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The symbolic model for algebra: functions and mechanisms

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Summary. The symbolic mode of reasoning in algebra, as it emerged during the sixteenth century, can be considered as a form of model-based reasoning. In this paper we will discuss the functions and mechanisms of this model and show how the model relates to its arithmetical basis. We will argue that the symbolic model was made possible by the epistemic justification of the basic operations of algebra as practiced within the abbaco tradition. We will also show that this form of model-based reasoning facilitated the expansion of the number concept from Renaissance interpretations of number to the full notion of algebraic numbers.

Key words: Model-based reasoning, symbolic algebra, concept expansion

1.1 Symbolic reasoning is model-based reasoning

We previously introduced the idea of considering algebraic problem solving as a model-based activity [16]. This allowed us to characterize the emergence of symbolic algebra in the sixteenth century as a transition from a geometrical model to a symbolic one. While the solution methods for algebraic problems, introduced in Europe by Latin translations from Arabic, are not by themselves geometrical, the validation for the rules of finding the roots of equations depended on a geometrical model. Geometrical proofs from the Arabic and abbaco tradition may have been derived from the practice of geometrical algebra which goes back to Old-babylonian algebra. Jens Høyrup has convincingly demonstrated how Old-Babylonian scribes did not solve equations as proposed by Neugebauer, but depended on a naive cut-and-paste geometrical model [18]. Jörgen Friberg has further shown that the geometrical algebra in book II of Euclid's *Elements* "appears instead to have been a direct translation into non-metric and non-numerical 'geometric algebra' of key results from Babylonian metric algebra" [12]. Greek geometric algebra can thus be considered a generalization of Babylonian metric algebra using the same geometric model.

While the geometrical model continued to provide an epistemic justification for the rules of algebra in the Arabic and abbaco tradition, it also had its limitations. Geometrical models lost their intuitive appeal once problems went beyond the three dimensions. Also notions such as negative quantities or negative surfaces are impossible or very difficult to represent geometrically. Within the abbaco tradition geometrical models were actually rarely used. We only find them in the treatises by Maestro Dardi di Pisa (w. 1344) Antonio de' Mazzinghi (c.1353-c.1383) , Maestro Bendetto da Firenze (1429-1479) and Piero della Francesca (1416-1492). During the abbaco period which is to be situated between 1300 and 1500, algebraic practice slowly moved towards a symbolic model. This is not immediately evident from the treatises they have left behind as there is little use of symbolism in these texts. However, we argue that symbolism was introduced into mathematics as a consequence of this process towards symbolic reasoning and not as a precondition. Algebraic symbolism developed into its present form during the sixteenth and early seventeenth century. The conditions for the transition towards a symbolic model were prepared by the practice of abbaco masters. The main condition was the epistemic justification of the basic operations of arithmetic. Once there was a strong belief that current mathematical practices had a general validity, it became possible to apply these operations in an abstract way, without accounting for the values they were dealing with. After several centuries of abbaco practice, this belief in the validity of the operations became so strong that it allowed for the acceptance of anomalous results, such as negative and imaginary quantities. The main mechanism of the symbolic model is that the practices of arithmetical operations were adopted within a model in which one makes abstraction of the actual values. This mechanism can be explained by the principle of permanence of equivalent forms.

