Fuzzy Modelling of Spatial Information

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Samenvatting

Wanneer men in de praktijk werkt met spatiale informatie, blijkt een groot deel van deze informatie onderhevig aan onzekerheid (als er twijfel bestaat over de gemodelleerde gegevens, bijvoorbeeld in het geval van voorspellingen) of aan onnauwkeurigheid (als de gemodelleerde gegevens slechts bij benadering gekend zijn). Deze onzekerheid of onnauwkeurigheid kan ofwel inherent zijn aan de reële situatie (bijvoorbeeld bij bodemsamenstelling) of kan te wijten zijn aan het feit dat metingen te complex of te duur zijn (bijvoorbeeld bij het registreren van neerslag). Bijgevolg kan het interessant zijn om deze onzekerheid of onnauwkeurigheid op te nemen in de gebruikte gegevensmodellen. Het doel van dit werk is spatiale modellen te ontwikkelen, waarmee onnauwkeurigheid of onzekerheid in rekening kunnen gebracht worden.

In het eerste hoofdstuk worden de concepten die verder in het werk aangewend zullen worden, geïntroduceerd. Allereerst betreft dit geografische informatiesystemen, in het bijzonder de representatiemodellen en bijhorende operaties. Er zijn twee grote categoriën van modellen: *entiteitgebaseerde* en *veldgebaseerde* modellen. In de entiteitgebaseerde modellen worden elementaire geometrische structuren (punten, lijnen en veelhoeken) gebruikt om geografische concepten voor te stellen; zo kunnen bijvoorbeeld punten dienen om locaties, lijnen om wegen of rivieren, en veelhoeken om percelen grond of gebieden (vb. begrensd door demografische informatie) voor te stellen. Een aantal operaties op entiteiten zijn bijvoorbeeld: bepalen van de kleinste omhullende rechthoek, bepalen van de convex omhullende, berekenen van de oppervlakte en berekenen van de afstand tussen entiteiten. Topologische relaties (welke, gebruik makend van de concepten binnengebied, grens en buitengebied, de relatieve ligging van gebieden uitdrukken) komen ook aan bod. Bij de veldgebaseerde modellen worden (gemeten) gegevens over een gebied gemodelleerd (bijvoorbeeld hoogtes). Hiervoor worden zowel bitmaps (waarbij het gebied wordt gepartitioneerd in een eindig aantal deelverzamelingen) als driehoeksnetwerken (waarbij data van een beperkt aantal punten wordt geïnterpoleerd) gebruikt. Bij veldgebaseerde modellen werken operatoren over het algemeen in op geassocieerde gegevens. Vermits in dit werk gebruik gemaakt wordt de theorie van de vaagverzamelingenleer (fuzzy set theory), bevat het eerste hoofdstuk ook een introductie tot zowel het concept van vaagverzamelingen als tot een aantal operaties. Het hoofdstuk wordt afgesloten met een overzicht van verwant onderzoek.

In hoofdstuk twee wordt de eerste originele bijdrage van ons werk voorgesteld: een conceptueel model voor vage gebieden. Dit model wijkt af van het traditionele model voor een gebied, waarbij een gebied gedefinieerd wordt aan de hand van zijn omtrek (voorgesteld door een veelhoek) en waarbij alle punten (locaties) binnen deze omtrek tot het gebied behoren. Het is ook mogelijk om een gebied als een verzameling van punten te beschouwen; deze visie wordt gehanteerd in ons werk en wordt uitgebreid tot een vaagverzameling (fuzzy set). Dit betekent dat met elk punt een een lidmaatschapsgraad uit het interval $[0, 1]$ wordt geassocieerd; de waarde 0 betekent dat het punt niet tot het gebied behoort, en hoe hoger de lidmaatschapsgraad hoe meer het punt tot het gebied behoort (de waarde 1 betekent bijgevolg dat het punt volledig tot het gebied behoort). Om een vaag gebied bruikbaar te maken, is het nodig om de geografische operatoren uit te breiden naar dit model. Dit impliceert aangepaste definities voor, onder andere, de kleinste omhullende rechthoek, de convex omhullende (beide zullen nu aanleiding geven tot een vaag gebied) en voor de berekening van de oppervlakte (twee interpretaties worden in acht genomen: de eerste resulteert in een vaaggetal dat de oppervlakte voorstelt, de tweede levert een scherp getal en is eigenlijk een uitbreiding van vage cardinaliteit) en van de oppervlakte. De topologie tussen vage gebieden wordt beschouwd gebruik makend van uitgebreide definities voor binnengebied, grens en buitengebied en een aanpassing van het traditionele 9-intersectie model (Egenhofer, [26]). Een aantal bijkomende uitbreidingen komen ook aan bod: vage locaties (vage punten), die van vage gebieden verschillen in interpretatie en in een aantal operatoren (vb. afstand), en een model om te werken met vage geassocieerde data (waarop type-2 vaagverzamelingen gebaseerd kunnen worden). Mogelijke toekomstige uitbreidingen die een bijkomend niveau van onzekerheid of onnauwkeurigheid toelaten met betrekking op de positie van de vage gebieden wordt ook aangehaald. Als laatste worden een aantal voorstellingswijzen voor vage gebieden vermeld.

Hoofdstuk drie handelt over de haalbaarheid van vage gebieden met betrekking tot hun gebruik in de praktijk: het concept van vage gebieden is een theoretisch concept, maar het is niet evident om dit in de praktijk toe te passen. De oorzaak hiervan ligt in het feit dat een conceptueel vaag gebied gebruikt maakt van een oneindig domein (een begrensde deelverzameling van de twee dimensionale ruimte) in combinatie met het extensie principe (een methode om operatoren uit te breiden naar vaagverzamelingen), wat wel een theoretische definitie oplevert, maar geen algoritme om het resultaat te berekenen. De tweede originele bijdrage van ons werk heeft als doel vage gebieden haalbaar te maken in de praktijk. Hiertoe werden drie modellen ontwikkeld, elk met hun eigen voor- en nadelen. Het eerste model maakt gebruik van contourlijnen: lijnen van gelijke lidmaatschapsgraad (vergelijkbaar met hoe isobaren op weerkaarten lijnen van gelijke druk voorstellen). Van deze methode worden een aantal varianten voorgesteld; de eerste variant maakt gebruik van één grens (waarbinnen alle punten de lidmaatschapsgraad 1 krijgen) en van een vormfunctie (shape-function); de vormfunctie wordt gebruik om het verloop van de lidmaatschapsgraad van deze ene grens naar buiten toe te modelleren in functie van de afstand tot deze grens. Bijgevolg zullen punten met eenzelfde lidmaatschapsgraad op eenzelfde afstand van de gegeven grens liggen. De tweede variant biedt meer vrijheid door gebruik te maken van twee grenzen; binnen de binnenste grens krijgen punten de lidmaatschapsgraad 1, buiten de buitenste grens de lidmaatschapsgraad 0, en tussen beide grenzen krijgen de punten een lidmaatschapsgraad bepaald door hun relatieve positie ten opzichte van beide grenzen, gebruik makend van de vormfunctie. Dit model wordt nog verder verfijnd door bijkomende tussenliggende grenzen toe te laten. Het model leent zich zeer goed tot een uitbreiding van het traditionele concept van buffergebieden (om zo tot vage buffergebieden te komen), maar is niet zo geschikt als model voor algemene vage gebieden: veel operaties zijn niet gesloten, wat betekent dat het type van resultaat van een operatie (vb. unie, intersectie) niet noodzakelijk hetzelfde is als dat van de operandi, waardoor de bruikbaarheid toch wel beperkt wordt.

In hoofdstuk vier komt een tweede model aan bod; dit is ons eerste model waarbij de operatoren gesloten zijn. Het werken met vage gebieden wordt praktisch haalbaar gemaakt door de twee dimensionale ruimte te benaderen door een discrete ruimte. Hiertoe wordt een gebied gepartitioneerd in een eindig aantal cellen, waarbij een cel beschouwd wordt als de kleinste eenheid in het model. Een lidmaatschapsgraad wordt dan geassocieerd met een cel, en representatief geacht als lidmaatschapsgraad voor alle punten binnen deze cel. Het gebruik van een discreet domein laat toe - in tegenstelling tot het gebruik van een continu domein - om het extensieprincipe rechtstreeks toe te passen om tot een berekenbaar resultaat te komen. De operatoren die in het conceptuele model aan bod kwamen, worden aangepast voor het bitmapmodel; ze worden niet alleen gedefinieerd in theoretische vorm en geverifieerd met de operatoren van het conceptuele model, maar ook gegeven in pseudo code, om de haalbaarheid van een implementatie aan te tonen. De verschillende uitbreidingen (zoals o.a. vage punten en vage geassocieerde waarden) worden ook aangehaald.

Hoofdstuk vijf handelt over een derde model dat ontwikkeld werd om de beperkingen van het contourlijnmodel tegen te gaan; dit is ons tweede model waarbij de operatoren gesloten zijn. In dit model worden lidmaatschapsgraden met een beperkt aantal punten (genoemd datapunten) geassocieerd. Uitgaande van deze punten wordt dan een driehoeksnetwerk, of TIN, opgebouwd met behulp van een Delaunay triangularisatie (de datapunten zijn de hoekpunten van het netwerk). Aan de hand van dit driehoeksnetwerk wordt de lidmaatschapsgraad van de overige punten bekomen door lineaire interpolatie op de lidmaatschapsgraden van de hoekpunten van de driehoek waarin een dergelijk punt ligt. Voor de operatoren is het mogelijk om het extensieprincipe toe te passen op de datapunten, maar er moet nagegaan worden of de interpolatie nog steeds het gewenste resultaat geeft; het kan nodig zijn om bijkomende datapunten toe te voegen aan het netwerk (bijvoorbeeld in het geval van de doorsnede). De definities voor de operatoren uit het conceptuele model worden aangepast om deze TIN structuur in rekening te brengen, en worden geverifieerd ten opzicht van het conceptuele model; net zoals bij het bitmap model worden de operatoren ook in pseudo code gegeven. Tot slot komen de bijkomende uitbreidingen uit de vorige hoofdstukken ook hier aan bod.

Het laatste hoofdstuk, hoofdstuk zes, begint met een overzicht van een aantal toepassingsgebieden, ook buiten het gebied van de spatiale databanken waarvoor de modellen initieel ontwikkeld werden. Vervolgens worden een aantal eigenschappen nuttig voor spatiale data structuren uit de literatuur geciteerd. Deze eigenschappen hebben niet alleen betrekking op de definities van objecten en operatoren, maar ook op de invloed van beperkingen door computerrepresentaties op het model (vb. de voorstelling van reële getallen in een computer). Deze eigenschappen vormen een geschikt referentiekader om conclusies betreffende de verschillende modellen te maken.

Summary

When working with spatial information, many information is prone to uncertainty (where the data to be modelled are not certain, as is the case for instance in predictions) or imprecision (where the data to be modelled are not accurate). This is either inherent to the real situation (e.g. soil composition) or is due to the fact that measurements are too complex or expensive (e.g. rainfall). For spatial information, it is therefore beneficial to allow for this uncertainty or imprecision to be incorporated in the models. The objective of this work is to develop models for spatial information in which uncertainty or imprecision is taken into account.

In the first chapter, concepts that will be used throughout the work are introduced. Attention goes to the geographic information systems and the representation models and related operations used. There basically are two types of models: *entity-based* models and *field-based* models. In entity based models, elementary geometric structures (point, lines and polygons) are used to represent geographic features, for instance points can represent locations, lines can represent roads or rivers and polygons can represent patches of land or regions (e.g. when representing demographic information). Some operations on entities are: minimum bounding rectangle, convex hull, calculation of the surface area and distance between entities. Topological relations for regions (which express the relative position, using the concepts interior, boundary and exterior) are also considered. In field based models, (measured) data are modelled over an entire region of interest (for instance altitudes). For this purpose, both bitmap models (in which the region is partitioned in a finite number of subsets) and triangular network models (in which data from a limited number of points - or vertices - are interpolated for other points) are used. Traditionally, the operations on field-based structures concern associated data. As fuzzy set theory will be used in this work, the first chapter also contains a section which introduces some preliminaries regarding fuzzy set theory; both the concept as well as some operations are considered. The chapter concludes with an overview of related work.

In chapter two, the first original contribution of this work is presented: a conceptual model for fuzzy regions. The concept of this model differs from the traditional model for regions, where a region is defined by means of its outline (which is specified by a polygon) meaning that all points (locations) inside this

outline belong to the region. It therefore is also possible to consider a region as a set of points; in this work, this point of view is adopted and the set is extended to a fuzzy set: every point is associated a membership grade in the range $[0, 1]$; a value 0 indicates that the point does not belong to the region and the higher the membership grade the more the point belongs to the region (a value 1 consequently indicates that the point fully belongs to the region). For a fuzzy region, the operations stemming from the geographic realm are extended to match this model. This implies appropriate definitions for the minimum bounding rectangle and convex hull (which both will now yield a fuzzy region) and for the computation of the surface area (two interpretations are considered: one yielding a fuzzy number to represent the surface area, and the other one yielding a crisp number, basically an extension of fuzzy cardinality) and the distance between fuzzy regions. The topology of fuzzy regions is considered using extended definitions for interior, boundary and exterior and an extension of the traditional 9-intersection model (Egenhofer, [26]). Some additional extended concepts are also put forward: fuzzy locations (fuzzy points), which differ from regions in both interpretation and some operations (e.g. distance); a model for working with fuzzy associated data (based on which for instance type-2 fuzzy regions can be defined). Also a possible future extension which allows for an additional level of fuzziness regarding the position of fuzzy regions is proposed. Finally, some representation methods for fuzzy regions are mentioned.

Chapter three concerns the manageability of fuzzy regions for practical uses. While the concept of fuzzy regions is a well defined concept, there are some issues when trying to use it in practice. These issues stem from the fact that the conceptual fuzzy region uses infinite domains (a limited subset of the two dimensional space) in combination with the extension principle (a technique to extend operations to work on fuzzy sets), which, while providing a theoretically sound definition does not provide for an algorithm to compute the result. The second original contribution of this work is intended to overcome these issues; for this purpose three models have been developed, each with its own benefits and drawbacks. The first model makes use of the concept of contourlines, which basically are lines of equal membership (similar to how for instance isobars on weather charts are lines of equal barometric pressure). In the chapter, different variants of the contourline approach are considered. The first one makes use of one boundary (inside which all points are given membership grade 1) and of a shape function; the shape-function is used to model the decrease in membership grade from this one boundary outward, depending on the distance to this boundary. Consequently, points with the same membership grade are all positioned at equal distance from the given boundary. The second variant provides for more degrees of freedom by allowing both an inner and an outer boundary to be specified; inside the inner boundary points are assigned membership grade 1, outside the outer boundary 0 and in between inner and outer boundary the membership grade for a point is determined based on its relative position to both boundaries and the shape-function. The model is extended further by also allowing intermediate boundaries to be specified. While the model lends itself quite well as an extension of the traditional concept for buffer regions (to yield fuzzy buffers), it is not very well suited as a model for fuzzy regions: many operations are not closed (meaning that the result of the operations (e.g. union, intersection) is not guaranteed to yield a result of the same type as the operands), which limits the usability.

In chapter four, a second model is presented; this is the first of our models in which the operations are closed. Fuzzy regions are made manageable in this concept by approximating the considered two dimensional space with a discrete space. For this purpose, a region is partitioned into a finite number of cells, which are considered to be the smallest unit in the model. A membership grade is then associated with a cell and is considered to be representative for the membership grades of all the points within this cell. The use of a discrete domain contrary to the use of a continuous domain (in the concept), allows for the extension principle to be applied directly, yielding a computational method. The definitions for the various operations from the conceptual model are considered again for the bitmap model; not only are they defined in a theoretical form and verified against the conceptual model but also provided in pseudo code, which illustrates the feasibility of an implementation. The different extensions (e.g. fuzzy points, fuzzy associated values, etc.) are also considered.

Next, in chapter five, a third model that was developed to overcome the limitations of the contourline model is presented; this is the second of our models in which the operations are closed. In this model, a limited number of points (called datapoints) are assigned a membership grade. Based on these points, a triangular irregular network (TIN for short) is constructed by means of a Delaunay triangulation (the datapoints will be the vertices of the network). Using this network, the membership grade for a point other than a datapoint is obtained through linear interpolation on the associated values of the datapoints of the triangle in which it is located. To extend the operations, it is possible to apply the extension principle to the datapoints. However, it is necessary that the interpolation still provides the correct result, which may require the addition of new datapoints to the TIN (for instance for the intersection, as explained in the chapter). Consequently, the definitions for the operations on fuzzy regions are adapted for fuzzy TINs, verified against the conceptual definitions, and provided in pseudo code. Again, the additional extensions are also considered for this model.

The concluding chapter, chapter six, starts with an overview of possible applications fields for the developed methods. While the models were developed with spatial information in mind, there are applications in other fields (e.g. image processing). Next, a number properties that are desired for spatial data structures, as found in literature, is provided. These properties not only concern the way objects and operations are defined, but also the way limited representations in computer systems (e.g. the computer representations of real numbers) have an impact on the model. These properties are ideally suited as a frame of reference to comment on the different models that we have developed.

Contents

List of Figures

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Preface

Introduction

Recently, there has been a growing interest in spatial information. While professional geographic information systems have been around for quite some time, some examples are GRASS, ArcGIS, Indrisi, AutoDesk Geospatial, MapInfo and InterGraph GeoMedia Suite; spatial information has gotten more widespread attention in the last decade. This interest occurs at the side of the end users, of companies that wish to use geographically related information to offer additional services, and of research facilities.

In the case of the end users, the growing interest has been triggered by the technological advancements which made GPS and navigation devices cheap and userfriendly, and by the internet, where broadband access made satellite imagery (e.g. Google Earth, Microsoft Visual Earth), routing applications (e.g. Map24, MapQuest) and even demographic information (e.g. "Giswest Vlaanderen" [1]) widely available. For most end users, navigation and nice images are all they really use.

For companies, a distinction is made between companies for which spatial information is part of their core business, and companies that intend to use spatial information to improve their core business. The former obviously have to combine a vast amount of data, gathered from measurements in the field, extracted from satellite and aerial photographs and provided by the authorities, to keep the information incorporated in their products up to date. The latter are companies that wish to adopt the geographically related information to improve their services, which mostly will concern applications of *location based services*, a system in which a user is targeted depending on his/her location. Currently, most people carry a mobile phone, and today's mobile phones have become quite advanced; it is also possible to determine the position of the users based on the GSM cell (the region covered by a GSM antenna) his/her phone is connected to. This opens the door to some basic positioning information, which is already used today in some places. In Kiev for instance, some GSM operators associate the name of a metro station with the GSM cell present at that station: this allows the user to see the name of the current station and (when riding the metro) the name of the upcoming station. A similar

[1] http://www.giswest.be

example can be found in Prague, where the name of the quarter is associated with the GSM cell, allowing the user to have some positional information. An application is also being developed by TomTom Navigation, where the intention is to use information from the location of various mobile phones to detect traffic jams. It is obvious that companies are seeing a possibility of sending promotions or advertisements to users in their vicinity. Given the fact that GPS receivers are becoming smaller (to the point where they are already starting to be integrated in mobile phones), the companies are anxiously looking forward to the new possibilities this development will offer. Not all applications concern advertisements and promotions: location based services can also help a user to find shops, restaurants or even doctors nearby.

In research, geographic information systems are used for much more than merely routing, positioning and navigation. Geographic information systems are used for land use analysis, geological information, climate studies, tracking hurricanes, modelling the spread of vegetation and wildlife, charting pollution sources and predicting how certain pollution spreads, traffic flow and numerous other applications. Furthermore, the systems are not only used to model the present information, but are also used to model changes in the data over time and even to predict trends and evolutions. As data are gathered from a multitude of sources (measurements in the field, information derived from satellite or aerial photographs, etc.), the systems have to deal with large amounts of data.

At the heart of any geographic information system (or GIS for short) is a database. To deal with the increased interest in spatial information, database vendors such as Oracle and IBM have introduced extensions to allow for the modelling of spatial data in such a way as to allow the system to query details about this spatial data: a spatial object is not treated as a BLOB[2], an object of which the contents are not interpreted by the database, but as a genuine object with which the database engine can interact. In all cases however, data are modelled as being crisp and well known, even though this is not always the case in reality. In reality, a lot of data are prone to imprecision or uncertainty, where imprecision refers to the fact that the data cannot be accurately defined and uncertainty refers to the fact that there is doubt (for instance in the case of predictions) concerning the data. This can either be inherent to the data themselves (meaning that the real life situation is not accurately defined) or due to limitations in obtaining the data (meaning that the real life situation is accurately defined, but this accurate definition cannot be determined, thus introducing the imprecision or uncertainty in the model). Inherent imprecision or uncertainty occurs for instance when considering the soil composition: the transition from one type of soil to another (e.g. from a sandy soil to a clay soil) will more likely be some gradual transition rather than a crisp one. Imprecision or uncertainty is introduced in the model when for instance rainfall is modelled: only the rainfall at a limited number of locations is considered, which is then extrapolated as the measured rainfall for a region (measuring at

[2]Binary Large OBject

every location is impossible). This is but one example; similar problems occur for many measurements that are carried out in the field (determining the layers of rock in the underground, counting of the number of species of an animal in a region), and this is moreso the case when considering predictions. At present the geographically related data are represented as crisp values (the users often know it is an approximation, but this approximation is not dealt with in the model) and regions are often considered larger than strictly necessary.

The purpose of this work is to develop techniques to include the uncertainty or imprecision of the data into the spatial models, as to allow a spatial information system to take the present uncertainty and imprecision into account in the representation, the computations and the analyses.

Objectives

Extending the current spatial databases to deal with imprecise and uncertain information, calls for research and changes in a number of fields. Databases have a three level structure consisting of an internal, a conceptual and an external layer [14]. The internal level is closest to the hardware and is concerned with how data are physically stored, the external level is closest to the users and is concerned with the way the data are viewed by the users; the conceptual level provides for a level of indirection between the two. Geographic systems can be viewed similarly: at the lowest level, it allows for basic geometric structures to be modelled; whereas at the conceptual level additional data are associated with these structures. These associated data then provides for a geographic interpretation: a line can represent a road, but also a river; a polygon can represent a physical patch of land, but also an artificial region (for instance defined by demographic information). The external layer then presents the information to the user.

Work has been done in the field of reasoning with regions that have undetermined boundaries; but this work can be considered to be at the external layer of the geographic information system (the assumption is made that a system has the ability to work with such uncertain and imprecise information). The research performed in our work is situated at the internal level: models for the representation of imprecise/uncertain regions are presented, regardless of what the region represents or where the uncertainty or imprecision stems from. These spatial structures should have a sound theoretical basis, but should also be suitable for implementation. The presented work is two-fold: first, a theoretical concept is developed, which, apart from defining the structures, also implies extending operators traditionally used on crisp regions and locations in GIS so that they are suited to deal with uncertain or imprecise regions and locations. Secondly, different implementable models are developed. This is necessary as the theoretical concept itself is not very well suited for direct implementation.

Structure

In chapter one, a short introduction to both spatial information and to fuzzy set theory is made, and the necessary concepts are introduced. The theoretical concept of fuzzy regions is defined and elaborated in chapter two. Traditionally, regions in spatial databases are defined by means of a polygon. However, a region can also be seen as a set of points (locations), more specifically the points that belong to the polygon used to represent the region. From this point of view, it is possible to adopt fuzzy set theory to consider a fuzzy set of points (locations); which basically means that with every point (location), a membership grade in the range $[0, 1]$ is associated. For each point, its membership grade represents the extent to which it belongs to the region (0 means that the point does not belong to the region; the higher the membership grade the more the point belongs to the region; hence 1 indicates that the point fully belongs to the region). Within this concept of fuzzy regions, operations stemming from the fuzzy realm (intersection, union, complement, α-cut), operations from the geographical realm (bounding rectangle, convex hull, surface area, distance), as well as topological relations (which require extensions of the concepts interior, boundary and exterior) are considered. This model is purely intended as a theoretical background, and is not suited for any direct implementation due to the use of an infinite domain in combination with the extension principle. Consequently, from this conceptual model, a number of models suitable for implementation have been derived. These implementation models are inspired on existing spatial structures.

The first implementable model is presented in chapter three. This model makes use of contourlines to outline an imprecisely defined or uncertain region. It is based on the concept of regions with broad boundaries, where an inner boundary and an outer boundary are used to define an undetermined boundary (inside the inner boundary, all the points belong to the region, outside the outer boundary points do not belong to the region). The contourline model as presented extends on this by allowing membership grades to be assigned to points inside this broad boundary, allowing this broad boundary to be specified in a more detailed way.

In chapter four, a totally different concept, which makes use of bitmap structures to represent imprecisely defined or uncertain regions, is presented. The bitmap structure simplifies the concept of fuzzy regions by grouping points together into cells. Each cell is assigned a membership grade which is considered to be the value of all points belonging to the cell. As there are only a limited number of cells, a manageable model for fuzzy regions is obtained. Apart from the representation of the regions, the various operations that have been defined in the conceptual model are adapted to suit this bitmap-based model. Each operator is defined theoretically and provided in pseudo code, and its functionality is verified against the conceptual model.

Next, in chapter five another different concept, making use of triangular networks for the representation of the fuzzy regions, is put forward. This model becomes manageable as only a limited number of points (called datapoints) are assigned a membership grade. Based on this limited number of points, a triangular network is constructed, such that the datapoints become the vertices of the network. This triangular network then allows for membership grades to be assigned to all points (not only datapoints) by applying linear interpolation to the membership grades of the datapoints. As is the case for the bitmap-model, the various operations from the theoretical concept are also considered for this triangular network model (both theoretically and in pseudo code) and verified against the conceptual model.

Finally, in chapter six, an overview of possible applications fields (both in the field of geographic information science as well as in other fields) is provided. A commonly used list of interesting properties for spatial datatypes is taken from literature; the presented models are then commented on, referring to these properties.

Acknowledgements

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Chapter 1

Preliminaries

1.1 Geographic Information Systems

1.1.1 Introduction

Systems dealing with spatial information are complex pieces of software, usually consisting of an advanced graphical user interface to allow maps to be drawn and information to be plotted; a database holding all the geographic information, both positional information (coordinates of a location) and associated data (features at a location), and a complex query engine that allows for both textual and graphical queries (allowing the user to select features on a map, or limit the features shown using constraints). The GIS must be aware of different coordinate systems used in the world, in order to cope with data coming from different sources, and allow for data from various sources to be combined and integrated.

To manage the huge amounts of data implied by spatial information, the data is represented in *layers* ([39], [41]), as illustrated on fig. 1.1. A *layer* is considered to group similar data together (e.g. a layer with road information, a layer with land usage, a layer with textual information, and so on). Layers can also be used to group data that are related. The user can choose to only view the layers he/she is interested in, this is both for usability reasons (displaying all data at once would become far too complex and is never really needed) and performance reasons (only the data in the selected layers need to be processed).

As spatial information systems deal with non-geographical and geographical data for different uses (and from different sources), in general they support two quite different data models: the entity-based model and the field-based model ([39], [41]); which are explained in the next sections.

1.1.2 Entity-based models

In the entity based model, real life objects (such as buildings, roads) are represented as objects in the GIS system ([39]). To achieve this, basic geometry

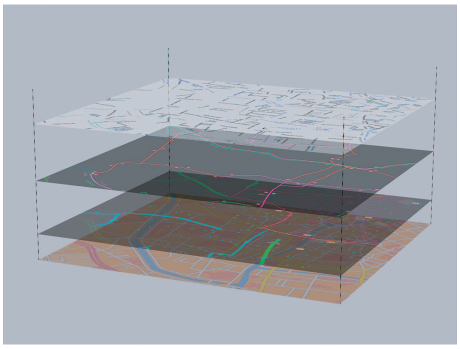


Figure 1.1: Illustration of layers in a GIS, from top to bottom: street names, bus routes and bus stops, indication of one way streets, colour coded map of different features (buildings, roads, water).

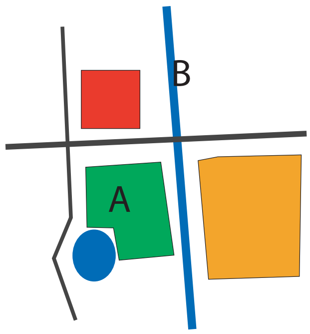
structures are used. These basic objects are usually a point (to represent a location), a line (straight line segment), and a circle. From these, more complex structures can be derived: a polyline (set of connected lines), a closed polyline (or a polygon), rectangle, arc and so on. Any number of these objects can in turn be used and combined to form an *entity*, a database object that represents a real world object (fig. 1.2a). It is possible for an entity to consist of several disconnected pieces.

Real life objects such as roads can then be represented using polylines (depending on the scale one is working at, both sides of the road may need to be modelled); regions (for instance patches of land) can be represented using closed polylines (the outline then represents the boundary of the region).

For an entity-based model, the underlying database not only needs to hold all the required coordinate information for an entity, but also information associated with this entity: the name and/or number of a road, the owner of the patch of land to name but a few (fig. 1.2b). The information associated with entities can differ depending on the entity: for a patch of land for instance (object A in fig. 1.2b), the land classification and address can be included; for a river (object B in fig. 1.2b), there can be a field indicating the type of object: river, canal, or even road (as all of these can be represented by means of a polyline).

The spatial information system has a number of specific operations to create, delete, modify, query and compare entities; both regarding topology (relations like *belonging to*, *touch* or *overlaps*), relative position (e.g. an entity is located north of another entity), and regarding geometric properties (e.g. surface calculation, distances, etc.). If the entity is stored in a database as a BLOB[1], the underlying database system cannot increase the performance

[1] Binary Large OBject, a term for an object of which the deeper structure is not known to - in this case - the database in which it is stored. As a result, the database cannot perform operations on this object, other than storing it and returning it.



(a)

object A

name	...
classification	"recreational area"
geometry	$[p_1(x_1, y_1), p_2(x_2, y_2),$ $p_3(x_3, y_3), p_4(x_4, y_4),$ $p_5(x_5, y_5), p_6(x_6, y_6)]$
address	...
city	...
vegetation	...
population	...

object B

name	Leie
type	"river"
geometry	[...]

(b)

Figure 1.2: Illustration of entities in a GIS: (a) polylines are used for roads, rivers, canals, etc.; closed polylines for regions, patches of water or land, etc. (b) additional data for an entity.

of the operations. To improve on this, various traditional database vendors (Oracle, IBM, etc.) have released spatial extensions: these allow for spatial objects to be stored in a database not as BLOBs but as structured objects, thus allowing the database query engine to interact with them.

1.1.3 Field-based models

Field-based models are the second type of models that are supported by spatial information systems ([41]). Contrary to entity-based models, field-based models are not used to represent real world objects, but to represent data that are associated with given locations. For many applications, it is necessary to model data spread over an entire region; this can concern physical properties of the locations (altitude, temperature, etc.), but also additional information (population density, land value, noise pollution, etc.).

The requirements for this type of data are quite different from the requirements in the entity-based model. In field-based models, a large amount of (associated) data needs to be stored. The different data could be represented by entities, but this would become far too complex (lots of entities) and require too much storage (e.g. coordinates for the different entities). To make data spread over vast regions manageable, two models are commonly used to represent this type of information: the bitmap model and the triangulated irregular network (TIN) model.

Bitmap model

The first model to be considered is the bitmap model. One of the major problems for field based methods is that the (theoretical) two dimensional space, denoted U, is infinite [2]. In the bitmap model, this space is partitioned in a finite number of subsets. With each of the resulting subsets, data are associated. The value associated with a subset is considered to be the associated value for every location of this subset. As the partitioning is in finite subsets, the system now has to deal with a finite and usually limited number of associated values, which becomes manageable.

Traditionally in GIS, an distinction is made between *bitmaps* and *choroplets*. A bitmap is usually defined by means of a regular distribution of regularly spaced sample points; and a choropleth is defined as a partitioning of the region of interest in a finite number of polygons. Resulting from these definitions, a bitmap consists of a finite number of resulting in cells of equal size and shape; whereas a choropleth map is comprised of a finite number of polygons possibly sharing common lines. Given their different nature, the implementation of both structures is quite different, both in representation (coordinates in a grid vs. individual polygons), storage and indexing. In this work, a bitmap is defined in a broader sense than the traditional bitmap, by allowing for irregular grids to be used. This in turn brings our definition closer to that of a choropleth map, but with additional limitations imposed on the occurring polygons: only rectangles which are distributed in an irregular grid are allowed. Depending on the implementation one pursues, it is still possible to choose either a traditional bitmap approach (by limiting to only use regular grids) or an approach closer to the choropleth.

