Practical initialization of homoclinic orbits from a Bogdanov-Takens point

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Abstract

In a recent paper [4], we improved the theoretical base for the initialization of homoclinic orbits. However, practical application of this method is not very robust without the consideration of some numerical issues. We deal with these issues and provide examples from a robust implementation of the initialization procedure in the software package MatCont [2].

Introduction

Consider the family of autonomous ODEs,

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^2. \tag{1}$$

The Bogdanov-Takens (BT) bifurcation occurs if the equilibrium of (1) has a double zero eigenvalue. The smooth normal form for this bifurcation is

$$\begin{cases} \dot{w}_0 = w_1, \\ \dot{w}_1 = \beta_1 + \beta_2 w_1 + (a + a_2\beta_2)\, w_0^2 + (b + b_2\beta_2)\, w_0 w_1 + d w_0^3 + e w_0^2 w_1, \end{cases} \tag{2}$$

where $w \in \mathbb{R}^2$ parametrizes the 2D parameter-dependent center manifold of (1), $\beta \in \mathbb{R}^2$ are the unfolding parameters and $\{a,\, a_2,\, b,\, b_2,\, d,\, e\}$ are the BT normal form coefficients. We say that the solution $x(t)$ of (1) is a homoclinic orbit to x_0 if it satisfies the following boundary value problem (BVP)

$$\begin{cases} \dot{x}(t) = f(x(t),\alpha), \\ f(x_0,\alpha) = 0, \\ \lim_{t\to\pm\infty} x(t) = x_0, \quad t \in \mathbb{R}, \\ \int_{-\infty}^{\infty} \langle x(t) - \hat{x}(t),\, \dot{\hat{x}}(t)\rangle \, \mathrm{d}t = 0, \end{cases} \tag{3}$$

where $\hat{x}$ is a reference solution.

Homoclinic Asymptotics

In [4] we showed that the homoclinic bifurcation curve satisfies the following asymptotic

$$\alpha(\varepsilon) = \varepsilon^2 \left(\frac{10b}{7a} K_{1,1}\right) + \varepsilon^4 \left(-\frac{4}{a}K_{1,0} + \frac{b^2 50}{a^2 49} K_{2,2} + \frac{b}{a}\left(\frac{1}{a}\left(\frac{100}{49}b_2 - 4\frac{e}{b}\right) + \frac{1}{a^2}\left(\frac{-50}{49}ba_2 + \frac{288}{2401}b^2 + \frac{146}{49}d\right)\right) K_{1,1}\right) + \mathcal{O}\left(\varepsilon^5\right), \tag{4}$$

$$x(t,\varepsilon) = \varepsilon^2 \left(\frac{10b}{7a} H_{01,1} + \frac{1}{a}\xi_0(\varepsilon t) q_0\right) + \varepsilon^3 \left(\frac{1}{a}\psi_0(\varepsilon t) q_1 + \frac{1}{a}\xi_1(\varepsilon t) q_0\right) + \varepsilon^4 \left(\frac{-4}{a^2} H_{01,0} + \frac{50b^2}{49a^2} H_{02,2} + \frac{b}{a}\left(\frac{1}{a}\left(\frac{100}{49}b_2 - 4\frac{e}{b}\right) + \frac{1}{a^2}\left(\frac{-50}{49}ba_2 + \frac{288}{2401}b^2 + \frac{146}{49}d\right)\right) H_{01,1} + \frac{1}{a}\xi_2(\varepsilon t) q_0 + \frac{1}{a}\psi_1(\varepsilon t) q_1 + \frac{1}{2a^2} H_{20,0}\xi_0^2(\varepsilon t) + \frac{10b}{7a^2} H_{21,0}\xi_0(\varepsilon t)\right) + \mathcal{O}(\varepsilon^5) \tag{5}$$