1.2 The principle of the permanence of equivalent forms

George Peacock was together with George Boole, August De Morgan and Duncan Gregory one of the founders of Cambridge's Analytical Society. This group laid the foundations of what they called algebra of logic, and would later become symbolic logic. This symbolic logic in turn would lay the new foundations for formalism in mathematics with Frege and Hilbert. Peacock was the first to coin the term 'symbolic algebra'. In 1830, he published his *Treatise on Algebra*, in two books. The first book is on *Arithmetical algebra*, the second on *Symbolic algebra*. Peculiarly, both works use symbols, but in arithmetical algebra: "we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions" [23, p. ix]. What this means is that Peacock formulates (arbitrary) restrictions on the operations of algebra so that the results always remain natural numbers. A quadratic equation is therefore not allowed in arithmetical algebra as it can lead to negative, irrational or imaginary roots. Symbolic algebra is then seen as a generalization of arithmetical algebra in which all its truths are preserved. For this property he coined the term "the principle of the permanence of equivalent forms". Forms which are equivalent within arithmetic (for any choice of natural numbers) therefore remain equivalent in symbolic algebra. These forms implicitly define the laws of associativity and commutativity for addition and multiplication and the law of distribution. Such an approach would later lead to the axiomatization of arithmetic and other branches of mathematics. Now, from the point of model-based reasoning, not only does Peacock's symbolic algebra use a symbolic model, so does his arithmetical algebra. The operations allowed in his arithmetical algebra preserve closure for the natural numbers, while his symbolic algebra allows for all the operations valid for the arithmetic of natural numbers, integers, irrational numbers and complex numbers. If we use the principle of permanence of equivalent forms to this last class of numbers, we can characterize the history of symbolic algebra until the advent of quaternions, introduced in 1843 by Hamilton. Quaternions do not preserve the commutative law for multiplication and lead to the idea of multiple possible algebras. We would like to demonstrate that Peacocks principle of the permanence of equivalent forms is a fruitful framework for studying changes in the history of the number concept. We will first show that the arbitrary limitations put on Peacock's arithmetical algebra also appear in the *Arithmetica* by Diophantus. Further we will demonstrate that such limitations on operations were gradually lifted from algebraic practice and that because of a process of epistemic justification of basic operations a symbolization of algebra became possible. We will show by two examples that some important developments and changes of the number concept can be explained as a form of model-based reasoning within this framework.

1.3 The arithmetical algebra of Diophantus

1.3.1 The myth of syncopated algebra

The *Arithmetica* by Diophantus has often been considered a transition point between rhetorical and symbolic algebra. In his study on Greek algebra [21], the German scholar Georg Heinrich Ferdinand Nesselmann coined the term 'syncopated algebra' for such intermediate phase. His tripartite distinction has become such a common-place depiction of the history of algebraic symbolism that modern-day authors even fail to mention their source. The repeated use of Nesselmann's distinction in three *Entwickelungstufen* (steps in the development) on the stairs to perfection is odd because it should be considered a highly normative view which cannot be sustained within our current assessment of the history of algebra. Its use in present-day textbooks can only be explained by an embarrassing absence of any alternative models. We have pointed out three serious problems with Nesselmann's approach [17] which we here summarize.

A problem of chronology

Firstly, if seen as steps within a historical development, as is most certainly the view by many who have used the distinction, the three phases suffer from some serious chronological problems. Nesselmann places Iamblichus, Arabic algebra, Italian abbacus algebra and Regiomontanus under rhetorical algebra ("Die erste und niedrigste Stufe") thereby covering the period from 250 to 1470. The second phase, called syncopated algebra, spans from Diophantus's *Arithmetica* to European algebra until the middle of the seventeenth century, including Viète, Descartes and van Schooten. The third phase is purely symbolic and constitutes modern algebra with the symbolism we still use today. Though little is known for certain about Diophantus, most scholars situate the Arithmetica in the third century which is about the same period as Iamblichus (c. 245-325). So, syncopated algebra overlaps with rhetorical algebra for most of its history. This raises serious objections and questions such as "Did these two systems influence each other?" With the discovery of the Arabic translations of the *Arithmetica* [26] [25] we now know that Diophantus was translated and discussed in the Arab world ever since Qustā ibn Lūqā's book (c. 860). So if the syncopated algebra of Diophantus was known by the Arabs why did it not affect their rhetorical algebra? If the Greek manuscripts used for the Arab translation of the *Arithmetica* contained symbols, we would expect to find some traces of it in the Arab version.

