Low Frequency Stability of the Mixed Discretization of the MFIE

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***Abstract*—Recently, a novel discretization for the magnetic field integral equation (MFIE) was presented [1]. This discretization involves both Rao-Wilton-Glisson (RWG) basis functions and Buffa-Christiansen (BC) basis functions and is dubbed 'mixed'. The scheme conforms to the functional spaces most natural to electromagnetics and thus can be expected to yield more accurate results. In this contribution, this intuition is corroborated by an analysis of the low frequency behavior of the classical and mixed discretizations of the MFIE. It is proved that the mixed discretization of the MFIE yields accurate results at very low frequencies whereas the classical discretization breaks down, as was already discussed extensively in literature [2].**

I. Introduction

Scattering of time-harmonic electromagnetic waves by perfect electrically conducting (PEC) surfaces can be modeled by many boundary integral equations, the electric and magnetic field integral equations (EFIE and MFIE) being the most prominent ones [3]. These equations typically are discretized by expanding the current density in terms of Rao-Wilton-Glisson (RWG) functions defined on a triangular mesh that approximates the scatterer's surface and testing the equations using those same RWG functions [4].

The EFIE often yields highly accurate results, is applicable to both open and closed structures, and extendable to impedance sheets. Regrettably, the linear systems that result from its discretization have unbounded condition numbers in the dense mesh regime, leading to prohibitive solution times. This problem can be solved by Calderón preconditioning (see [5] and references therein), i.e. by exploiting the EFIE operator's self-regularizing property. A Calderón preconditioned EFIE system basically represents a discretized second kind integral equation and thus is amenable to efficient iterative solution. Alternatively, a multi-resolution basis can be used for the finite element spaces of expansion and testing functions (see [6] and references therein). The singular value spectrum of the resulting linear system once again is bounded from above and below, thus facilitating its efficient iterative solution.

The MFIE, in contrast, yields upon discretization well-conditioned systems without further manipulations. Unfortunately, the MFIE's solution is less accurate than that of the EFIE, with a numerical error that can be up to several orders of magnitude larger than that of the EFIE. A lot of attention has been given to this problem in the literature. Roughly speaking, the origin of the classically discretized MFIE's inaccuracy has been traced back to two causes: (i) the equation's inability to deal with objects that comprise sharp corners [7]–[11]. Solution strategies include alternative testing procedures, accurate computation of interaction integrals, and enrichment of the finite element space. Good results can be obtained using these methods, although it is hard to predict in which circumstances these methods excel. (ii) the equation's inability to predict the correct frequency dependence of the current solution's Helmholtz components [2]. The solution strategy proposed in [2] amounts to computing the correct current solution as a perturbation of the static solution.

Recently, a novel discretization strategy for the MFIE was presented [1]. In this scheme, aptly called 'mixed MFIE', the current is approximated as a linear combination of Rao-Wilton-Glisson functions, whereas the testing is done with Buffa-Christiansen functions. The use of two sets of finite element spaces results in a discretization that is conforming with regard to the function spaces most natural to electromagnetics *and* that yields well-conditioned system matrices allowing for the efficient iterative solution of the system. Numerical examples showed that the resulting algorithm gives solutions that are significantly more accurate than that of the classically discretized MFIE. This remains true when applied to sharp-edged geometries.

In this contribution, the algorithm introduced in [1] is further analyzed. In particular, it is proved that the classically discretized MFIE's low frequency breakdown reported in [2] is not present in the novel discretization. This is done by showing that the Helmholtz components of the current solution of the mixed discretization of the MFIE scale, as a function of the frequency, in the same manner as those of the continuous equation.

II. The classically discretized MFIE at low frequencies

Consider a closed PEC scatterer with simply connected surface Γ and exterior normal $\hat{\boldsymbol{n}}$, embedded in a background medium with permittivity ϵ and permeability μ. The scatterer is illuminated by an incident electromagnetic field $\left(\boldsymbol{e}^i, \boldsymbol{h}^i\right)$. Enforcing the boundary condition for the tangential trace of the magnetic field on Γ leads to the MFIE for the induced current density $\boldsymbol{j}$:

$$\hat{\boldsymbol{n}} \times \boldsymbol{h}^i(\boldsymbol{r}) = M[\boldsymbol{j}](\boldsymbol{r}) = \left\{\frac{1}{2} - K\right\}[\boldsymbol{j}](\boldsymbol{r}) \tag{1}$$

$$:= \frac{1}{2}\boldsymbol{j}(\boldsymbol{r}) - \hat{\boldsymbol{n}} \times \frac{1}{4\pi} p.v. \int_\Gamma \nabla \frac{e^{-jkR}}{R} \times \boldsymbol{j}(\boldsymbol{r}')\, dS'.$$