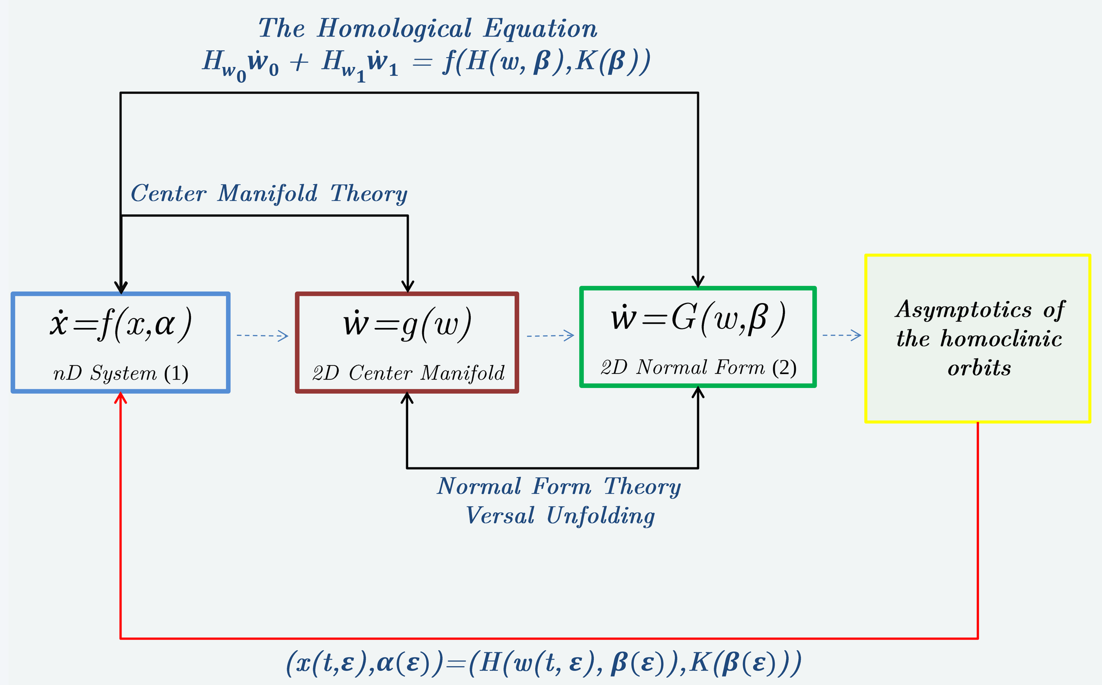
where

$$\xi_0(t) = 2\left(1 - 3\operatorname{sech}^2(t)\right), \quad \psi_0(t) = \dot{\xi}_0(t),$$

$$\xi_1(t) = -\frac{72b}{7a}\frac{\sinh(t)\log(\cosh(t))}{\cosh^3(t)}, \quad \psi_1(t) = \dot{\xi}_1(t),$$

$$\xi_2(t) = -\frac{216}{49}\frac{b^2}{a^2}\frac{\log^2(\cosh(t))(\cosh(2t) - 2)}{\cosh^4(t)} - \frac{216}{49}\frac{b^2}{a^2}\frac{\log(\cosh(t))(1 - \cosh(2t))}{\cosh^4(t)} - \frac{18}{49}\frac{b^2}{a^2}\frac{(6t\sinh(2t) - 7\cosh(2t) + 8)}{\cosh^4(t)} - \frac{(10ba_2 + 14d)}{7a^2} + \frac{t\left(\frac{30}{7}ba_2 + 12d\right)\sinh(t)}{\cosh^3(t)a^2} + \frac{\left(\frac{30}{7}ba_2 - 30d\right)}{\cosh^2(t)a^2} + \frac{27d}{\cosh^4(t)a^2},$$

$\{q_0,\, q_1,\, K_{1,0},\, K_{1,1},\, K_{2,2},\, H_{01,0},\, H_{01,1},\, H_{02,2},\, H_{21,0},\, H_{20,0}\} \in \mathbb{R}^n$ and $0 < \varepsilon \ll 1$.