The role of scribes

The earliest extant Greek manuscript, once in the hands of Planudes and used by Tannery, is the thirteenth-century Codex Matritensis 4678 (ff. 58-135). The extant Arabic translation published independently by Jacques Sesiano and Roshdi Rashed was completed in 1198. So no copies of the *Arithmetica* before the twelfth century are extant. The ten centuries separating the original text from the earliest Greek copy is a huge distance. Two important revolutionary changes took place around the ninth century: the transition of papyrus to paper and the replacement of the Greek uncial or majuscule script by a new minuscule one. The transition to the new script was very uniform and drastic to a degree which puzzles today's scholars. From about 850 every scribe copying a manuscript would almost certainly adopt the minuscule script. Transcribing an old text into the new text was a laborious and difficult task, certainly not an undertaking to be repeated when a copy in the new script was already somewhere available. It is therefore very likely that all extant manuscript copies are derived from one Byzantine archetype copy in Greek minuscule. Although contractions where also used in uncial texts, the new minuscule much facilitated the use of ligatures. This practice of combining letters, when performed with some consequence, saved considerable time and therefore money. Imagine the time savings by consistently replacing ἀριθμὸς, which appears many times for every problem, by ϛ in the whole of the *Arithmetica*. The role of professional scribes should therefore not be underestimated. Although we find some occurrences of shorthand notations in papyri, the paleographic evidence we now have on a consistent use of ligatures and abbreviations for mathematical words points to a process initiated by mediæval scribes rather than to an invention by classic Greek authors. Whatever syncopated nature we can attribute to the *Arithmetica* it is mostly an unintended achievement of the scribes.

Symbols or ligatures?

A third problem concerns the interpretation of the qualifications 'rhetorical' and 'syncopated'. Many authors of the twentieth century attribute a highly symbolic nature to the *Arithmetica*. Let us take Cajori as the most quoted reference on the history of mathematical notations. Typical for Cajori's approach is the methodological mistake of starting from modern mathematical concepts and operations and looking for corresponding historical ones. He finds in Diophantus no symbol for multiplication, and addition is expressed by juxtaposition. For subtraction the symbol is $\pitchfork$. As an example he writes the polynomial $x^3 + 13x^2 + 5x + 2$ as $\kappa^{\upsilon}\,\overline{\alpha}\,\varsigma\,\overline{\eta}\pitchfork\delta^{\upsilon}\,\overline{\iota\gamma}\,\mathring{\mu}\,\overline{\beta}$ where κ^{υ}, δ^{υ}, ἀριθμὸς are the third, second and first power of the unknown and $\mathring{\mu}$ represents the units. Higher order powers of the unknown are used by Diophantus as additive combination of the first to third powers.

Cajori makes no distinction between symbols, notations or abbreviations. In fact, his contribution to the history of mathematics is titled *A History of Mathematical Notations*. In order to investigate the specific nature of mathematical symbolism one has to make the distinction between symbolic and non-symbolic mathematics. This was, after all, the purpose of Nesselmann's threefold phases. We take the position together with Thomas Heath, Paul Ver Eecke and Jacob Klein, that the letter abbreviations in the *Arithmetica* should be understood purely as ligatures [19, p. 146]:

> We must not forget that all the signs which Diophantus uses are merely word abbreviations. This is true, in particular for the sign of "lacking", $\pitchfork$, and for the sign of the unknown number, ς, which (as Heath has convincingly shown) represents nothing but a ligature for ἀριθμὸς.

Even Nesselmann acknowledges that the 'symbols' in the *Arithmetica* are just word abbreviations ("sie bedient sich für gewisse oft wiederkehrende Begriffe und Operationen constanter Abbreviaturen statt der vollen Worte"). In his excellent French literal translation of Diophantus, Ver Eecke consequently omits all abbreviations and provides a fully rhetorical rendering of the text as in "Partager un carré proposé en deux carrés" (II.8, "Divide a given square into two squares"), which makes it probably the most faithful interpretation of the original text.

This objection marks our most important critique on the threefold distinction: symbols are not just abbreviations or practical short-hand notations. Algebraic symbolism is a sort of representation which allows abstractions and new kinds of operations. This symbolic way of thinking can use words, ligatures or symbols. The distinction between words, word abbreviations and symbols is in some way irrelevant with regards to the symbolic nature of algebra.

