The synthesis of a quantum circuit

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Abstract

As two basic building blocks for any quantum circuit, we consider the 1-qubit `NEGATOR`(θ) circuit and the 1-qubit `PHASOR`(θ) circuit, extensions of the `NOT` gate and `PHASE` gate, respectively: `NEGATOR`(π) = `NOT` and `PHASOR`(π) = `PHASE`. Quantum circuits (acting on w qubits) consisting of controlled `NEGATOR`s are represented by matrices from $\mathrm{XU}(2^w)$; quantum circuits (acting on w qubits) consisting of controlled `PHASOR`s are represented by matrices from $\mathrm{ZU}(2^w)$. Here, $\mathrm{XU}(n)$ and $\mathrm{ZU}(n)$ are subgroups of the unitary group $\mathrm{U}(n)$: the group $\mathrm{XU}(n)$ consists of all $n \times n$ unitary matrices with all line sums equal to 1 and the group $\mathrm{ZU}(n)$ consists of all $n \times n$ unitary diagonal matrices with first entry equal to 1. We conjecture that any $\mathrm{U}(n)$ matrix can be decomposed into four parts: $U = e^{i\alpha} Z_1 X Z_2$, where both Z_1 and Z_2 are $\mathrm{ZU}(n)$ matrices and X is an $\mathrm{XU}(n)$ matrix. For $n = 2^w$, this leads to a decomposition of a quantum computer into simpler blocks.

1 Introduction

A classical reversible logic circuit, acting on w bits, is represented by a permutation matrix, i.e. a member of the finite matrix group $\mathrm{P}(2^w)$. A quantum circuit, acting on w qubits, is represented by a unitary matrix, i.e. a member of the infinite matrix group $\mathrm{U}(2^w)$. The classical reversible circuits form a subgroup of the quantum circuits. This is a consequence of the group hierarchy

$$\mathrm{P}(n) \subset \mathrm{U}(n) ,$$

where n is allowed to have any (positive) integer value.

Below, we will construct an arbitrary quantum circuit according to a bottom-up approach. For this purpose, we start from the simplest logic operation possible on a single (qu)bit (i.e. $w = 1$ and thus $n = 2^w = 2$), being the `IDENTITY` operation $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Next, we consider two different square roots of that 2×2 matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The former is a permutation matrix and thus represents a classical logic gate, i.e. the `NOT` gate; the latter is not a permutation matrix, but is a unitary matrix and therefore represents a quantum logic gate, called the `PHASE` gate.

Next, we interpolate between the `IDENTITY` u and an as of yet arbitrary unitary matrix q:

$$m = (1-t)u + tq ,$$

where t is a parameter interpolating between u (for $t = 0$) and q (for $t = 1$). We impose that m is a unitary matrix. If $q^2 = u$, then this leads to the condition that t is complex and of the form

$$t = \frac{1}{2} \left(1 - e^{i\theta} \right) ,$$

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where θ is a real parameter [1]. Note that $t = 0$ for $\theta = 0$ and $t = 1$ for $\theta = \pi$. For $\theta = 2\pi$, the value of t has returned to 0. Now, by choosing $q =$ NOT and $q =$ PHASE, respectively, this leads to two different 1-parameter single-qubit operations:

$$\begin{pmatrix} \cos(\theta/2)e^{-i\theta/2} & i\sin(\theta/2)e^{-i\theta/2} \\ i\sin(\theta/2)e^{-i\theta/2} & \cos(\theta/2)e^{-i\theta/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \; .$$

We denote them with the schematics

$$-\boxed{N(\theta)}- \quad \text{and} \quad -\boxed{\Phi(\theta)}- \; ,$$

respectively. The former operation, we call the `NEGATOR` gate [2]; the latter, we call the `PHASOR` gate. Each of these two sets of matrices constitutes a continuous group, i.e. a 1-dimensional Lie group. Both groups contain the `IDENTITY` circuit. Indeed: $\texttt{NEGATOR}(0) = \texttt{PHASOR}(0) =$ `IDENTITY`. Additionally, by construction, the `NOT` gate is a `NEGATOR` and the `PHASE` gate is a `PHASOR`. Indeed:

$$\texttt{NEGATOR}(\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \texttt{PHASOR}(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ,$$

sometimes abbreviated to `X` and `Z` gate, respectively [4]. For $\theta = \pi/2$, we have the square root of `NOT` and the square root of `PHASE`:

$$\texttt{NEGATOR}(\pi/2) = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \quad \text{and} \quad \texttt{PHASOR}(\pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \; .$$