Continuation of homoclinic orbits in MatCont [1, 3]

MatCont uses a set of equations that consists of several parts to solve the BVP (3):

1. Truncate: $\dot{x}(t) - 2Tf(x(t),\alpha) = 0, \quad t \in [-T, T]$
2. The equilibrium condition: $f(x_0,\alpha) = 0$
3. The phase condition: $\int_0^1 \langle x(t) - \hat{x}(t),\, \dot{\hat{x}}(t)\rangle \, \mathrm{d}t = 0$
4. The invariant subspaces (*Ricatti equation*):
$$\begin{cases} T_{22U}(s)Y_U(s) - Y_U(s)T_{11U}(s) + T_{21U}(s) - Y_U(s)T_{12U}(s)Y_U(s) = 0 \\ T_{22S}(s)Y_S(s) - Y_S(s)T_{11S}(s) + T_{21S}(s) - Y_S(s)T_{12S}(s)Y_S(s) = 0 \end{cases}$$
5. The boundary conditions:
$$\begin{cases} \left\langle x(0) - x_0,\, Q^{U^\perp}(s)\right\rangle = 0, \quad Q^{U^\perp}(s) = Q_U(0)\begin{pmatrix} -Y_U^*(s) \\ I \end{pmatrix} \\ \left\langle x(1) - x_0,\, Q^{S^\perp}(s)\right\rangle = 0, \quad Q^{S^\perp}(s) = Q_S(0)\begin{pmatrix} -Y_S^*(s) \\ I \end{pmatrix} \end{cases}$$
where the columns of $Q^{U^\perp}(s)$ $\left(Q^{S^\perp}(s)\right)$ form a basis for the orthogonal complement of the unstable (stable) invariant subspace.
6. The distance between the end points and x_0:
$$\begin{cases} \|x(0) - x_0\| - \varepsilon_0 = 0 \\ \|x(1) - x_0\| - \varepsilon_1 = 0 \end{cases}$$

The homoclinic parameters

T, ε_0 and ε_1 are the homoclinic continuation parameters.

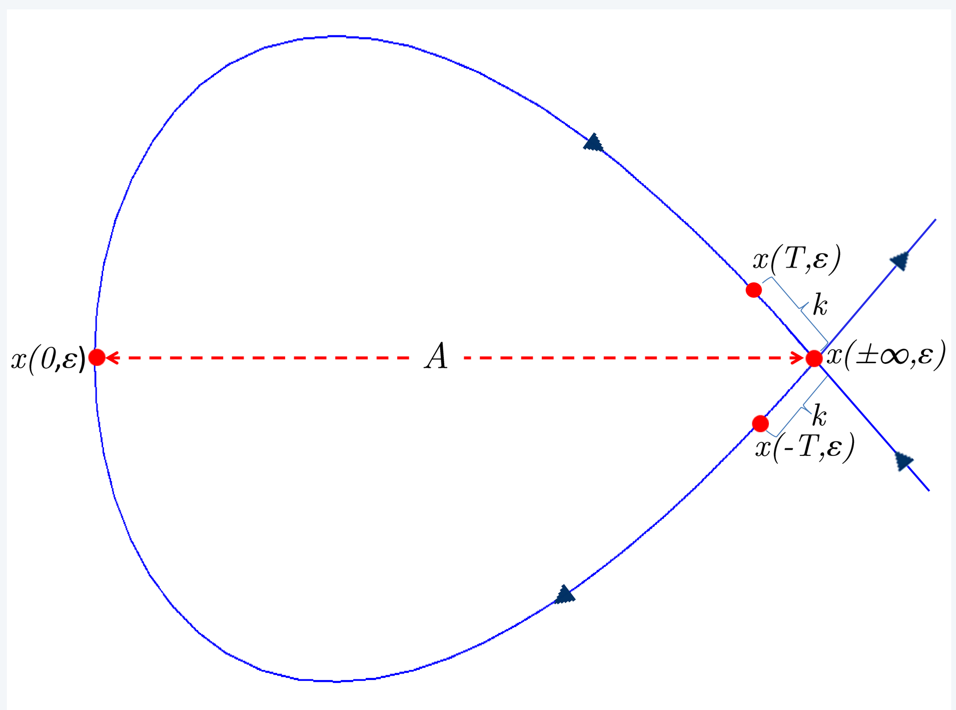
Problem

To start continuation of homoclinic orbits, we need the initial value of (x_0, α_0), $x(t)$, $\hat{x}(t)$, Y_S, Y_U, T, ε_0, ε_1.