To define a bitmap, first the a number of concepts need to be introduced.

Definition 1 (convex set)
A non-empty subset A of $\mathbb{R}^n$, such that for any elements $x, y \in A$ and for any number c such that $0 \leq c \leq 1$ the element $cx + (1-c)y$ of $\mathbb{R}^n$ belongs to A, is called a convex set.

Definition 2 (cell)
A cell *is the smallest unit considered in the bitmap; it is a bounded, convex (see 1) subset of the universe U.*

Definition 3 (grid)
A grid *— in GIS — is a finite collection $G \subseteq \wp(U)$ such that*

$$\begin{array}{ll} \textit{(i)} & (\forall c, c' \in G)(c \cap c' = \emptyset) \\ \textit{(ii)} & \displaystyle\bigcup_{c \in G} c = U \end{array}$$

[2] In practice, the two dimensional space will be limited both by a lower and and upper boundary as well as in resolution.

Consequently, G is a partition of the universe U. Although not required by the above definition, the cells of a bitmap commonly have the same shape and size. Rectangular cells are most used, but bitmaps are not limited to this; sometimes even hexagonal cells are more appropriate. For the bitmap model, the *universe* U is a two-dimensional space with perpendicular reference axes: the horizontal X-axis and the vertical Y-axis.

The *resolution* of a bitmap is considered to be the number of cells of the bitmap and is usually specified in a horizontal resolution (the number of cells along to the horizontal axis) and a vertical resolution (the number of cells along the vertical axis)[3]. The resolution of the bitmap will determine the amount of data required: a high resolution will provide for a more accurate approximation, at the cost of increased amount of data; a low resolution will yield a lower cost, but will not provide as good an approximation.

Definition 4 (bitmap)
A bitmap associates data (from a domain D) with each cell of a grid G:

$$B = (G, f_G) \tag{1.1}$$

using the mapping function f_G for this grid:

$$\begin{aligned} f_G : G &\rightarrow D && (1.2)\\ c &\mapsto f_G(c) && (1.3) \end{aligned}$$

The domain D is dependent on the features being modelled; mostly it is a numerical domain (to have the bitmap represent temperatures or other measured values).

As G is a partition of U, $\forall p \in U \exists! c \in G : p \in c$. Consequently, the associated value for a point p (representing a location) can be obtained by

$$\begin{aligned} f : U &\rightarrow D\\ p &\mapsto f_G(c), p \in c \end{aligned}$$

Triangulated Irregular Network model

Unlike the bitmap model, which tries to simplify the field based information by considering a finite number of cells, the Triangulated Irregular Network models continue to work with an infinite number of locations. However, data in all the locations is no longer independent of one another: only a limited number of measured (and thus stored) data is considered, while the rest is obtained through an interpolation process.

A Triangulated Irregular Network (TIN) is based on a partition of the two-dimensional space in non-overlapping triangles. This structure is often used

[3]This definition matches the definition of the size of an image in imaging processing; its resolution is expressed in pixels per inch. In a GIS, every bitmap is considered to cover the entire map; so in a way all bitmaps have the same physical dimension: resolution is directly linked with size.

in digital elevation models (DEMs). TINs use a vector-mode approach [39], more specifically their basic structures are points, edges and triangles. No assumption is made about the distribution and location of the vertices of the triangles [39]. It is defined by a non-empty finite set of points (called *data-points*), connected by non-intersecting straight line segments thereby covering the plane completely with non-overlapping triangles. This can be denoted by means of a triplet containing three finite sets: a set P of points (the vertices of the triangles), a set E of edges (the straight line segments that are the sides of the triangles) and a set T of non-overlapping triangles including their interior (called the *tiles* of the TIN)[4].

Definition 5 (TIN)
A TIN is defined as a triplet consisting of a set P of points, a set E of edges, and a set T of triangles

$$Tin = (P, E, T) \tag{1.4}$$

For a given set of points, a number of triangulations are possible; the *Delaunay triangulation* [42],[43] is commonly used. Given a set of points P, its Delaunay triangulation will be a network (a set of edges (E) which in this case will form a number of triangles, contained in T) such that for every triangle in the network, its circumscribing circle does not contain additional points of the set P. Apart from a few trivial cases (e.g. four points, each located on the corners of a square), the resulting TIN network is completely and uniquely defined on a given set of points; appropriate definitions also eliminate the trivial cases. Due to this definition, the triangles in the TIN will resemble the equilateral triangle (having all sides equal) as closely as possible [43]. This property will have a beneficial effect on the interpolation, as degenerate cases caused by narrow, sharp triangles will be avoided.

Various algorithms exist to create a Delaunay triangulation, some algorithms work on the entire set P, others work by adding points one by one. For a more detailed explanation on the algorithms, we refer to [42],[43], [39]. For illustration purposes, one algorithm is added in Appendix A.1.

An interesting extension of the TIN is obtained through the *constrained Delaunay triangulation* [42]. Instead of defining a network on merely a set P of points, this method offers the possibility to specify a set $E' \subseteq E$ of edges to be part of the final triangulation. The resulting triangulated irregular network does not necessarily satisfy the definition of a regular Delaunay triangulation: now the circumscribing circle of a triangle in T might contain additional points of P.

Modification of a TIN network is also possible: algorithms exist to add points to or remove points from the point-set P; the changes required to E and T in order to maintain a Delaunay or constrained Delaunay network are localized around the added or removed points, and can be made quite performantly. The algorithms to perform the addition or the removal of points extend beyond

[4] Only one part of the triplet is required to uniquely define the TIN, but both for referencing to edges and triangles as well as for future extensions, it is convenient to consider this triplet.

the scope of this work; an example is given in Appendix A.1. For more details we refer to [39].

Definition 6 (datapoints, mapping function)
The defining points of the TIN are called the datapoints. *Numerical data are associated with each of these points, using a mapping function* f_1*:*

$$\begin{aligned} f_1 : P &\rightarrow [0,1] \\ p(x,y) &\mapsto f_1(p(x,y)) \end{aligned}$$

These points are contained in the set P*.*

A triangulation is performed in two dimensions, the notation $p(x,y)$ refers to points considered in the two dimensional space. For some operations and calculations, it is interesting to consider the value associated with each of the points as a third dimension, for which the notation $p(x,y,z)$ will be used, which is a shorthand for $p(x,y,f_1(p(x,y)))$.

The mapping function f_1 provides for associated values for the datapoints only. The associated values for other points are obtained using the linear interpolation (as is applied on a TIN) and the mapping function f_1. From these, a function f_2 is derived, by means of which the interpolated values can be obtained.

$$\begin{aligned} f_2 : U &\rightarrow [0,1] \\ p(x,y) &\mapsto \begin{cases} f_1(p(x,y)) & \text{if } p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } p(x,y) \notin P \end{cases} \end{aligned}$$

Where A, B, C and D are the parameters of the equation $Ax + By + Cz + D = 0$ of the plane containing the three points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ and $p_3(x_3, y_3, z_3)$ (with the understanding that $z_j = f(x_j, y_j)$, $j = 1, 2, 3$). The triangle formed by $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ is a triangle of the TIN and $p(x, y, 0)$ is inside or on an edge of this triangle.

$$\begin{aligned} A &= y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2) \\ B &= z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2) \\ C &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ D &= -Ax_1 - By_1 - Cz_1 \end{aligned}$$

The points $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ and $p_3(x_3, y_3)$ in the XY-plane should not be co-linear, which is guaranteed by the fact that no Delaunay triangulation (or even in a constrained Delaunay triangulation) would result in a triangulation containing such a degenerate case.

Similarly as before, it is possible to consider the associated data with each of the points as a third dimension. In this case, the notation $p(x, y, z)$ will be used as a shorthand notation for $p(x, y, f_2(x, y))$. The notation does not allow a distinction between datapoints and non-datapoints, but this poses no problem.

For the remainder of this work, only networks obtained through a Delaunay triangulation as well as through a constrained Delaunay triangulation will both be referred to as TINs.

1.1.4 Operations in GIS

Minimum bounding rectangle

An important operation in geographic systems is the *minimum bounding rectangle*, or *MBR* for short.

Definition 7 (minimum bounding rectangle)
Given an entity, the operator returns the smallest rectangle (with sides equidistant to the reference axes) circumscribing the entity.

This is illustrated on fig. 1.3. The operation is useful for indexing the entities (classifying the location of rectangles is easier than classifying varying shapes), but also can be used to speed up certain tests: e.g. if the MBRs of two regions don't intersect, the regions themselves cannot intersect. Determining if two rectangles, which are equidistant to the reference axes, intersect is not computationally intensive. An intersection between the MBR only implies that further testing is required. As such, MBRs are also used to optimize spatial query implementations ([39]): in the first step (the *filter phase*), the objects whose MBRs satisfy the spatial predicate are selected. In this step, spatial tests are performed on rectangles, which is not computationally intensive but still allows for the elimination of inappropriate objects: if an MBR does not satisfy the predicate, neither will the object. However, this is not true the other way around: if an MBR does satisfy a spatial predicate, additional tests are needed to verify that the object itself satisfies the predicate. These additional tests are performed in the next step; the *refinement step*.

The minimum bounding rectangle of a region A will be denoted $MBR(A)$.

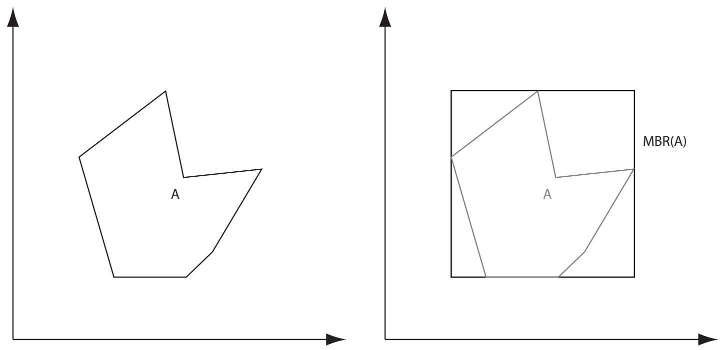
Convex hull

To introduce the concept of the convex hull, first the notion *convex* needs to be defined.

Definition 8 (convex polygon)
A polygon A is called convex if for any two points (p_1, p_2) chosen inside the polygon, the straight line segment connecting them is also contained inside the polygon.

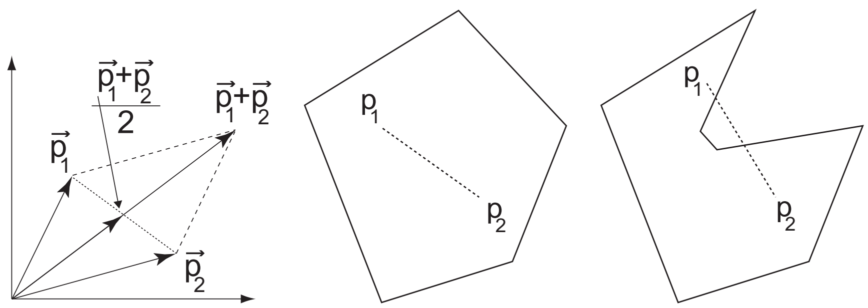
$$\forall p_1, p_2 \in A : \frac{\overrightarrow{p_1} + \overrightarrow{p_2}}{2} \in A \tag{1.5}$$

This definition treats the points p_i as vectors $\overrightarrow{p_i}$, which is illustrated on fig. 1.4.



(a) (b)

Figure 1.3: Illustration of the crisp MBR: (a) sample polygon, (b) the MBR of the polygon in (a).



(a) (b) (c)

Figure 1.4: Illustration of the convex polygons: (a) vectorcalculus, (b) a convex polygon, (c) a non-convex polygon.

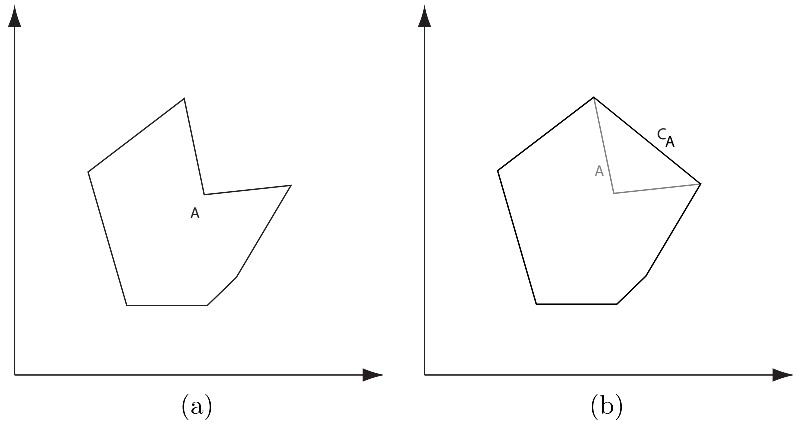


Figure 1.5: Illustration of the convex hull: (a) sample polygon, (b) the convex hull of the polygon in (a).

Definition 9 (convex hull)
The convex hull of a given set A is the smallest convex set in which the original set is contained. The notion smallest *refers to the fact that no proper subset of the convex hull is convex and contains the original set at the same.*

In two dimensions, the convex hull of n points can be computed in $O(n \ln n)$. Naturally, the convex hull of a convex polygon is the polygon itself. Some examples can be seen on fig. 1.5. Similarly to the minimum bounding rectangle, the convex hull can be used to optimize other operations.

Surface area

The surface area of a region is very straightforward: a region is delimited by a polygon, the surface of the region is the surface of the polygon. Not can the surface area be used as a property of a region, but the operation is also a basic operation which can help determine if three points are co-linear or if two segments intersect. This area can be calculated by first triangulating the polygon (not to be confused with Delaunay triangulation), and then adding up the surface areas of all the triangles.

Polygon triangulation is the division of a plane polygon into a set of triangles, usually with the restriction that each triangle side (apart from sides that coincide with a side of the polygon) is entirely shared by two adjacent triangles. For simple polygons, this always requires a finite number of triangles, and can theoretically be done in $O(n)$ [8], where n is the number of vertices of the polygon[5]. An illustration of polygon triangulation can be seen on fig. 1.6.

[5] In [44] it was shown that the $O(n)$ algorithm is quite hopeless to implement; the algorithm commonly used is $O(\ln \ln n)$ [46]

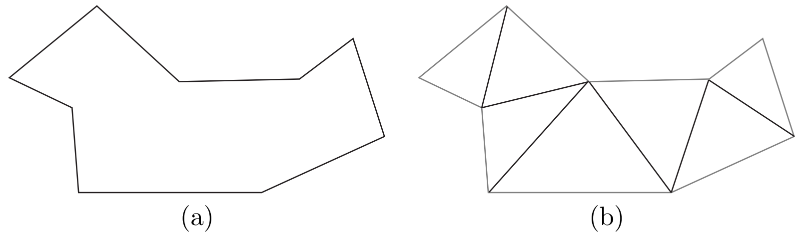


Figure 1.6: Illustration of the triangulation of a polygon: (a) sample polygon, (b) the triangulation (a).

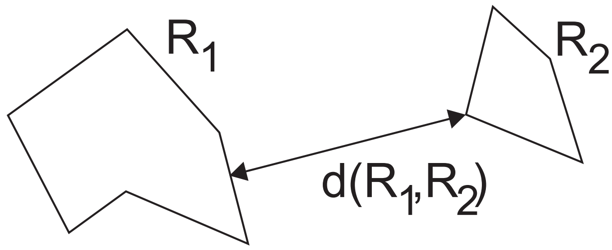


Figure 1.7: Illustration of the distance between two regions R_1 and R_2.

Distance calculation

Before introducing the distance between two regions, first the distance between two points is considered. As a distance measure, the Euclidean distance is mostly used.

Definition 10 (Euclidean distance between points)
Consider two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$. The Euclidean distance is defined as:

$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.6}$$

Definition 11 (Euclidean distance between regions)
The Euclidean distance between two regions R_1 and R_2 is defined as the shortest (Euclidean) distance between any two points of both regions.

$$d(R_1, R_2) = min(d(p_1, p_2), \forall p_1 \in R_1 \wedge \forall p_2 \in R_2) \tag{1.7}$$

This is illustrated on fig. 1.7.

Topology

Topological relations — simply put — refer to the relative positions of regions. The definitions of the various topology cases for two regions are based on the concepts of interior, exterior and boundary; for which first some additional definitions are required.

Definition 12 (open set)
A set $A \subset \mathbb{R}^2$ is open *if any point $x \in A$ is contained in A together with an open n-ball centered at x of radius* r, *i.e. $B_x^r = \{y \in \mathbb{R}^2 : |y - x| < r\}$.*

$$A \text{ is open if } \forall x \in A, \exists r : B_x^r = \{y \in \mathbb{R}^2 : |y - x| < r\} \subset A \tag{1.8}$$

Definition 13 (complement of a set)
If A is a set contained in a universe U, the complement of A, denoted $co(A)$ is the set of all elements in U that do not belong to A.

$$co(A) = x \in U : x \notin A \tag{1.9}$$

Definition 14 (closed set)
The complement of an open set, is a closed *set:*

$$A \text{ is closed if } co(A) \text{ is open} \tag{1.10}$$

or

$$\begin{gathered} A \text{ is closed if} \\ \forall x \in co(A), \exists B \subseteq co(A) : B_x^r = \{y \in \mathbb{R}^2 : |y - x| < r\} \subset B \end{gathered} \tag{1.11}$$

Definition 15 (neighbourhood of a point)
The neighbourhood *of a point p is a set containing an open set that contains p:*

$$\begin{gathered} A \text{ is a neighbourhood for } p \\ \text{if} \\ p \in A \wedge \exists B \subseteq A : B_x^r = \{y \in \mathbb{R}^2 : |y - x| < r\} \subset B \end{gathered} \tag{1.12}$$

Definition 16 (closure of a set)
The closure *$\overline{A}$ of a set A is the smallest closed set containing A.*

Using the above concepts, the interior, exterior and boundary of regions can be defined.

Definition 17 (interior A° of a region A)
The interior *A° of a region A is defined as the set of points $p \in A$ such that A contains a neighbourhood of p.*

$$A^\circ = \{p \in A \mid \exists B \subset A, B \text{ is a neighbourhood for } p\} \tag{1.13}$$

In practice, the interior of a region holds all the points that are inside the region.

Definition 18 (boundary ∂A of a region A)
The boundary *∂A of a region $A \subset U$ is*

$$\partial A = \overline{A} - A^\circ \tag{1.14}$$

where $\overline{A}$ is the closure, and A° is the interior of A in U. For fuzzy regions, the notation ΔA will be used to indicate the boundary of a fuzzy region. This notation was introduced by Clementini (for regions with broad boundaries, 1.3.2) to indicate that the boundary encompasses more than a single line; appropriate definitions for ΔA are in the related sections.

Definition 19 (exterior A^- of a region A)
The exterior *A^- of a region A is defined as the complement of ∂A:*

$$A^- = \mathbb{R}^2 - \partial A \tag{1.15}$$

where $-$ is the notation for set-minus.

In practice, the exterior of a region encompasses all the points that are not part of the region.

To determine the topological relations between two regions, the intersections between the interior, boundary and exterior of both regions are considered: the intersection between the interiors of both A and B, the intersection between the interior of region A and the boundary of region B, and so on. In total, there are *nine* possible combinations; which are grouped in the nine-intersection matrix ([25], [26]):

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \tag{1.16}$$

By assigning each matrix element 0 if the intersection is empty, and 1 if the intersection is not empty, $2^9 = 512$ relations can be deduced. For crisp regions in a two-dimensional space $\mathbb{R}^2$, only eight such intersection matrices are meaningful. These yield the eight possible relations [25]: disjoint, contains, inside, equal, meet, covers, coveredBy and overlap; which are illustrated on fig. 1.8.

The nine-intersection model can not only be used for regions, but also for regions with holes ([39]) and for objects with a lower co-dimension[6] [39]. For objects with co-dimension 0, the nine-intersection model yields the same result as the four-intersection model (which only uses the concepts interior and boundary); for objects of co-dimension 1 the nine-intersection model provides more detail than the four-intersection model.

The topological relations can be grouped in a *conceptual neighbourhood graph*: it has a node for each of the 8 relations, and an arc between every 2 relations that make up a gradual transition, [25]. A "gradual transition" implies a minimal number of changes in the matrix elements from one relation to another relation. The topological neighbourhood graph is shown on fig. 1.9.

[6]The co-dimension represents the difference between the dimension of the space and the dimension of the object; a line for instance has co-dimension 1, whereas a point for instance has co-dimension 0

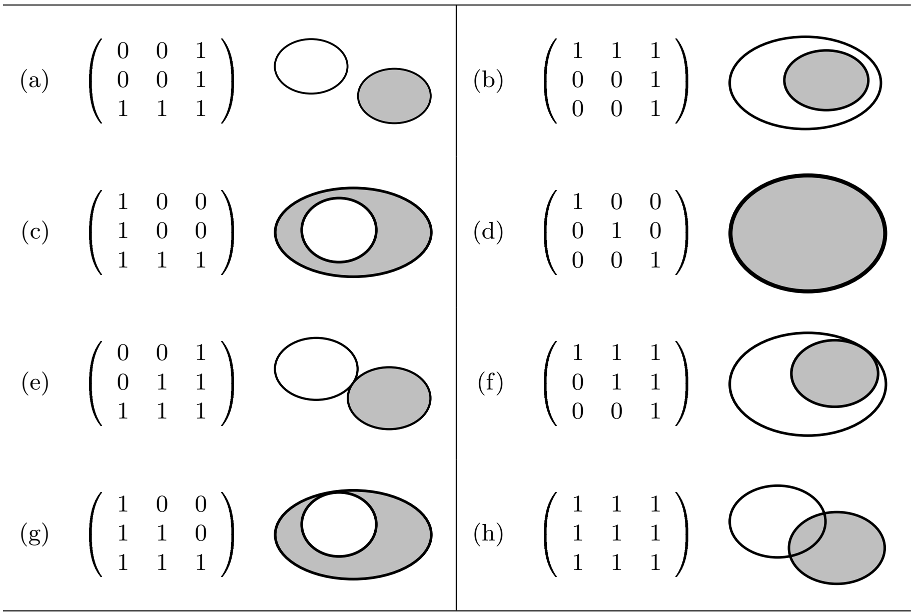


Figure 1.8: Topological relations for crisp regions: disjoint (a), contains (b), inside (c), equal (d), meet (e), covers (f), coveredBy (g) and overlap (h), with their intersection matrices.

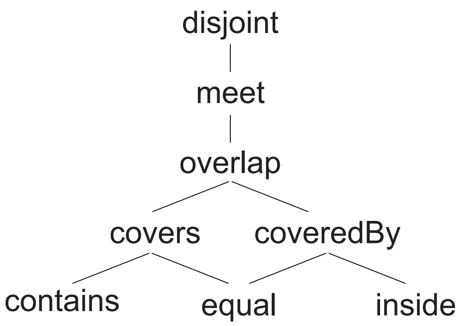


Figure 1.9: Conceptual neighbourhood graph for crisp topology relations.

Buffer

Most GIS support the buffer operation for entities. Given an entity and a (Euclidean) distance, the buffer is a new entity that surrounds the original entity at the given distance. Because of this definition, the corners of the buffer region have a rounded appearance (as illustrated on fig. 1.10).

However, as not all commercial GIS are equally capable of working with arcs, the buffer region is sometimes simplified by defining the buffer region as a region made up of straight line segments parallel to the line segments defining

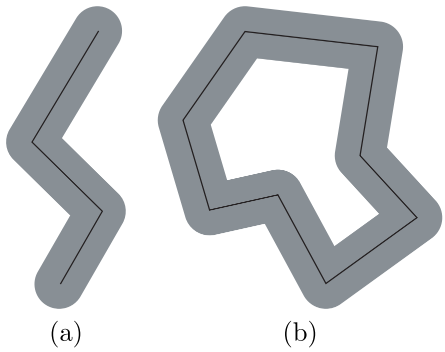


Figure 1.10: Buffers with a rounded appearance: (a) around a polyline, (b) around a closed polyline, (c) around a polygon.

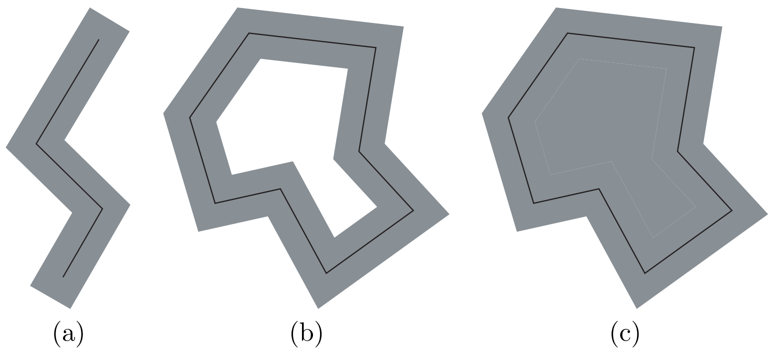


Figure 1.11: Buffers with a square appearance: (a) around a polyline, (b) around a closed polyline, (c) around a polygon.

the original region, as illustrated on fig. 1.11. The advantage of defining the buffer like this is that it also is a polygon, which for some GIS is an easier structure to work with and makes the buffer operation a closed operation in those systems (guaranteeing that the result can be used as an argument for another operation).

This function can be used to define a buffer zone (or safety-zone) around a region, the behaviour in the corner points is sometimes considered to be less important: the buffer without sharp corners overestimates the buffer with rounded corners (which is adequate for safety zones).

1.1.5 Limitations

While the above models (entity based and field based) are commonly used, and are adequate for many applications, they do have some negative aspects. One of them is that all information modelled is crisp and perfect. While this is sufficient for many applications, quite often the source data are prone to uncertainty or imprecision, either inherently (in the case of soil composition for instance) or due to limitations in observations (measurements can only be performed at some locations, not at every possible location). Furthermore, gathered information can be missing for some locations, or it can be inconsistent with other knowledge ([36]).

These imperfections can occur either in the coordinate information (most likely regarding the position or the outline of an entity in an entity based model), but the associated data can also be uncertain or imprecise. The next section will elaborate on the sources of this imprecision and uncertainty, and on related work to include it in the models.

1.2 Fuzzy Set Theory

1.2.1 Concept

Fuzzy set theory has been presented by Zadeh in ([52]). Contrary to traditional set theory, fuzzy set theory allows for non crisp relations: an element can belong to a set to a given extent, or it can belong to a set with a certain possibility. This is accomplished by associating a membership grade to each element. This membership grade is a number in the range $[0, 1]$; it can have a number of interpretations ([22]), as explained in 1.2.2.

Definition 20 (fuzzy set)
Consider the universe U. The fuzzy set $\tilde{A}$ has a membership function $\mu_{\tilde{A}}$, defined by

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow]0, 1] \\ x &\mapsto \mu_{\tilde{A}}(x) \end{aligned}$$

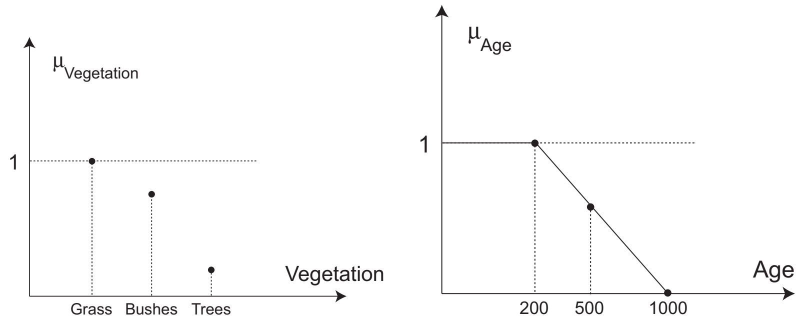
The membership function associates a membership grade with each element. Elements that are not part of the set are considered to have a membership grade 0. A set $\tilde{A}$ with a finite number of elements x, y and z, can be denoted as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), (y, \mu_{\tilde{A}}(y)), (z, \mu_{\tilde{A}}(z))\}$.

Definition 21 (normed fuzzy set)
A fuzzy set $\tilde{A}$ is called normed if there as at least one element with membership grade 1, i.e.

$$\exists x \in U : \mu_{\tilde{A}}(x) = 1 \qquad (1.17)$$

1.2.2 Interpretation of fuzzy sets

Membership grades can have a number of interpretations ([22]). They can have a *veristic* interpretation, a *possibilistic* interpretation, or an interpretation as degrees of truth. Other interpretations for membership grades are possible, but it was shown in [22] that these are equivalent with one of the above interpretations.



(a) (b)

Figure 1.12: Examples of fuzzy sets: (a) fuzzy set over a discrete domain, (b) fuzzy set over a continuous domain.

In a veristic interpretation, the membership grade indicates a degree of belonging to the set. In other words: all elements with a membership grade greater than 0 belong to the set, but some more than others; elements with membership grade 1 are said to *fully* belong to the set. The higher the membership grade, the more the element belongs to the set. Such an interpretation is also called *conjunctive*, and is often illustrated with the example of "the languages a person speaks"; a GIS example can concern the vegetation in a given region: there can be different types of vegetation, but some types more prominently present than others. The set $\{(\text{grass}, 1), (\text{bushes}, 0.8), (\text{trees}, 0.2)\}$ indicates that there is a lot of grass, less bushes, and only few trees. This set is illustrated in fig. 1.12a. Another example is for instance the set of "young landscapes", as illustrated in fig. 1.12b. An age of 200 is considered to be completely young, but an age of 500 is only considered young to an extent of 0.5. This means that a landscape that is 500 years old, will only partly be considered to be young.

In a possibilistic interpretation, there is uncertainty about which elements belong to the fuzzy set. The membership grade indicates this possibility: a value 0 indicates that the element does not belong to the set, a value 1 indicates the element certainly belongs to the set. The higher the membership grade, to higher the possibility the element belongs to it; all elements with a value greater than 0 belong to the set. This is also called *disjunctive*. An example is when the age of a landscape is unknown, but it is known it is young. Using the same membership function as in fig. 1.12b, it can be concluded that an age up to 200 years is completely possible, but an older ages is less likely (as indicated by the membership grade). The main difference with the veristic interpretation is that the landscape only has one age: only one age is valid, it is just unknown which is the correct age.

1.2.3 Operations

Zadeh's extension principle

The extension principle is fundamental in fuzzy set theory. It provides a mechanism to define operations that work on fuzzy sets based on the operations on crisp sets. A crisp relation

$$R : U_1 \times ... \times U_n \rightarrow Y \tag{1.18}$$

is *fuzzified* if it interacts with fuzzy sets defined over the sets $U_1, ..., U_n$ and Y and is of the form

$$\tilde{R} : \tilde{\wp}(U_1) \times ... \times \tilde{\wp}(U_n) \rightarrow \tilde{\wp}(Y) \tag{1.19}$$

Where $\tilde{\wp}(U)$ is the notation for the fuzzy powerset of U, the set of all fuzzy sets defined over the domain U.

