Chaotic pulsations in variable stars with harmonic mode coupling

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Some variable stars show multi–periodic behaviour with, among others, peaks in their power spectra at harmonically spaced frequencies with ratios 1:2:4. Such modes are nonlinearly coupled by two second–harmonic interactions and their amplitude equations are shown by a Painlevé analysis to be nonintegrable in a hamiltonian sense. Chaotic phenomena are thus expected, especially when other modes and dissipation are included. An example of stars to which this might apply is G191–16 among the variable white dwarfs.

1. Introduction

Some variable stars, such as certain white dwarfs, have individual pulsation amplitudes which vary with time. Power spectra of *e.g.* nonradial oscillations in certain *ZZ Ceti* stars contain, among others, prominent peaks at harmonically spaced frequencies with ratios 1:2:3, 1:2:4 or 1:2:3:4. Simultaneous nonlinear interactions between regularly spaced frequencies can theoretically be modelled by very special cases of mode coupling, in that selection rules combine three–mode coupling with degenerate cases of second harmonic generation (SHG), where two modes coalesce at half the frequency of another one. The nonlinear amplitude equations are fundamentally different from the usual three–mode or SHG equations, but can be brought into hamiltonian form for the regime without dissipation. Painlevé analysis shows that they are *not* integrable, in contrast to the simple three–mode or SHG cases. The 1:2:3 frequency spacing is covered elsewhere (*Verheest, Hereman & Serras* 1990), while the case 1:2:4 will be addressed here. Eventually, the system evolves chaotically, depending very sensitively upon initial conditions.

Before applying these findings to real stars, two inherent restrictions have to be discussed. First of all, other stellar modes have been left out, and secondly the analysis was done in a conservative framework. Real stars are dissipative, but the motivation for nevertheless concluding something about real stars is that if a simpler, hamiltonian model already points to chaotic behaviour, then the inclusion of additional modes and/or dissipation cannot improve matters (*Verheest, Hereman & Serras* 1990). Other researches reached similar conclusions concerning the possibility of chaotic pulsations, especially for the case of harmonic ratios 1:2:4 indicative of period–doubling phenomena, notably for the variable white dwarfs PG1351+489 (*Goupil, Auvergne & Baglin* 1988) and G191–16 (*Vauclair et al.* 1989).

2. Basic formalism

Adhering to a model of three interacting modes, where the second mode is the second harmonic of the fundamental and the third mode is the second harmonic of the second, the combined selection rules for the angular frequencies ω_j $(j = 1, 2, 4)$ read

$$\omega_2 = 2\omega_1, \qquad \omega_4 = 2\omega_2 = 4\omega_1. \tag{1}$$

Consequently, the third mode is the fourth harmonic of the fundamental, hence the mnemonic use of the index 4. The equations governing the slow time changes in the complex mode amplitudes a_j are different from either the usual three–mode or the simple SHG cases, and also from the 1:2:3 frequency spacing case (*Verheest, Hereman & Serras* 1990), and are of the form (*Verheest* 1976)

$$\dot{a}_1 = 2i\lambda\overline{a}_1 a_2, \qquad \dot{a}_2 = i\lambda a_1^2 + 2i\mu\overline{a}_2 a_4, \qquad \dot{a}_4 = i\mu a_2^2, \tag{2}$$

plus their complex conjugates. These equations are derivable from the Hamiltonian $H = \lambda(a_1^2\overline{a}_2 + \overline{a}_1^2 a_2) + \mu(a_2^2\overline{a}_4 + \overline{a}_2^2 a_4)$, in a description where complex conjugate variables are at the same time canonically conjugate (*Verheest* 1987). Besides the Hamiltonian, there is a second independent first integral $E = a_1\overline{a}_1 + 2a_2\overline{a}_2 + 4a_4\overline{a}_4$, a measure for the global mode energy, but that is not yet enough for complete integrability. Indications about integrability are given through a Painlevé analysis, see *e.g. Menyuk, Chen & Lee* (1983).

3. Painlevé analysis

We formally rewrite the system (2) as a set of ODEs in real variables a_j and A_j (coming from $\overline{a}_j$) by taking $\tau = it$ as a new independent variable, hence

$$\begin{aligned} \dot{a}_1 &= 2\lambda A_1 a_2, & \dot{A}_1 &= -2\lambda a_1 A_2, \\ \dot{a}_2 &= \lambda a_1^2 + 2\mu A_2 a_4, & \dot{A}_2 &= -\lambda A_1^2 - 2\mu a_2 A_4, \\ \dot{a}_4 &= \mu a_2^2, & \dot{A}_4 &= -\mu A_2^2. \end{aligned} \tag{3}$$