We will now show that the solution method of Diophantus often reflects the characteristics of Peacock's arithmetical algebra in the way solutions are guided by arbitrary limitations on the possible solutions.

1.3.2 Diophantus's number concept

The definition of number states that "all numbers are made up of some multitude of units, so that it is manifest that their formation is subject to no limit" [14, p. 7]. Thus zero and one were not considered numbers, only natural numbers higher than one are multitudes. Negative numbers were considered "absurd". Irrational solutions do not appear at all since they were not considered numbers. Fractions are acceptable as they can be brought to the same denominator and thus become multitudes. The value of the *arithmos* can only be a number which satisfies this concept of numbers. Therefore zero can never be a solution. Where quadratic problems lead to a positive and a negative root, Diophantus always takes the positive solution. In case of two positive roots, the smaller one is used. Other types of solutions are not allowed in Diophantus's arithmetical algebra.

1.3.3 The restrictions of arithmetical algebra

We will now demonstrate by means of two examples from book IV that Diophantus's algebra resembles Peacock's arithmetical algebra in putting arbitrary restrictions on the operations to avoid irrational and negative solutions. Not only is Diophantus avoiding such solutions, the process of resolving the indeterminacy of many of his problems precisely depends on these restrictions.

Avoiding non-rational solutions

The first problem IV.10 asks to find two cubes the sum of which is equal to the sum of their sides [14, p. 172]. The solution starts with the choice of two and three *arithmoi* for the sides of the two cubes. If we use x for *arithmos* then we arrive at the identity $5x = 35x^3$, with an irrational result for x. In Arabic algebra or abbaco algebra this would pose no problem at all. Now the next step is interesting. Diophantus remarks that the solution would become rational if we find "two cubes the sum of which has to the sum of their sides as the ratio of a square to square". The cubes 5^3 and 8^3 satisfy this condition and this choice leads to the rational solution $(\frac{125}{343}, \frac{125}{343})$. Heath adds a long footnote to this problem that a general solution can be obtained by dividing the equation $x^3 + y^3 = x + y$ by $(x + y)$. This observation is true of course, but is a typical approach of symbolic algebra which would not be endeavored by Diophantus. The diophantine approach is to first 'probe' the problem with the most simple choice of $2x$ and $3x$ for the sides of the cubes. He then notices that this leads to a non-rational solution and uses the problematic expression to find the condition which guarantees a rational solution. It is precisely the restriction of "the ratio of a square to square" which guarantees a rational solution that resolves the indeterminacy of the problem. This way of reasoning is not coincidental but systematic to many problems of the *Arithmetica*. Problems 9, 10, 11, 12, 14, 18, 24, 28, 31 and 32 of book IV explicitly state conditions to make the result rational.

Avoiding negative solutions

Problem 27 of book IV shows how Diophantus adds conditions to avoid a negative even when the final solution would be positive. The problems asks for two numbers such that their product minus either gives a cube [14], 168. He takes for the first number $8x$ and for the second $x^2 + 1$ so that the product minus the first $8x^3 + 8x - 8x$ is a cube. Then he notices that the product minus the first $(8x^3 + 8x - x^2 - 1)$ becomes a problem as it should be equated to $(2x - 1)^3$ to get rid of the cube term. He therefore calls this "impossible". Heath remarks that the expression can be equated to either $(2x - \frac{1}{12})^3$ or $(\frac{1}{12}x - 1)^3$ with a positive rational solution for both. However, this is not the point. The salient point is that Diophantus chooses new initial conditions in order to guarantee a positive result with $8x + 1$ for the first and x^2 for the second. Now $(8x^3 + 8x - x^2 - 1)$ can be equated with $(2x - 1)^3$ and the result will be positive. Here again the choice of conditions to resolve the indeterminacy is guided by his limitations on the conception of number.