To solve (1) via a boundary element method, the surface Γ is approximated by a mesh comprising V vertices, E edges, and F faces. The current $\boldsymbol{j}$ is approximated as

$$\boldsymbol{j}(\boldsymbol{r}) \approx \sum_{i=n}^{N} I_n \boldsymbol{f}_n(\boldsymbol{r}), \tag{2}$$

where the functions $\boldsymbol{f}_n$ are RWG basis functions defined on the N interior edges of the mesh [4]. Approximation (2) is substituted in (1) and the discretization procedure is completed by testing the equation with suitable testing functions. In the literature, RWG functions are usually chosen as testing functions because this scheme gives rise to a well-conditioned system matrix. However, this discretization of the MFIE leads to a solution current that has an incorrect frequency scaling. To show this, the finite element space spanned by the RWGs will be split into two subspaces. The first subspace comprises divergence free functions and is spanned by the so-called loops $\boldsymbol{f}^l_n, n = 1, ..., V-1$ around all vertices (except one). The second subspace is the complement of the first subspace and is spanned by the so-called stars $\boldsymbol{f}^s_n, n = 1, ..., F-1$ [12]. In this basis, the discretized MFIE is given by

$$\begin{pmatrix} \mathsf{M}^{ll} & \mathsf{M}^{ls} \\ \mathsf{M}^{sl} & \mathsf{M}^{ss} \end{pmatrix} \cdot \begin{pmatrix} \mathsf{I}^{l} \\ \mathsf{I}^{s} \end{pmatrix} = -\begin{pmatrix} \mathsf{H}^{l} \\ \mathsf{H}^{s} \end{pmatrix}, \tag{3}$$

where

$$\mathsf{M}^{ab}_{mn} = \left(\boldsymbol{f}^a_m, M\boldsymbol{f}^b_n\right), \tag{4}$$

$$\mathsf{H}^{a}_{m} = \left(\boldsymbol{f}^a_m, \hat{\boldsymbol{n}} \times \boldsymbol{h}^{inc}\right), \tag{5}$$

and I^a_m are the expansion coefficients of the current solution with regard to the basis of loops and stars.

A simple numerical evaluation of M^{ab}_{mn} and H^a_m in the low-frequency regime shows that these quantities converge to nonzero constants in the low-frequency limit, i.e. they do not depend on the frequency at very low frequencies:

$$\mathsf{M}^{ab}_{mn} = \mathcal{O}(1), \tag{6}$$

$$\mathsf{H}^{a}_{m} = \mathcal{O}(1). \tag{7}$$

As a consequence, the current solution of equation (3) does not depend on the frequency either, i.e. $\mathsf{I}^l = \mathcal{O}(1)$ and $\mathsf{I}^s = \mathcal{O}(1)$. However, such a scaling for I^s is not physical, because the charge density $\rho(\boldsymbol{r})$ generated by this current is proportional to the inverse of the frequency:

$$\rho(\boldsymbol{r}) = \frac{j}{\omega} \nabla \cdot \boldsymbol{j}(\boldsymbol{r}). \tag{8}$$

Therefore, the charge corresponding to the classically discretized MFIE current solution explodes when the frequency is lowered. This in turn leads to large errors in the scattered fields, as is discussed extensively in literature [2].

III. THE MIXED MFIE AT LOW FREQUENCIES

Unlike the classically discretized MFIE, the mixed MFIE is tested by means of the so-called 'rotated' Buffa-Christiansen functions $\hat{\boldsymbol{n}} \times \boldsymbol{g}_n, n = 1, ..., E$ [1], [13]. The finite element space of the BC functions also contains subspaces comprising divergence free functions and a complementary space. The divergence free subspace is spanned by loops $\boldsymbol{g}^l_n, n = 1, ..., F-1$ around all faces (except one). The complementary space is spanned by the so-called stars $\boldsymbol{g}^s_n, n = 1, ..., V-1$ [12]. Using these testing functions, the mixed discretization of the MFIE reads

$$\begin{pmatrix} \mathsf{P}^{ll} & \mathsf{P}^{ls} \\ \mathsf{P}^{sl} & \mathsf{P}^{ss} \end{pmatrix} \cdot \begin{pmatrix} \mathsf{I}^{l} \\ \mathsf{I}^{s} \end{pmatrix} = -\begin{pmatrix} \mathsf{G}^{l} \\ \mathsf{G}^{s} \end{pmatrix}, \tag{9}$$

where

$$\mathsf{P}^{ab}_{mn} = \left(\hat{\boldsymbol{n}} \times \boldsymbol{g}^a_m, M\boldsymbol{f}^b_n\right), \tag{10}$$

$$\mathsf{G}^{a}_{m} = \left(\hat{\boldsymbol{n}} \times \boldsymbol{g}^a_m, \hat{\boldsymbol{n}} \times \boldsymbol{h}^{inc}\right). \tag{11}$$