Initializing (x_0, α_0), T, $x(t)$, ε_0 and ε_1

It follows from (4) and (5) that:

1. $x_0 = x(\pm\infty) = \varepsilon^2\left(\frac{10b}{7a}H_{01,1} + \frac{2}{a}q_0\right)$, $\quad \alpha_0$
2. The initial amplitude: $A = \varepsilon^2\left(\frac{6}{|a|}\right)$. The parameter A is chosen by the user.
Therefore, $\varepsilon = \sqrt{A\frac{|a|}{6}}$ defines the initial (*perturbation*) parameter ε.
3. The initial T can be obtained by solving
$$\|x(\pm\infty,\varepsilon) - x(\pm T,\varepsilon)\| = k \overset{\approx}{\Longrightarrow} \operatorname{sech}(\varepsilon T) = \frac{1}{\varepsilon}\sqrt{k\frac{|a|}{6}}.$$
The parameter k is chosen by the user.
4. The initial T with (5) can be used to compute $\{x(t), \varepsilon_0, \varepsilon_1\}$.



Remark

In the GUI input window of MatCont, A is denoted as `amplitude` while k is denoted as `TTolerance`.

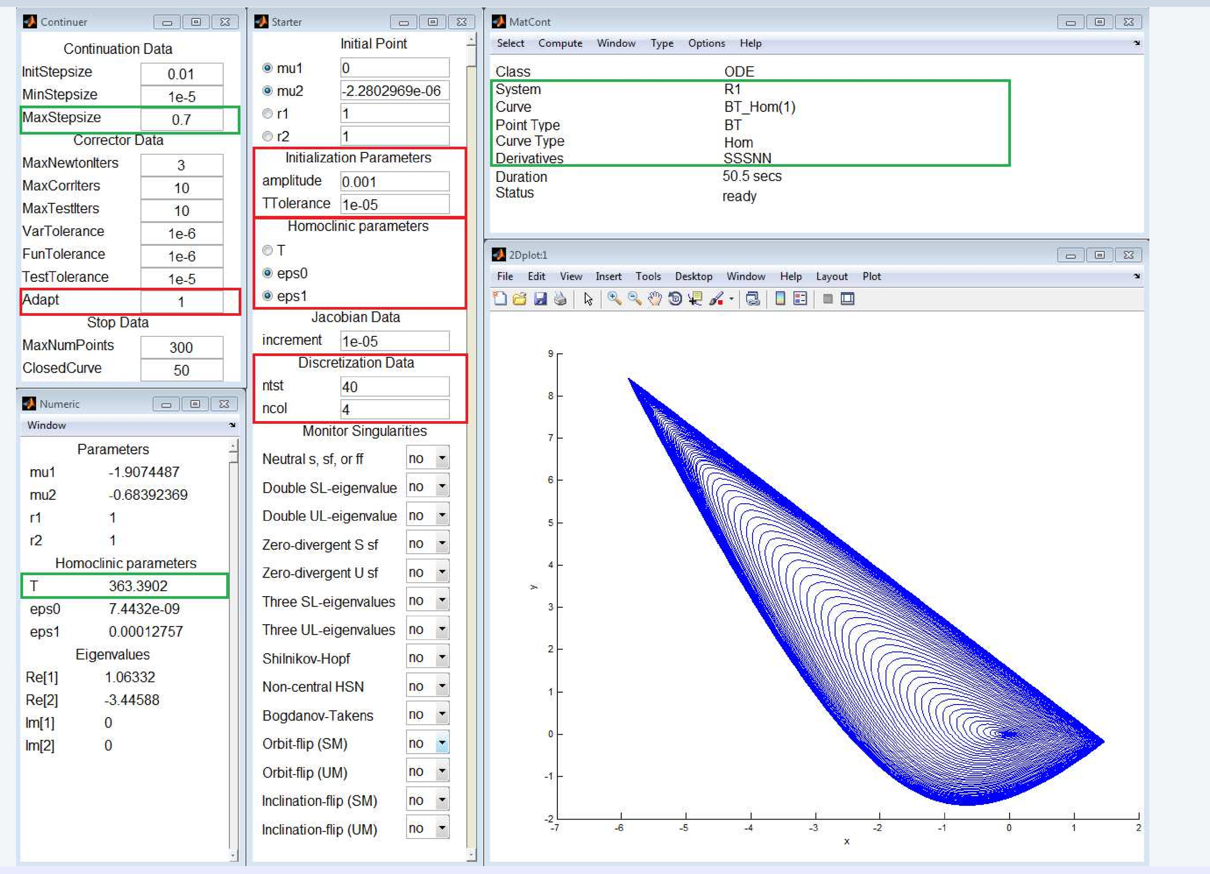
Remark

The initial Y_S and Y_U are set to 0. The initial $\hat{x}(t) = x(t)$.

The user has to active:

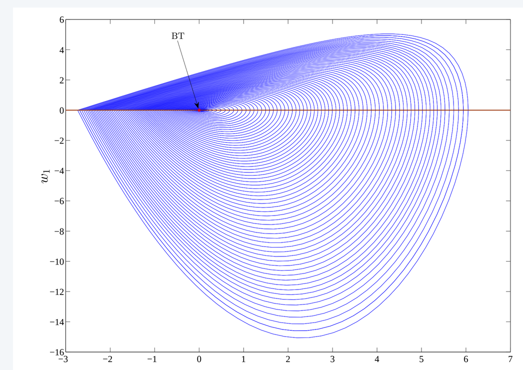
α_1,α_2 and one or two out of three homoclinic parameters T, ε_0, ε_1.

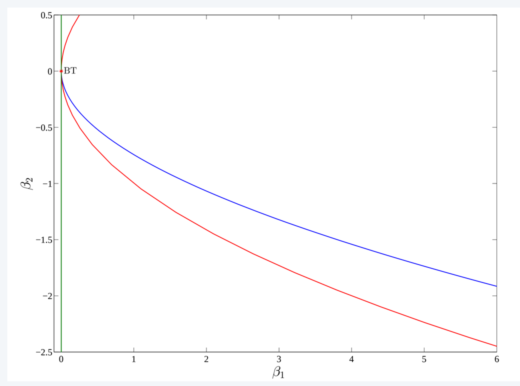
The GUI of MatCont



Examples

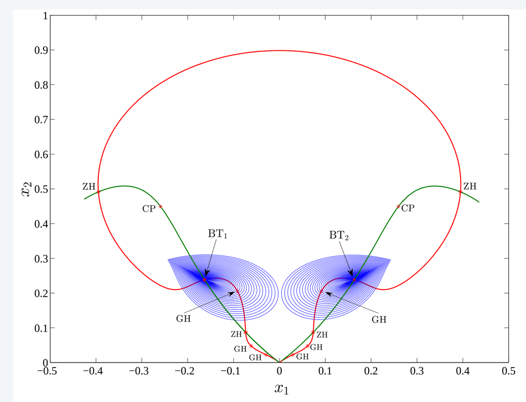
1. The BT normal form system (2) with
$$\{a = -1,\, b = 1,\, a_2 = b_2 = d = e = 0\}$$