1.3.4 Expanding the number concept

Jacob Klein, a student of Heidegger and interpreter of Plato, wrote a long treatise in 1936 on the number concept starting with Plato and the development of algebra from Diophantus to Viète [19]. It became very influential for the history of mathematics after its translation into English in 1968. For Klein it is not the evolution of solution methods for solving equations which follows some logical path but the ontological transformation of the underlying concepts within an ideal Platonic realm. He restricts all other possible understandings of the emergence of symbolic algebra by formulating his research question as follows: "What transformation did a concept like that of *arithmos* have to undergo in order that a 'symbolic' calculating technique might grow out of the Diophantine tradition?" [19, p. 147]. According to Klein it is ultimately Viète who "by means of the introduction of a general mathematical symbolism actually realizes the fundamental transformation of conceptual foundations" [19, p. 149]. Klein places the historical move towards the use of symbols with Viète and thus ignores important contributions by the abbaco masters, by Michael Stifel [28] [29], Girolamo Cardano [7] [8] and the French algebraists Jacques Peletier [24], Johannes Buteo [6] and Guillaume Gosselin [13]. The new environment of symbolic representation provides the opportunity to "the ancient concept of arithmos" to "transfer into a new conceptual dimension" [19, p. 185]. As soon as this happens, symbolic algebra is born: "As soon as 'general number' is conceived and represented in the medium of species as an 'object' in itself, that is, symbolically, the modern concept of 'number' is born"[19, p. 175].

Of course, Klein is right that the expansion of the number concept is crucial to the emergence of symbolic algebra but we do not endorse a philosophy where concepts realize themselves with the purpose to advance mathematics. The line of influence is in the opposite direction. The algebraic practices of abbaco masters facilitated the expansion of the number concept. But before we will demonstrate this for negative and imaginary numbers we first have to understand the primary conditions for such a process. The acceptance of a new kind of solutions to algebraic problems becomes possible only when there is a strong belief in the validity of accepted practices. We will now discuss how these practices were epistemically justified.

1.4 Epistemic justification of basic operations

1.4.1 Example of abstraction: multiplying binomials

A good example of the process of abstraction as a necessary condition for the transition to a symbolic mathematics is found in the multiplication procedure for two binomials. The procedure of crosswise multiplication, "multiplicare in croce" is a recurring topic in almost all abbaco treatises. The interesting aspect is that the method applies to a wide variety of 'numbers' and still follows the same procedure. The procedure is often accompanied by a diagram showing the terms in a crosswise fashion. In our interpretation the diagram functions as a validation for the procedure rather than being an essential element in the application of the procedure. Let us look at an example in the abbaco treatise by Paolo Gherardi where he multiplies two rational numbers [4, p. 16]:

> Se noi avessimo a multipricare numero sano e rocto contra numero sano e rocto, sì dovemo multipricare l'uno numero sano contra l'altro e possa li rocti in croce. Asempro a la decta regola. $12\frac{1}{2}$ via $15\frac{1}{4}$ quanto fa? Però diremo: 12 via 15 fa 180. Or diremo: 12 via $\frac{1}{4}$ fa [3], echo 183. Or prendi il $\frac{1}{2}$ di $15\frac{1}{4}$ ch'è $7\frac{5}{8}$, agiustalo sopra 183 e sono $190\frac{5}{8}$ e tanto fa $12\frac{1}{2}$ via $15\frac{1}{4}$. Ed è facta.

Here the multiplication of two fractional numbers $12\frac{1}{2}$ and $15\frac{1}{4}$ is surprisingly treated as the product of two binomials $(12+\frac{1}{2})(15+\frac{1}{4})$ instead of the product of two fractions $\frac{25}{2}$ and $\frac{61}{4}$. In the pseudo-Paolo dell'abbaco treatise, the author explicitly refers to the two methods, one by multiplication of binomials and the other as a multiplication of two fractions [1, p. 28]. We therefore understand the method of multiplying binomials as a general procedure for multiplying all kinds of entities that can be expressed as binomials. It suffices to identify the elements of the two binomials in the crosswise diagram to justify the method. We thus find it applied to the multiplication of surds and the multiplication of polynomials. As shown in the figure above, Maestro Dardi uses the crosswise

Fig. 1.0. Maestro Dardi's scheme for crosswise multiplication of surd binomials (from Chigi M.VIII.170 $f.7^r$).

multiplication for calculating the square of $(\sqrt{5}+\sqrt{7})$ as $12+\sqrt{140}$. Many abbaco treatises also treat the multiplication of two algebraic binomials in the same way, such as $(x-2)(x-3)$.