Definition 22 (fuzzy powerset $\tilde{\wp}(U)$)
The set of all fuzzy sets defined over a domain U is denoted as $\tilde{\wp}(U)$. Consider a fuzzy set $\tilde{A}$:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}) \mid x \in U \wedge \mu_{\tilde{A}}(x) > 0\} \tag{1.20}$$

The fuzzy powerset of U then is

$$\tilde{\wp}(U) = \{\tilde{A} \mid \tilde{A} \text{ is defined as in (1.20)}\} \tag{1.21}$$

Definition 23 (Zadeh extension principle)
To define the Zadeh extension principle, consider the sets $U_1, ..., U_n$ and Y and a crisp relation R

$$R : U_1 \times ... \times U_n \rightarrow Y \tag{1.22}$$

then the fuzzified relation $\tilde{R}$ of R is defined as

$$\begin{aligned} \tilde{R} : \tilde{\wp}(U_1) \times ... \times \tilde{\wp}(U_n) &\rightarrow \tilde{\wp}(Y) \\ \tilde{V}_1, ..., \tilde{V}_n &\mapsto \tilde{R}(\tilde{V}_1, ...\tilde{V}_n) \end{aligned}$$

where $\tilde{R}(\tilde{V}_1, ...\tilde{V}_n)$ is the fuzzy set in $\tilde{Y}$ defined by

$$\tilde{R}(\tilde{V}_1, ...\tilde{V}_n) :$$

$$\begin{aligned} Y &\rightarrow [0, 1] \\ y &\mapsto \begin{cases} \sup\limits_{(x_1,...,x_n) \in V_{(R,y)}} \min(\mu_{\tilde{V}_1}(x_1), ...\mu_{\tilde{V}_1}(x_n)) & \forall y \in wd(R) \\ 0 & \forall y \notin wd(R) \end{cases} \end{aligned}$$

where $wd(R)$ is the notation for the co-domain of crisp the relation R:

$$wd(R) = \{y \mid y \in Y \cap \exists (x_1, ..., x_n) \in U_1 \times ... \times U_n : (x_1, ..., x_n)Ry\} \tag{1.23}$$

and $V_{(R,y)}$ is the set of n-tuples that are mapped by the crisp relation R to the value y:

$$V_{(R,y)} = \{(x_1, ..., x_n) \mid (x_1, ..., x_n) \in U_1 \times ... \times U_n \wedge (x_1, ...x_n)Ry\} \tag{1.24}$$

Set operations

In fuzzy set theory, set operations are defined using t-norms and t-conorms. A t-norm is a function with two arguments (denoted *T(x,y)*) that satisfies the following properties:

$$\begin{array}{ll} T(x,y) = T(y,x) & \text{(commutativity)} \\ T(a,b) \leq T(c,d) \text{ if } a \leq c \text{ and } b \leq d & \text{(monotonicity)} \\ T(a,T(b,c)) = T(T(a,b),c) & \text{(associativity)} \\ T(a,1) = a & \text{(identity element)} \end{array} \quad (1.25)$$

T-conorms are in a sense dual to t-norms; for any t-norm, its complementary conorm is defined by $S(a,b) = 1 - T(1-a, 1-b)$. A t-conorm is defined by the following properties:

$$\begin{array}{ll} S(x,y) = S(y,x) & \text{(commutativity)} \\ S(a,b) \leq S(c,d) \text{ if } a \leq c \text{ and } b \leq d & \text{(monotonicity)} \\ S(a,S(b,c)) = S(S(a,b),c) & \text{(associativity)} \\ S(a,0) = a & \text{(identity element)} \end{array} \quad (1.26)$$

For both t-norms and t-conorms, an infinite number of functions satisfy the properties. Commonly some specific functions are used for t-norms and t-conorms: Zadeh introduced the minimum as the t-norm ($min(a,b)$) and the maximum as t-conorm ($max(a,b)$), Lukasiewicz introduced $max(a+b-1,0)$ as T-norm and $min(a+b,1)$ as S-norm. Others are product ($a \times b$) and limited sum ($a + b - a \times b$) as t-norm and t-conorm respectively, and lastly the drastic t-norm (yields a if $b = 1$, b if $a = 1$ and 0 otherwise) and drastic t-conorm (yields a if $b = 0$, b if $a = 0$ and 1 otherwise). The algebras obtained by considering certain t-norms and t-conorms share properties, but also differ in other properties. As this is outside of the scope of this work, we refer to [23], [32].

Definition 24 (intersection of fuzzy sets)
The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined using a t-norm.

$$\tilde{A}\tilde{\cap}\tilde{B} = \{(x, \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x)) \mid \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x) = T((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)))\} \quad (1.27)$$

Definition 25 (union of fuzzy sets)
The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined similarly to the intersection, but now a t-conorm is used.

$$\tilde{A}\tilde{\cup}\tilde{B} = \{(x, \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(x)) \mid \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(x) = S((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)))\} \quad (1.28)$$

To indicate that the operation works on fuzzy sets, the operations are denoted as $\tilde{\cap}$ and $\tilde{\cup}$ instead of $\cap$ and $\cup$.

Definition 26 (complement of a fuzzy set)
The complement of a fuzzy set $\tilde{A}$ is defined as

$$co(\tilde{A}) = \{(x, 1 - \mu_{\tilde{A}}(x))\} \quad (1.29)$$

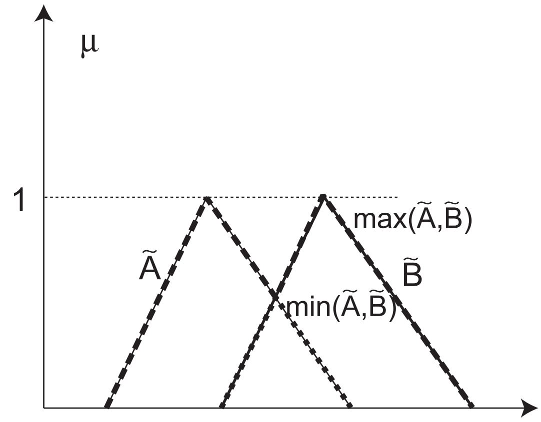


Figure 1.13: Examples of the minimum and maximum as set operations on two fuzzy sets $\tilde{A}$ and $\tilde{B}$.

As illustrated on fig. 1.13, the intersection contains the elements that occur to some degree in each of the fuzzy sets. For each element x, the membership grade $\mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x)$ is calculated by means of the t-norm: if the minimum is used as the t-norm, the associated value for x becomes the smallest of the respective values $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{A}}(x)$. Similarly, the union holds all the intersection elements from $\tilde{A}$ and $\tilde{B}$. The associated value is calculated using an t-conorm: if the maximum is used, the associated value for a location x in the union is the maximum of the values $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{A}}(x)$.

α-cut

Determination of the α-cut of a fuzzy set is an operation commonly used for defuzzification: it removes all aspects of fuzziness and reverts its (fuzzy) argument to a crisp set. The α-cut of a fuzzy set is a crisp set that contains all the elements of the fuzzy set for which a constraint it satisfied: a *strong α-cut* contains the elements with membership grades strictly greater than a given α; a *weak α-cut* contains the elements with membership grades greater than or equal to a given α. This is illustrated on fig. 1.14

Strong α-cut The *strong α-cut* is defined as:

$$\tilde{A}_{\overline{\alpha}} = \{x \mid \mu_{\tilde{A}}(x) > \alpha\} \tag{1.30}$$

Support A special case of a strong α-cut is the *support*; this is the strong alpha-cut with threshold 0. This is an important α-cut, as it results all the elements that belong to some extent to the fuzzy set.

$$\tilde{A}_{\overline{0}} = \{x \mid \mu_{\tilde{A}}(x) > 0\} \tag{1.31}$$

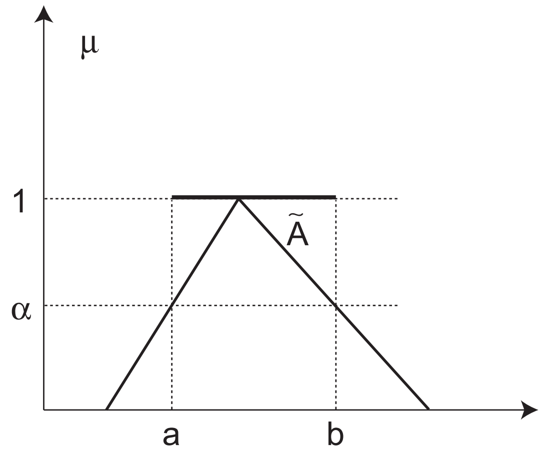


Figure 1.14: Example of the α-cut of a fuzzy region $\tilde{A}$.

Weak α-cut The *weak α-cut* is defined as:

$$\tilde{A}_\alpha = \{(x, 1) \mid \mu_{\tilde{A}}(x) \geq \alpha\} \tag{1.32}$$

Core Similarly to the strong α-cut, the weak α-cut has a special case, now for a threshold 1. This α-cut is called the *core*, and returns all the elements that fully belong (membership grade 1) to the given fuzzy set.

$$\tilde{A}_1 = \{x \mid \mu_{\tilde{A}}(x) = 1\} \tag{1.33}$$

height

The *height* of a fuzzy set returns the highest membership grade that occurs in a fuzzy set. For normed fuzzy sets, this will always equal 1(by definition), but for non-normed sets this can take any value in the range $[0, 1]$. Formally, the height of a fuzzy set $\tilde{A}$ is defined ([23]) as:

$$height(\tilde{A}) = \sup_x(\mu_{\tilde{A}}(x)) \tag{1.34}$$

1.2.4 Possibilistic Truth Values

As mentioned in (1.2.2), membership grades of fuzzy sets can be interpreted as degrees of truth. However, to represent (fuzzy) truth values, the concept of possibilistic truth values poses an alternative theory ([38], [16], [17]), and this will be adopted further on. A possibilistic truth value is a possibility distribution over the boolean range $I = \{True, False\}$ (sometimes denoted $\{T, F\}$). Consequently, a single possibilistic truth value for a proposition p

holds both a degree $\mu_{True}(p)$, which represents the possiblity of p being true and a degree $\mu_{False}(p)$, which represents the possibility of p being false.

For this generalization, consider P the representation of the universe of all propositions and $\tilde{\wp}(I)$ the set of all the fuzzy sets, which can be defined over the universe $I = \{True, False\}$.

Definition 27 (possibilistic truth value)
The possibilistic truth value $\tilde{t}(p)$ of a proposition $p \in P$ is formally defined by means of the mapping function $\tilde{t}$:

$$\tilde{t} : P \rightarrow \tilde{\wp}(I) : p \mapsto \tilde{t}(p) \tag{1.35}$$

which associates with each $p \in P$ a fuzzy set $\tilde{t}(p)$. The semantics of the associated fuzzy set $\tilde{t}(p)$ is defined in terms of a possibility distribution:

$$\forall\, x \in I : \Pi_{t(p)}(x) = \mu_{\tilde{t}(p)}(x) \tag{1.36}$$

i.e.

$$\forall\, p \in P : \Pi_{t(p)} = \tilde{t}(p) \tag{1.37}$$

To illustrate this, the possibilistic truth value $\tilde{t}(\text{“the land value is low”}) = \{(True, 1.0), (False, 0.7)\}$ of the proposition “the land value is low” is interpreted as

$$\Pi_{t(\text{“the land value is low”})} = \{(True, 1.0), (False, 0.7)\}$$

i.e.

$$Pos(t(\text{“the land value is low”}) = True) = 1.0$$
$$Pos(t(\text{“the land value is low”}) = False) = 0.7$$

To cope with propositions for which a truth value is undefined, the concept of extended possibilistic truth values is introduced ([19]). For this generalization, the set $\tilde{\wp}(I^*)$ of all the fuzzy sets, which can be defined over the universe $I^* = \{True, False, \perp_{Boolean}\}$ is considered. The newly added $t^*(p)$ equals to $\perp_{Boolean}$, if (some of) the elements of the proposition p are not applicable, undefined, nonexistent or not supplied (in this case it is not meaningful to decide whether p corresponds to the reality or not, i.e. p is neither true nor false, but undefined).

Definition 28 (extended possibilistic truth value)
The extended possibilistic truth value $\tilde{t}^*(p)$ of a proposition $p \in P$ is formally defined by means of the mapping function $\tilde{t}^*$:

$$\tilde{t}^* : P \rightarrow \tilde{\wp}(I^*) : p \mapsto \tilde{t}^*(p) \tag{1.38}$$

which associates with each $p \in P$ a fuzzy set $\tilde{t}^*(p)$. The semantics of the associated fuzzy set $\tilde{t}(p)$ is defined in terms of a possibility distribution:

$$\forall\, x \in I^* : \Pi_{t^*(p)}(x) = \mu_{\tilde{t}^*(p)}(x) \tag{1.39}$$

i.e.

$$\forall\, p \in P : \Pi_{t^*(p)} = \tilde{t}^*(p) \tag{1.40}$$

With this previous definition, the extended possibilistic truth value $\tilde{t}^*(p)$ of a proposition $p \in P$ has to be interpreted as follows:

$$\begin{aligned} Pos(t^*(p) = True) &= \mu_{\tilde{t}^*(p)}(True) \\ Pos(t^*(p) = False) &= \mu_{\tilde{t}^*(p)}(False) \\ Pos(t^*(p) = \perp_{Boolean}) &= \mu_{\tilde{t}^*(p)}(\perp_{Boolean}) \end{aligned}$$

where Pos denotes a possibility measure. For more details on possibilistic truth values and extended possibilistic truth values, we refer to [19].

1.3 Uncertainty and Imprecision in GIS

1.3.1 Why uncertainty and imprecision in GIS?

Many data to be modelled in GIS systems are prone to uncertainty or imprecision. This can either be attributed to limitations in the data sources, or it can be due to inherent uncertainty/imprecision in the real world situation. Imprecision refers to the fact that the data cannot be accurately defined (e.g. when measurements cannot be accurately made, for instance because it is too complex or too expensive), whereas uncertainty refers to the fact that there is doubt (e.g. in the case of predictions) concerning the data. Some authors [40] refer to the former as *fuzziness*. The different interpretations yield no difference in the representation of the regions affected by fuzziness, but does yield a difference in operations. Adopting the terminology from [54], in this book the word *fuzzy* is used to indicate the presence of either uncertainty or imprecision.

In an entity-based approach, both the position of objects and the outline of regions can be uncertain or imprecise. For a position, the cause can either be the position is not accurately known (for instance in predicting the location of a person, given some past location; when modelling wildlife; or when using limited gps systems), or when the position can only be determined from other features (i.e. finding a person when knowing he/she is close to a river, a water tower and a highway). For a region, it can occur either because the real boundary is imprecise (i.e. soil composition: the transition from one type of soil to another will hardly ever be a crisp boundary), because the actual boundary - though crisp - cannot be determined accurately (i.e. underground rocks or caves), or because the outline is an estimate (i.e. the spread of wildlife).

In a field-based approach, uncertainty and imprecision can occur at the level of the associated data. The causes here can be inaccurate measurements (i.e. population densities, average rainfall for an area), estimated information, predictions (i.e. about future situations) and extrapolations (i.e. to consider historic information).

While a GIS that supports the notions of imprecision and uncertainty in its data structures would allow for richer modelling, just supporting imprecision and uncertainty in the query engine (even though the database itself only holds crisp information) would allow for a better querying and might make the system more user friendly. By providing such a richer query system, the user would be able to ask the system for locations *close to* another location, or at *walking distance*; or for *interesting* patches of land (where *interesting* can be determined from land usage, pollution, vicinity of specific roads, etc.).

Currently, GIS do not provide adequate support to include uncertainty or imprecision in the representations. On many occasions, buffers are used to overestimate the actual boundary, making sure it is present in the model. Of course as using this techniques adds more points around the boundary (without indication of what most likely is the boundary), analyses following it are skewed. By performing the same analyses with different sized buffers, some alternate solutions can be obtained, but it is a cumbersome process. By allowing a GIS to work with uncertain or imprecise information a more accurate model of reality can be obtained ([6], [15], [37]). When using it for analyses and decision making, it stands to reason that this will improve the results returned, either by providing more solutions or by documenting the returned solutions with indications on what criteria they are better and on what criteria they are worse than others.

1.3.2 Related work

In literature, there has been some work in the field of incorporating fuzziness in geographic systems. Most of the developed fuzzy methods find their application in classification (mainly for land use analysis). In the next subsections, related research by different authors is briefly explained, but with an emphasis on the use of fuzzy set theory. The different subsections each only hold the basics of the approaches, but enough to grasp the concepts and to illustrate how their approaches related to the presented approach in this work. The referenced articles provide for an excellent starting point for more details on the different approaches.

Broad boundaries

The broad boundary model was presented by E. Clementini in [10]; it extends the model for regions (in an entity based model). The concept extended the boundary of a region: instead of a single polyline representing the boundary, two lines were used. This yielded what he called an *inner* and an *outer* boundary. Inside the inner boundary are all the points that certainly belong to the region (or the points that completely belong to the region - the broad boundary model makes no distinction between imprecision and uncertainty); outside the outer boundary are the points that do not belong to the region. In between remains the *broad boundary*, containing points that may or may not belong to the region (or that only belong partly to the region). No other information concerning points in the broad boundary is provided.

Definition 29 (simple region)
A simple region is defined as a closed, homogeneously two-dimensional simply connected subset of $\mathbb{R}^2$.

Definition 30 (region with a broad boundary)
A region A with a broad boundary is made up of two simple regions A_1 and A_2, with $A_1 \subseteq A_2$, where ∂A_1 is the inner boundary and ∂A_2 is the outer boundary of A.

The regions A_1 and A_2 are simple regions; for crisp regions, the notions interior (denoted A_1° and A_2°), boundary (∂A_1 and ∂A_2), exterior (A_1^- and A_2^-) and closure ($\overline{A_1}$ and $\overline{A_2}$) have been defined in 1.1.4.

Definition 31 (boundary ΔA of a region A with broad boundary)
The broad boundary ΔA of a region A with broad boundary is a closed connected subset of $\mathbb{R}^2$ with a hole. ΔA is comprised between the inner boundary A_1 and the outer boundary A_2 of A, such that $\Delta A = \overline{A_2 - A_1}$ or equivalently $\Delta A = A_2 - A_1^\circ$. If $A_1 \subset A_2$, then ΔA is two-dimensional; in the limit case $A_1 = A_2$, ΔA is a closed line. If $\partial A_1 \cap \partial A_2 \neq \emptyset$, then ΔA is not homogeneously two-dimensional and may present one-dimensional parts and separations in its interior.

Definition 32 (interior A° of a region A with broad boundary)
The interior A° of a region A with broad boundary is defined as:

$$A^\circ = A_2 - \Delta A = A_1^\circ \tag{1.41}$$

Definition 33 (exterior A^- of a region A with broad boundary)
The exterior A^- of a region A with broad boundary is defined as:

$$A^- = \mathbb{R}^2 - A_2 \tag{1.42}$$

Definition 34 (closure $\overline{A}$ of a region with broad boundary)
The closure $\overline{A}$ of a region A with broad boundary is defined as:

$$\overline{A} = A^\circ \cup \Delta A = A_2 \tag{1.43}$$

On fig. 1.15, an example of a region with a broad boundary is shown. The inner boundary A_1 and outer boundary A_2 are illustrated, as well as the interior A°, boundary ΔA and exterior A^-. The two arrows indicate one-dimensional parts of the broad boundary.

Consequently, the interior and exterior of a region with broad boundary are open sets, whereas the boundary is a closed set. Simple regions are a special

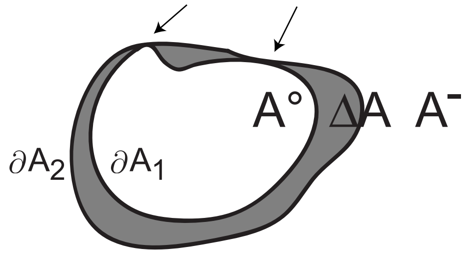


Figure 1.15: A region A with a broad boundary. The arrows indicate portions of the boundary that are one-dimensional parts.

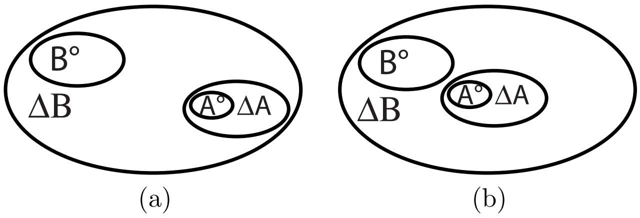


Figure 1.16: Illustration of two different cases for *meet* in broad boundary topology.

case, where inner and outer boundary coincide, thus $\Delta A = \partial A$. Furthermore, Clementini assumes that the extension of the boundary is smaller than its interior: $\Delta A << A^\circ$ (the *small boundaries assumption*). This assumption will imply minor differences with the egg-yolk model 1.3.2. It is possible for the boundary to have 1-dimensional parts, to model entities that fit this behaviour (as indicated by the arrows on fig. 1.15).

To define the topology model for regions with broad boundaries, a nine-intersection matrix similar to the crisp intersection matrix shown in 1.16 is used.

$$\begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \Delta B & A^\circ \cap B^- \\ \Delta A \cap B^\circ & \Delta A \cap \Delta B & \Delta A \cap B^- \\ A^- \cap B^\circ & A^- \cap \Delta B & A^- \cap B^- \end{pmatrix} \tag{1.44}$$

In [10], Clementini shows that of the 2^9 possible matrices in the intersection model, 44 cases are now valid (compare this to the eight valid cases in the crisp topology, 1.1.4). The *small boundaries assumption* eliminates 4 of these cases. The 44 cases are illustrated on fig. 1.17, both the intersection matrices and a graphical representation is shown. The 4 cases eliminated by the small boundaries assumption are cases 14-17. Similarly as for the crisp topology, the conceptual neighbourhood graph can be constructed for the broad boundary topology.

In the broad boundary approach however, the model for the region provides for a broader boundary but not for additional information regarding points in this broad boundary: points are still either inside the region, outside the region, or on the boundary. As a result, no distinction is made for the regions in figure

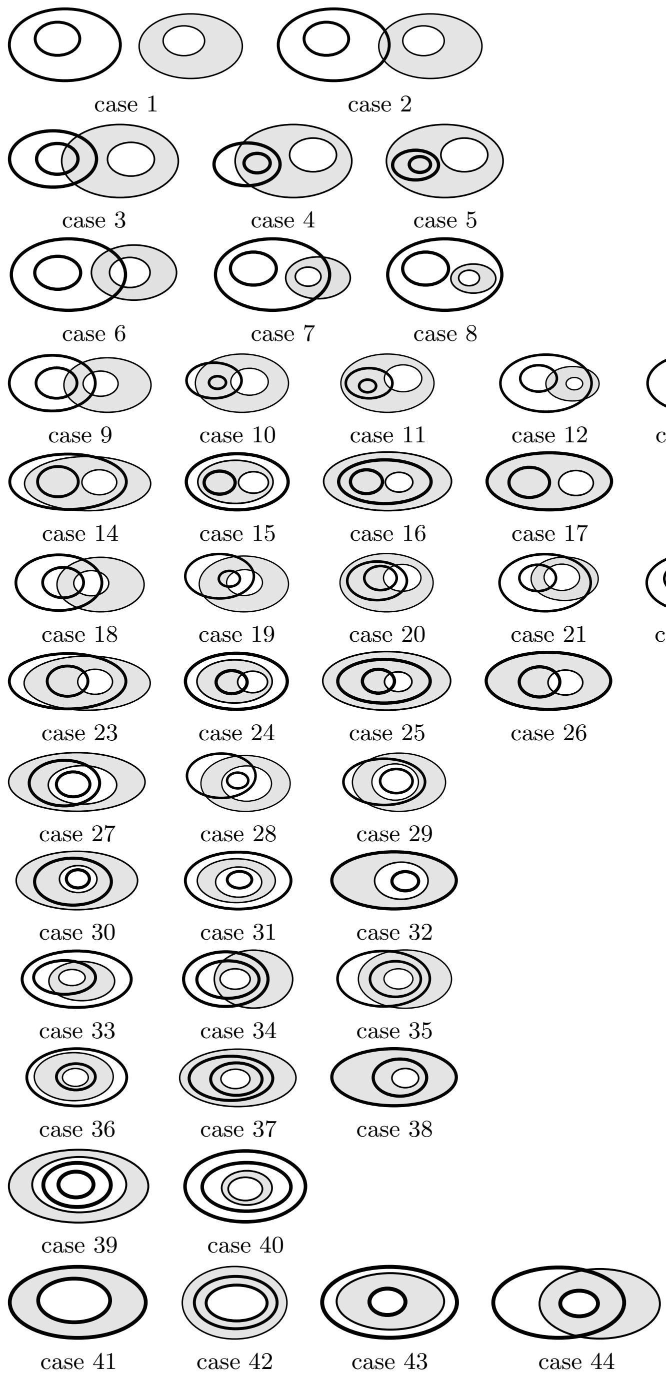


Figure 1.17: Topology cases in the broad boundary model (redrafted after [10])).

1.16a and 1.16b. Intuitively, one might consider a point closer to the inner boundary to belong more to the region, in which case 1.16b is more *overlap* and less *meet* than 1.16a.

Egg-Yolk

The egg-yolk approach is presented by Cohn and Gotts ([13], [28]), and is similar to the broad boundary model mentioned before. The model also makes use of two crisp boundaries: an inner boundary (called the *yolk*) inside which points belong to the region and an outer boundary (called the *egg*) outside which points do not belong to the region. The main difference is that the authors took a logical approach to define the topology; whereas Clementini (1.3.2) took a geometrical approach. This causes them to define a boundary as an open set, which allows them to make a distinction between cases that would fall in the same category in the broad boundary model. Because of this, their intersection model has 46 possible cases. Furthermore, they don't adopt the *small boundaries assumption*, causing this model to have 4 more intersection cases than the broad boundary model.

Technically, these differences are minor, the interesting aspect is that both the geometrical and the logical approach lead to similar conclusions: topology model for the egg-yolk model is quite similar to the broad boundary model; it has the same remarks with regard to this work.

Fuzzy minimum bounding rectangles (FMBR)

In [45], the authors present an approach to derive fuzzy regions from a crisp region. The approach makes use of bounding rectangles: the minimum bounding rectangle serves as the outer boundary (outside of which points do not belong to the fuzzy region, and have membership grade 0), whereas the maximum bounded rectangle servers as the inner boundary (in which points completely belong to the region and have membership grade 1). In between both the inner and outer rectangle, a freely chosen number of rectangles can be considered. These rectangles are then assigned membership grades decreasing from the inside out. This is illustrated on fig. 1.18.

For the definition of a fuzzy set, we refer to 20.

Definition 35 (map space S)
A map space S is a bounded subset of $\mathbb{R}^2$

An *FMBR* is a fuzzy region, and can be represented as a fuzzy set delimited by an MBR (with edges parallel to the reference axes X and Y) in the universe (or map space).

Definition 36 ($FMBR(\tilde{A})$)
Let $\tilde{A} \in F(U)$ be a fuzzy region and $FMBR(\tilde{A})$ its approximation, then $FMBR(\tilde{A}) \subset U$, where.

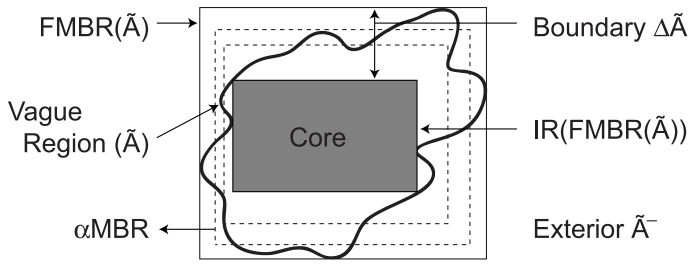


Figure 1.18: Fuzzy regions as MBRs (redrafted after [45]).

- $FMBR(\tilde{A}) = \{u \in U \mid 0 < \mu_{FMBR}(u)\}$
- $core(FMBR(\tilde{A})) = IR(FMBR(\tilde{A})) = \{u \in U \mid \mu_{FMBR(\tilde{A})}(u) = 1\}$
- $\Delta\tilde{A} = \{u \in U \mid 0 < f_{FMBR(\tilde{A})}(u) < \sup_{w \in U} \mu_{FMBR(\tilde{A})}(w)$

The αMBR are generated from the $FMBR$, using the diagonal of the $FMBR$ ($FMBR_{diagonal}$)and the ratio of generation of the α-cuts (r_α):

$$FMBR_{diagonal} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.45}$$

where (x_1, y_1) and (x_2, y_2) are coordinates for respectively the bottom left and the top right corner of the FMBR.

$$r_\alpha = \frac{FMBR_{diagonal}}{k} \tag{1.46}$$

where k is a constant which determines the separation between the α-cuts: the greater k, the greater the number of αMBR and the better the approximation.

The formula to define the αMBR, depends on the morphology of the $FMBR$ ([45]), for illustration purposes only the case where the core is central (not touching the $FMBR$ is shown).

$$\begin{aligned} FMBR_{\alpha_i} &= (X_{2_{\alpha_i}} = X_{2_{\alpha_{i-1}}} - r_\alpha, Y_{2_{\alpha_i}} = Y_{2_{\alpha_{i-1}}} - r_\alpha), \\ &\quad (X_{1_{\alpha_i}} = X_{1_{\alpha_{i-1}}} + r_\alpha, Y_{1_{\alpha_i}} = Y_{1_{\alpha_{i-1}}} + r_\alpha) \end{aligned}$$

Definition 37 (membership function for an FMBR)
The membership function for the FMBR is defined as

$$\mu_{FMBR(\tilde{A})}(x,y) = \begin{cases} 1 & \text{if } (x,y) \in IR(FMBR(\tilde{A})) \\ \frac{\lambda - d((\alpha, MBR), edgeIR(FMBR(\tilde{A})))}{\beta} & \text{if } (x,y) \in \Delta\tilde{A} \\ 0 & \text{if } (x,y) \in \tilde{A}^- \end{cases} \tag{1.47}$$

where

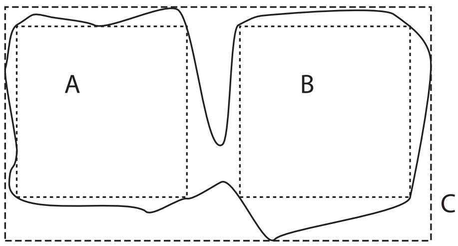


Figure 1.19: Illustration of a possible ambiguity for the maximum bounded rectangle. Rectangles A and B are both candidates for the maximum bounded rectangle; rectangle C is the minimum bounding rectangle.

- *λ is the maximum distance between an edge of $FMBR(\tilde{A})$ and the edge of $IR(FMBR(\tilde{A}))$*
- *$d((\alpha, MBR), edgeIR(FMBR(\tilde{A})))$ is the maximum distance between an edge of the αMBR and the edge of $IR(FMBR(\tilde{A}))$*
- *β is the maximum distance between the centroid of $IR(FMBR(\tilde{A}))$ (this is the center of the rectangle representing the core) and an edge of $IR(FMBR(\tilde{A}))$*

The original main purpose for using fuzzy bounding rectangles, is to approximate fuzzy regions automatically and performantly. While this is an interesting train of thought, there also are a number of downsides to working with rectangles: the approach only allows for limited approximations (by definition, only rectangular ones), regardless of the original shape of the object, and there can be ambiguities as illustrated on fig. 1.19. The maximum inscribed rectangle is not always uniquely defined, or can be open to interpretation (in fig. 1.19, both rectangle A and rectangle B are possible candidates for the maximum inscribed rectangle). While both these issues can be overcome by having an expert intervene, his/her intervention negates the possibility of performing it automatically.

Realm/ROSE extension

In [40], Markus Schneider extended the Realm/ROSE approach to model fuzzy and uncertain regions. The idea is to consider determined zones surrounding the undetermined borders of the objects,and expressing its minimal and maximal extensions. Before introducing the Realm/ROSE extension, first a short introduction to the Realm/ROSE model will be made.

Definition 38 (realm)
A realm used as a basis for spatial data types is a finite set of points and non-intersecting line segments over a discrete domain, which can be viewed (from a graph-theoretical point of view) as a planar graph over a finite resolution grid.

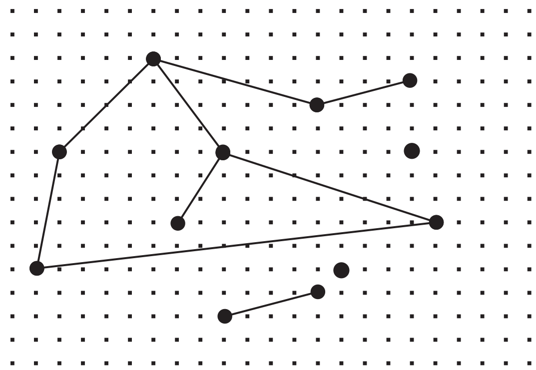


Figure 1.20: A number of objects defined using a realm (redrafted after [40]).

All spatial objects (points, lines, regions) can be defined in terms of points and line segments in the realm. In such spatial databases, no objects are defined directly, but are always defined as suitably selecting some realm elements. All spatial objects are thus realm-based, this is illustrated on fig. 1.20. The underlying grid arises from the fact that numbers always have a finite representation in computer memory: the representations will be of fixed length and correspond to integer or real data types.

The realm concept serves a number of purposes: it guarantees closure properties (all operations are closed), it shields geometric computations from numeric correctness and robustness problems (no new points are ever needed, all numeric problems are dealt with below and within the realm), it provides for a precise specification (lends itself to a correct implementation) and it enforces geometric consistency (common parts of different realm objects are exactly the same).

Definition 39 (R-cycle, R-face, R-unit in a realm)
Consider a realm as a planar graph. An R-cycle *is a cycle of this graph. An* R-face *is an R-cycle possibly enclosing some other disjoint R-cycles corresponding to regions with holes. An* R-unit *is a minimal R-face: any R-face within the R-unit is equal to the R-unit.*

These notions support the definition of regions with holes.