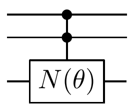
The former is sometimes referred to as the `V` gate [3] [5], the latter is sometimes called the `S` gate [6]. Finally, for $\theta = \pi/4$, we have the quartic roots

$$\texttt{NEGATOR}(\pi/4) = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1+i & \sqrt{2}-1-i \\ \sqrt{2}-1-i & \sqrt{2}+1+i \end{pmatrix} \quad \text{and} \quad \texttt{PHASOR}(\pi/4) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1+i \end{pmatrix} \; ,$$

sometimes called the `W` gate [5] and the `T` gate [6] [7], respectively.

Now, we consider multiple-qubit (say, w-qubit) circuits. For this purpose, we introduce both the controlled `NEGATOR` gates and the controlled `PHASOR` gates. As an example, we give here the $w = 3$ schematic of the positive-polarity twice-controlled `NEGATOR` (the lowermost quantum wire being the target line), represented by the block-diagonal matrix

$$\begin{pmatrix} \mathbf{1}_{6\times 6} & & \\ & \cos(\theta/2)e^{-i\theta/2} & i\sin(\theta/2)e^{-i\theta/2} \\ & i\sin(\theta/2)e^{-i\theta/2} & \cos(\theta/2)e^{-i\theta/2} \end{pmatrix} =$$



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where $\mathbf{1}_{a\times a}$ denotes the $a \times a$ unit matrix. Of course, we equally introduce controlled `PHASOR`s, negative-polarity controls, a target on a higher-positioned wire, etc...

It turns out [9] that all possible `NEGATOR`s and controlled `NEGATOR`s together generate a group $\mathrm{XU}(2^w)$, subgroup of the unitary group $\mathrm{U}(2^w)$. They cannot generate the full $\mathrm{U}(2^w)$ group, because the matrix representing a (controlled) `NEGATOR` has all line sums (i.e. all row sums and all column sums) equal to 1. The multiplication of two matrices with all line sums equal to 1 yields again a unit-line-sum matrix. Therefore a quantum circuit composed exclusively of (controlled) `NEGATOR`s cannot synthesize a unitary matrix with one or more line sums different from unity. Whereas the unitary goup $\mathrm{U}(n)$ has n^2 dimensions, the group $\mathrm{XU}(n)$ has only $(n-1)^2$ dimensions and is isomorphic to $\mathrm{U}(n-1)$ [8] [9] [10]. Analogously, a quantum circuit composed exclusively of (controlled) `PHASOR`s can only generate matrices from $\mathrm{ZU}(2^w)$, where $\mathrm{ZU}(n)$ is the group of diagonal unitary matrices with unit entry at the upper-left corner. The group $\mathrm{ZU}(n)$ has only $(n-1)$ dimensions [10].

We can summarize as follows: we find two subgroups of the unitary group $\mathrm{U}(n)$:

- $\mathrm{XU}(n)$, i.e. all $n \times n$ unitary matrices with all of their $2n$ line sums equal to 1;
- $\mathrm{ZU}(n)$, i.e. all $n \times n$ diagonal unitary matrices with upper-left entry equal to 1.

Whereas the infinite unitary group U(n) describes quantum computing, the finite permutation group P(n) describes classical reversible computing. Whereas XU(n) is both supergroup of P(n) and subgroup of U(n), in contrast, ZU(n) is a subgroup of U(n) but not a supergroup of P(n):

$$\mathrm{P}(n) \subset \mathrm{XU}(n) \subset \mathrm{U}(n) \tag{1}$$

$$\mathrm{ZU}(n) \subset \mathrm{U}(n) \; . \tag{2}$$

The XU circuits therefore can be considered as circuits 'between' classical and quantum circuits, whereas the ZU circuits are truly non-classical circuits.