For the weak Painlevé test, we expand all variables as

$$a_j = c_j\tau^{p_j} + d_j\tau^{p_j+r} + \cdots, \qquad A_j = C_j\tau^{P_j} + D_j\tau^{P_j+r} + \cdots \tag{4}$$

and try to find the most singular terms in each equation (with $c_j \neq 0$ and $C_j \neq 0$). Using (4) in (3) gives

$$\begin{aligned} p_1 c_1\tau^{p_1-1} &= 2\lambda C_1 c_2\tau^{P_1+p_2}, \\ p_2 c_2\tau^{p_2-1} &= \lambda c_1^2\tau^{2p_1} + 2\mu C_2 c_4\tau^{P_2+p_4}, \\ p_4 c_4\tau^{p_4-1} &= \mu c_2^2\tau^{2p_2}, \\ P_1 C_1\tau^{P_1-1} &= -2\lambda c_1 C_2\tau^{p_1+P_2}, \\ P_2 C_2\tau^{P_2-1} &= -\lambda C_1^2\tau^{2P_1} - 2\mu c_2 C_4\tau^{p_2+P_4}, \\ P_4 C_4\tau^{P_4-1} &= -\mu C_2^2\tau^{2P_2}. \end{aligned} \tag{5}$$

All equations, except the second and the fifth, are easy to balance and give

$$\begin{aligned} p_2 &= p - P - 1, & P_2 &= -p + P - 1, \\ p_4 &= 2p - 2P - 1, & P_4 &= -2p + 2P - 1, \end{aligned} \tag{6}$$

if we call $p_1 = p$ and $P_1 = P$. Whether in the second and the fifth equation of (5) all terms are dominant or not (in the latter case $p + P + 2 > 0$ is inferred), we find already useful results in the form

$$c_4 = \frac{\mu c_2^2}{2p - 2P - 1}, \qquad C_4 = \frac{\mu C_2^2}{2p - 2P + 1}, \qquad c_2 C_2 = -\frac{pP}{4\lambda^2}. \tag{7}$$

In the case where $p + P + 2 > 0$, we try to determine the values for c_j and C_j from (5) reduced to its most singular terms. We calculate that

$$c_2 C_2 = \frac{(p - P + 1)(2p - 2P + 1)}{2\mu^2} = \frac{(p - P - 1)(2p - 2P - 1)}{2\mu^2} \tag{8}$$

and have now three expressions for c_2C_2. The two expressions in (8) are only compatible provided $p = P$, in which case (7) and (8) yield

$$c_2 C_2 = -\frac{p^2}{4\lambda^2} = \frac{1}{2\mu^2}. \tag{9}$$

There are thus no acceptable values for p, λ or μ, and we are led to the case where all terms are dominant, implying that $P = -p - 2$. The values for c_j and C_j now have to be found from

$$\begin{array}{rclcrcl} pc_1 &=& 2\lambda C_1 c_2, & \qquad & (p+2)C_1 &=& 2\lambda c_1 C_2, \\ (2p+1)c_2 &=& \lambda c_1^2 + 2\mu C_2 c_4, & & (2p+3)C_2 &=& \lambda C_1^2 + 2\mu c_2 C_4, \\ (4p+3)c_4 &=& \mu c_2^2, & & (4p+5)C_4 &=& \mu C_2^2. \end{array} \tag{10}$$

Combining some of these equations in a judicious way gives us two expressions for c_1C_1, namely

$$c_1 C_1 = \frac{p}{2\lambda^2}\left\{2p + 1 - \xi\frac{p(p+2)}{4p+3}\right\} = \frac{p+2}{2\lambda^2}\left\{2p + 3 - \xi\frac{p(p+2)}{4p+5}\right\}, \tag{11}$$

having put for brevity $\xi = \mu^2/2\lambda^2$. Both expressions in (11) are only compatible if either $p = -1$, so that all the weights become equal to -1, or

$$p = -1 \pm \sqrt{\frac{1-\xi}{16-\xi}} \equiv -1 \pm q. \tag{12}$$

This requires that $1-\xi$ and $16-\xi$ have the same sign, in other words that $0 < \xi \leq 1$ or that $16 < \xi$. We will return to this case further on, but first address the simpler case where all the weights are -1. From (7), (10) and (11) we obtain

$$\begin{array}{lll} c_2 = -\dfrac{\lambda c_1^2}{1+\xi}, & C_2 = \dfrac{\lambda C_1^2}{1+\xi}, & c_1 C_1 = \dfrac{1+\xi}{2\lambda^2}, \\ c_4 = -\dfrac{\mu\lambda^2 c_1^4}{(1+\xi)^2}, & C_4 = \dfrac{\mu\lambda^2 C_1^4}{(1+\xi)^2}. & \end{array} \tag{13}$$

We note that only one of the constants c_j and C_j can be taken arbitrary, either c_1 or C_1. The determination of the leading terms in (4) is thus complete and we move on to the next step in the Painlevé analysis, the determination of the resonances r. Keeping only the terms linear in d_j and D_j results in

$$\begin{array}{lcl}(1-r)d_1 + 2\lambda(C_1 d_2 + c_2 D_1) & = & 0,\\ (1-r)d_2 + 2\lambda c_1 d_1 + 2\mu(C_2 d_4 + c_4 D_2) & = & 0,\\ (1-r)d_4 + 2\mu c_2 d_2 & = & 0,\\ (1-r)D_1 - 2\lambda(c_1 D_2 + C_2 d_1) & = & 0,\\ (1-r)D_2 - 2\lambda C_1 D_1 - 2\mu(c_2 D_4 + C_4 d_2) & = & 0,\\ (1-r)D_4 - 2\mu C_2 D_2 & = & 0.\end{array} \tag{14}$$