1.4.2 Epistemic justification of the rules of signs

The justification for the rules of signs build further on the justification schemes for the multiplication of binomials. These rules define the result of arithmetical operations on combinations of positives and negatives. These rules were common in cultures that recognized and calculated with negative quantities such as China and India. The Brāhmaspuṭhasiddhānta of c. 628 includes all the rules of sign for addition, subtraction, multiplication and division [10]. They also appear in Arabic works from the eleventh century. In Europe there was no recognition of negative quantities and therefore a formal treatment of the rules of signs appeared much later. These rules were known implicitly and were applied within the abbacus tradition, for example in the multiplication of irrational binomials in Fibonacci (1202; [5, p. 370] [27, p. 510]).

Its epistemic validation stems from correctly applying the rules for multiplying binomials by cross-wise multiplication in which you add all the subproducts. The first of such proofs in European mathematics appeared in a treatise of c.1343 by Maestro Dardi titled *Aliabraa argibra* ($f.4^v$, [11, p. 44]). It explains why a negative multiplied by a negative makes a positive. It is repeated in various other manuscripts dealing with algebra during the fifteenth century. The reasoning goes as follows: we know that 8 times 8 makes 64. Therefore $(10-2)$ times $(10-2)$ should also result in 64. You multiply 10 by 10, this makes 100, then 10 times -2 which is -20 and again 10 times -2 or -20 leaves us with 60. The last product is $(-2)(-2)$ but as we have to arrive at 64, this must necessarily be $+4$. Therefore a negative multiplied by a negative always makes a positive. The strong belief by abbaco masters

Fig. 1.1. Maestro Dardi's scheme for the justification of the rules of signs by cross-wise multiplication (from Chigi M.VIII.170 $f.4^v$).

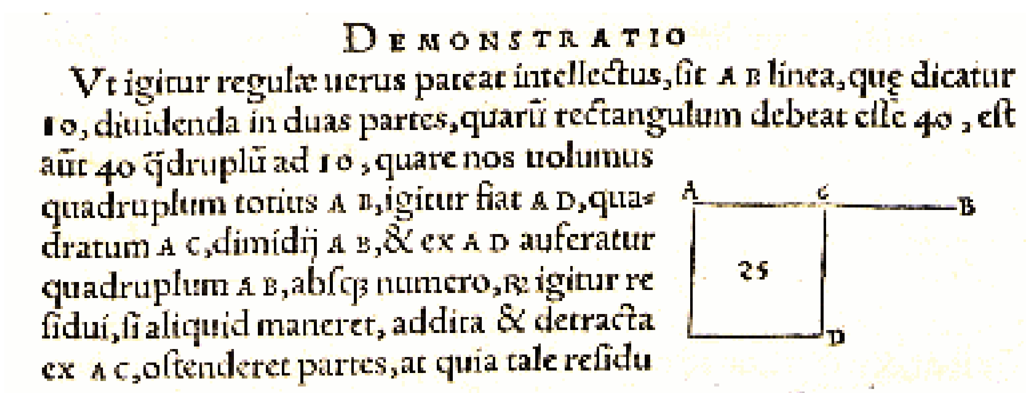
in the correctness of the operations made it possible to extend algebra from the domain of natural numbers to a domain which includes negative numbers. Using the principle of permanence of equivalent forms, the "proof" is the only possible justification for the rules of signs which preserves the distributive law and the law of identity for multiplication (1 times any number n equals n). Reasoning within a symbolic model which was epistemologically justified and led to the creation of new objects on the object level: negative numbers in arithmetic. Luca Pacioli lists the rules of signs for all basic operations in his *Summa* of 1494 [22]. Interestingly, he does not refer to whole numbers, fractions, surds or cossic numbers. The rules are formulated abstractly, as in " a negative divided by a negative makes a positive", which we would expect within a symbolic context.