Definition 40
An R-block *is a maximal connected component of a realm graph (it supports the definition of the lines data type).*

The ROSE algebra — where ROSE stands for RObust Spatial Extension — contains very general data types (points, lines and regions).

Definition 41 (points, lines, regions in a realm)
Let R be a realm.
A points *object is a set of R-points.*
A lines *object is a set of disjoint R-blocks.*
A regions *object is a set of (edge-) disjoint R-faces.*

To extend the Realm/ROSE approach, Schneider introduces the concepts *vpoint*, *vline* and *vregion* (v stands for “vague”). These vague objects are defined using sharp (or crisp) means. The central idea is to approximate each of the undetermined boundaries of a region object: its outer boundary line, and the boundary lines of its possibly existing holes; *zones* will be modelling a kind of “irregular spatial intervals”, called a *border zone* and a *hole zone*. A *border zone* is modelled by two or more simple cycles, one representing its *outer border* and one or more representing its *inner border(s)*. A *hole zone* is modelled by two simple cycles, one representing its inner border and one representing its outer border. Matching inner and outer borders C^{in} and c^{out} define the undetermined border, so that for a vague region they express the vagueness of the real, undetermined boundary lines which lay somewhere in between.

Definition 42 (vague region in a realm)
Let $C = (c^{out}, C^{in})$ denote a pair of a single R-cycle c^{out} and a non-emtpy set of R-cycles $C^{in} = (c_1^{in}, ..., c_n^{in})$ and let $H = (H^{out}, H^{in})$ denote a pair of (possibly emtpy) sets of R-cycles $h^{out} = (h_1^{out}, ..., h_n^{out})$ and $H^{in} = (h_1^{in}, ..., h_n^{in})$. Then a vague region vr *is a pair (C, H) so that the following conditions are satisfied:*

$$
\begin{array}{rl}
i) & \forall i \in \{1, ...n\} : c_i^{in} \text{ area-inside } c^{out} \\
ii) & \forall k, l \in \{1, ...n\}, k \neq l : c_k^{in} \text{ edge-disjoint } c_l^{in} \\
iii) & \forall k \in \{1, ...m\} \exists l \in \{1, ..., n\} : h_k^{out} \text{ edge-inside } c_l^{in} \\
iv) & \forall k, l \in \{1, ...m\}, k \neq l : h_k^{out} \text{ edge-disjoint } h_l^{out} \\
v) & \text{there exist two bijective functions } f : \{1, ..., m\} \rightarrow H^{out} \text{ and} \\
 & g : \{1, ..., m\} \rightarrow H^{in} \text{ such that } \forall i \in \{1, ..., m\} : g(i)\ area - inside\ f(i)
\end{array}
\tag{1.48}
$$

The concepts *area-inside*, *edge-disjoint* and *edge-inside* express the relations between Realm/ROSE objects: area-inside means that one area is entirely comprised inside the other (edges can be shared), edge-disjoint means that no edges are shared and edge-inside means that one edge is completely inside the area (it may not overlap with the border of the area); for the formal definitions we refer to [40].

As a result of this definition, the vague region inherits the properties of normal regions in a realm [40], with regards to algebraic laws (commutativity, associativity, ...).

While this model is richer than the broad boundary model, in that it allows for multiple disconnected interiors and holes, the same remark as for broad boundary regions still holds: the model does not provide for additional information about the borders. The use of a realm does however prevent issues due to rounding errors from occurring and improves on perfomance, which could make the realm approach complementary to the TIN based approach 5.

Fuzzy object-based data model for imperfect spatial information

In [4], Bordogna and Chiesa propose a representation of imperfect spatial information using fuzzy set theory (see 1.2), and suggest an approach to represent and manage such information in a fuzzy object-based data model.

Definition 43 (fuzzy spatial object A)
A fuzzy spatial object A is modelled by a fuzzy set on a universe that is a spatial domain X (which is a bounded subset of $\mathbb{R}^2$); for the definition of a fuzzy set, we refer to 20.

For two fuzzy objects A and B, topological relations are also considered. For symmetrical topology relations (*disjoint*, *overlap*), a similarity is defined. This similarity uses the basic set $[0, 1]$ as the domain for the similarity measure between fuzzy sets.

Definition 44 (similarity between two regions)
The similarity between two regions is defined by

$$S(A(x), B(x)) = \frac{|A \cap B|}{|A \cup B|} \tag{1.49}$$

where $|A|$ denotes the cardinality of the fuzzy set A.

The value of S increases in [0,1] as the two fuzzy objects more and more overlap.

The linguistic value *overlapped* and its compound values (e.g. *almost overlapped*) are defined as non-decreasing function on the domain S; while the linguistic value *disjoint* and its compound values (e.g. *almost disjoint*) are defined as non-increasing functions.

For asymmetric topological relations (*outside*, *inside*), a fuzzy inclusion measure is defined. This inclusion measure also uses the basic set $[0, 1]$ as the domain of fuzzy inclusion.

Definition 45 (inclusion of region $A(x)$ in $B(x)$)
The inclusion of region $A(x)$ in $B(x)$ is defined by

$$I(A, B) = \frac{|A \cap B|}{|A|} \tag{1.50}$$

where $|A|$ denotes the cardinality of the fuzzy set A.

The value of I increases as more portions of A are included in B. Similarly to above, the linguistic value *inside* and its compound values are defined as non-decreasing functions over the domain of $I(A, B)$; while *outside* and its compound values are defined as non-increasing functions. The relationship *contains* is defined by the equivalence: "*A contains B*" $\equiv$ "*B inside A*".

To represent imperfect spatial information, a Fuzzy Object-Based Datamodel, which is based on the Fuzzy Object Oriented Database approach, is used. The authors distinguish between *field-based* (where data is considered to be global and to vary continuously and smoothly over the domain) and *exact object-based* (where entities are modelled) models. In this model, the following spatial data types are defined:

- the *Fuzzy Object Class* (FOC) whose instances are fuzzy objects representing spatial entities characterized by fuzziness,
- the *Indeterminate Object Class* (IOC) whose instances are indeterminate objects representing crisp spatial entities whose position is partially or vaguely known,
- the *Fuzzy and Indeterminate Object Class* (FIOC) whose instances are fuzzy and indeterminate objects representing vague spatial entities whose position is partially known,
- the *Indeterminate Spatial Relationship Class* (ISRC) whose instances are indeterminate topological, directional and distance relationships between pairs of objects.

The approach is complementary to what is presented in this work, in that it provides a framework in which the presented results can be applied. The presented practical approaches (4, 5) can be used to define objects that could be instances of the *Fuzzy Object Class*, or they can be used to represent the partial or vague knowledge regarding instances of the *Indeterminate Object Class*. Furthermore, the extension proposed in 2.5.2 can be used to represent instances of the *Fuzzy and Indeterminate Object Class*.

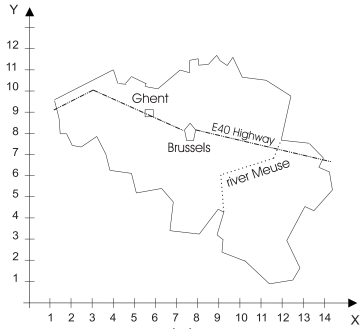
General constraints in spatio-temporal database modelling and querying

Traditionally in GIS, objects themselves are defined directly (by means of some defining points and lines). It is however also possible to define a constraint (which can be seen as a relation that has to be satisfied) on geographic data, and define the object as "consisting of the points that satisfy the constraint". This can even be done in a fuzzy framework, as presented in [21]. The authors present a fuzzy constraint based approach, which is both suitable for data modelling and querying fuzzy spatial and temporal information.

A simple example of the use of constraints to model crisp objects, is given on fig 1.21. This example makes use of traditional, crisp constraints (equality, less than, greater than, etc.).

Definition 46 (unconditional generalized constraint on a variable X) *An unconditional generalized constraint on a variable X is defined by*

$$X isr R \tag{1.51}$$



(a)

Ghent:
$(x \geq 5.5) \wedge (x \leq 5.9) \wedge (y \geq 8.8) \wedge$
$(y \leq 9.2)$

E-40 highway (in Belgium):
$((y \geq 9.1) \wedge (1.8y - x = 14.8) \wedge (y \leq 10))$
$\vee ((y \leq 10) \wedge (x + 2y = 23) \wedge (y \geq 8.2))$
$\vee ((y \leq 8.2) \wedge (x + 4.9y = 48.2) \wedge (y \geq 7))$

River Meuse (in Belgium):
$((y \geq 4.4) \wedge (x + 0.2y = 10.2) \wedge (y \leq 6))$
$\vee ((y \geq 6) \wedge (2.9y - x = 8.4) \wedge (y \leq 6.9))$
$\vee ((y \geq 6.9) \wedge (x - 0.4y = 8.8) \wedge (y \leq 8))$

(b)

Figure 1.21: Illustration of the use of constraints to represent crisp objects: (a) illustration on a map, (b) list of the constraints used (redrafted after [21]).

where R is the constraining relation and isr is a variable copupla in which the discrete-valued variable r defines the way in which R constrains X.

To model truth values that represent the degrees of truth for objects satisfying constraints, extended possibilistic truth values (defined in 1.2.4) are used. The authors then construct a type system, in which an object is characterized by a number of properties to describe its structure and explicitly defined operators that define its behaviour. A property is either an attribute or a binary relationship.

Within the type system, not only the type specification is expressed by means of a generalized constraint (denoted isr), but also the domain value is expressed using a generalized constraint (denoted isr').

In order to describe the fuzzy sets of $\tilde{\wp}(\{(x)|x \in \mathbb{R}\} \cup \{\perp_{Time}\})$ and of $\tilde{\wp}(\{(x_1, x_2, \ldots, x_n)|x_1, x_2, \ldots, x_n \in dom_{SpaceDim}\} \cup \{\perp_{Space}\}))$ the concept linear arithmetic constraint has been generalized.

In order to come to the definition of a generalized linear arithmetic constraint, the definitions for the comparison operators $=$, $\leq$ and $\geq$ are generalized. Traditionally, these comparison operators allow to describe crisp subsets of an n-dimensional space. For the generalization, a normalized fuzzy set $\tilde{V}$ has been associated with each operator. This fuzzy set is defined over the universe of valid distances, i.e., the set $\mathbb{R}^*$ of positive real numbers, and the boundary condition $\mu_{\tilde{V}}(0) = 1$ must hold for it.

Definition 47 (generalized comparison operators)
If

$$d((x_1, x_2, \ldots, x_n), (x'_1, x'_2, \ldots, x'_n))$$

denotes the Euclidean distance between the regular elements $(x_1, x_2, \ldots, x_n)$ *and* $(x'_1, x'_2, \ldots, x'_n)$ *of the considered space, i.e.*

$$d((x_1, x_2, \ldots, x_n), (x'_1, x'_2, \ldots, x'_n)) =$$

$$\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + \ldots + (x_n - x'_n)^2}$$

then the membership functions of the fuzzy sets described by the generalized operators $=_{\tilde{V}}$, $\leq_{\tilde{V}}$ *and* $\geq_{\tilde{V}}$ *are defined as follows:*

- *Temporal type* $Time$. $\forall\ (x) \in dom_{Time} \setminus \{\perp_{Time}\}, \forall\ t \in \mathbb{R}$:

  - $\mu_{x =_{\tilde{V}} t}((x)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x),(x'))|(x') \in dom_{Time} \wedge x' = t\}$$

  - $\mu_{x \leq_{\tilde{V}} t}((x)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x),(x'))|(x') \in dom_{Time} \wedge x' \leq t\}$$

  - $\mu_{x \geq_{\tilde{V}} t}((x)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x),(x'))|(x') \in dom_{Time} \wedge x' \geq t\}$$

- *Spatial type*

  $$Space\ id(id_1 : SpaceDim;\ id_2 : SpaceDim; \ldots ; id_n : SpaceDim)$$

  $\forall\ (x_1, x_2, \ldots, x_n) \in dom_{id} \setminus \{\perp_{id}\}, \forall\ (a_1, a_2, \ldots, a_n, b) \in \mathbb{R}^{n+1}$:

  - $\mu_{a_1x_1+a_2x_2+\ldots+a_nx_n =_{\tilde{V}} b}((x_1, x_2, \ldots, x_n)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x_1, x_2, \ldots, x_n),(x'_1, x'_2, \ldots, x'_n)) \mid (x'_1, x'_2, \ldots, x'_n) \in dom_{id} \wedge a_1x'_1 + a_2x'_2 + \ldots + a_nx'_n = b\}$$

  - $\mu_{a_1x_1+a_2x_2+\ldots+a_nx_n \leq_{\tilde{V}} b}((x_1, x_2, \ldots, x_n)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x_1, x_2, \ldots, x_n),(x'_1, x'_2, \ldots, x'_n)) \mid (x'_1, x'_2, \ldots, x'_n) \in dom_{id} \wedge a_1x'_1 + a_2x'_2 + \ldots + a_nx'_n \leq b\}$$

  - $\mu_{a_1x_1+a_2x_2+\ldots+a_nx_n \geq_{\tilde{V}} b}((x_1, x_2, \ldots, x_n)) = \mu_{\tilde{V}}(d')$, *where*

    $$d' = \min\{d((x_1, x_2, \ldots, x_n),(x'_1, x'_2, \ldots, x'_n)) \mid (x'_1, x'_2, \ldots, x'_n) \in dom_{id} \wedge a_1x'_1 + a_2x'_2 + \ldots + a_nx'_n \geq b\}$$

The entire model allows for representation of geographic objects which are uncertain or imprecise. Consider the following example: in the two-dimensional space

$$Space\ id(x : SpaceDim;\ y : SpaceDim)$$

of the map of Ghent in Figure 1.22, the fuzzy set of points representing 'In Ghent, in the environment of a winding of the river Leie' can be described by the generalized linear constraint

$$(\tilde{c}_1\ \tilde{\wedge}\ \tilde{c}_2\ \tilde{\wedge}\ \tilde{c}_3\ \tilde{\wedge}\ \tilde{c}_4\ \tilde{\wedge}\ \tilde{c}_5\ \tilde{\wedge}\ \tilde{c}_6\ \tilde{\wedge}\ \tilde{c}_7\ \tilde{\wedge}\ \tilde{c}_8\ \tilde{\wedge}\ \tilde{c}_9)\ \tilde{\wedge}\ (\tilde{c'}_1\ \tilde{\vee}\ \tilde{c'}_2\ \tilde{\vee}\ \ldots\ \tilde{\vee}\ \tilde{c'}_n)$$

where the constraints

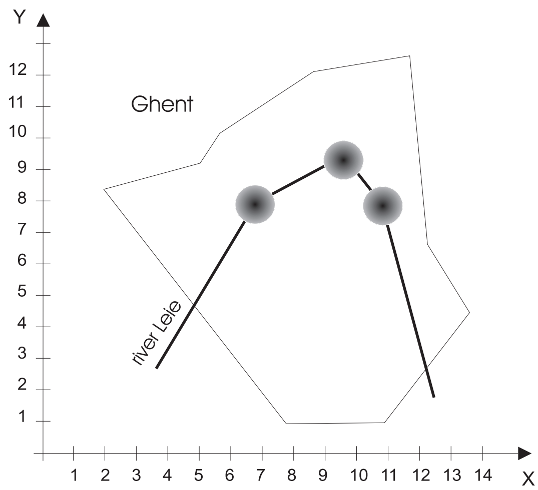


Figure 1.22: Illustration of the use of generalized constraints to model imperfect two-dimensional data.

- $\tilde{c}_1 = (y \geq_{\tilde{U}} 1)$,
- $\tilde{c}_2 = (y + 1.24x \geq_{\tilde{U}} 10.68)$,
- $\tilde{c}_3 = (y - 0.21x \leq_{\tilde{U}} 7.78)$,
- $\tilde{c}_4 = (y - 3.25x \geq_{\tilde{U}} -8.65)$,
- $\tilde{c}_5 = (y - 0.63x \leq_{\tilde{U}} 6.57)$,
- $\tilde{c}_6 = (y - 0.17x \leq_{\tilde{U}} 10.67)$,
- $\tilde{c}_7 = (y + 21.67x \leq_{\tilde{U}} 272.7)$,
- $\tilde{c}_8 = (y + 1.15x \leq_{\tilde{U}} 20.39)$ and
- $\tilde{c}_9 = (y - 1.42x \geq_{\tilde{U}} -14.65)$

with appropriate fuzzy set $\tilde{U}$ are used to model Ghent and the constraints c'_i, $i = 1, 2, \ldots, n$ are used to model the environments of the windings of the river Leie. For example, with an appropriate fuzzy set $\tilde{V}$, the three windings depicted in the figure can be described by the generalized linear constraints

- $\tilde{c'}_k = (x =_{\tilde{V}} 7.2 \;\tilde{\wedge}\; y =_{\tilde{V}} 7.8)$,
- $\tilde{c'}_l = (x =_{\tilde{V}} 9.8 \;\tilde{\wedge}\; y =_{\tilde{V}} 9)$ and
- $\tilde{c'}_m = (x =_{\tilde{V}} 11 \;\tilde{\wedge}\; y =_{\tilde{V}} 7.8)$

The model is quite extensive and supports both temporal and spatial information using a strong theoretical basis. The use of constraints to represent objects is interesting, but might require a lot storage and/or computations when analysis are performed, especially for complex objects (a lot of different constraints may be needed to define the object). As it was built from a query-engine point of view, many traditional geographic operations have not yet been considered. However, as CAD systems make use of a constraint spatial data models (the shapes are simpler than in GIS), this approach could be used to extend those systems. The query-engine side of the approach is however quite complementary to our proposed models.

Natural objects with indeterminate boundaries

An overview of possible issues with data, followed by a way of applying fuzzy sets to represent fuzzy regions a GIS is presented in ([7]). In the introduction, the authors mention that their can be issues regarding the data acquisition of data that is assumed to be perfect for complex objects, boundaries, attribute values, topology and continuous field information; and briefly illustrates how to minimize the issues while still maintaining a crisp model. For inexact objects, the need for inexact data models is illustrated, and a model using fuzzy set theory to represent inexact objects (focussing on regions with indeterminate boundaries) is presented. A fuzzy object is defined by means of a crisp object of which the boundary is *fuzzified*. The authors put forward a general expression for a membership functions to achieve this fuzzification.

$$\mu_{\tilde{A}}(z) = \begin{cases} \frac{1}{1+\left(\frac{z-b_1-d_1}{d_1}\right)^2} & \text{if } z < b_1 + d_1 \\ 1 & \text{if } b_1 + d_1 \leq z \leq b_2 - d_2 \\ \frac{1}{1+\left(\frac{z-b_2+d_1}{d_2}\right)^2} & \text{if } z > b_2 + d_2 \end{cases} \tag{1.52}$$

where b_1 and b_2 define the exact boundaries of the object A and d_1 and d_2 are parameters to shape the function.

The fuzzy objects are then used in classification techniques, in which the operations *and* and *or* are performed using the fuzzy equivalents min and max. No other operations or uses are considered for the model, nor how such regions can be represent in practical implementations.

Conceptual framework and fuzzy set implementation for geographic features

In [47], the authors present a framework for fuzziness in a feature-based geographic information system. Features are defined as fuzzy sets in the two dimensional space.

Definition 48 (map space V)
A map space V is a bounded subset of $\mathbb{R}^2$.

Definition 49 (fuzzy feature)
A fuzzy feature is a fuzzy set whose universe is a map space.

In the examples, a fuzzy feature is defined by means of its core (where the membership grade equals 1) and a linear, decreasing function of the distance to the core to define the membership grades less than 1.

The only considered operations are core and boundary:

Definition 50 (core)

$$core(\tilde{A}) = \{v \in V : \mu_{\tilde{A}}(v) = \sup_{\omega \in V}\{\mu_{\tilde{A}}(\omega)\}\} \tag{1.53}$$

Definition 51 (boundary)

$$boundary(\tilde{A}) = \{v \in V : 0 < \mu_{\tilde{A}}(v) < \sup_{\omega \in V}\{\mu_{\tilde{A}}(\omega)\}\} \tag{1.54}$$

In general, $\sup_{\omega \in V}\{\mu_{\tilde{A}}(\omega)\} = 1$.

The authors however do not provide other operations for this model. A major downside to the supplied boundary definition is the fact that it yields a crisp result. For example, if the $\sup_{\omega \in V}\{\mu_{\tilde{A}}(\omega)\} = 1$, the boundary can contain both points with a membership grade 0.1 and points with a membership grade 0.9, but in the boundary both these points are treated the same.

Analytic Hierarchy Process

In [2], the author explains Saaty's Analytic Hierarchy Process, a method that combines concepts from fuzzy set theory with multi-criteria methodology. In classic decision making, land classification is performed using boolean logic; defying this often leads to problems regarding loss of information or error propagation. Furthermore, there can be situations in which there exists uncertainty regarding the classification. The developed techniques mainly serve a purpose in land classification, for use in multi criteria decision making. The Analytic Hierarchy Process is just one technique to derive membership grades in a geographic context, other techniques also exist. The uncertainty modelled is uncertainty in the attribute data, not in the spatial data itself.

The basis of the Analytic Hierarchy Process (or AHP for short) is fuzzy set theory; an introduction to this can be found in 1.2.

The authors use the notation $X = \{x\}$ to denote a finite set of n points (object/elements/properties), for example:

$$\text{soil drainage property } X \equiv \{x_1, x_2, x_3\}$$

respectively

$$\text{“low”, “moderate” and “extreme”}$$

impermeability. The fuzzy set S in X associates each x_i with a membership grade $\mu_S(x_i)$, representing the extent to which x_i belongs to S. For three elements, this yields

$$S \equiv \{(x_1, \mu_S(x_1)), (x_2, \mu_S(x_2)), (x_3, \mu_S(x_3))\} \tag{1.55}$$

To assign membership grades in the context of soil suitability analysis, Burrough adopted a membership function of the form:

$$\mu_A(x) = \frac{1}{1 + a(x-c)^2} \tag{1.56}$$

Here, a is used to determine the spread and c is used to determine the centre of the distribution of the soil property. As this membership function might not always be suitable for different applications, the AHP provides a way of deriving the membership grades using pairwise comparison of n elements. For this, the matrix A is defined:

$$A = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix} \tag{1.57}$$

The coefficients w_i represent an estimate for the magnitude of the elements x_i with respect to a given property P. The coefficients w_i can be calculated by solving the characteristic value problem

$$A \cdot w = n \cdot w \tag{1.58}$$

where $w = [w_1, w_2, ..., w_n]$. There is only one independent row in A; all rows of A differ by a constant multiple from the first row, so all the characteristic values λ_i $(i = 1, \dots, n)$ are zero except one, denoted λ_{max}. The system can be restated as

$$A \cdot w = \lambda_{max} \cdot w \tag{1.59}$$

The characteristic vector solution w is recovered from any column of A; by normalizing a unique solution is found.

The AHP method provides for assigning membership grades that can be used in classification or satisfaction of properties. These membership grades can then be applied in multi criteria decision making, to overcome the shortcomings imposed by the binary logic.

This approach is different from our goal, which is to model spatial uncertainty or imprecision. Extensions later on (in 2.5.3) will allow for the modelling of fuzzy associated data. Finding appropriate membership grades for this could be done using the described AHP method; our intention is to provide a model to represent and work with this kind of information, regardless of the source of the vagueness.

Intelligent expert system shell for knowledge-based GIS

In [33], an expert system to develop knowledge based GIS systems is presented. The system is called *FLESS*, which stands for Fuzzy Logic Expert System Shell.

We'll provide a short introduction to this system. The system creates, modifies or deletes *objects*; an object is a basic entity, for instance "temperature", "slope" and "population density". An object can be binary (boolean), e.g. polygon X has *no vegetation* or it can be fuzzy, e.g. polygon X has *high temperature*. Along with each object, a number of properties are stored; the most important are a list of rules (see below) whose antecedent parts contain the object and a list of rules whose consequent parts contain the object.

If an object is fuzzy, its associated values are represented by fuzzy subsets. These are usually stored as finite dimensional vectors with default values, for instance "hot" can be stored as

$$(0\ 0.125\ 0.250\ 0.375\ 0.547\ 0.675\ 0.828\ 0.875\ 1\ 1) \tag{1.60}$$

whose values are determined with respect to points, such as 24, 25, 26, 27, 28, 29, 30, 31, 32, 33 (C). In a more common notation, this means:

$$\begin{gathered}\{(24,0),(25,0.125),(26,0.250),(27,0.375),(28,0.547),\\(29,0.675),(30,0.828),(31,0.875),(32,1),(33,1)\}\end{gathered} \tag{1.61}$$

This determination stems from a subjective feeling or an objective function.

A fact is a data proposition of the form

```
<OBJECT> is <VALUE> (fuzzy/non-fuzzy certainty factor)
```

The value can either be a linguistic expression (hot) or a numeric value. The system is capable of converting a number into a linguistic expression.

A rule is an implication statement expressing the relationship between antecedent and consequent propositions. Attached to each rule is a fuzzy/non-fuzzy uncertainty factor, indicating the confidence in this rule. The general structure of a rule is

```
(rule <rule-name>
if <object 1> <operator 1> <value 1> and/or
   <object 2> <operator 2> <value 2> and/or
     ...
   then <object n> is <value n>
) certainty is <certainty factor>
```

The operators can be ordinary inequalities or fuzzy inequalities; the certainty factor can be a precise value in $[-1,+1]$, a fuzzy number or a linguistic probability.

An example of a rule in GIS is

```
IF the slope is more or less gentle
  AND (the precipitation is moderate OR
```

```
      the underground water table is sufficiently high)
  AND the temperature is warm
THEN the piece of land is suitable for cultivating crop X
 with certainty 0.9
```

The FLESS system provides for a system to reason with fuzzy attribute data. The goal of this study is to develop models to represent fuzzy spatial information and (to a lesser extent) fuzzy attribute data. The FLESS system is complementary to the techniques described in 2.5.3.

Error propagation in cartographic modelling using boolean logic and continuous classification

In [31], Heuvelink and Burrough study the error propagation when a fuzzy (continuous) classification is used. The authors use the membership function MF^C

$$\begin{array}{ll} MF^C(z) = \frac{1}{1+\left(\frac{z-b_1-d_1}{d_1}\right)^2} & \text{if } z < b_1 + d_1 \\ MF^C(z) = 1 & \text{if } b_1 + d_1 \leq z \leq b_2 - d_2 \\ MF^C(z) = \frac{1}{1+\left(\frac{z-b_2+d_2}{d_2}\right)^2} & \text{if } z > b_2 - d_2 \end{array} \tag{1.62}$$

Where, the parameters d_i determine the width of the *transition* zone. The transition zones are such that the continuous membership function equals 0.5 at b_i. If $d_1 = 0$ and $d_2 = 0$, the membership function yields the boolean function.

The error propagation is illustrated by the authors on a land classification example. This example shows that the fuzzy classification is less influenced by errors in the input than the traditional classification is (the results are illustrated using a bitmap representation in which the land classification is coloured).

Just like the previous methods, this paper presents interesting work concerning working with fuzzy attribute data (esp. in the land classification), but has little to do with fuzzy locational information. The results from [] are interesting when considering to use our fuzzy region model for fuzzy associated data, as presented in 2.5.3.

Chapter 2

Fuzzy regions

2.1 Concept of fuzzy regions

In this chapter, the concept of fuzzy regions will be introduced. Subsequent chapters will introduce models that are based on this concept, but that are more adequate for implementation. Fuzzy regions are a concept related to an entity based approach (1.1.2). Traditionally regions are defined by specifying their outline, and implicitly all locations (points) inside the outline are considered to be part of the region ([41]). But one can turn this around: a set of locations can be said to make up a region. This is the first step in defining fuzzy regions, from there on it is a small step to extend a region to a fuzzy region: a fuzzy set of locations (where each location is represented by a point and each location has a membership grade associated). The membership grades for regions are interpreted in a veristic way: all locations belong to the region, but some more than others (1.2.2).

This concept differs from the earlier approaches (see 1.3.2) for a number of reasons: firstly, the region is not defined by its outline, but by means of its elements, i.e. the locations that belong to this region. Secondly, it provides for an infinite number of membership grades to indicate the extent to which each of the locations belong to the region. This is illustrated on fig. 2.1: both the broad boundary model, as the egg-yolk model would treat the points p_1 and p_2 alike. However, p_1 is much closer to the interior than p_2 is; for some applications it should be possible to make a distinction between them.

Finally, in the concept, no assumption is made on the distribution of the points: the Fuzzy Minimum Bounding Rectangles (FMBR) and similar models assume some simple layout of boundaries (the rectangle become smaller with increasing membership grades). While this can still be accomplished with fuzzy regions, the concept is not limited to this.

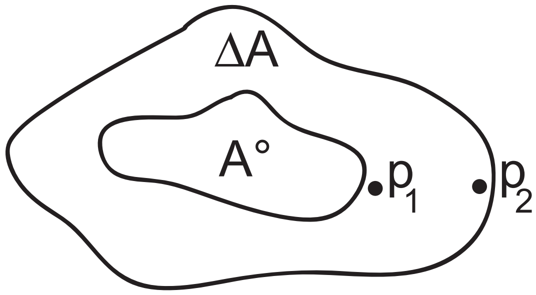


Figure 2.1: Broad boundary and egg-yolk regions consider p_1 and p_2 the same: both belong to the boundary.

2.2 Definition of a fuzzy region

Consider $A \subseteq U$ the set of all the points that belong to the region (this is a crisp set). The crisp set A is then generalized to a fuzzy set $\tilde{A}$, defined as follows.

Definition 52 (fuzzy region $\tilde{A}$)
A fuzzy region $\tilde{A}$ is defined as:

$$\tilde{A} = \{(p, \mu_{\tilde{A}}(p)) \mid p \in U, \mu_{\tilde{A}}(p) > 0\} \tag{2.1}$$

where

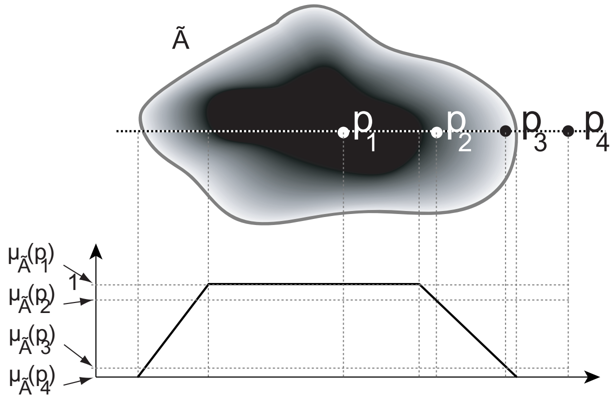
$$\begin{aligned} \mu_{\tilde{A}} : U &\to]0,1] \\ p &\mapsto \mu_{\tilde{A}}(p) \end{aligned}$$

Here, U is the universe of all locations p; the membership grade $\mu_{\tilde{A}}(p)$ expresses the extent to which p belongs to the fuzzy region.

On fig. 2.2, an example of a fuzzy region is shown.

The main difference between fuzzy regions and more common fuzzy sets, is that the domain of the fuzzy region U is in itself a two dimensional domain (theoretically $\mathbb{R}^2$, but normally limited depending on the representation method of the coordinates). For clarity, the notation $\tilde{A}$ will be used for fuzzy regions (and fuzzy sets in general), whereas A will be used for crisp regions (and crisp sets).

The notion of a *simple* fuzzy region is sometimes used. By a simple fuzzy region a fuzzy region of which the membership grades decrease from the inside towards outside; in other words: every α-level is simple region. This definition makes use of the α-level, as defined for fuzzy sets in 1.2.3 and adopted for fuzzy regions in 2.4.1.



$\mu_{\tilde{A}}(p_1) = 1$
$\mu_{\tilde{A}}(p_2) = 0.8$
$\mu_{\tilde{A}}(p_3) = 0.1$
$\mu_{\tilde{A}}(p_4) = 0$
...

Figure 2.2: A fuzzy region, for illustration purposes the fuzzy region is delimitted by a light grey line. The membership grades for points belonging to the region are shaded, ranging from black (membership grade 1) to white (membership grade 0). A cross section shows an how the membership grades along the drawn line evolve. On the right are possible membership grades for the points illustrated.