2 First decomposition of a unitary matrix

In Reference [1], the following theorem is proved: any U(n) matrix U can be decomposed as

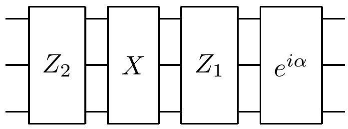
$$U = e^{i\alpha}\, Z_1 X_1 Z_2 X_2 Z_3 ... Z_{p-1} X_{p-1} Z_p \; ,$$

with $p \leq n(n-1)/2 + 1$ and where all Z_j are ZU(n) matrices and all X_j are XU(n) matrices. In Reference [10], it is proved that a shorter decomposition exists: with $p \leq n$. Finally, in Reference [11], it is conjectured that an again shorter decomposition exists: with $p \leq 2$.

In the present paper, we investigate what would be the consequences of the conjecture that each U(n) matrix can be decomposed as

$$U = e^{i\alpha}\, Z_1 X Z_2 \; . \tag{3}$$

Reference [11] provides a numerical algorithm to find the number α and the matrices Z_1, X, and Z_2 for a given matrix U, based on a Sinkhorn-like approach. According to the conjecture, a quantum schematic (here for $w = 3$ and thus $n = 8$) looks like



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If n is even, then we note the identity

$$\mathrm{diag}(a, a, a, a, a, ..., a, a) = \; P_0 \; \mathrm{diag}(1, a, 1, a, 1, ..., 1, a) \; P_0^{-1} \; \mathrm{diag}(1, a, 1, a, 1, ..., 1, a) \; ,$$

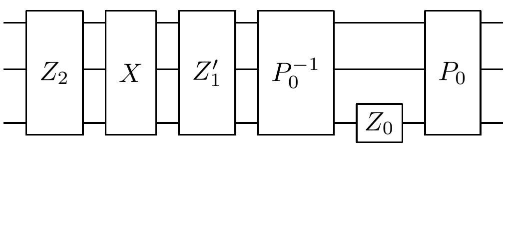
where a is a short-hand notation for $e^{i\alpha}$ and P_0 is the (circulant) permutation matrix

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 \end{array}\right) ,$$

i.e. the P matrix called the cyclic-shift matrix, which can be implemented with classical reversible gates (i.e. one `NOT` and $w - 1$ controlled `NOT`s [12] [13]). We thus can transform (3) into a decomposition containing exclusively XU and ZU matrices:

$$U = P_0 Z_0 P_0^{-1} Z_1' X Z_2 \; ,$$

where $Z_0 = \mathrm{diag}(1, a, 1, a, 1, ..., 1, a)$ is a ZU matrix which can be implemented by a single (uncontrolled) `PHASOR` gate and where Z_1' is the product $Z_0 Z_1$:



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3 Further decomposition of a unitary matrix

For convenience, we rewrite eqn (3) as

$$U = e^{i\alpha_n}\ L_n X_n R_n\ ,$$

where the left matrix L_n and the right matrix R_n are members of ZU(n) and X_n belongs to XU(n). As a member of the $(n-1)^2$-dimensional group XU(n), X_n has the following form [9]:

$$X_n = T_n \begin{pmatrix} 1 & \\ & U_{n-1} \end{pmatrix} T_n^{-1}\ ,$$

where U_{n-1} is a member of U($n-1$) and T_n is an $n \times n$ generalized Hadamard matrix. Reference [9] provides the algorithm to find the U_{n-1} matrix corresponding to a given XU(n) matrix X_n.