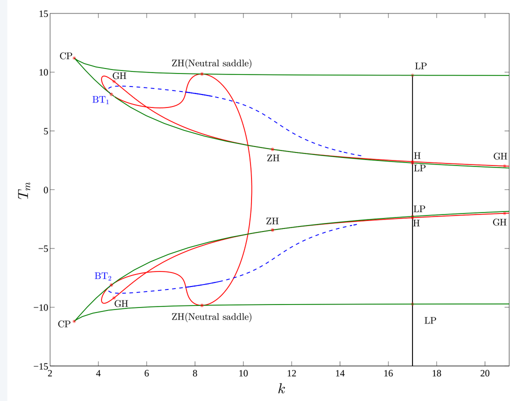




2. Indirect field oriented control system

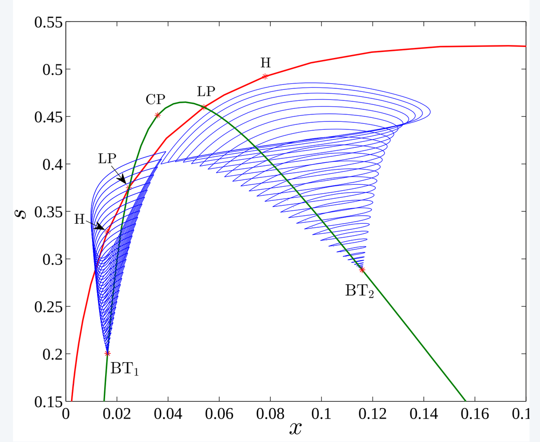
$$\begin{cases} \dot{x}_1 = -c_1 x_1 + c_2 x_4 - \frac{kc_1}{u_2^0} x_2 x_4, \\ \dot{x}_2 = -c_1 x_2 + c_2 u_2^0 + \frac{kc_1}{u_2^0} x_1 x_4, \\ \dot{x}_3 = -c_3 x_3 - c_4 c_5 \left(x_2 x_4 - u_2^0 x_1\right) + \left(c_4 T_m + c_3 w_{ref}\right), \\ \dot{x}_4 = (k_p c_3 - k_i) x_3 - k_p c_4 c_5 \left(x_2 x_4 - u_2^0 x_1\right) + k_p \left(c_4 T_m + c_3 w_{ref}\right). \end{cases} \tag{6}$$

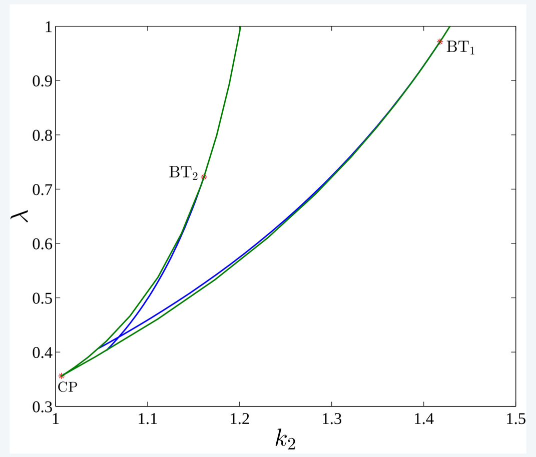




3. CO-oxidation in a platinum model

$$\begin{cases} \dot{x} = 2k_1 z^2 - 2k_{-1}x^2 - k_3 xy, \\ \dot{y} = k_2 z - k_{-2} y - k_3 xy, \\ \dot{s} = k_4 (z - \lambda s), \\ z = 1 - x - y - s, \end{cases} \tag{7}$$