Definition 53 (simple fuzzy region / complex fuzzy region)
A fuzzy region $\tilde{A}$ is called simple when

$$\forall \alpha \in]0, 1] : \tilde{A}_\alpha \textit{ is a simple region without holes} \tag{2.2}$$

where $\tilde{A}_\alpha$ is the weak α-cut of $\tilde{A}$ at level α (2.4.1), which yields a crisp region . A complex *fuzzy region is a fuzzy region that is not simple, meaning that there can be α-levels that contain holes, or α-levels that are made up from disconnected regions.*

An illustration of simple and complex fuzzy regions can be seen on fig. 2.3. The region in fig. 2.3a is complex, as the core is disconnected. While the region in fig. 2.3b appears simple, it is not due to the fact that the membership grades do not decrease from the inside outward (there are α-levels that yield regions with a hole). The region in fig. 2.3c is a simple fuzzy region (note that the region does not have to be convex).

Fuzzy regions that are simple are a more intuitive concept to work with; a lot of situations will yield fuzzy regions where membership grades will increase toward the core.

2.3 Interpretation and examples

As mentioned before, membership grades in fuzzy regions are considered to have a veristic interpretation, meaning that all points belong to the set to some extent. Various practical applications would benefit from allowing a partial-belonging-to relation. Consider for example the soil composition, more specifically the boundary between for instance a sandy soil and a clay soil. This

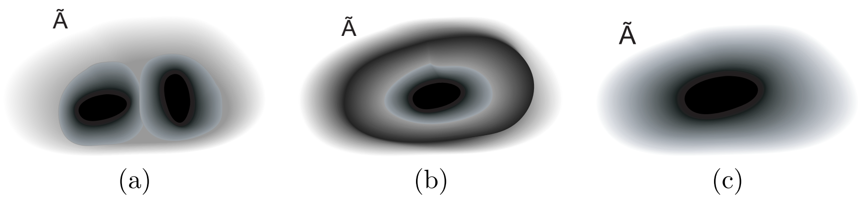


Figure 2.3: Illustration of the notion *simple*: (a) and (b) are examples of complex fuzzy regions, (c) is an example of a simple fuzzy region.

boundary is currently derived from a number of measurements and represented by a crisp line. However, in reality, this is more of a "zone", gradually changing from all sand and no clay, to slightly more clay, to an equal distribution, to more clay than sand and finally all clay and no sand.

Slightly similar is the area affected by a natural disaster (earthquake, hurricane, etc.). There will be a region that is worst affected, but moving away from this core there will still be damages, though not as bad as in the core region.

In the above cases, the imprecision occurs in reality, and would occur even if perfect measurements are possible. Somewhat different it the case where a limitation imposed on measurements can require an approximation for the boundary. An example of this is the hunting territory of animals, or even boundary of a lake: as an entity in which all levels are represented, one can consider the lake to include the locations that are flooded in the event of high water levels (in which case those locations are assigned lower membership grades with a veristic interpretation). However if one considers the actual boundary the lake ought to be represented as a region delimited with a possibilistic line. This later interpretation is not yet possible with the presented model, but is also under investigation. In general, these situations occur when the boundaries in reality are crisp, but when it is too difficult (e.g. too expensive, too labour-intensive, too technically challenging) to measure these crisp boundaries. The real crisp boundary can then be approximated by a fuzzy boundary.

2.4 Operations on fuzzy regions

There are quite a number of interesting operations possible on fuzzy regions: some operations stem from the fuzzy realm, others have a geographic origin. In the following subsection, an overview of the considered operations as well as their definitions are provided.

2.4.1 Operations from the fuzzy realm

The fuzzy regions are defined as fuzzy sets over a two dimensional domain. Consequently, the operations from the fuzzy realm are the same as their counterparts for regular fuzzy sets. For completeness, the definitions are repeated below, but with the notations adapted for fuzzy regions.

Set-operations

Consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$. The set operations (intersection, union) on both fuzzy regions is the same as the definition for fuzzy sets, using t-norms (1.25) and t-conorms (1.26).

Definition 54 (intersection of two fuzzy regions $\tilde{A}$ and $\tilde{B}$)

$$\tilde{A}\tilde{\cap}\tilde{B} = \{(x, \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x)) \mid \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x) = T((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)))\} \tag{2.3}$$

Definition 55 (union of two fuzzy regions $\tilde{A}$ and $\tilde{B}$)

$$\tilde{A}\tilde{\cup}\tilde{B} = \{(x, \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(x)) \mid \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(x) = s((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)))\} \tag{2.4}$$

Definition 56 (complement of a fuzzy region $\tilde{A}$)

$$co\tilde{A} = \{(x, \mu_{co\tilde{A}}(x)) \mid \mu_{co\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)\} \tag{2.5}$$

The intersection holds those locations that are present in both regions. The associated membership grade for the locations is calculated by means of the t-norm; if the minimum is used as the T-norm, the associated value for a locations x becomes the smallest of the respective values $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$. Similarly, the union holds all the locations that exist in both regions; now the associated value is calculated using an S-norm. If the maximum is used, the associated value for a location x in the union is the maximum of the values $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$. The complement holds all those points that were not present in the original region; the occurring membership grades are computed using the definition of the complement of a fuzzy set.

α-cut

When working with fuzzy sets, it can be necessary to extract crisp sets from the fuzzy information, which can for instance be needed to process the data using non-fuzzy techniques. This is often called “defuzzifying” a fuzzy set, for which the α-cut operation is commonly used. The α-cut holds the elements for which the membership grades are greater than a given value α (the α-level). As a fuzzy region is fundamentally a fuzzy set, the α-cut is easily defined, along with its special cases. These definitions are very straight forward, and are only repeated here for completeness.

Weak α-cut The *weak α-cut* is defined as:

$$\tilde{A}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2.6}$$

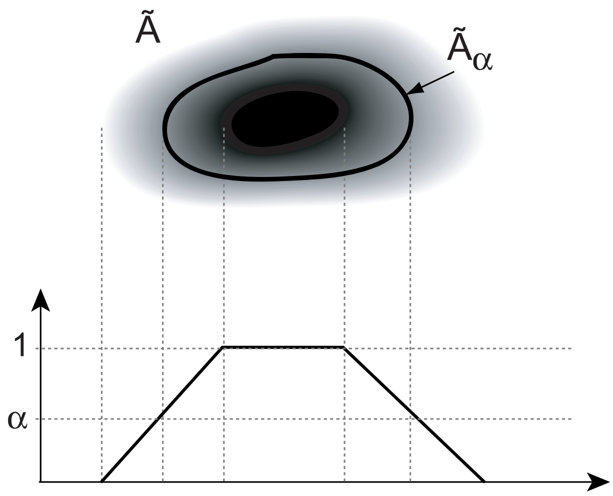


Figure 2.4: An example for the α-cut of $\tilde{A}$ is illustrated by the black outline. α-cuts using lower α values will yield a larger region (up to the support); α-cuts using higher α values will yield a smaller region (down to the core).

Core A special case of a weak α-cut is the *core*; this is the weak α-cut with threshold 1. This is an important α-cut, as it results all the elements that belong completely to the fuzzy set.

$$\tilde{A}_1 = \{x \mid \mu_{\tilde{A}}(x) \geq 1\} \tag{2.7}$$

Strong α-cut The *strong α-cut* is defined as:

$$\tilde{A}_{\overline{\alpha}} = \{x \mid \mu_{\tilde{A}}(x) > \alpha\} \tag{2.8}$$

Support Similarly to the weak α-cut, the strong α-cut has a special case, now for a threshold 0. This α-cut is called the *support*, and returns all the elements that belong to some extent to the given fuzzy set.

$$\tilde{A}_{\overline{0}} = \{x \mid \mu_{\tilde{A}}(x) > 0\} \tag{2.9}$$

It is important to note that while the α-cut of a fuzzy region results in a crisp set of locations, an additional step still needs to be performed before this crisp set of locations can be represented as a traditional, crisp region; as illustrated on fig. 2.4. This is due to the fact that traditionally, a region is defined by its outline. Consequently, when the crisp set of locations has been found, the outline of all these locations needs to be determined.

2.4.2 Operations from the geo-spatial realm

Minimum bounding rectangle

The minimum bounding rectangle for crisp regions has been explained in 7. Traditionally, MBRs are used for both indexing as for optimizing query evaluation. Both uses are still appropriate for fuzzy regions: in an index, the MBR at different α-levels can be taken into account. Furthermore, just as MBRs are used to optimize query evaluations, similar evaluations can be made for fuzzy regions; two fuzzy regions will not intersect to a degree greater than e.g. 0.8 if the MBRs at that α-level don't intersect.

Before defining the fuzzy minimum bounding rectangle [1], consider the desired properties of this operator. For a crisp region, one can state it simplifies the outline to a rectangle. For a fuzzy region, it would be interesting if the simplified outline would still incorporate some of the fuzziness of the original region (which implies that the bounding rectangle will also be a fuzzy region). Using the α-cuts (2.4.1), it is possible to defuzzify a fuzzy region. It would be interesting if the fuzzy bounding rectangle were compatible with defuzzified regions. Combining these desired properties has led to the definition of the fuzzy minimum bounding rectangle of a fuzzy region $\tilde{A}$ (denoted $\tilde{mbr}(\tilde{A})$) as a new fuzzy region for which every weak α-cut is the crisp minimum bounding rectangle of the same weak α-cut of the original fuzzy region.

Definition 57 (minimum bounding rectangle of a fuzzy region $\tilde{A}$)

$$\tilde{mbr}(\tilde{A}) = \{(p, \mu_{\tilde{mbr}(\tilde{A})}(p))\} \tag{2.10}$$

where

$$\begin{aligned} \mu_{\tilde{mbr}(\tilde{A})} : U &\rightarrow \;]0,1] \\ p &\mapsto \sup\{\alpha_i \mid \alpha_i \in]0,1] \wedge p \in mbr(\tilde{A}_{\alpha_i})\} \end{aligned}$$

Following the definition, fuzzy bounding rectangle is simple (53) and rectangular; as is clearly illustrated on fig. 2.5.

Convex hull

The convex hull of a crisp region 1.1.4 returns the smallest convex polygon that holds a given polygon. The concept of a fuzzy convex polygon should resemble this for crisp regions (which are in essence a special case of fuzzy regions). Consequently, the fuzzy convex hull has similar properties to a fuzzy minimum bounding rectangle: it should take into account the fuzziness of the region, while at the same time also be compatible with the crisp convex hull of a defuzzified fuzzy region. As a result, a fuzzy convex hull is defined as being a fuzzy region, of which every α-cut is the crisp convex hull for the matching α-cut of the original fuzzy region. This leads to the definition:

[1] Not to be confused with the Fuzzy MBR approach as presented by Somodevilla and Petry in [45], see 1.3.2.

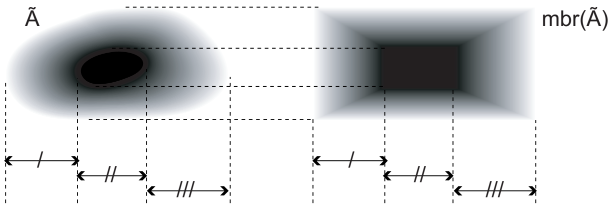


Figure 2.5: Illustration of a the fuzzy MBR of a fuzzy region: the MBR is a rectangular region, that at each α-level holds the MBR corresponding with the polygon obtained by considering the same α-level in the original fuzzy region.

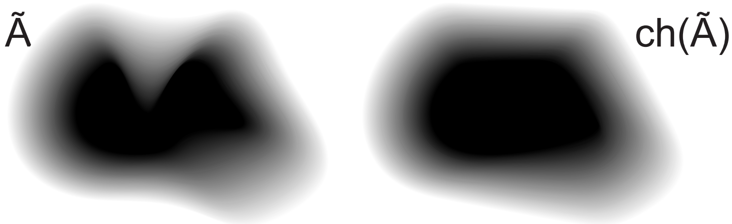


Figure 2.6: Illustration of the fuzzy convex hull of a fuzzy region: (a) fuzzy region $\tilde{A}$, (b) the fuzzy convex hull of $\tilde{A}$.

Definition 58 (convex hull (ch) of a fuzzy region $\tilde{A}$)

$$\tilde{ch}(\tilde{A}) = \{(p, \mu_{\tilde{ch}(\tilde{A})}(p))\} \tag{2.11}$$

where

$$\begin{aligned} \mu_{\tilde{ch}\tilde{A}} : U &\rightarrow \;]0,1] \\ p &\mapsto \sup\{\alpha_i \mid \alpha_i \in]0,1] \wedge p \in ch(\tilde{A}_{\alpha_i})\} \end{aligned}$$

Similar to the fuzzy minimum bounding rectangle, the convex hull is a simple fuzzy region (53), illustrated on fig. 2.6.

Surface area

The surface area of a fuzzy region lends itself to two interpretations, depending on the interpretation of the fuzzy region. The first interpretation is best illustrated with the aforementioned example of the boundary of a lake (where there is a veristic interpretation), the surface area of the lake as an entity depends on the size of the lake, which in turn is modelled with different membership grades: locations with a membership grade less than 1 contribute to the surface area only if the lake has the size that matches with this membership grade. Consequently, the surface area will represent this as a fuzzy number, where the possibility distribution is dependent of the locations in the fuzzy region. On the other hand, if the fuzziness is used to represent intrinsic fuzziness, there usually will be no regarding the surface area, hence it will yield a crisp number. This crisp number is an extension of the concept of fuzzy cardinality.

Interpretation 1: fuzzy number The fuzzy surface area $\tilde{S}$ in the first interpretation will result in a fuzzy number that represents the possible surface areas. By definition ([32]), a fuzzy number $\tilde{B}$ is defined as a fuzzy set over $\mathbb{R}$ that satisfies the properties:

- $\tilde{B}$ is normed, i.e. there is at least one element x for which $\mu_{\tilde{B}}(x) = 1$
- $\forall \alpha \in]0,1] : \tilde{B}_{\overline{\alpha}}$ is a closed interval
- the support $\tilde{B}_0$ of $\tilde{B}$ must be bounded

For future computations, it is useful to have a fuzzy number as the result of a fuzzy surface area: calculations with different surface areas can be performed using fuzzy arithmetic ([32]).

Both the weak and the strong α-cut of a fuzzy region (2.4.1) $\tilde{A}$ yield a crisp region, denoted $\tilde{A}_\alpha$ respectively $\tilde{A}_{\overline{\alpha}}$. For these crisp regions and for every $\alpha \in]0,1]$, the following property holds:

$$S(\tilde{A}_{\overline{\alpha}}) \leq S(\tilde{A}_\alpha) \tag{2.12}$$

where S is the notation for the calculation of the surface area of a crisp region. The equality only occurs if $S(\tilde{A}_\alpha - \tilde{A}_{\overline{\alpha}}) = 0$; this happens if the points p for which $\mu_{\tilde{A}}(p) = \alpha$ form a one dimensional object (i.e. a line). This allows us to define the surface area as:

Definition 59 (fuzzy surface area $\tilde{S}^f$)

$$\tilde{S}^f(\tilde{A}) = \{(x, \mu_{\tilde{S}^f(\tilde{A})}(x)), x \in U\} \tag{2.13}$$

where U is the considered universe and

$$\begin{aligned} \mu_{\tilde{S}^f(\tilde{A})}(x) : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \begin{cases} 1 & \textit{if } x = S(\tilde{A}_1) \\ \sup\{\alpha \mid \\ \qquad S(\tilde{A}_{\overline{\alpha}}) \leq x \leq S(\tilde{A}_\alpha)\} & \textit{if } S(\tilde{A}_{\overline{\alpha}}) \leq x \leq S(\tilde{A}_\alpha) \\ 0 & \textit{elsewhere} \end{cases} \end{aligned}$$

This is illustrated on fig. 2.7.

This result is normed if the arguments are normed. Furthermore, every α-cut for $\alpha \in]0,1]$ yields a closed interval. This is even the case if the membership function of a fuzzy region contains discontinuities (as illustrated on fig. 2.8a). The membership function for a fuzzy region that has a discontinuous membership function is illustrated in fig. 2.8b. The result will be a decreasing membership function, which is piecewise if a finite number of α-cuts is considered.

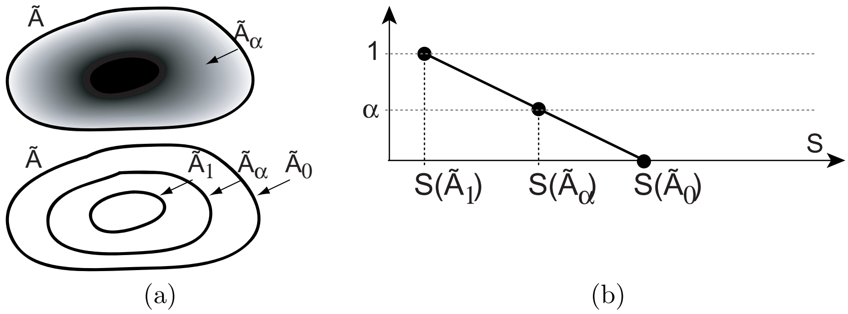


Figure 2.7: Illustration of the fuzzy surface area yielding a fuzzy number: (a) fuzzy region $\tilde{A}$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy surface area of $\tilde{A}$.

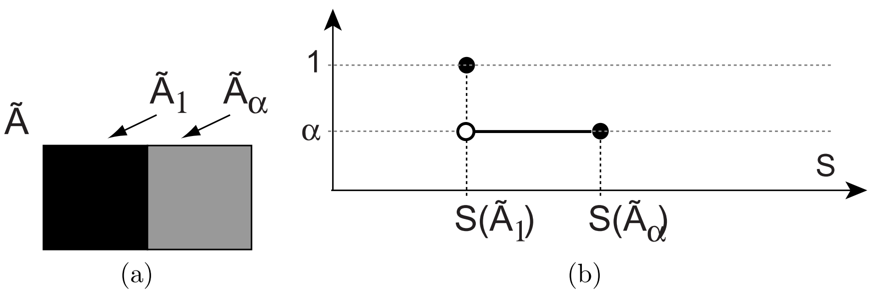


Figure 2.8: Illustration of the fuzzy surface area of a region with a discontinuous membership function: (a) fuzzy region $\tilde{A}$, (b) the fuzzy surface area of $\tilde{A}$.

Interpretation 2: crisp number In the second interpretation, the fuzziness is used to indicate the intrinsic vagueness of a region. The surface area therefore becomes a crisp number. This number takes all points into consideration, where the membership grade for each point determines how much is will contribute: a point with a membership grade 0.5 will only contribute half of what a point with membership grade 1 will contribute. In a discrete set, this number resembles the fuzzy cardinality, but for infinite sets this needs to be extended.

Definition 60 (fuzzy surface area $\tilde{S}^c$)

$$\tilde{S}^c(\tilde{A}) = \int_{(x,y)\in U} \mu_{\tilde{A}}(p(x,y)))d(x,y) \tag{2.14}$$

This interpretation can be used when the results need to be processed by a non-fuzzy system.

Distance calculation

The distance to a fuzzy region is a complicated concept. Consider for instance the distance form Spain to the United Kingdom. Gibraltar is part of the United Kingdom, but this distance might not always be desired: for a criminal evading the Spanish police, Gibraltar might be the easiest way out, but most people will mean the main island of the UK. The distance for complex fuzzy regions (53) might yield weird results; we propose two possible distance measurements that are related to the crisp concept of the shortest distance between regions. Without having further information on how the membership grades of the fuzzy region are interpreted, it is virtually impossible to pinpoint the most appropriate definition.

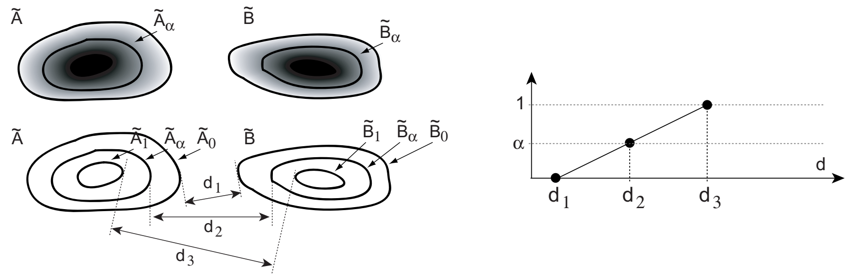
α-level approach The first definition is based on α-levels. With this definition, the distance to a fuzzy region will be represented by a fuzzy number, which models all the possible distances between the corresponding α-levels. To define the distance calculation, consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$. The distance between crisp regions is defined as the shortest distance between them (1.1.4). The fuzzy distance $\tilde{d}(\tilde{A},\tilde{B})$ is a fuzzy number representing the possible distances between the α-cuts of A and B.

Definition 61 (distance between fuzzy regions)
(in an α-level approach)

$$\tilde{d}(\tilde{A},\tilde{B}) = \{(x, \mu_{\tilde{d}(\tilde{A},\tilde{B})}(x)) \mid x \in \mathbb{R}\} \tag{2.15}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup\{\alpha \mid d(\tilde{A}_\alpha, \tilde{B}_\alpha) \le x \le d(\tilde{A}_{\overline{\alpha}}, \tilde{B}_{\overline{\alpha}})\} \end{aligned}$$



(a) (b)

Figure 2.9: Illustration of the fuzzy distance yielding a fuzzy number: (a) fuzzy regions $\tilde{A}$ and $\tilde{B}$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy distance between $\tilde{A}$ and $\tilde{B}$.

In the case of fuzzy regions with discontinuous membership functions, the definitions provides for a result similar to the result obtained by the definition of the fuzzy surface (2.4.2). The definition is illustrated on fig. 2.9.

In this definition, the distance to points with a membership grade 1 is the only distance which in turn receives membership grade 1. Intuitively, one might wonder about this: points with a membership grade greater than 0.5 already are considered to belong more to the region than to be outside of it.

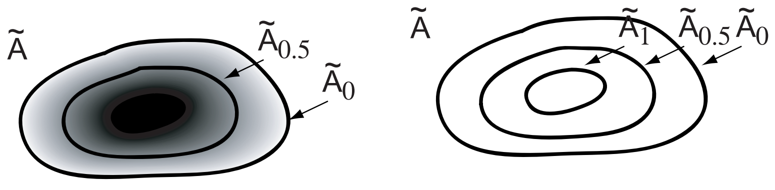
Topological approach The second interpretation makes use of the definitions for interior, exterior and boundary of a region; the distance between two fuzzy regions $\tilde{A}$ and $\tilde{B}$ in this interpretation will be denoted $\tilde{d}^{\Delta}(\tilde{A}, \tilde{B})$. The topology concepts will first be explained, after which the topological approach for distances is considered (66).

2.4.3 Topology

Traditionally, topology between regions is defined using the boundary, interior and exterior of the regions involved. For fuzzy regions, which are not defined by means of their boundary, these concepts need to be defined. Furthermore, the elements of the intersection matrices and the interpretation of the matrices themselves differs from the traditional approach. To illustrate the fuzzy topology concepts, a region $\tilde{A}$ as shown in figure 2.10 is used.

Defining the extended concepts

Boundary The boundary-concept in the fuzzy region model is not part of the definition of a region; the boundary has to be extracted from the definition of the region. For regions with broad boundaries, and in the egg-yolk model (1.3.2), the boundary was a new region, more specifically the region between both the inner and outer boundaries. For fuzzy regions, the analogy makes sense: the boundary will be a new fuzzy region, where points will be assigned



(a) (b)

Figure 2.10: Sample region to illustrate the fuzzy topology concepts: (a) represented using grey scales (b) represented with a number of contour lines.

membership grades to indicate to what extent they belong to this boundary, as shown on fig. 2.11a and fig. 2.11b. Points p with membership grade $\mu_{\tilde{A}}(p) = 0$ or $\mu_{\tilde{A}}(p) = 1$ in the original region do not belong to the boundary, as they are completely outside, respectively completely inside the region. Points with a membership grade less than 0.5 belong less to the region than points with a membership grade greater than 0.5; consequently, 0.5 will play a crucial part. Points p with membership grade $\mu_{\tilde{A}}(p) = 0.5$ will be considered to completely belong to the boundary (and will be assigned membership grade $\mu_{\Delta\tilde{A}}(p) = 1$ in the boundary). The more the original membership grade differs from 0.5, the lower the membership grade it will be assigned in the boundary. This is accomplished with the function

$$2(0.5 - |0.5 - x|), \forall x \in [0, 1] \tag{2.16}$$

This is illustrated on fig. 2.11c.

As crisp regions (and broad boundary regions) can be considered to be special cases of fuzzy regions, the definition must be such that it reverts back to the boundary definitions for those situations (this is also required if for instance a region's boundary is crisp at some part). This is achieved by considering the boundaries $\partial\tilde{A}_\alpha$ at every α-level α. The boundary of a fuzzy region $\tilde{A}$ will be defined as:

Definition 62 (boundary $\Delta\tilde{A}$ of a fuzzy region $\tilde{A}$)

$$\Delta\tilde{A} = \bigcup_{\alpha \in]0,1]} \{(p, 2(0.5 - |0.5 - \alpha|)) \mid p \in \partial\tilde{A}_\alpha\} \tag{2.17}$$

This basically means that the fuzzy boundary $\Delta\tilde{A}$ (the same notation Δ as in [10] is used) holds all the points that are not in completely in the interior, nor completely in the exterior. The membership grades are associated in a way that points for which $\mu_{\tilde{A}}(p) = 0.5$ are assigned the membership grade $\mu_{\Delta\tilde{A}}(p) = 1$.

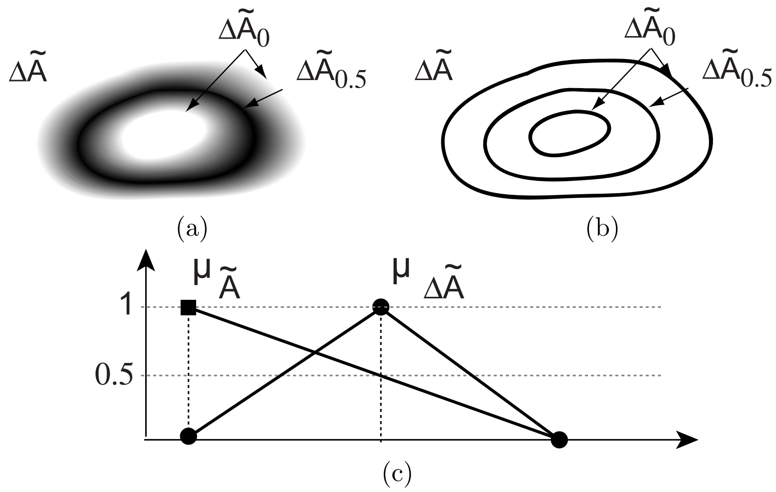


Figure 2.11: Illustration of the fuzzy boundary: (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\Delta\tilde{A}$.

Interior The concept of the interior is similar to the concept of the boundary and it also is a fuzzy region. The membership grades of a fuzzy region are in the range $[0, 1]$; 1 indicates the points that are completely part of the region; this will also be the core of the interior. Points p just outside this core, but still belonging to a substantial extent to the region (i.e. $\mu_{\tilde{A}}(p) > 0.5$ are also considered to be part of the interior to a lesser extent. Points p with a membership grade $\mu_{\tilde{A}}(p) \leq 0.5$, are considered not to belong to the interior.

Definition 63 (interior $\tilde{A}^\circ$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^\circ = \{(p, \mu_{\tilde{A}^\circ}(p)\} \tag{2.18}$$

Where

$$\begin{aligned} \mu_{\tilde{A}^\circ} : U &\rightarrow [0, 1] \\ p &\mapsto \begin{cases} 0 & \mu_{\tilde{A}}(p) \leq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(p) & \textit{elsewhere} \end{cases} \end{aligned}$$

The interior is defined using the membership grades of the points. The interior $\tilde{A}^\circ$ is a fuzzy region, containing only those points that have a membership grade strictly greater than 0.5 in the original region $\tilde{A}$. The membership grades $\mu_{\tilde{A}}(p)$ are all in the range $]0.5, 1]$, these are rescaled so that the interior has membership grades in the range $[0, 1]$, see figure 2.12.

Note that some relations that are valid in the crisp case, are no longer valid: traditionally, $\tilde{A}^\circ = \tilde{A} \backslash \partial\tilde{A}$, but $\tilde{A}^\circ \neq \tilde{A} \backslash \Delta\tilde{A}$.

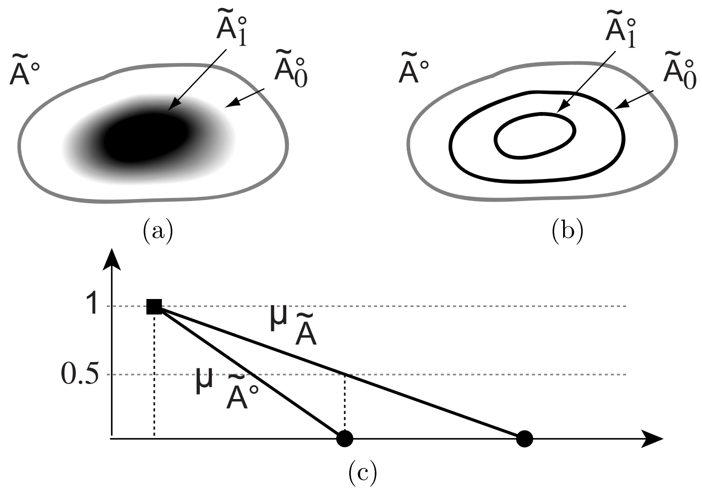


Figure 2.12: Illustration of the fuzzy interior (the outline of $\tilde{A}$ is shown): (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\tilde{A}°$.

Exterior The exterior is defined similarly to the interior; only points p for which $\mu_{\tilde{A}}(p) < 0.5$ are now considered.

Definition 64 (exterior $\tilde{A}^-$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^- = \{(p, \mu_{\tilde{A}^-}(p)\} \tag{2.19}$$

Where

$$\begin{aligned} \mu_{\tilde{A}^-} : U &\rightarrow [0,1] \\ p &\mapsto \begin{cases} 0 & \mu_{\tilde{A}}(p) \geq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(p) & \textit{elsewehere} \end{cases} \end{aligned}$$

Similarly to the interior, the exterior is defined using the membership grades of the points. The exterior $\tilde{A}^-$ also is a fuzzy region, containing only those points that have a membership grade smaller than 0.5 in the original region $\tilde{A}$. The original membership grades $\mu_{\tilde{A}}(p)$ for these points are in the range $[0, 0.5]$, the membership grades $\mu_{\tilde{A}^-}(p)$ are in the range $[0, 1]$, as can bee seen in figure 2.13. Points outside the outline are also assigned a membership grade 1.

Due to this definition, some relationships with regards to the boundary are lost, similarly to the interior.

Matrix elements In the crisp nine-intersection model, the matrix elements are considered to be 0 if the intersection is empty, and 1 if it is not. In our

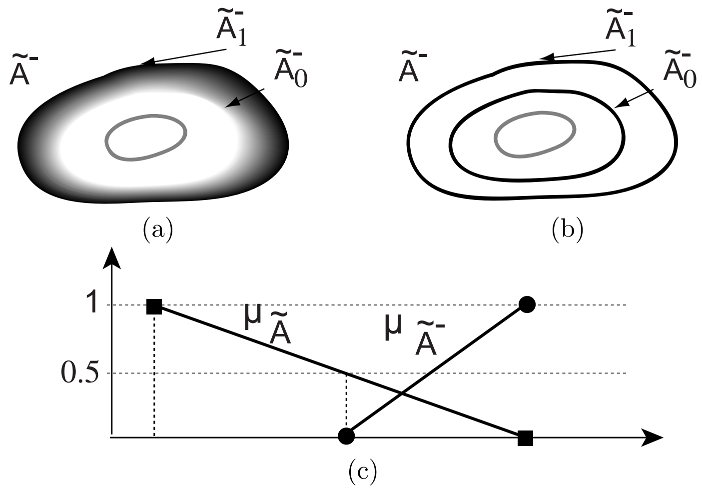


Figure 2.13: Illustration of the fuzzy exterior (the outline of the core of $\tilde{A}$ is also shown): (a) represented using grey scales, (b) represented using some contour lines, (c) an example of the membership functions for both $\tilde{A}$ and $\tilde{A}^{\circ}$.

approach, the matrix elements are deduced from each intersection: each matrix element is the value of the highest membership grade occurring in the intersection. An example of such a matrix element is:

$$height(\mu_{\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ}}) \tag{2.20}$$

Where *height* of a fuzzy set X is defined [23] as in (1.2.3) before:

Definition 65 (height)

$$height(X) = \sup_{p}(\mu_X(p)) \tag{2.21}$$

Note that matrix elements are no longer limited to $\{0, 1\}$, but can have any value in the range $[0, 1]$. This in turn will impact how the intersection matrices ought to be interpreted.

