Conditional probability and Van Lambalgen's theorem

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Van Lambalgen's theorem states that a pair (α, β) of sequences is Martin-Löf random if and only if α is Martin-Löf random and β is Martin-Löf random relative to α. The definition of Martin-Löf randomness can be extended to define randomness of (pairs of) sequences relative to any computable (bivariate) measure P.

In this talk we discuss generalizations of Van Lambalgen's theorem for such measures P. Following the original proof, one can show that if P is a product of computable measures, the theorem still holds. However, to generalize the theorem for measures P that are non-product measures, one needs to define conditional probability. Let λ be the empty string and let

$$P(x|\alpha) = \lim_{n} \frac{P(x, \alpha_1 \dots \alpha_n)}{P(\lambda, \alpha_1 \dots \alpha_n)}.$$

This measure is a Radon-Nikodym derivative of $P(x, \alpha)$ relative to $P(\lambda, \alpha)$. In [3] it is shown that for computable P for which $P(\cdot|\alpha)$ exists and is computable uniformly in α, Van Lambalgen's theorem holds.

Unfortunately, computability of P does not imply computability of $P(\cdot|\alpha)$, or even existence of $P(\cdot|\alpha)$. In [1], Hayato Takahashi showed that $P(\cdot|\alpha)$ is defined for all α that are $P(\cdot, \lambda)$-Martin-Löf random. Moreover, he showed in [2] that if for a fixed α the function $P(\cdot|\alpha)$ is computable, then for this α, some version of Van Lambalgen's theorem holds.

The goal of the talk is to overview results from Hayato Takahashi and to argue that some natural versions of Van Lambalgen's theorem fail unless it is assumed that $P(\cdot|\alpha)$ is computable.

References

[1] H. Takahashi. On a definition of random sequences with respect to conditional probability. *Information and Computation*, 206(12):1375–1382, 2008.

[2] H. Takahashi. Algorithmic randomness and monotone complexity on product space. *Information and Computation*, 209(2):183–197, 2011.

[3] V. G. Vovk and V. V. V'yugin. On the empirical validity of the bayesian method. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 253–266, 1993.

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