For this linear and homogeneous system in d_j and D_j to have a non–trivial solution we must equate the determinant of the coefficient matrix to zero. Using (13), the possible values for r are then given by

$$(r+1)r(r-2)(r-3)(r-1-\xi)(r-1+\xi) = 0. \tag{15}$$

A resonance $r = -1$ corresponds to an arbitrary shift in the origin of τ, and $r = 0$ to an arbitrary constant (c_1 *or* C_1) in the most singular terms. For the system to be integrable, the other resonances have to be non–negative integers and realisable, as we are expanding in ascending powers of τ. This requires that

$$1+\xi \geq 0, \qquad 1-\xi \geq 0, \tag{16}$$

therefore $\xi = 1$ or $\xi = 0$. A value $\xi = 1$ would lead to *two* resonances zero, imposing that one could choose *two* of the c_j and C_j arbitrary, which cannot be done, however. The other possibility, $\xi = 0$ or $\mu = 0$, corresponds to simple SHG, long known to be integrable. Hence, we must conclude that the double SHG studied here is not integrable. Numerical computations have yielded positive Lyapunov exponents, even when initially all the energy is in the fundamental, indicating chaotic behaviour (see *e.g. Steeb, Louw & Villet* 1987).

We return now to the case where $p = -1 \pm q$, so that (7), (10) and (11) give

$$\begin{array}{lll} c_2 = -\dfrac{\lambda c_1^2}{2(1+q)}, & C_2 = \dfrac{\lambda C_1^2}{2(1-q)}, & c_1 C_1 = \dfrac{1-q^2}{\lambda^2},\\ c_4 = \dfrac{\mu\lambda^2 c_1^4}{4(4q-1)(1+q)^2}, & C_4 = \dfrac{\mu\lambda^2 C_1^4}{4(4q+1)(1-q)^2}. & \end{array} \tag{17}$$

As in the previous case, where all the weights were -1, only one of the constants c_j and C_j can be taken arbitrary and we arrive at the system

$$\begin{array}{lcl}(1-r-q)d_1 + 2\lambda(C_1 d_2 + c_2 D_1) & = & 0,\\ (1-r-2q)d_2 + 2\lambda c_1 d_1 + 2\mu(C_2 d_4 + c_4 D_2) & = & 0,\\ (1-r-4q)d_4 + 2\mu c_2 d_2 & = & 0,\\ (1-r+q)D_1 - 2\lambda(c_1 D_2 + C_2 d_1) & = & 0,\\ (1-r+2q)D_2 - 2\lambda C_1 D_1 - 2\mu(c_2 D_4 + C_4 d_2) & = & 0,\\ (1-r+4q)D_4 - 2\mu C_2 D_2 & = & 0,\end{array} \tag{18}$$

for the d_j and D_j. The resonances r are obtained from

$$(r+1)r(r-2)(r-3)(r-1-\sqrt{1+60q^2})(r-1+\sqrt{1+60q^2}) = 0, \tag{19}$$

leading to nonintegrability on similar grounds as in the previous case. Because the system studied is not integrable, over long time periods we expect irregular phenomena, in sharp contrast to the usual periodic three–mode interactions. The presence of other modes would only increase the complexity and hence enforce the nonintegrability of the model, and so would dissipation.

4. Chaotic pulsations

Stars in which the power spectrum *includes* peaks at a fundamental frequency and its second and fourth harmonics (and hence for which our conclusions might be relevant) include certain *ZZ Ceti* stars. These are single, normal (hydrogen) DA white dwarfs with luminosity variations and hence denoted by DAV. There are also pulsating helium white dwarf (DBV) stars (see *Winget* (1988) for a review of these compact pulsators). The most pronounced of the relevant DAV stars is G191–16, with a light curve dominated by a frequency $\nu_0 = 1.12$ mHz and its harmonics at $2\nu_0$, $3\nu_0$ and $4\nu_0$ (*Vauclair et al.* 1989). Another example is the DBV star PG1351+489 with $\nu_0 = 1.028$ mHz (*Goupil, Auvergne & Baglin* 1988). Since both the special cases with frequency ratios 1:2:3 and 1:2:4 have now been shown to be nonintegrable, the general case with 1:2:3:4 spacing cannot be integrable either. Other *ZZ Ceti* stars which include in their spectra the ratios 1:2:3:4 are *VY Hor* (= BPM31594) (with frequencies at 1.620, 3.240, 4.864 and 6.484 mHz) and its northern hemisphere twin *BG CVn* = GD154 (*O'Donoghue* 1986). The conclusions about deterministic low–order chaos can at this stage only be indicative, in view of the few stars studied so far observationally in any serious detail (*Perdang* 1990).

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