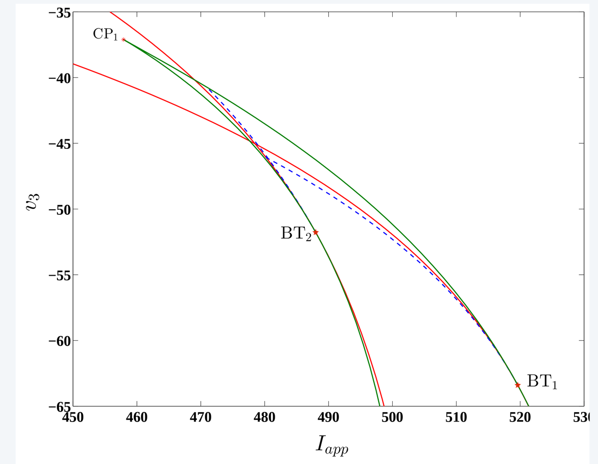


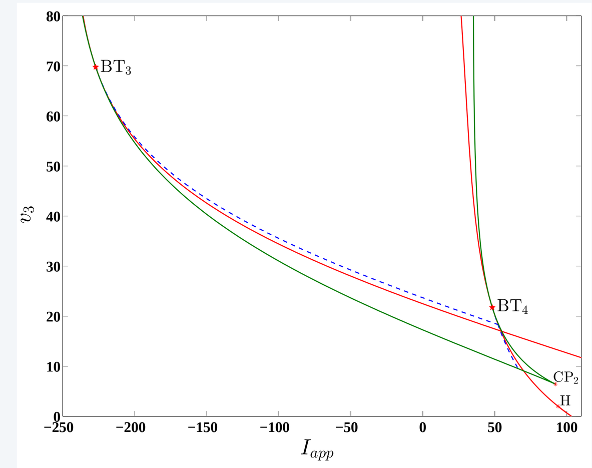
4. Morris-Lecar neuron Model

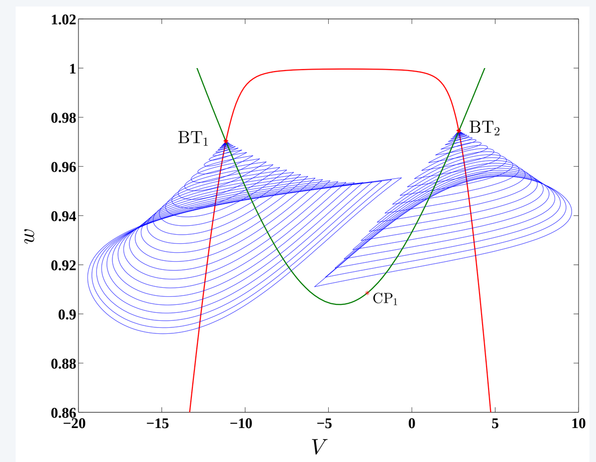
$$\begin{cases} C\dot{V} = I_{app} - I_{ion}, \\ \dot{w} = \frac{\phi(w_\infty - w)}{\tau}, \end{cases} \tag{8}$$

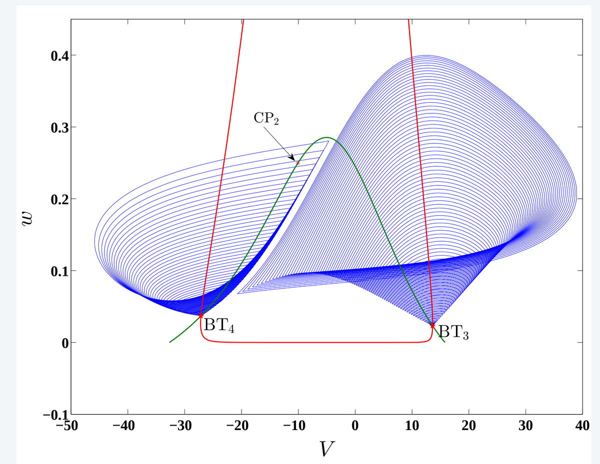
where

$$\begin{cases} I_{ion} = g_{Ca} m_\infty (V - V_{Ca}) + g_K w (V - V_K) + g_L (V - V_L), \\ m_\infty = 0.5\left(1 + \tanh\left(\frac{V - v_1}{v_2}\right)\right), \\ w_\infty = 0.5\left(1 + \tanh\left(\frac{V - v_3}{v_4}\right)\right), \\ \tau = \frac{1}{\cosh\left(\frac{V - v_3}{2v_4}\right)}. \end{cases}$$









Conclusion

We recommend to keep T fixed and use `eps0` and `eps1` as variable homoclinic parameters. In the MatCont continuer window set `Adapt` $= 1$. Then start to increase/decrease the `amplitude` value. This works for most studied models. However, this choice is not an absolute rule and it takes some trial-and-error to set all parameters (including `TTolerance`, continuation parameters and adaptation (`Adapt`)) for the continuation. Note that in each case both **Compute|Forward** and **Compute|Backward** should be tried.

References

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