \pi, P \rangle$ meets $\mathcal{O}$ in more than nine points, a contradiction.

Case 2. The three points X_1, X_2, X_3 are collinear. The line h through these is a hermitian line. Consider $P \in A$. Then P lies in a solid $\langle \pi, U \rangle$ for some point $U \in \pi^{\perp}$, and hence P lies in $\langle \pi^{\perp}, X \rangle$ for some hermitian point $X \in \pi$. As $P \in \mathcal{O}$, then $X \notin \mathcal{O}$. Hence each of the 27 points P of $\mathcal{O}$ with $P \notin \pi$ lies in $\langle \pi^{\perp}, X_i \rangle$ for some $i = 1, 2, 3$. Hence these 27 points lie in the 4-space $V := \langle \pi^{\perp}, h \rangle$. Consider two points in $V \cap \mathcal{O}$. As there are 21 planes on these two points in V, we find a plane π' on these two points with at least four and hence six points in $\mathcal{O}$. As before this gives rise to a 4-space V' that contains the 27 points of $\mathcal{O}$ that are not in π'. Then V and V' are distinct 4-spaces, each containing 27 points of $\mathcal{O}$. As the 3-space $V \cap V'$ can meet $\mathcal{O}$ in at most nine points, it follows that $|\mathcal{O}| \geq 2 \cdot 27 - 9$, which is a contradiction. $\square$

Lemma 3.6 *If S is a 3-space S with $S \cap H(5,4) = H(3,4)$, then $|S \cap \mathcal{O}| < 6$*

Proof. Assume that $|S \cap \mathcal{O}| \geq 6$. Consider a point V of $H(5,4)$ that lies in the perp of S. Then $\langle V, S \rangle \cap H(5,4)$ is a cone with vertex V over the $H(3,4)$ in S, and thus $\langle V, S \rangle$ meets $\mathcal{O}$ in precisely nine points. In fact, if we project these nine points from V onto the $H(3,4)$ in S, we obtain an ovoid $\mathcal{O}'$ of the $H(3,4)$. Six of the points of $\mathcal{O}'$ are also points of $\mathcal{O}$. Investigating the two possible types for $\mathcal{O}'$, we find a plane π of S that meets $\mathcal{O}'$ in six or nine points and at least four of which belong to $\mathcal{O}$. This contradicts the previous lemma. $\square$

Lemma 3.7 $H(5,4)$ *does not have an ovoid.*

Proof. Assume $\mathcal{O}$ is an ovoid of $H(5,4)$. By our results, every hermitian plane meets $\mathcal{O}$ in at most three points. Hence, every plane meets $\mathcal{O}$ in at most three points.

Consider a hermitian point V with $V \notin \mathcal{O}$, and the induced variety $H(3,q^2)$ in $V^\perp/V$. Then $V^\perp \cap \mathcal{O}$ is an *ovoid* of this $H(3,q^2)$. Investigating the two types of ovoids in $H(3,4)$ (of the quotient geometry), we see that some hermitian plane of the $H(3,4)$ meets the ovoid in at least six points. Hence, there exists a cone with vertex V over a hermitian curve that meets $\mathcal{O}$ in at least six points. In this cone we find a hermitian plane with more than two points in $\mathcal{O}$.

Hence, there exists a hermitian plane π with exactly three points in $\mathcal{O}$. These three points of $\pi \cap \mathcal{O}$ are not collinear, since otherwise they generate with every other point of $\mathcal{O}$ a hermitian plane with more than three points. Thus, exactly three hermitian lines h_1, h_2, h_3 of π miss $\mathcal{O}$. These three lines form a triangle and pairwise meet in an hermitian point. Use the notation of A and B from Lemma 3.3. Then $|B| = 12$. For each point $P \in B$, we have that $h_P := P^\perp \cap \pi$ is a hermitian line; as $P \in \mathcal{O}$, then $h_P \cap \mathcal{O} = \emptyset$ and hence $h_P = h_i$ for some i.

Let x_i be the number of points $P \in B$ with $h_P = h_i$. Then $x_1 + x_2 + x_3 = |B| = 12$. We may assume that $x_1 \geq 4$. Consider the hermitian point C of h_1 that does not lie on h_2 and not on h_3. Then C is perpendicular to exactly x_1 points of B. Part (b) of Lemma 3.3 gives thus $x_1 \geq 6$. Hence $h_1^\perp$ contains $x_1 \geq 6$ points of $\mathcal{O}$. As $h_1^\perp$ meets $H(5,4)$ in a $H(3,4)$, this contradicts the preceding lemma. $\square$

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