1.5 Expansion of the number concept

If our first example were negative numbers, our second example comes as no surprise. Cardano was the first to perform a calculation with - what is now known as - imaginary numbers, in chapter 37 of the *Ars Magna* named de *regula falsum ponendis*, or the "Rule of Postulating a Negative" [9, III, p. 287] [8, English p. 219].

Here the context is slightly different. In the early sixteenth century quadratic problems in algebra were reduced to a standard form for which a canonical rule could be applied. Depending on the sign of the coefficients, three different rules were applied, already known from the first Arabic work on algebra. Furthermore, a geometrical proof was known for these rules, so there was a justified belief in their correctness. Now Cardano was confronted with a quadratic problem which leads to imaginary roots. However, he does not reduce the problem to an equation but tries to reason geometrically. The problem is as follows:

> The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of -15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $(5+\sqrt{-15})$ and $(5+-\sqrt{-15})$.


DEMONSTRATIO

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarũ rectangulum debeat esse 40, est aũt 40 q̃druplũ ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residu

Fig. 1.2. Cardano picturing a negative surface (from the *Ars Magna*, 1545, $f.66^r$).

By using a geometrical demonstration, he tries to get a grasp on the new concept (Figure 3). He proceeds as with the standard demonstration of the rule for solving quadratic problems. Let AB be the 10 to be divided. Divide the line at C into two equal parts. Square AC to AD. Since 40 is four times 10 this corresponds with the rectangle $4AB$. Now $\sqrt{-15}$ corresponds with the subtraction of AD by the larger $4AB$. Thus, Cardano finds that this strange new object is a negative or a missing surface. This makes no sense to him and he therefore writes that the problem is impossible. Cardano still struggled with the interpretation of $\sqrt{-15}$, but was putting aside the mental tortures involved and performed the multiplication of the surd binomials

$$(5+\sqrt{-15})(5+\sqrt{-15})$$

correctly to arrive at 25+15 or 40. Multiplying the two binomials produces four terms. The first is evidently 25. The second and third $(5(\sqrt{-15})+5+(-\sqrt{-15})$

are cancelled out by their signs, whatever their value is. The innovation lies in the fourth:

$$(\sqrt{-15})(-\sqrt{-15}) = -(-15)$$

Cardano was well aware of the proofs of the rules of signs, as the one by Maestro Dardi we discussed, and proceeds in a similar way. Here the product of the two terms must be 15 to arrive at the sum of 40. The multiplication of a positive root with a negative must lead to something negative. However, the result must be +15, therefore the product of two roots of minus 15 must be minus 15. Again, the reasoning takes place within the symbolic model. No reasonable interpretation could be given to the root of minus 15. Cardano attempted a geometrical interpretation, but a negative surface makes no sense either. Actually, Cardano is here using an abductive reasoning step which is explained by the chapter heading "Posing a negative". The occurrence of the root of a negative is an anomaly to the Renaissance conception of number. To make the reasoning acceptable he opts the most convenient hypothesis which fits into the rhetoric of abbaco algebra: begin by posing a negative value for the cosa. The operations become perfectly acceptable when one poses that the cosa stands for -15 as:

$$(\sqrt{x})(-\sqrt{x}) = -(x)$$

It took another three centuries before a sensible geometrical interpretation was established. Using the law of distribution and the rules of signs, valid on the symbolic level, Cardano defined the first operations on imaginary numbers. In 1572, Bombelli would formulate all possible operations on imaginary numbers by which the number concept was again extended. In both examples new mathematical objects were created by reasoning within the symbolic model. Through the epistemological justification of correctly performing the operations, these new objects, negative numbers and imaginary numbers, became accepted.

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