Interpreting the nine-intersection model for fuzzy regions

Using the above definitions of a fuzzy region (consider regions $\tilde{A}$ and $\tilde{B}$, the appropriate definitions for interior $\tilde{A}^{\circ}$, exterior $\tilde{A}^{-}$ and boundary $\Delta\tilde{A}$ and the intersection of fuzzy regions $\tilde{\cap}$, the nine-intersection matrix becomes:

$$\begin{pmatrix} h(\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ}) & h(\tilde{A}^{\circ}\tilde{\cap}\Delta\tilde{B}) & h(\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-}) \\ h(\Delta\tilde{A}\tilde{\cap}\tilde{B}^{\circ}) & h(\Delta\tilde{A}\tilde{\cap}\Delta\tilde{B}) & h(\Delta\tilde{A}\tilde{\cap}\tilde{B}^{-}) \\ h(\tilde{A}^{-}\tilde{\cap}\tilde{B}^{\circ}) & h(\tilde{A}^{-}\tilde{\cap}\Delta\tilde{B}) & h(\tilde{A}^{-}\tilde{\cap}\tilde{B}^{-}) \end{pmatrix} \tag{2.22}$$

Where $h(X)$ is a shorthand notation for the $height(X)$ of a fuzzy set X.

A major difference between this intersection matrix, and the aforementioned intersection matrices (for crisp regions, 1.16; for regions with undetermined boundaries,1.44) is that in the above matrix, the elements are no longer limited to $\{0, 1\}$, but are in the range $[0, 1]$.

While at first this might seem to yield an infinite number of cases, it is possible to categorize them based on their values. When a matrix element is 0, it means there is no intersection whatsoever. For elements in the range $]0, 1[$, this means there is some intersection, but not between the cores of the regions. Finally, when the matrix element is 1, it means that there is an intersection with the core of one of the regions. Basically, the fuzzy region consists of three subregions: the interior, the exterior and the boundary; just like the broad boundary model (1.3.2). Consequently, the 44 cases of the broad boundary model (1.3.2) and of the egg-yolk model (1.3.2) will serve as a starting point for the case study of the fuzzy region topology.

Topological approach for distance calculation

The distance of a point to a fuzzy region (or by extension, between two fuzzy regions) is a difficult concept. One might consider that points with a degree greater than 0.5 already belong more to the region than points with less than 0.5. This would imply that these points contribute less of the distance of the region; this can be accomplished by defining the distance using, both the distance to its (fuzzy) boundary and the distance to its (fuzzy) interior. [2]

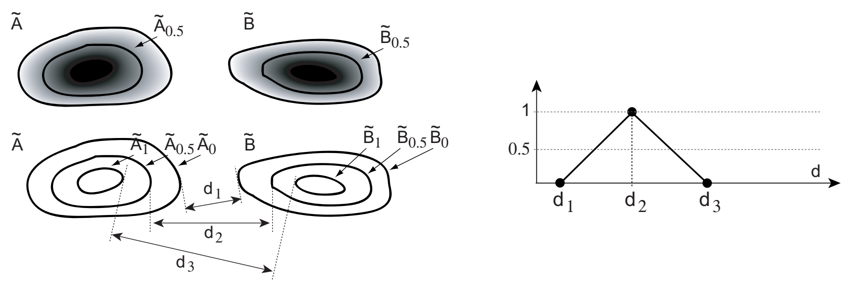
Definition 66 (distance between fuzzy regions)
(in a topological approach)

$$\tilde{d}^{\Delta}(\tilde{A}, \tilde{B}) = \{(x, \mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})}(x))\} \tag{2.23}$$

$$\begin{array}{rcl} \mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})} : \mathbb{R} & \rightarrow & [0,1] \\ x & \mapsto & \begin{cases} \mu_{\tilde{d}(\Delta\tilde{A},\Delta\tilde{B})}(x) & \textit{if } \ x < d(\Delta\tilde{A}_{0.5}, \Delta\tilde{B}_{0.5}) \\ 1 - \mu_{\tilde{d}(\tilde{A}^{\circ},\tilde{B}^{\circ})}(x) & \textit{if } \ x \geq d(\Delta\tilde{A}_{0.5}, \Delta\tilde{B}_{0.5}) \end{cases} \end{array}$$

Note that the membership grade for the distance decreases as the membership for the region increases from 0.5 to 1. This reflects the fact from the crisp case where the distance between a point that belongs to a region is 0; points with a membership grade greater than 0.5 are considered to be more inside the region than outside the region. The distance between for such points is therefore assigned lower membership grade. The choice for 0.5 is in a sense arbitrary,

[2] This definition of distance should not be confused with the topological distance as defined by Egenhofer, which expresses the distance between two different topology cases in a conceptual neighbourhood graph.



(a) (b)

Figure 2.14: Illustration of the fuzzy distance yielding a fuzzy number: (a) fuzzy regions $\tilde{A}$ and $\tilde{B}$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy distance between $\tilde{A}$ and $\tilde{B}$.

but it stands to reason to consider the halfway-point of the membership grades. This is illustrated on fig. 2.14.

This definition yields a nice intuitive result for regions that are represented by normalized fuzzy sets (21) and that are simple fuzzy regions (53), but also for regions that are a union of normalized, simple fuzzy regions.

Case study of fuzzy region topology

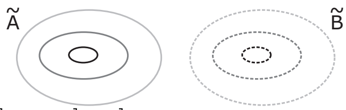
For the case study, the different matrix elements will be used to identify different topological relations. For each case, a matrix element will match a specific value or be inside a specific interval. To illustrate which values or intervals are possible for the matrix elements, consider the fuzzy regions $\tilde{A}$ and $\tilde{B}$. Given the definition of the boundary $\Delta\tilde{A}$, intersections between this boundary and the interior $\tilde{B}^\circ$ will only yield 1 if there is a point p in the intersection for which $\mu_{\Delta\tilde{A}}(p) = 1$ and $\mu_{\tilde{B}^\circ}(p) = 1$; in other words for which $\mu_{\tilde{A}}(p) = 0.5$ and $\mu_{\tilde{B}}(p) = 1$. As no assumption is made regarding the membership grades of both regions, there is no guarantee that there are points for which $\mu_{\tilde{A}}(p) = 0.5$. The same reasoning can be made for $\Delta\tilde{B}$. Consequently, as the presence of an element with membership grade 0.5 in a fuzzy region cannot be guaranteed (which yields a membership grade 1 in the boundary of that fuzzy region), the intersection with this boundary might never equal 1. Thus for the matrix elements that make use of a boundary element, i.e. $h(\tilde{A}^\circ\tilde{\cap}\Delta\tilde{B})$, $h(\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ)$, $h(\Delta\tilde{A}\tilde{\cap}\Delta\tilde{B})$, $h(\Delta\tilde{A}\tilde{\cap}\tilde{B}^-)$ and $h(\tilde{A}^-\tilde{\cap}\Delta\tilde{B})$, a distinction will only be made between the value 0 (no intersection) and the intervals $]0,1[$ (intersection but not full) and $]0,1]$ (possibly a full intersection).

The interior and exterior of $\tilde{A}$ yield membership grades 1 for points where $\mu_{\tilde{A}}(p) = 1$ and $\mu_{\tilde{A}}(p) = 0$ respectively. Following the definition of a region, an intersection between any combination of interior and exterior of 2 regions can therefore yield any value in the range $[0,1]$, however, this time the distinction will be made between three cases: the matrix element equals 0 (no intersection), the element is in the range $]0,1[$ (intersection, but not full intersection), or the element equals 1 (full intersection). Cases can of course be combined (e.g. $]0, 1]$ to indicate an intersection which can be a full intersection).

Consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$.

Case 1 The first considered case in the case study is when both regions $\tilde{A}$ and $\tilde{B}$ are completely disjoint, which yields the following.

$$\tilde{A} \text{ disjoint } \tilde{B} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$

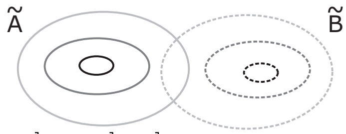


where $a_1 \in]0, 1]$, $a_2 \in]0, 1]$

This matrix is completely similar to the nine-intersection matrix of disjoint crisp regions, and to the nine-intersection matrix of disjoint regions with broad boundaries.

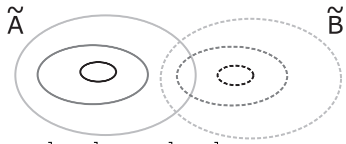
Case 2 If the regions meet slightly, this means that there is an overlap between the two boundaries. Consequently, $h(\Delta\tilde{A}\tilde{\cap}\Delta\tilde{B}) > 0$ (otherwise both regions would not have overlapping boundaries, and it would yield the above case). It is also possible for the boundary $\Delta\tilde{A}$ to intersect partially with the interior $\tilde{B}^\circ$. There is a symmetrical case, when the boundary $\Delta\tilde{B}$ intersects the interior $\tilde{A}^\circ$; both cases can occur at the same time. As a result, this case yields 4 subcases.

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\Delta\tilde{B} = \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset \end{array} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



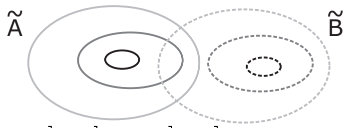
where $b \in]0,1]$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\Delta\tilde{B} = \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset \end{array} \qquad \begin{pmatrix} 0 & c_1 & 1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



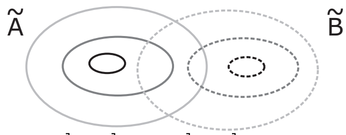
where $b \in]0,1]$, $c_1 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\Delta\tilde{B} \neq \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset \end{array} \qquad \begin{pmatrix} 0 & 0 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_2 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\Delta\tilde{B} \neq \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset \end{array} \qquad \begin{pmatrix} 0 & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1, c_2 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

The above cases can be summarized in one matrix:

$$\begin{pmatrix} 0 & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.24}$$

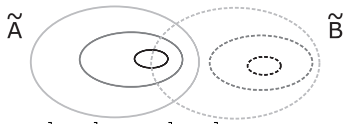
where $b \in]0,1]$, $c_1, c_2 \in [0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$.

Case 3 & case 6 Case 3 occurs when the broad boundary $\Delta\tilde{A}$ intersects with the broad boundary $\Delta\tilde{B}$ and when the interior $\tilde{A}^\circ$ also intersects with the boundary $\Delta\tilde{B}$ (or vice versa: the symmetrical case 6). First, an intersection between the boundaries implies that $h(\Delta\tilde{A}\tilde{\cap}\Delta\tilde{B}) > 0$. It is possible for this matrix element to equal 1, to indicate that the boundaries fully intersect (meaning that there are points for which $\mu_{\tilde{A}}(p) = \mu_{\tilde{B}}(p) = 0.5$.

Second, it is possible for $\Delta\tilde{A}$ to intersect with $\tilde{B}^\circ$. Even further, if the interior of $\tilde{A}$ intersects with the boundary of $\tilde{B}$, this means that $h(\tilde{A}^\circ\tilde{\cap}\Delta\tilde{B}) > 0$. It is possible for this matrix element to equal 1, if there are points p such that $\mu_{\tilde{A}}(p) = 1 \wedge \mu_{\tilde{B}}(p) = 0.5$; it is not possible of this element to equal 0 (as this would yield the previous case).

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset$

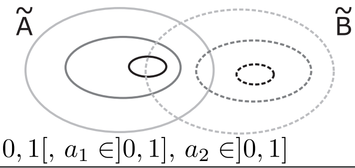
$$\begin{pmatrix} 0 & c_1 & 1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset$

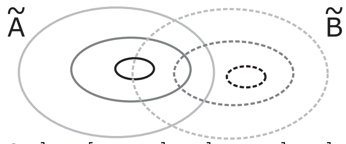
$$\begin{pmatrix} 0 & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset$

$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

The above matrices can be summarized in one single matrix:

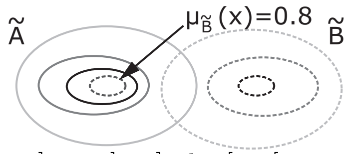
$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.25}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $d \in [0,1[$.

This matrix includes additional cases that are obtained with complex fuzzy regions (Definition 53), which makes it possible for $\tilde{A}^\circ$ to intersect with $\tilde{B}^\circ$, even when $\Delta\tilde{A}$ does not intersect with $\tilde{B}^\circ$. This particular situation is elaborated on in further detail, to illustrate what kind of *additional* cases are included.

This situation would yield the following intersection matrix:

$$\tilde{A}^\circ \cap \tilde{B}^\circ \neq \emptyset \qquad \Delta\tilde{A} \cap \tilde{B}^\circ = \emptyset \qquad \begin{pmatrix} d & c_1 & 1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $d \in [0,1[$

In this example, as illustrated on the figure, the fuzzy region $\tilde{B}$ is a complex region: all the points inside the portion indicated by the arrow have membership grade 0.8 in the region $\tilde{B}$; causing it to have α-cuts for which the result consists of several disconnected crisp regions (in this example, all the α-cuts yield a result that consists of two disconnected regions). These points belong both to the interior $\tilde{B}^\circ$, but also to the boundary $\Delta\tilde{B}$. As the membership grade is less than 1, the intersection with the interior $\tilde{A}^\circ$ will never equal 1. As there are no points for which both the membership grade in $\tilde{B}$ is greater than 0.5 and for which the membership grade in $\tilde{A}$ is in $]0,1[$, the intersection between the boundary $\Delta\tilde{A}$ and the interior $\tilde{B}^\circ$ is empty.

This particular case is not comprised in any of the above three cases, but is an additional case that complies with the summarizing matrix. Due to the fact that these fuzzy regions are not simple fuzzy regions, reasoning with them is often counter-intuitive. The combination of the different cases for simple fuzzy regions in a summarizing matrix however, yields a summarizing matrix in which the additional cases for complex fuzzy regions are included.

The symmetrical case (case 6) has the following summarized intersection matrix:

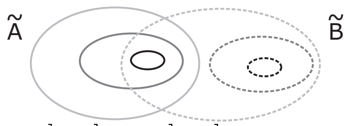
$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.26}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $d \in [0,1[$.

Case 4 & case 7 In case 4, the boundary $\Delta\tilde{A}$ intersects with the boundary $\Delta\tilde{B}$ and the interior $\tilde{A}^\circ$ is entirely located inside the broad boundary $\Delta\tilde{B}$ (case 7 is the symmetrical case). First, an intersection between the boundaries has the same implications as before: $h(\Delta\tilde{A}\tilde{\cap}\Delta\tilde{B}) > 0$, the element can equal 1, and the boundary $\Delta\tilde{A}$ can or cannot with the interior $\tilde{B}^\circ$ (to a degree strictly less than 1).

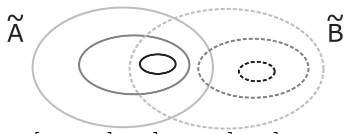
Second, if the interior $\tilde{A}^\circ$ is inside the boundary $\Delta\tilde{B}$, it can intersect (to a degree strictly less than 1) with the interior $\tilde{B}^\circ$, with the exterior $\tilde{B}^-$, or with both.

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ\tilde{\cap}\tilde{B}^- = \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & 0 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



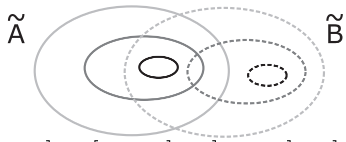
where $b \in]0,1]$, $c_1 \in]0,1]$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & e_1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



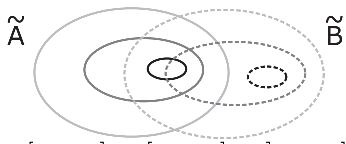
where $b \in]0,1]$, $c_1 \in]0,1]$, $e_1 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



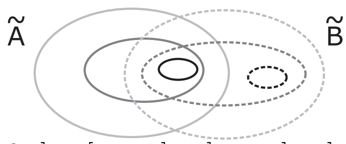
where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $e_1 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset \end{array} \quad \begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $e_1 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset \\ \Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ\tilde{\cap}\tilde{B}^- = \emptyset \end{array} \quad \begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

The above cases can be summarized in one single matrix:

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.27}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $d \in [0,1[$, $e_1 \in [0,1[$.

As before, this matrix also includes the additional cases for complex fuzzy regions.

The symmetrical case is case 7, which is characterized by the following matrix:

$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.28}$$

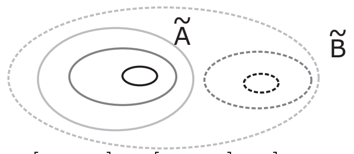
where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_2 \in [0,1[$.

Case 5 & case 8 In case 5, the boundary $\Delta\tilde{A}$ and a fortiori the interior $\tilde{A}^\circ$ are completely inside the boundary $\Delta\tilde{B}$ (case 8 is the symmetrical case).

The boundary $\Delta\tilde{A}$ can intersect (to a degree strictly less than 1) the interior $\tilde{B}^\circ$, the exterior $\tilde{A}^-$, or neither, or both. Same holds for the interior $\tilde{A}^\circ$.

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset$
$\Delta\tilde{A}\tilde{\cap}\tilde{B}^- \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset$

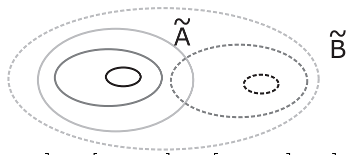
$$\begin{pmatrix} 0 & c_1 & e_1 \\ 0 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $e_1 \in]0,1[$, $a_1 \in]0,1[$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset$
$\Delta\tilde{A}\tilde{\cap}\tilde{B}^- \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset$

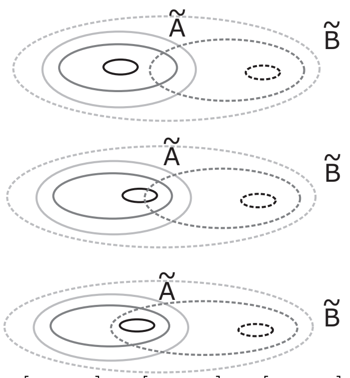
$$\begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & a & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $e_1 \in]0,1[$, $a_1 \in]0,1[$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\Delta\tilde{A}\tilde{\cap}\tilde{B}^- \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset$

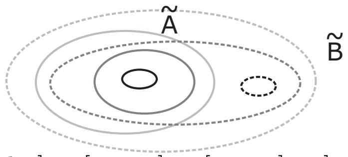
$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $e_1 \in]0,1[$, $a_1 \in]0,1[$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\Delta\tilde{A}\tilde{\cap}\tilde{B}^- \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^- = \emptyset$

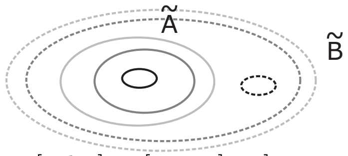
$$\begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $a_1 \in]0,1[$, $a_2 \in]0,1]$

$\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset$
$\Delta\tilde{A}\tilde{\cap}\tilde{B}^- = \emptyset$
$\tilde{A}^\circ\tilde{\cap}\tilde{B}^- = \emptyset$

$$\begin{pmatrix} d & c_1 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $d \in]0,1[$, $a_2 \in]0,1]$

Additionally, when the fuzzy regions are complex, it is possible for $\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ \neq \emptyset$ even if $\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset$. Similarly, it is possible for $\tilde{A}^\circ\tilde{\cap}\tilde{B}^- \neq \emptyset$ if $\Delta\tilde{A}\tilde{\cap}\tilde{B}^- = \emptyset$.

Apart from complex regions, a new special case can occur: the boundary $\Delta\tilde{B}$ can have a large area where the membership grade equals 0.5. If the region $\tilde{A}$ happens to be entirely located in this area, the following conditions are true at the same time: $\Delta\tilde{A}\tilde{\cap}\tilde{B}^\circ = \emptyset$, $\tilde{A}^\circ\tilde{\cap}\tilde{B}^\circ = \emptyset$, $\Delta\tilde{A}\tilde{\cap}\tilde{B}^- = \emptyset$ and $\tilde{A}^\circ\tilde{\cap}\tilde{B}^- = \emptyset$.

These additional cases are automatically included in the summarizing matrix:

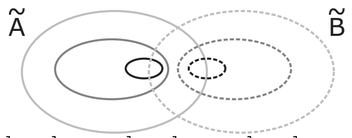
$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.29}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $d \in [0,1[$, $e_1 \in [0,1[$.

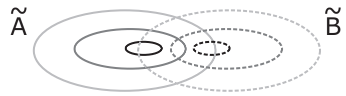
Case 9 In case 9, the boundary $\Delta\tilde{A}$ intersects with the interior $\tilde{B}^\circ$ and vice versa.

It is possible for $\tilde{A}^\circ$ to intersect (to a degree strictly less than 1) with $\tilde{B}^\circ$.

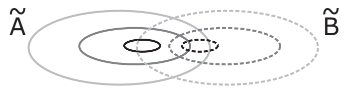
$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \qquad \begin{pmatrix} 0 & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$

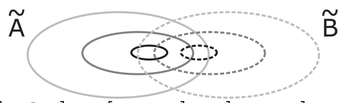


where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $a_1 \in]0,1]$, $a_2 \in]0,1]$



$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset \qquad \begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$





where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

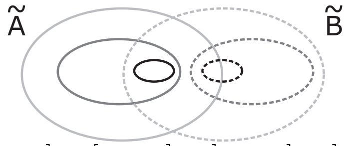
The summarizing matrix for case 9 is

$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.30}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$.

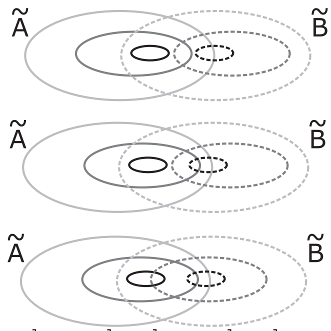
Case 10 & case 12 In case 10, the boundary $\Delta\tilde{A}$ intersects with the interior $\tilde{B}^\circ$, the boundary $\Delta\tilde{B}$ and the exterior $\tilde{B}^-$. The interior $\tilde{A}^\circ$ can intersect with either $\Delta\tilde{B}$ (to a degree of up to 1) and with $\tilde{B}^\circ$, $\tilde{B}^-$ (to a degree strictly less than 1).

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \end{array} \qquad \begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



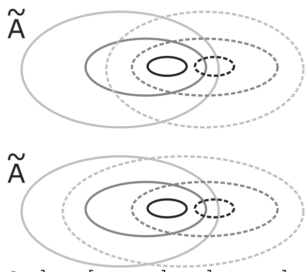
where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \end{array} \qquad \begin{pmatrix} 0 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \end{array} \qquad \begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1[$, $a_1 \in]0,1]$, $a_2 \in]0,1]$

The summarizing matrix for case 10 is

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.31}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$.

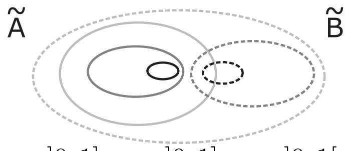
The symmetrical case 12 yields the following summarizing matrix

$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.32}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_2 \in [0,1[$.

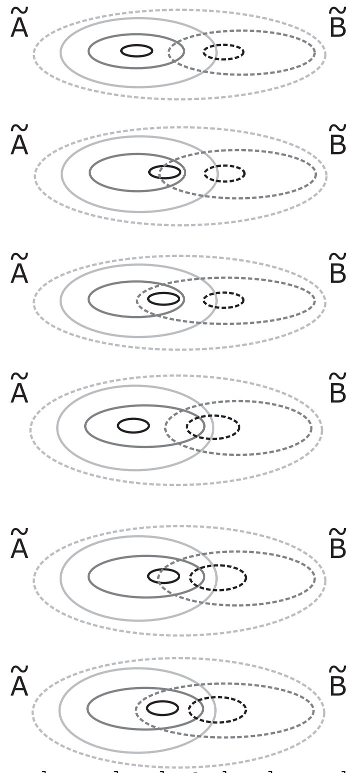
Case 11 & case 13 In case 11, the boundary $\Delta\tilde{A}$ intersects with the interior $\tilde{B}^\circ$ and the boundary $\Delta\tilde{B}$. An intersection with the exterior $\tilde{B}^-$ is possible, but only to a degree strictly less than 1. The interior $\tilde{A}^\circ$ can intersect with either $\tilde{B}^\circ$, $\Delta\tilde{B}$ and $\tilde{B}^-$.

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \end{array} \qquad \begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



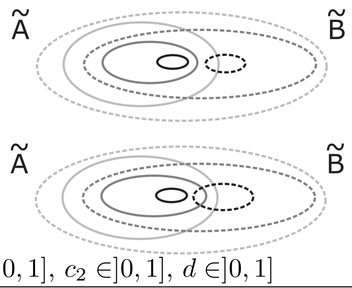
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \end{array} \qquad \begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



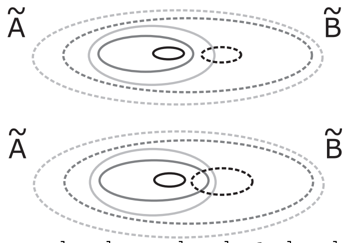
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1]$, $e_1 \in]0,1[$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \end{array} \qquad \begin{pmatrix} d & c_1 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset \end{array} \quad \begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where , $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1]$

Theoretically, it is also possible for the $\tilde{B}$ to have a large region where it has membership grade 0.5. Consequently, it is possible for the boundary $\Delta\tilde{A}$ not to intersect with the exterior $\tilde{B}^-$, for the interior $\tilde{A}^\circ$ not to intersect with the exterior $\tilde{B}^-$, or for both at the same time. This yields three additional cases.

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$

where $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$

where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \end{array} \quad \begin{pmatrix} 0 & c_1 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$

where $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

All of the above cases can be summarized in one matrix, which additionally takes into account the cases for complex fuzzy regions:

$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.33}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

The symmetrical case 13 is summarized by the matrix

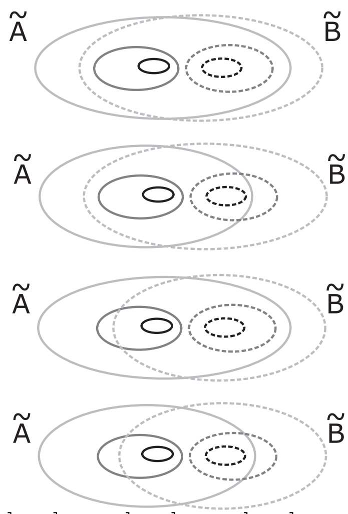
$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.34}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

Case 14 In case 14, the boundary $\Delta\tilde{A}$ intersects with the boundary $\Delta\tilde{B}$; both interiors are inside this intersection.

It is possible for $\tilde{A}^\circ$ to intersect (to a degree strictly less than 1) with the interior $\tilde{B}^\circ$ or the exterior $\tilde{B}$-, or with both at the same time. Similarly, this is also possible for $\tilde{B}^\circ$.

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset \quad \begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in]0,1[$

If there is an intersection (to a degree strictly less than 1) possible between $\tilde{A}^\circ$ and $\tilde{B}^\circ$, the number of subcases increases dramatically, even for simple fuzzy regions. To illustrate, consider the intersections between the α-levels 0.5 and 1 for both $\tilde{A}$ and $\tilde{B}$:

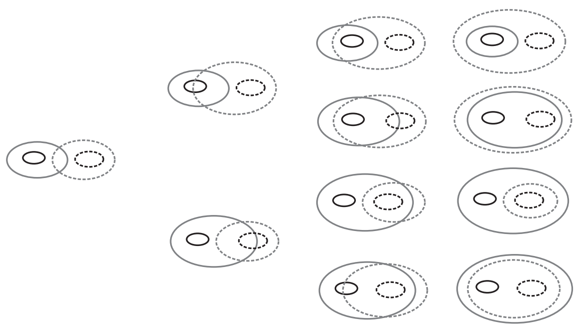


Figure 2.15: Possible topology cases for the α-levels 0.5 and 1 if no full intersection is possible.

Each of above 11 cases should be combined similarly to the 4 mentioned before, with the limitation that both crisp regions $\tilde{A}_1$ and $\tilde{B}_1$ should be inside the intersection of $\tilde{A}_{\overline{0}}$ and $\tilde{B}_{\overline{0}}$. For complex regions, there even are additional cases. All of these cases can be summarized in one matrix, which additionally takes into account the cases for complex fuzzy regions:

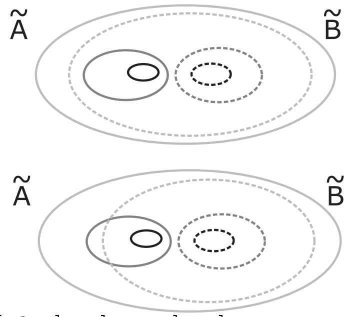
$$\begin{pmatrix} d & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.35}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

Case 15 & case 16 In case 15, the boundary $\Delta\tilde{B}$ is completely inside the boundary $\Delta\tilde{A}$ (no full intersection with the exterior $\tilde{A}^-$ is possible). The case is more or less similar to case 14.

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ = \emptyset$$
$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$$

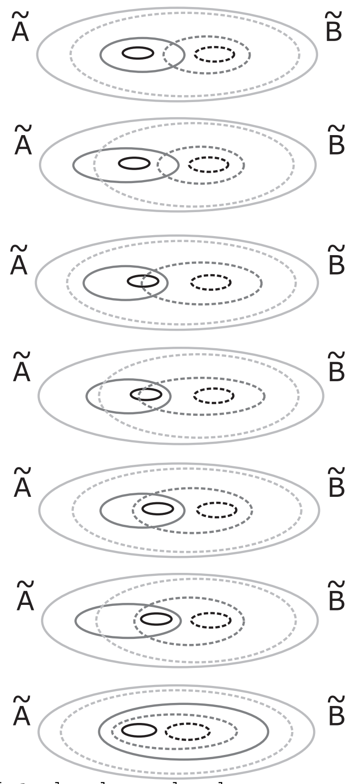
$$\begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



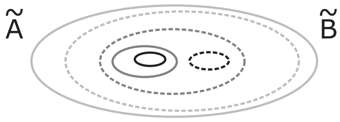
where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset$$
$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$$

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$

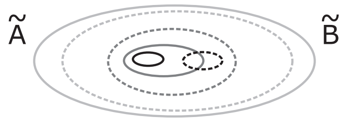


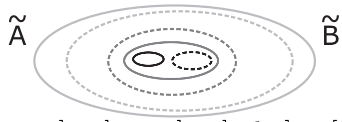
where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1[$, $e_1 \in]0,1[$



$\tilde{A}^\circ \tilde{\cap} \tilde{B}^\circ \neq \emptyset$
$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$

$$\begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$





where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1[$

All the above cases, as well as the additional cases for complex fuzzy regions are summarized as

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix} \tag{2.36}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$.

Case 16 is the symmetrical case, which has the summarizing matrix

$$\begin{pmatrix} d & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.37}$$

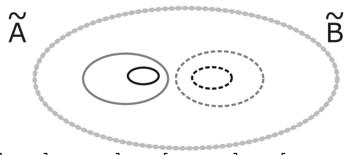
where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_2 \in [0,1[$.

Case 17 In case 17, the outer boundaries match. Depending of definitions of the regions, $\tilde{A}^{-}\tilde{\cap}\Delta\tilde{B}$ can be empty or can intersect to a degree strictly smaller than 1 (and vice versa).

The interior $\tilde{A}^{\circ}$ can intersect with $\tilde{B}^{\circ}$ (only to a degree strictly less than 1). The interior $\tilde{A}^{\circ}$ can intersect the exterior $\tilde{A}^{-}$ (only to a degree strictly less than 1) and vice versa. As not to overload the number of cases, their matrix elements will be considered in the range $[0,1[$.