Again according to the De Vos–De Baerdemacker conjecture [11], U_{n-1} can be decomposed as $e^{i\alpha_{n-1}}\, l_{n-1} x_{n-1} r_{n-1}$, a product of a scalar, a ZU($n-1$) matrix, an XU($n-1$) matrix, and a second ZU($n-1$) matrix. We thus obtain for X_n the product $T_n L_{n-1} X_{n-1} R_{n-1} T_n^{-1}$, where

$$L_{n-1} = \begin{pmatrix} 1 & \\ & e^{i\alpha_{n-1}}\, l_{n-1} \end{pmatrix}\ ,\ \ X_{n-1} = \begin{pmatrix} 1 & \\ & x_{n-1} \end{pmatrix}\ ,\ \ \text{and}\ \ R_{n-1} = \begin{pmatrix} 1 & \\ & r_{n-1} \end{pmatrix}\ .$$

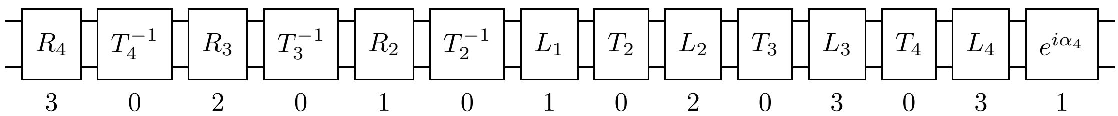
Hence, we have $U = e^{i\alpha_n}\ L_n T_n L_{n-1} X_{n-1} R_{n-1} T_n^{-1} R_n$. By applying such decomposition again and again, we find a decomposition $e^{i\alpha_n}\ L_n T_n L_{n-1} T_{n-1} L_{n-2} ... T_2 L_1 X_1 R_1 T_2^{-1} R_2 ... R_{n-2} T_{n-1}^{-1} R_{n-1} T_n^{-1} R_n$ of an arbitrary member of XU(n). As automatically X_1 and R_1 equal the unit matrix $\mathbf{1}_{n\times n}$, we thus obtain

$$U = e^{i\alpha_n}\ L_n T_n L_{n-1} T_{n-1} L_{n-2} ... T_2 L_1 T_2^{-1} R_2 ... R_{n-2} T_{n-1}^{-1} R_{n-1} T_n^{-1} R_n\ , \tag{4}$$

where all n matrices L_j and all $n-1$ matrices R_j belong to the $(n-1)$-dimensional group ZU(n). The $n-1$ matrices T_j are block-diagonal matrices of the form

$$T_j = \begin{pmatrix} A & \\ & S_j \end{pmatrix}\ ,$$

where A is an arbitrary $(n-j) \times (n-j)$ unitary matrix and S_j is a $j \times j$ generalized Hadamard matrix. An obvious choice consists of A equal to $\mathbf{1}_{(n-j)\times(n-j)}$ and S_j equal to the $j \times j$ discrete Fourier transform. For $w = 2$ (and thus $n = 4$), eqn (4) thus looks like the following cascade of six constant matrices, seven ZU circuits, and one overall phase:



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where the T_j blocks represent the $n-1$ constant matrices

$$T_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\ ,\ T_3 = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ & 1/\sqrt{3} & \omega/\sqrt{3} & \omega^2/\sqrt{3} \\ & 1/\sqrt{3} & \omega^2/\sqrt{3} & \omega/\sqrt{3} \end{pmatrix}\ ,$$

$$\text{and}\quad T_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\ ,$$

with ω equal to the cubic root of unity ($\omega = e^{i\,2\pi/3} = -1/2 + i\,\sqrt{3}/2$). Beneath each of the $4n-2$ blocks is displayed the number of real parameters of the block. These numbers sum to 16, i.e. exactly n^2, the dimensionality of U(n).

Hence, the synthesis problem of an arbitrary U(2^w) matrix is reduced to two smaller problems. First, for the given value of w, we have to synthesize the $2^w - 1$ circuits T_j. Then, for the particular matrix U, we have to synthesize the $2^{w+1} - 1$ circuits of type ZU(2^w). The synthesis of an arbitrary ZU(2^w) circuit is discussed in the next section.