The main difference between this testing scheme and the classical one is the fact that the BC functions are rotated. As a consequence, the rotation by means of the surface normal can be eliminated from (10) and (11) (except for the first term in the right hand side of equation (1)). For example equation (11) can be simplified to

$$\mathsf{G}^{a}_{m} = \left(\boldsymbol{g}^a_m, \boldsymbol{h}^{inc}\right). \tag{12}$$

The elimination of the surface normal from the formulas allows the exploitation of some remarkable properties of loop currents. For example, when $a = l$ in (12), it is shown in [14] that the right hand side becomes proportional to the frequency:

$$\mathsf{G}^{l}_{m} = \mathcal{O}(\omega). \tag{13}$$

Also, it can be shown that the block P^{ll} in (9) is proportional to the square of the frequency [14]

$$\mathsf{P}^{ll} = \mathcal{O}(\omega^2). \tag{14}$$

When these two properties are used in (9), the following is obtained

$$\begin{pmatrix} \mathcal{O}(\omega^2) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \cdot \begin{pmatrix} \mathsf{I}^{l} \\ \mathsf{I}^{s} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\omega) \\ \mathcal{O}(1) \end{pmatrix}. \tag{15}$$

Dividing the first row by ω and multiplying the second column with ω yields a more easily interpretable form:

$$\begin{pmatrix} \mathcal{O}(\omega) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(\omega) \end{pmatrix} \cdot \begin{pmatrix} \mathsf{I}^{l} \\ \frac{1}{\omega}\mathsf{I}^{s} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(1) \end{pmatrix}. \tag{16}$$

From this, it is easily shown that

$$\mathsf{I}^{s} = \mathcal{O}(\omega), \tag{17}$$

which is the correct frequency scaling for the star currents. Apparently the mixed MFIE automatically leads to the correct frequency scaling of the star currents. As a consequence the mixed MFIE yields accurate results at very low frequencies, in contrast to the classically discretized MFIE.

To further corroborate these theoretical results, a numerical test was conducted. Consider the scattering by a PEC sphere with a radius of 1m, discretized using 42 faces. This is a very coarse mesh, but that is not a problem since the results shown in the above are valid regardless of the mesh density. A plane wave excitation at various frequencies is used. Figure 1 shows the norm of I^s as a function of the frequency. As can be clearly seen, the norm of the star current converges to a constant for the classically discretized MFIE. For the mixed MFIE, however, the norm of the star current converges to zero when the frequency is lowered. The proportionality of the star current with ω is also easily recognized.

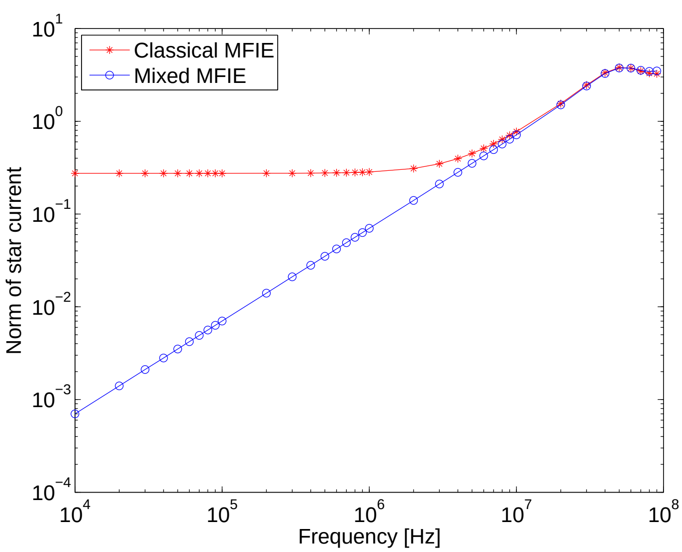

Fig. 1. The norm of the star current as a function of the frequency. The star current in the classically discretized MFIE converges to a constant in the low frequency limit, whereas the star current in the mixed MFIE scales correctly, i.e. is proportional to the frequency.

IV. Conclusion

The low frequency behavior of the solution current to the boundary integral equations of electromagnetics are known (e.g. [2]). Any valid discretization of these boundary integral equation should yield approximate solutions that mimic the frequency scaling of the exact solution. In this contribution, it was shown that the recently introduced mixed discretization of the MFIE exhibits the correct frequency scaling, whereas the classical discretization does not.

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