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ}=\emptyset$

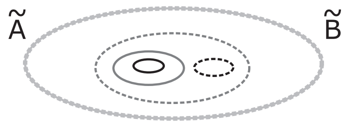
$$\begin{pmatrix} 0 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$

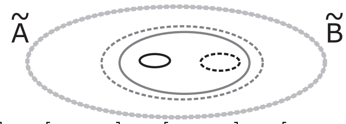


where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1[$, $e_1 \in]0,1[$, $e_2 \in]0,1[$

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ}\neq\emptyset$
$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-}=\emptyset$

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$

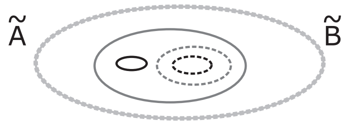


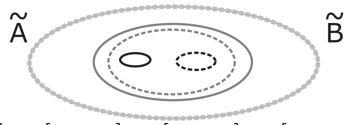


where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1[$, $c_1 \in]0,1[$, $c_2 \in]0,1[$, $d \in]0,1[$, $e_1 \in]0,1[$

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ}\neq\emptyset$
$\tilde{A}^{-}\tilde{\cap}\tilde{B}^{\circ}=\emptyset$

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$

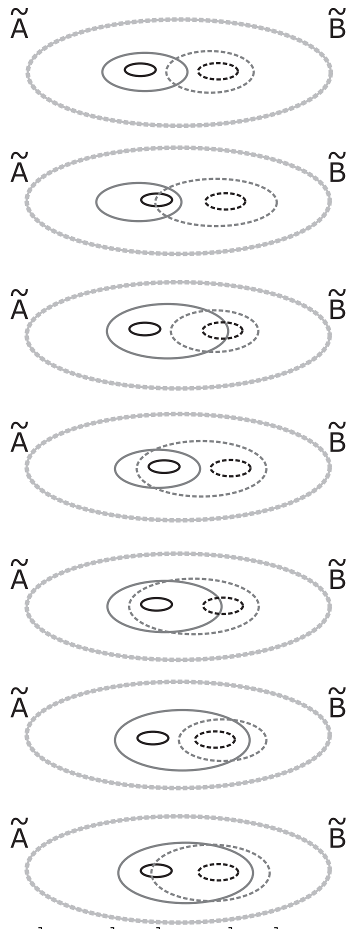




where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1[$, $c_1 \in]0,1[$, $c_2 \in]0,1[$, $d \in]0,1[$, $e_1 \in]0,1[$

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{\circ} \neq \emptyset$
$\tilde{A}^{-}\tilde{\cap}\tilde{B}^{\circ} \neq \emptyset$
$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-} \neq \emptyset$

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in]0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

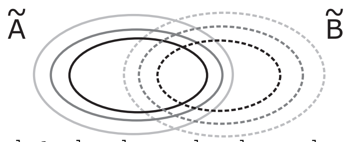
All of the above cases can be summarized in one intersection matrix, which also includes the additional cases for non-topological reasons

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.38}$$

where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $d \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

Case 18 In case 18, the interior $\tilde{A}^\circ$ intersects to a degree 1 with both the interior $\tilde{B}^\circ$ and the exterior $\tilde{B}^-$, and vice versa. The interior $\tilde{A}^\circ$ intersects also with the boundary $\Delta\tilde{B}$ and vice versa. For simple fuzzy regions, this only yields one case.

$$\begin{pmatrix} 1 & c_1 & 1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$

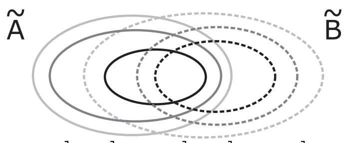


where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

The summarizing matrix is exactly the same as the above matrix.

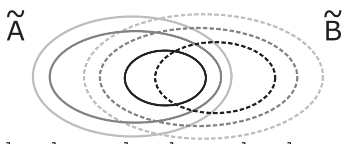
Case 19 & 21 Case 19 is similar to case 18, apart from the fact that no intersection to a degree 1 is possible between the interior $\tilde{A}^\circ$ and the exterior $\tilde{B}^-$. This yields two subcases: either there is a partial intersection between $\tilde{A}^\circ$ and $\tilde{B}^-$, or there is no intersection possible.

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \quad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \quad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

The summarizing matrix is

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.39}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$.

Case 21 is the symmetrical case, with summarizing matrix

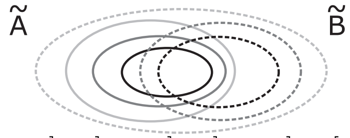
$$\begin{pmatrix} 1 & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.40}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_2 \in [0,1[$.

Case 20 & 22 Case 20 is similar to case 19, but now no intersection to a degree 1 is possible between the boundary $\Delta\tilde{A}$ and the exterior $\tilde{B}^-$.

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$

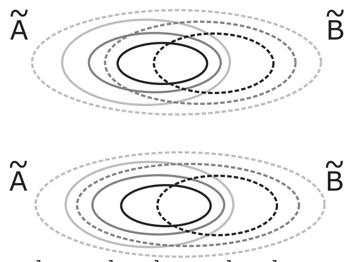
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$
$\Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset$

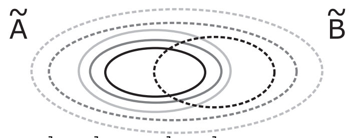
$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$
$\Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset$

$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$

The summarizing matrix is

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.41}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$.

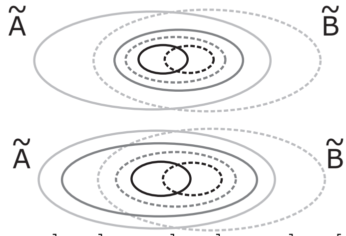
Case 22 is the symmetrical case, which has summarizing matrix

$$\begin{pmatrix} 1 & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.42}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_2 \in [0,1[$.

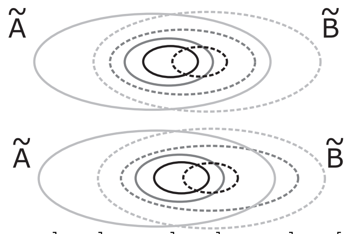
Case 23 In case 23, the interior $\tilde{A}^\circ$ intersects to a degree 1 the interior $\tilde{B}^\circ$; it also intersects with the exterior $\tilde{B}^-$, but only to a degree strictly less than 1. The interior $\tilde{A}^\circ$ also intersects with the boundary $\Delta\tilde{B}$. Similar conditions hold for $\tilde{B}^\circ$.

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \\ \tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset \end{array} \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$$\begin{array}{l} \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset \end{array} \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_2 \in]0,1[$

The cases where both $\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$ and $\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$, yield a large number of alternatives. To illustrate, consider the remaining possible intersections between the α-levels 0.5 and 1 of both $\tilde{A}$ and $\tilde{B}$.

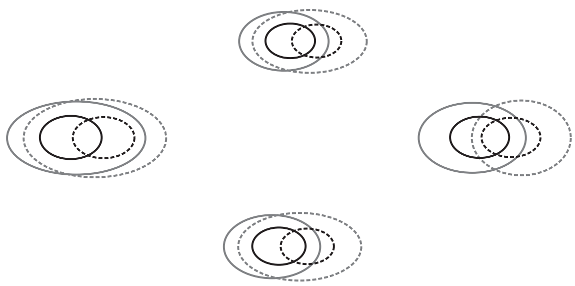


Figure 2.16: Possible topology cases for the α-levels 0.5 and 1 if a full intersection is required, and neither level at 0.5 is completely contained in the other.

The possible variants for case 23 all have in common that there is no full intersection possible between the interior $\tilde{A}^\circ$ and the exterior $\tilde{B}^-$ (or vice versa). For each of the 4 above cases, the boundary of one region may or may not have a full intersection with the exterior of the other region. In total, this yields 16 additional cases, all represented by the intersection matrix

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-} \neq \emptyset$ $\tilde{A}^{-}\tilde{\cap}\tilde{B}^{\circ} \neq \emptyset$	$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$
where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in]0,1[$	

The summarizing matrix combines the above cases, as well as additional special cases:

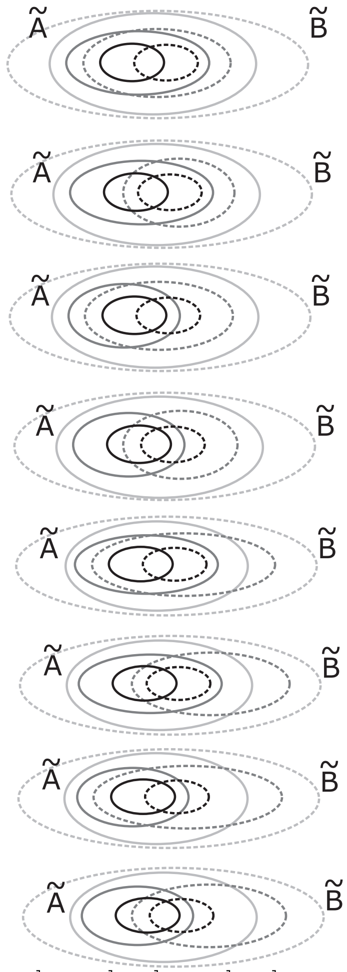
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.43}$$

where where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

Case 24 & 25 In case 24, the boundary $\Delta\tilde{A}$ is completely inside $\Delta\tilde{B}$: the intersection between $\Delta\tilde{A}$ and $\tilde{B}^-$ is strictly less than 1. There is a full intersection between the interiors $\tilde{A}^\circ$ and $\tilde{B}^\circ$.

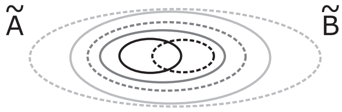
$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

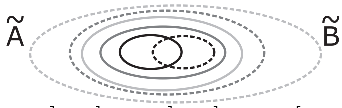
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

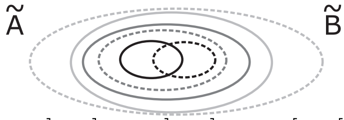
$\tilde{A}^{\circ} \tilde{\cap} \tilde{B}^{-} = \emptyset$ $\quad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$





where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$

$\tilde{A}^{-} \tilde{\cap} \tilde{B}^{\circ} = \emptyset$ $\quad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_2 \in [0,1[$

The matrix combining all the above cases is:

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.44}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

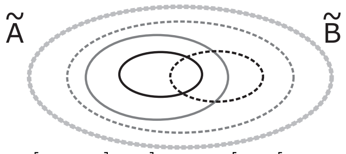
Case 25 is the symmetrical case, which is summarized by:

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.45}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

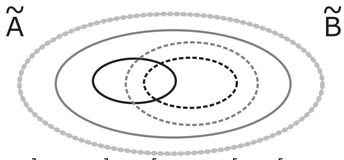
Case 26 Case 26 is similar to case 17; but now there is an intersection to a degree 1 between the interiors $\tilde{A}^\circ$ and $\tilde{B}^\circ$. The interior $\tilde{A}^\circ$ can intersect the exterior $\tilde{A}^-$ (only to a degree strictly less than 1) and vice versa. As not to overload the number of cases, their matrix elements will be considered in the range $[0,1[$.

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



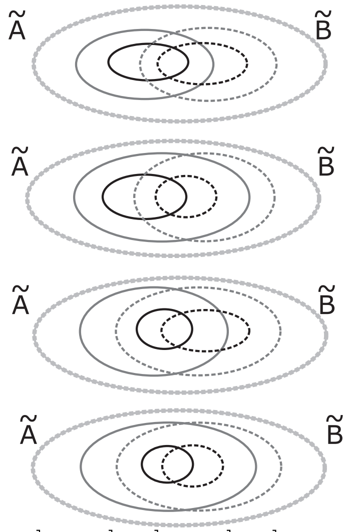
where $a_2 \in]0,1[$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1]$, $e_2 \in [0,1[$

$$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1[$, $e_1 \in [0,1[$

$$\begin{matrix} \tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset \end{matrix} \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$,
$e_1 \in [0,1[$, $e_2 \in [0,1[$

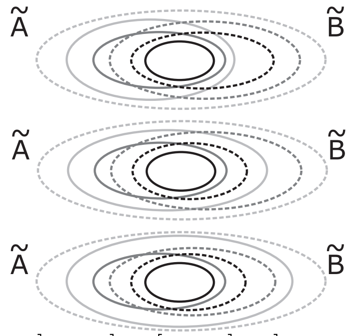
The summarizing matrix for all of the above cases is:

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.46}$$

where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

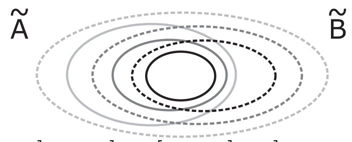
Case 27 & 33 Case 27 is similar to case 20, but now the interior $\tilde{A}^\circ$ is completely inside the interior $\tilde{B}^\circ$.

$$\begin{array}{l}\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \\ \tilde{A}^- \tilde{\cap} \Delta\tilde{B} \neq \emptyset\end{array} \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



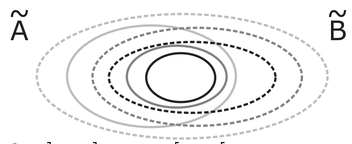
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1]$

$$\begin{array}{l}\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} \neq \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset\end{array} \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



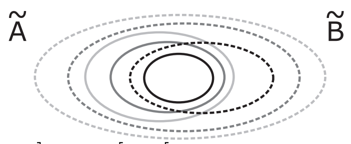
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1]$

$$\begin{array}{l}\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset\end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



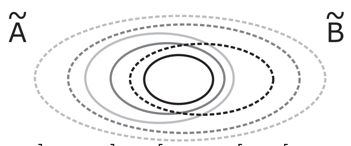
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_2 \in [0,1[$

$$\begin{array}{l}\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} = \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset\end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_2 \in]0,1]$, $b \in]0,1]$, $c_2 \in [0,1[$

$$\begin{array}{l}\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \\ \tilde{A}^\circ \tilde{\cap} \Delta\tilde{B} \neq \emptyset \\ \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset\end{array} \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & 0 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in [0,1[$

The above matrices (including the special cases) can be summarized as

$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.47}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$.

Case 33 is the symmetrical case, with summary

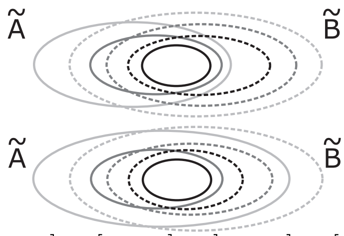
$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.48}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in]0,1]$.

Case 28 & 34 Case 28 is similar to case 27, apart from the fact that the boundary $\Delta\tilde{A}$ intersects differently with the exterior $\tilde{B}^{-}$.

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-} \neq \emptyset$
$\tilde{A}^{\circ}\tilde{\cap}\Delta\tilde{B} \neq \emptyset$

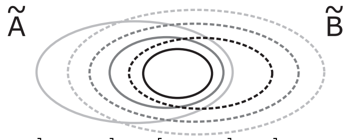
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-} = \emptyset$
$\tilde{A}^{\circ}\tilde{\cap}\Delta\tilde{B} \neq \emptyset$

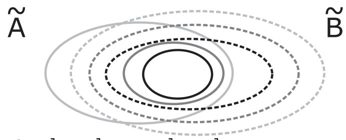
$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1[$, $c_2 \in]0,1]$

$\tilde{A}^{\circ}\tilde{\cap}\tilde{B}^{-} = \emptyset$
$\tilde{A}^{\circ}\tilde{\cap}\Delta\tilde{B} = \emptyset$

$$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_2 \in]0,1]$

The summarizing matrix (which also includes special cases) for the above cases is

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix} \tag{2.49}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in [0,1[$

Case 34 is the symmetrical case, which is summarized by

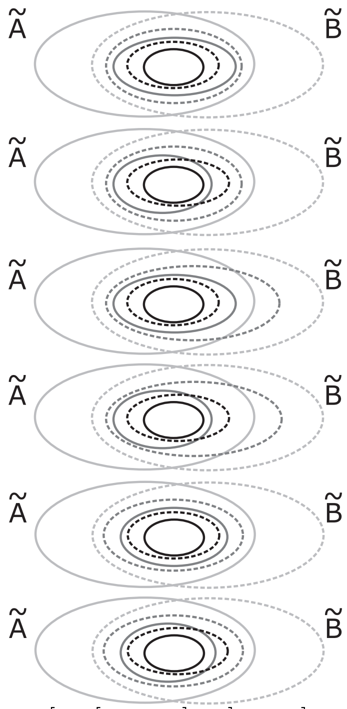
$$\begin{pmatrix} 1 & c_1 & 1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.50}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $e_2 \in [0,1[$

Case 29 & 35 Case 29 is similar to case 28, but the interior $\tilde{B}^\circ$ is completely inside the boundary $\Delta\tilde{A}$.

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

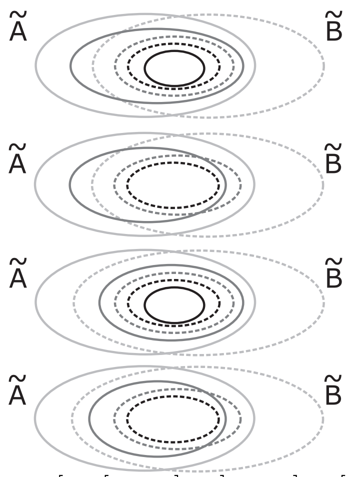
$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



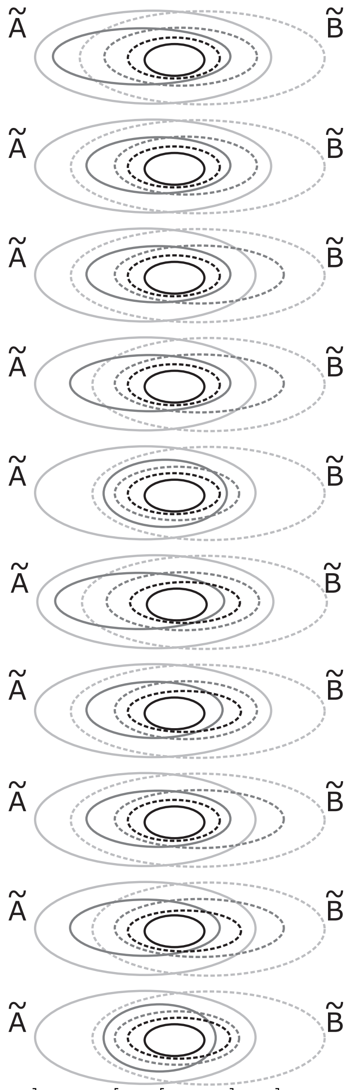
where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, , $c_2 \in]0,1]$, $e_2 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset$

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$ $\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$	$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$	

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in]0,1[$

The matrix summarizing the above cases, and adding the cases for complex fuzzy regions and special cases is

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.51}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

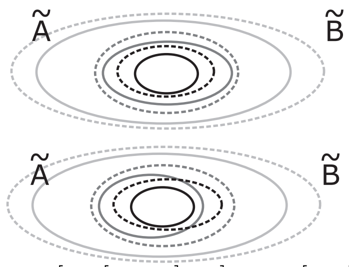
Case 35 is the symmetrical case, summarized by

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.52}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

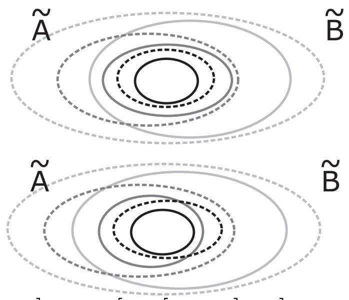
Case 30 & 36 In case 30, the interior $\tilde{A}^\circ$ is inside the interior $\tilde{B}^\circ$, and the boundary $\Delta\tilde{A}$ is inside the boundary $\Delta\tilde{B}$.

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$ $\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$



where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_2 \in [0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$ $\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$



where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in [0,1[$

The above cases, including additional special cases, are summarized in

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.53}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

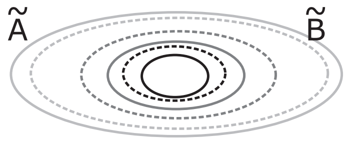
Case 36 is the symmetrical case, summarized by

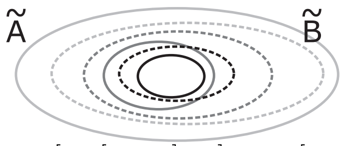
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.54}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

Case 31 & 37 In case 31, the interior $\tilde{A}^\circ$ is inside the interior $\tilde{B}^\circ$, and the boundary $\Delta\tilde{B}$ is inside the boundary $\Delta\tilde{A}$.

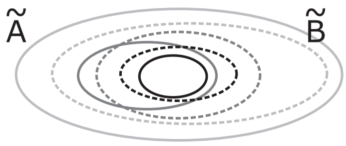
$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$

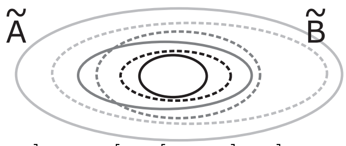




where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_2 \in [0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$





where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in [0,1[$

The above cases, including additional special cases, are summarized in

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.55}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

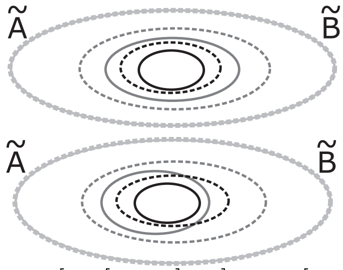
Case 37 is the symmetrical case, summarized by

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.56}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

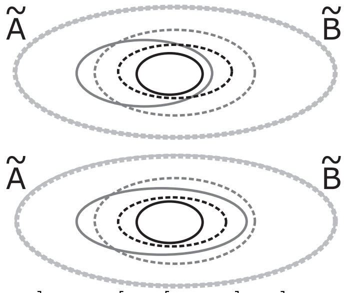
Case 32 & 38 In case 32, the interior $\tilde{A}^\circ$ is inside the interior $\tilde{B}^\circ$, and the boundaries $\Delta\tilde{A}$ and $\Delta\tilde{B}$ overlap.

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \qquad \begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_2 \in [0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in [0,1[$

The above cases, including additional special cases, are summarized in

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.57}$$

where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

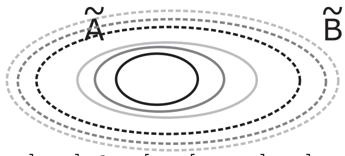
Case 38 is the symmetrical case, summarized by

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.58}$$

where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in]0,1]$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$.

Case 39 & 40 In case 39, both the interior $\tilde{A}^\circ$ and the boundary $\Delta\tilde{A}$ are inside $\tilde{B}^\circ$.

$$\begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & a_1 \\ 1 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in [0,1[$, $c_2 \in]0,1]$

Case 40 is the symmetrical case, which has the intersection matrix

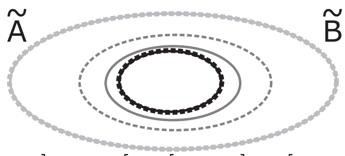
$$\begin{pmatrix} 1 & c_1 & 1 \\ 0 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix} \tag{2.59}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in [0,1[$, $c_1 \in]0,1]$

Case 41 In case 41, the interiors $\tilde{A}^\circ$ and $\tilde{B}^\circ$ match; as do the boundaries $\Delta\tilde{A}$ and $\Delta\tilde{A}$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

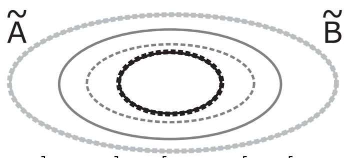
$$\begin{pmatrix} 1 & 0 & e_1 \\ 0 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset$

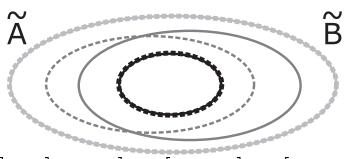
$$\begin{pmatrix} 1 & 0 & e_1 \\ 0 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in [0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

$$\begin{pmatrix} 1 & 0 & e_1 \\ 0 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $e_1 \in]0,1[$, $e_2 \in]0,1[$

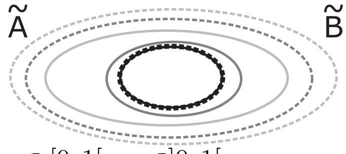
The above cases are summarized in one matrix:

$$\begin{pmatrix} 1 & 0 & e_1 \\ 0 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.60}$$

where $a_1 \in [0,1[$, $a_2 \in [0,1[$, $b \in]0,1]$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

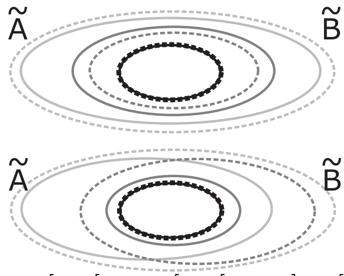
Case 42 & case 43 In case 42, the interiors $\tilde{A}^\circ$ and $\tilde{B}^\circ$ match and the boundary $\Delta\tilde{A}$ is completely inside the boundary $\Delta\tilde{B}$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \quad \tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset \quad \Delta\tilde{A} \tilde{\cap} \tilde{B}^- = \emptyset \qquad \begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & 0 \\ e_2 & a_2 & 1 \end{pmatrix}$$



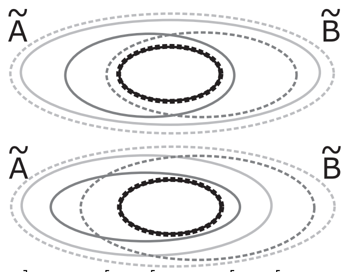
where $a_2 \in]0,1]$, $b \in]0,1[$, $c_2 \in [0,1[$, $e_2 \in]0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \quad \tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset \quad \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



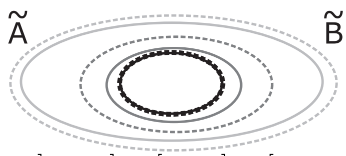
where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in]0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset \quad \tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset \quad \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \qquad \begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in]0,1[$, $e_2 \in]0,1[$

$$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset \quad \tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset \quad \Delta\tilde{A} \tilde{\cap} \tilde{B}^- \neq \emptyset \qquad \begin{pmatrix} 1 & 0 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_2 \in]0,1[$ $e_2 \in]0,1[$

The summarizing matrix holding all the above cases and additional special cases is:

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.61}$$

where $a_1 \in [0,1[$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

The symmetrical case is case 43, with matrix:

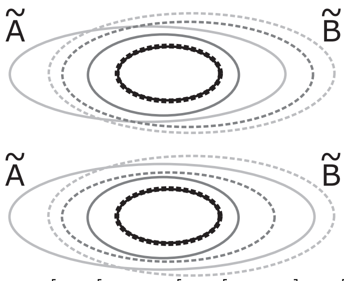
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.62}$$

where $a_1 \in]0,1]$, $a_2 \in [0,1[$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

Case 44 In case 44, the interiors $\tilde{A}^\circ$ and $\tilde{B}^\circ$ match; the boundaries intersect.

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ = \emptyset$

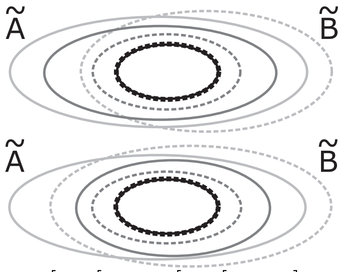
$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ 0 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- = \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

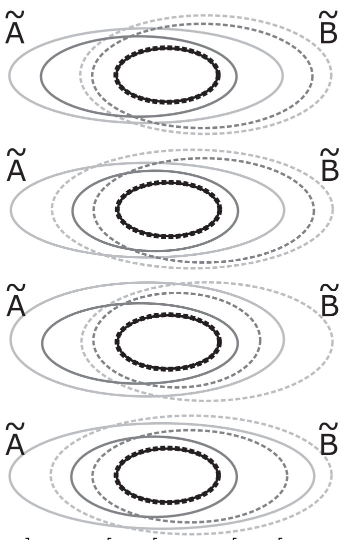
$$\begin{pmatrix} 1 & c_1 & 0 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_2 \in]0,1[$

$\tilde{A}^\circ \tilde{\cap} \tilde{B}^- \neq \emptyset$
$\tilde{A}^- \tilde{\cap} \tilde{B}^\circ \neq \emptyset$

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix}$$



where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$,
$e_1 \in]0,1[$, $e_2 \in]0,1[$

The above cases can be summarized as

$$\begin{pmatrix} 1 & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} \tag{2.63}$$

where $a_1 \in]0,1]$, $a_2 \in]0,1]$, $b \in]0,1]$, $c_1 \in [0,1[$, $c_2 \in [0,1[$, $e_1 \in [0,1[$, $e_2 \in [0,1[$

Remark

Cases caused by complex fuzzy regions In each of the above case of the case study, the summarizing matrix actually allows for more cases than the ones drawn. One reason for this is that the drawings are made for simple fuzzy regions (for ease of understanding), while complex regions can have additional relative positions: it is for instance possible for the interiors of two regions to intersect, without having the interior of one region intersect with the boundary of a second region. This is illustrated in case 3.

Cases caused by particular distributions of membership grades A second reason why there can be more cases than those drawn in the above case study, is when a fuzzy region has a particular distribution of membership grades. In the considered examples, the membership grades was strictly decreasing for the interior outward. However, even in this situation there can be additional cases: assume for instance two fuzzy regions with membership grades only in $\{0.5, 1\}$. In this situation, it is possible for instance for the boundary of one region to fully intersect with the interior of the other region, while there is no intersection between the interior of the former with the boundary of the latter.

Note that this particular distribution of membership grades is very interesting: the interior, exterior and boundary are in fact crisp regions. Furthermore, when the regions are simple, the topology completely matches the topology model as presented by Clementini ([10]) as well as the model presented by Cohn and Gotts ([13], [28]). Consequently, the broad boundary model and the egg-yolk model can be considered to be a special case of the fuzzy region model.

Similarities with the broad boundary model and the crisp model When the only occurring membership grades are in $\{0, 1\}$, the definitions for interior, exterior and boundary revert back to the definition of crisp regions. Just like broad boundary regions are a special case, crisp regions are also a special case of fuzzy regions (as they already are a special case of the broad boundary model).

Notice how the intersection matrix for *meet* of fuzzy regions tells us quite a lot about the relative position of two fuzzy regions. In both traditional topology and the Clementini terminology, *meet* is symmetrical. Using fuzzy regions, it is possible to distinguish between A *meets* B (if $b_1 > b_2$) and B *meets* A (if $b_2 > b_1$). However, due to the large number of cases, becomes quite difficult to name them.

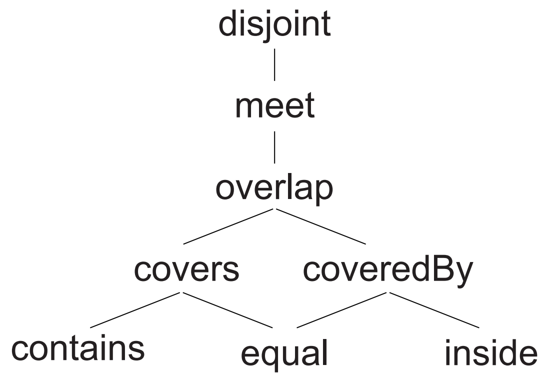


Figure 2.17: Conceptual neighbourhood graph for crisp regions.

Conceptual neighbourhood graph for fuzzy topology

Commonly, topological relations are grouped using a *conceptual neighbourhood* graph. Each topology case is a node of the graph; different nodes are connected provided their topology matrices differ in only one element. The conceptual neighbourhood graph (using the traditional terminology) for crisp regions is given in fig. 2.17. Clementini provided the conceptual neighbourhood graph for broad boundary regions.

The conceptual neighbourhood graph for fuzzy regions can also be constructed, but the concept needs to be tweaked somewhat. Two nodes are considered to be connected, when a natural transition from one case to another case exists: when moving the two regions relatively from each other, at some point a new case will be obtained. Such a transition is said to be natural. These natural transitions are the ones that occur with a minimal change (a change from 0 to $[0,1[$, $[0,1[$ to $]0,1]$, $]0.1]$ to 1; or vice versa) to a minimal number of matrix elements. In the broad boundary model, this was defined by considering a single value change; this could not be maintained due to the fact we have limited the number of cases by grouping similar cases. If each individual case was considered, the original premise could have been maintained, but it would have vastly increased the number of cases. Clementini grouped different cases in his conceptual neighbourhood graph based on a subjective interpretation, and provided a name for each group. For fuzzy regions, it is possible to group several cases based on the value range of each of the 9 matrix elements; for any value this results in up to 5 groups: 0, $[0,1[$, $]0,1[$, $]0,1]$, 1.

The conceptual neighbourhood graph for fuzzy regions is illustrated on fig. 2.18. As illustrated in the overview of the topology cases, most cases consist of a number of sub-cases, which in turn can also be grouped in a conceptual neighbourhood graph.

For each of the values a, b_1, b_2, c, d, e_1, e_2, the cases are grouped on different figures; for a, fig. 2.19, for b_1 and b_2 fig. 2.20, for c, fig. 2.21, for d, fig. 2.22 and for e_1 and e_2 fig. 2.23.