We close the present section by deriving from (4) a dual decomposition. By introducing the matrices $L'_j = T_{j+1} L_j T_{j+1}^{-1}$ and $R'_j = T_{j+1} R_j T_{j+1}^{-1}$, for $j < n$, as well as $L'_n = T_n L_n T_n^{-1}$ and $R'_n = T_n R_n T_n^{-1}$, we indeed find

$$U = e^{i\alpha_n} \; T_n^{-1} L'_n T_n L'_{n-1} T_n L'_{n-2} T_{n-1} ... T_3 L'_1 T_3^{-1} R'_2 ... T_{n-1}^{-1} R'_{n-2} T_n^{-1} R'_{n-1} T_n^{-1} R'_n T_n \; ,$$

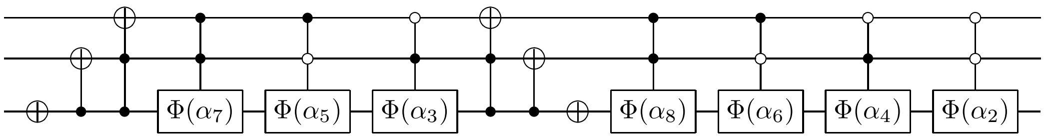
where all matrices L'_j and R'_j belong to an $(j-1)$-dimensional subgroup of XU(n). If, in particular, each T_j is composed of the $(n-j)\times(n-j)$ unit block combined with the $j \times j$ discrete Fourier transform, then this subgroup consists of block-diagonal matrices with an $(n-j)\times(n-j)$ unit block and a $j \times j$ circular matrix from XU(j) [12] [14].

4 Synthesizing a ZU circuit

The decomposition of a matrix Z, arbitrary member of ZU(n), is straigthforward. Indeed, for even n, the matrix can be written as the following product of four matrices:

$$\text{diag}(1, a_2, a_3, a_4, a_5, a_6, ..., a_n) =$$
$$\text{diag}(1, a_2, 1, a_4, 1, a_6, ..., 1, a_n) \; P_0 \; \text{diag}(1, 1, 1, a_3, 1, a_5, ..., 1, a_{n-1}) \; P_0^{-1} \; ,$$

where a_j is a short-hand notation for $e^{i\alpha_j}$. If n equals 2^w, then the diagonal matrix diag$(1, a_2, 1, a_4, 1, a_6, ...)$ represents 2^{w-1} `PHASOR`s, controlled $(w-1)$ times, and the diagonal matrix diag$(1, 1, 1, a_3, 1, a_5, ...)$ represents $2^{w-1}-1$ `PHASOR`s, controlled $(w-1)$ times. E.g. for $w = 3$, we obtain



We thus have a total of $2^w - 1$ controlled `PHASOR`s. According to Lemma 7.5 of Barenco et al. [15], each multiply-controlled gate $\Phi(\alpha)$ can be replaced by classical gates and three singly-controlled `PHASOR`s $\Phi(\pm\,\alpha/2)$. According to De Vos and De Baerdemacker [10], each singly-controlled `PHASOR` $\Phi(\beta)$ can be decomposed into two controlled `NOT`s and three uncontrolled `PHASOR`s $\Phi(\pm\,\beta/2)$. We thus obtain a circuit with a total of $9(2^w-1)$ uncontrolled `PHASOR`s.

5 Conclusion

We have demonstrated that, provided the ZXZ-conjecture of De Vos and De Baerdemacker is true, an arbitrary quantum circuit, acting on w qubits, can be decomposed into $2^{w+1}-1$ blocks, each described by a $2^w \times 2^w$ matrix from the (2^w-1)-dimensional Lie group ZU(2^w), subgroup of U(2^w), separated by $2(2^w-1)$ `FOURIER` circuits. The ZU blocks can be further decomposed into classical gates and a total of $9(2^{w+1}-1)(2^w-1)$ uncontrolled `PHASE` gates. As $\Phi(\theta) = \mathtt{H}\, N(\theta) \mathtt{H}$, each uncontrolled `PHASE` gate can be substituted by two `HADAMARD` gates and one uncontrolled `NEGATOR` gate. Taking into account that the `HADAMARD` gate is a `FOURIER` circuit, we thus have provided two synthesis algorithms, based on two different (dual) gate libraries:

- classical gates + `FOURIER` circuits + `PHASE` gate and
- classical gates + `FOURIER` circuits + `NEGATOR` gate.

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