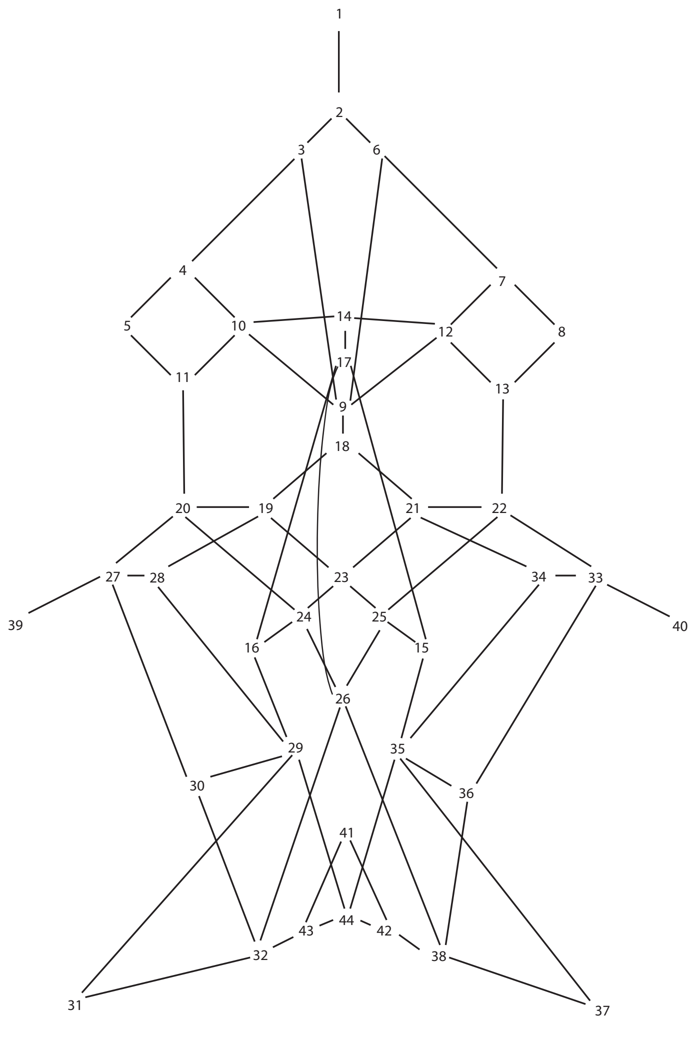


Figure 2.18: Conceptual neighbourhood graph for fuzzy regions.

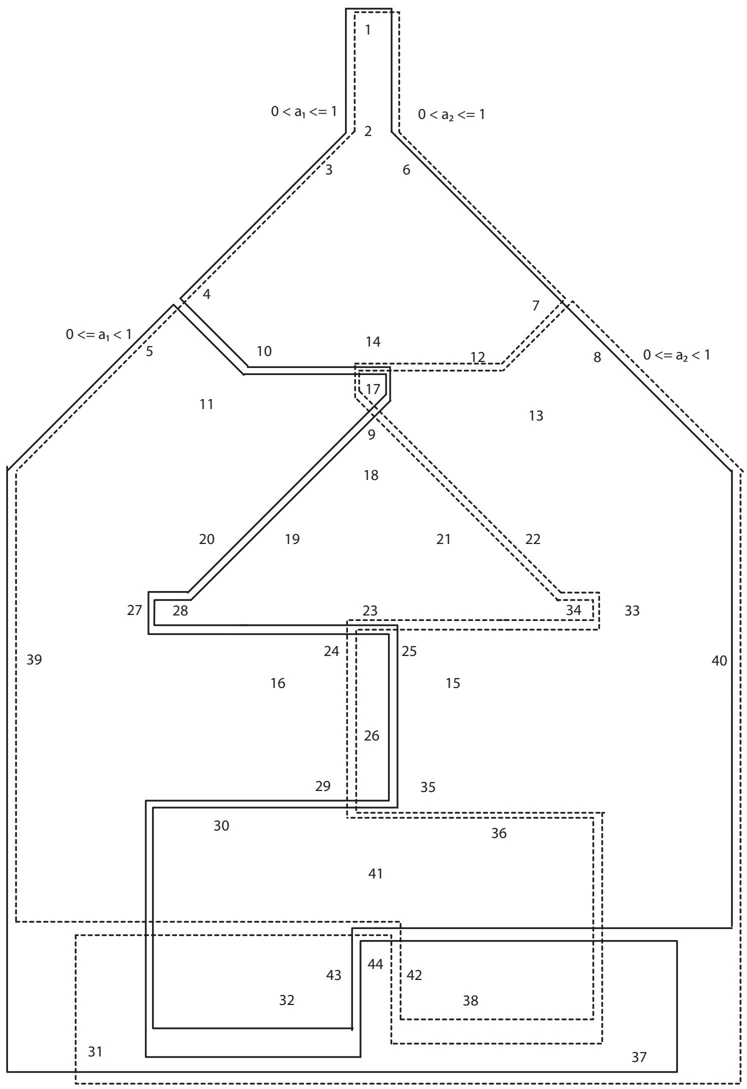


Figure 2.19: Groups associated with value a.

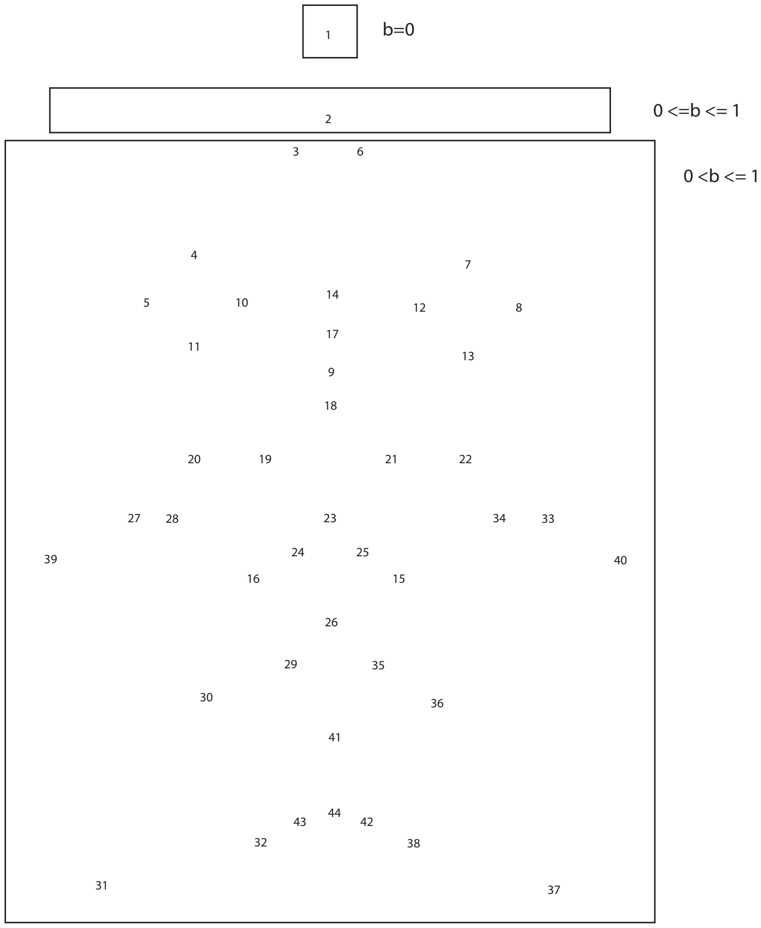


Figure 2.20: Groups associated with value b.

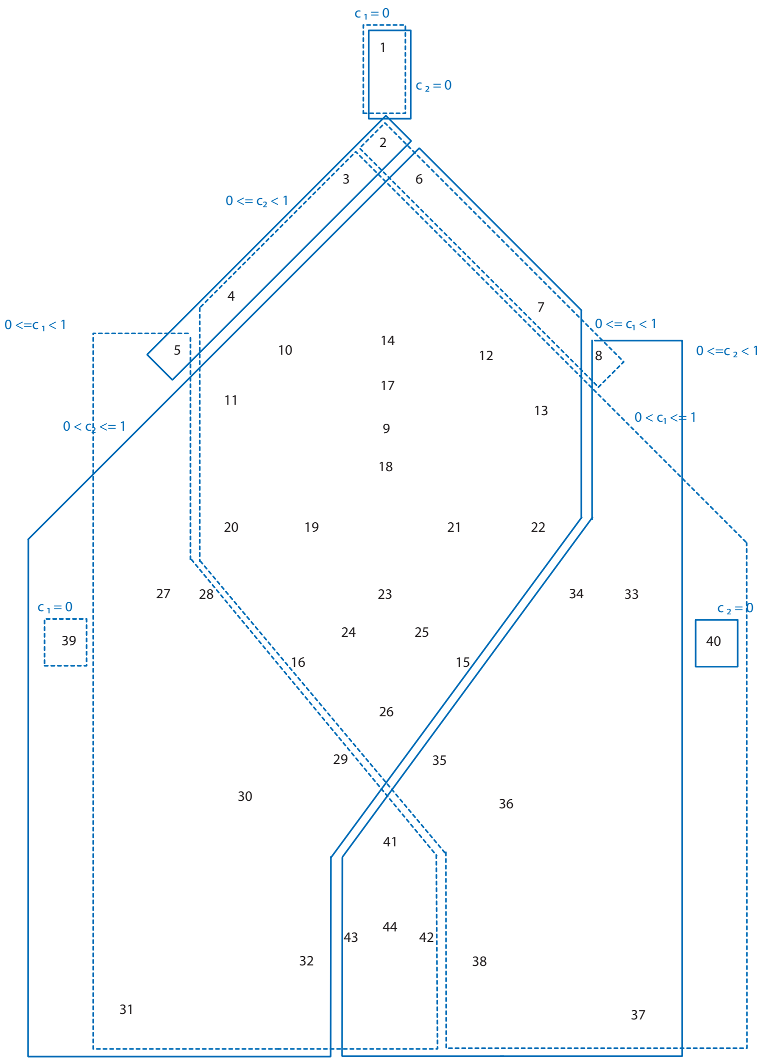


Figure 2.21: Groups associated with values c_1 and c_2.

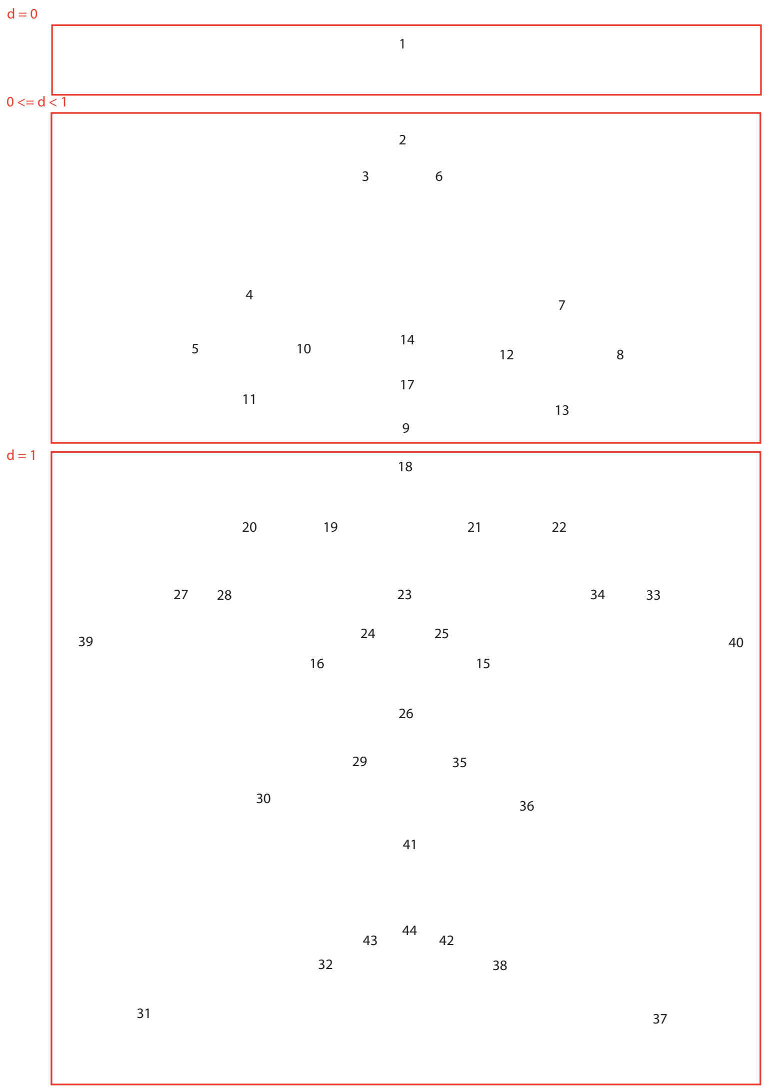


Figure 2.22: Groups associated with value d.

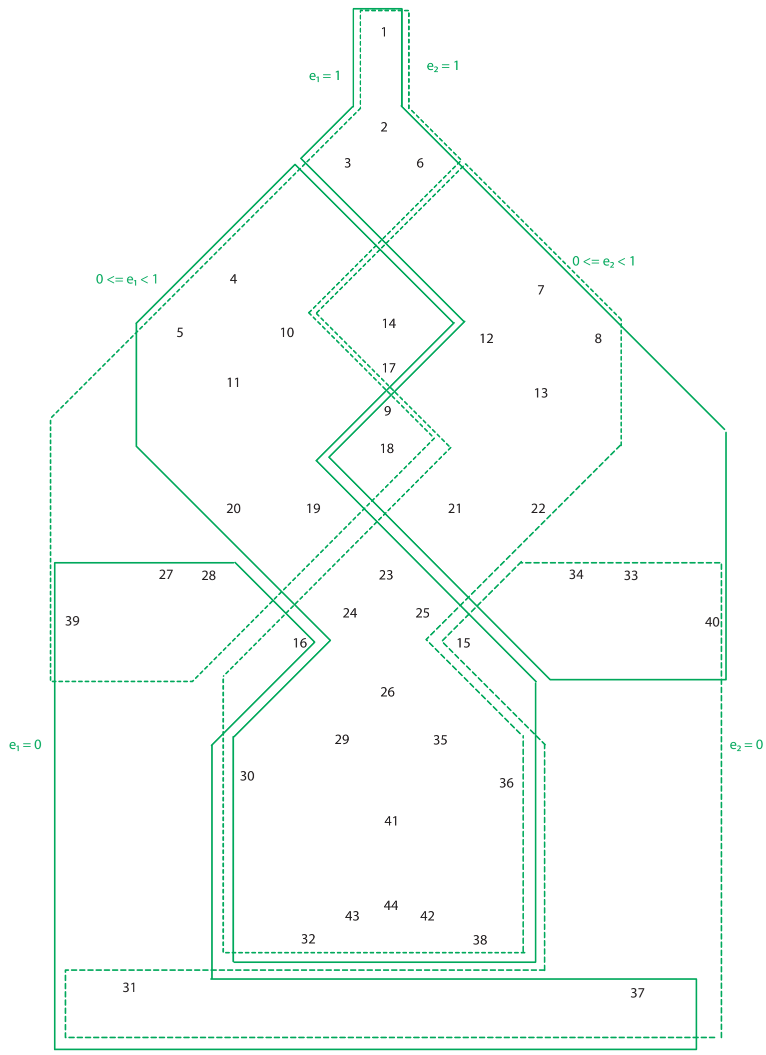


Figure 2.23: Groups associated with values e_1 and e_2.

Interpreting the matrices

Contrary to the traditional intersection matrices, both for crisp regions as for broad boundaries, the interpretation of the fuzzy intersection matrices differ. A consequence of the fact that the regions are truly fuzzy, is that the different cases are no longer mutually exclusive. This is best illustrated by means of an example; two given regions could for instance yield the following intersection matrix:

$$\begin{pmatrix} d & c_1 & e_1 \\ c_2 & b & a_1 \\ e_2 & a_2 & 1 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.7 & 0.4 \\ 0.4 & 0.6 & 0.3 \\ 1 & 0.6 & 1 \end{pmatrix} \tag{2.64}$$

Now, the case at hand will be determined using the graphs in which the different cases are grouped.

- a_1 and a_2
  Using the graph on fig. 2.19, there are two groups for the value a_1: a group where $0 < a_1 \leq 1$ and a group where $0 \leq a_1 < 1$. For the value a_1, both groups are possible. Using the same graph leads us to the same conclusion for a_2. The possible cases with these values are: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44\}$

- b
  For the value of b, the graph on fig. 2.20, shows that there are three groups: $b = 0$, $0 \leq b \leq 1$ and $0 < b \leq 1$. Obviously, the value for b is not 0, so case 1 is no match (not even partial) for this example. This leaves the following cases: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44\}$

- c_1 and c_2
  For the value of c_1, the possible groups are defined by the values $c_1 = 0$, $0 \leq c_1 < 1$, $0 < c_1 \leq 1$, and are shown on fig. 2.21 Only the groups where $c_1 = 0$ are impossible, which leave out the cases 1 and 39. For c_2 the groups and the conclusion is similar, and the cases 1 and 40 are omitted. This leaves: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 41, 42, 43, 44\}$

- d
  For the value d, there are three groups: $d = 0$, $0 \leq d < 1$, $d = 1$ as illustrated on fig. 2.22. In the example, $d = 0.6$; only the group for which this is possible is retained, which leaves the cases: $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17\}$

- e_1 and e_2
  For e_1, the groups are shown in fig. 2.23. The groups are defined by: $e_1 = 1$, $0 \leq e_1 < 1$, $e_1 = 0$. For the value of $e_1 = 0.4$, only the second group is applicable; this leaves the cases: $\{4, 5, 10, 11, 14, 17, 19, 20, 23, 24, 25, 26,$

$29, 30, 32, 35, 36, 38, 41, 42, 43, 44\}$.
The conclusion for $e_2 = 1$ is similar, but now the group for which $e_2 = 1$ is required. This retains the cases: $\{1, 2, 3, 4, 5, 6, 9, 10, 11, 18, 19, 20, 27, 28, 39\}$

The intersection of all of the above cases that are retained for each of the values leaves the cases which are more or less appropriate for the example. This intersection is:

$$\{4, 5, 10, 11\} \tag{2.65}$$

Consequently, this example matches with 4 cases. The grouping as explained before makes it easy to pinpoint the cases at hand; the conceptual neighbourhood graph links the cases that are related (notice how the 4 cases are closely linked in fig. 2.18).

It is possible to further distinguish between the cases that occur, using the values that occur in the intersection matrix. The rule we use is that values smaller than 0.5 belong more to ranges of the form $[0, 1[$ than to the range $]0, 1]$; whereas values greater than 0.5 have the opposite property (this is an intuitive rule). To work with this, match values are assigned for every matrix element x that distinguishes two groups and for every case i that is in either of the two groups.

Definition 67 (Match value m_x^i for a case i and a matrix element x)

$$m_x^i = \begin{cases} x & \textit{if } x < 0.5 \textit{ and range of case i} =]0,1] \\ 1-x & \textit{if } x < 0.5 \textit{ and range of case i} = [0,1[\\ x & \textit{if } x \geq 0.5 \textit{ and range of case i} =]0,1] \\ 1-x & \textit{if } x \geq 0.5 \textit{ and range of case i} = [0,1[\end{cases} \tag{2.66}$$

Consider the example, the differences between the cases 4, 5, 10 and 11 is in the values of $a_1 = 0.3$, $c_1 = 0.7$ and $c_2 = 0.4$. The match values for the value a_1 are:

$$\begin{array}{llll} a_1 = 0.3 & \text{cases 4,10:} & 0 < a_1 \leq 1 & \Rightarrow m_{a_1}^4 = m_{a_1}^{10} = 0.3 \\ & \text{cases 5,11:} & 0 \leq a_1 < 1 & \Rightarrow m_{a_1}^5 = m_{a_1}^{11} = 0.7 \end{array} \tag{2.67}$$

This indicates that cases 5 and 11 are a better match for the example than cases 4 and 10. For the values c_1 and c_2, match values are calculated similarly:

$$\begin{array}{llll} c_1 = 0.7 & \text{case 5:} & 0 \leq c_1 < 1 & \Rightarrow m_{c_1}^5 = 0.3 \\ & \text{cases 4,10,11:} & 0 < c_1 \leq 1 & \Rightarrow m_{c_1}^4 = m_{c_1}^{10} = m_{c_1}^{11} = 0.7 \\ c_2 = 0.4 & \text{cases 4,5:} & 0 \leq c_2 < 1 & \Rightarrow m_{c_2}^4 = m_{c_2}^5 = 0.6 \\ & \text{cases 10,11} & 0 < c_2 \leq 1 & \Rightarrow m_{c_2}^{10} = m_{c_2}^{11} = 0.4 \end{array} \tag{2.68}$$

According to the match values for c_1, cases 4,10,11 are a better match than case 5; whereas the match values for c_2 leads us to conclude that cases 4 and 5 are a better match than cases 10 and 11. As all the match values are in the range $[0,1]$, a t-norm can be used to aggregate them. The use of the minimum yields:

$$\begin{aligned}
\text{case } 4 &: \min\{m^4_{a_1}, m^4_{c_1}, m^4_{c_2}\} = \min\{0.3, 0.7, 0.6\} = 0.3 \\
\text{case } 5 &: \min\{m^5_{a_1}, m^5_{c_1}, m^5_{c_2}\} = \min\{0.7, 0.3, 0.6\} = 0.3 \\
\text{case } 10 &: \min\{m^{10}_{a_1}, m^{10}_{c_1}, m^{10}_{c_2}\} = \min\{0.3, 0.7, 0.4\} = 0.3 \\
\text{case } 11 &: \min\{m^{11}_{a_1}, m^{11}_{c_1}, m^{11}_{c_2}\} = \min\{0.7, 0.7, 0.4\} = 0.4
\end{aligned}$$

As the aggregated match value is the highest for case 11, the topology for the example is closer to this case, than to any of the other three cases. However, the differences between the different averages are very small, so the regions in the example still resemble the other three cases quite closely.

2.5 Extensions to fuzzy regions

2.5.1 Fuzzy Locations

Concept of fuzzy locations

Fuzzy regions so far have been interpreted in a veristic way: the set represents a region, and every point belongs to some extent (indicated by the membership grade) to this region. However, fuzzy sets can also be interpreted in a possibilistic way: only one element is representative, but it is not known which element; the membership grade associated with an element indicates the possibility of this element. Using this interpretation, a fuzzy region becomes a set of possible locations for a point; each of the possible locations has a membership grade associated to indicate just how possible it is. The fuzzy region in this interpretation can be said to represent a fuzzy point, which in turn can be used to represent a fuzzy location.

The possibilistic interpretation matches the interpretation of fuzzy sets for fuzzy numbers: modelling *approximately 5* means that a single number is modelled, the value of which is *close to 5*, but it is not exactly known what the number is. Fuzzy points are similar: an *approximate location* is modelled, it is somewhere in the provided (fuzzy) region, but the exact location is not known.

Definition of fuzzy points

Consider the set $A \subseteq U$ the set of all the points that are possible points for the uncertainly or imprecisely known location. The crisp set A is then extended to a fuzzy set $\tilde{p}^A$ with a possibilistic interpretation, defined completely similar to the way a fuzzy region and a fuzzy set are defined.

Definition 68 (fuzzy point $\tilde{p}^A$)
A fuzzy point $\tilde{p}^A$ is defined as:

$$\tilde{p}^A = \{(p, \mu_{\tilde{p}^A}(p)\} \tag{2.69}$$

Where

$$\begin{aligned} \mu_{\tilde{p}^A} : U &\rightarrow]0,1] \\ p &\mapsto \mu_{\tilde{p}^A}(p) \end{aligned}$$

Here, U is the universe of all locations p; the membership grade $\mu_{\tilde{p}^A}(p)$ expresses the possibility that p is the inaccurately or imprecisely known location $\tilde{p}^A$.

Operations on fuzzy points

Along with the change in interpretation, some operations also will change. While the concepts *union* and *intersection* have no meaning for points, they can still be used on fuzzy regions representing possible locations for a point. Similar remarks are valid for the bounding rectangle, convex hull and surface area. The distance will yield a different result though.

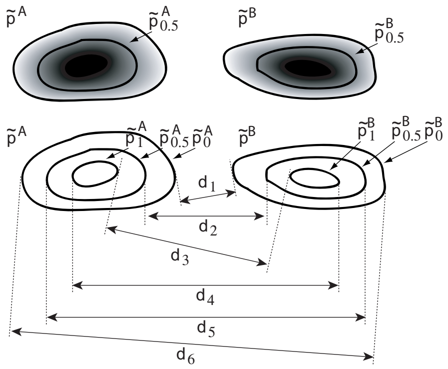
Set-operations The set operations are meaningful, they can be used to combine information from different sources regarding the modelled location. For example, suppose it is known that the point is *close to a given river* and *close to a given city*. The intersection between the fuzzy region that represent the points that are close to the river and the fuzzy region that represents the points that are close to a given city will provide the fuzzy region for points that satisfy both criteria.

α-cut operation Determining the α-cut is also still meaningful, and can be used to determine the crisp region in which the locations have at least a given membership grade.

Minimum bounding rectangle and convex hull Both the operations yielding the minimum bounding rectangle and the convex hull can be used to approximate the outline of the region of possible locations.

Surface area calculation By definition, the surface area of a single point is 0. The definition of the surface area for fuzzy regions can still be used to serve as an indication over which area the imprecisely known or inaccurately known point is located. It does not indicate the surface area of the point.

Distance calculation The distance between two fuzzy points differs from the definition of the distance between fuzzy regions. The reason for this is that for fuzzy regions the distance could not exceed the distance between both cores (the definition for the distance between regions is the minimum distance between them). The imprecisely or uncertainly known point can however be positioned at any point of the region. This will also be reflected in the definition of the distance between two fuzzy points. Basically, for every possible distance,



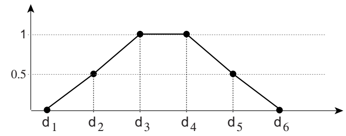


Figure 2.24: Illustration of the fuzzy distance between fuzzy points: (a) fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$ (illustrated using grey scales, and using some contourlines), (b) the fuzzy distance between $\tilde{p}^A$ and $\tilde{p}^B$.

the largest α value for which this distance is still possible between the α-cuts of both regions is assigned, as illustrated on fig. 2.24.

Definition 69 (distance $\tilde{d}(\tilde{p}^A, \tilde{p}^B)$ between two fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$)

$$\tilde{d}(\tilde{p}^A, \tilde{p}^B) = \{(x, \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)}(x))\} \tag{2.70}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup_{\alpha \in]0,1]} \{\alpha \mid p_1 \in \tilde{p}^A_\alpha, p_2 \in \tilde{p}^B_\alpha \wedge d(p_1, p_2) = x\} \end{aligned}$$

Note that there is only one possible interpretation for the distance, unlike in the case of the distance between fuzzy regions: the definition of distance that makes use of topological concepts no meaning for fuzzy points, as the topological concepts themselves have no meaning. The above definition is therefore closest to the α-level definition of distances between fuzzy regions.

This definition does not yield a fuzzy number for fuzzy points represented by complex fuzzy regions (53). However, as the result is to be interpreted in a possibilistic way, the result makes sense, as illustrated on fig. 2.25.

Topology The topology of the fuzzy points can be used similar to the set operations. The operations don't work on the points as such, but on the region delimiting the point.

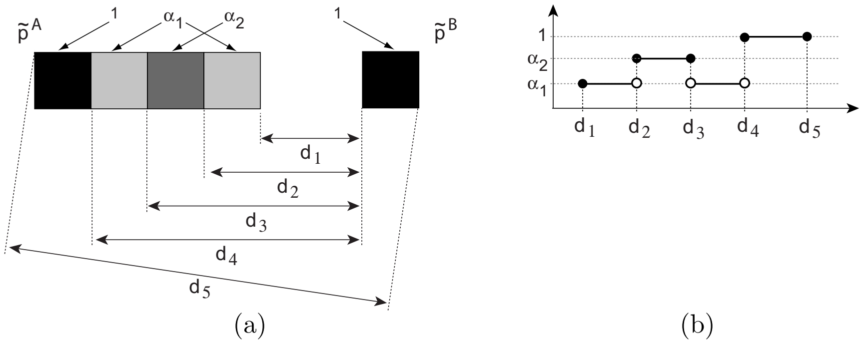


Figure 2.25: Illustration of the fuzzy distance between points: (a) fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$ (illustrated using grey scales, (b) the fuzzy distance between $\tilde{p}^A$ and $\tilde{p}^B$.

2.5.2 Regions at fuzzy positions

In the current approach for fuzzy regions, fuzzy regions are considered to be at crisp positions: the membership grade indicates to which each point belongs to the region. It can however be interesting to represent a region (either crisp or fuzzy) for which its position is unknown or uncertain: a region at a fuzzy position. This could for instance be used to model possible locations for building a new airport, where the uncertainty of the postion stems from environmental limitations (noise pollution, wildlife areas, safety zones, ...).

To model this, the concepts of fuzzy points and fuzzy regions can be combined. The position of a region is then determined by an *anchor-point*: every point of the region is specified relatively to this anchor-point. The anchor-point is a fixed point of the region, traditionally, the center of gravity is used. However, as a region can be fuzzy, its center of gravity should also be a fuzzy point; this would no longer allow for it to be used as a point of reference for the region. Consequently, another point is used. The concept is illustrated on fig. 2.26a, where a fuzzy point $\tilde{A}$ holds all the possible positions for region R (two of the infinite number of possible positions are illustrated).

There is however no reason to keep this limited to crisp regions. A fuzzy region can therefore also be located at a fuzzy position, which is illustrated on fig. 2.26. Incorporating this adds an additional level of uncertainty or fuzziness, requiring the use of *level-2* fuzzy sets: a level-2 fuzzy set $\tilde{\tilde{V}}$ defined over a universal set U is a fuzzy set which elements are ordinary fuzzy sets, all being defined over the same universal set U. The membership function of a level-2 fuzzy set has the form

$$\mu_{\tilde{\tilde{V}}} : \tilde{\wp}(U) \rightarrow [0, 1] \tag{2.71}$$

where $\tilde{\wp}(U)$ denotes the fuzzy power set of the universal set U. Originally, level-2 fuzzy sets were presented by Zadeh [53] and were more elaborately studied

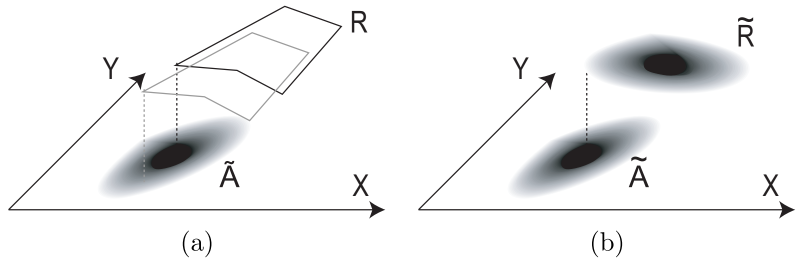


Figure 2.26: Examples of the use of an anchor-point: (a) the fuzzy point $\tilde{A}$ determines the position of region R, the fuzzy point $\tilde{A}$ determines the position of the fuzzy region $\tilde{R}$.

by Gottwald [29]. A formal definition of the concept "level-2 fuzzy set" is given as:

Definition 70 (level-2 fuzzy set)
A level-2 fuzzy set $\tilde{\tilde{V}}$ defined over a universal set U is defined by:

$$\tilde{\tilde{V}} = \{(\tilde{V}, \mu_{\tilde{\tilde{V}}}(\tilde{V})) \mid \forall\, \tilde{V} \in \tilde{\wp}(U) : \mu_{\tilde{\tilde{V}}}(\tilde{V}) > 0\} \tag{2.72}$$

where each ordinary fuzzy set $\tilde{V}$ is defined by:

$$\tilde{V} = \{(x, \mu_{\tilde{V}}(x)) \mid \forall\, x \in U : \mu_{\tilde{V}}(x) > 0\}\,. \tag{2.73}$$

For further information on the use of level-2 fuzzy sets in databases, we refer to [20]. The concept of level-2 fuzzy sets should not be confused with the concept of type-2 fuzzy sets, as explained in 2.5.3.

2.5.3 Fuzzy associated data

Fuzzy regions as explained before can be used to model uncertainty or imprecision regarding locations. It is however also possible for associated data to be prone to uncertainty or imprecision, consider the example of the weather (uncertain temperature predictions, imprecise temperature readings).

Considering regions with fuzzy associated data is quite different from the concept of fuzzy regions. In the concept of fuzzy regions, the entire region was considered as a two dimensional fuzzy set; each point has a membership grade associated. When the associated data is considered to be fuzzy, this means that every point of a crisp region now has a fuzzy set associated, but the region itself is not a fuzzy set; it is merely a set of points with fuzzy associated data.

This associated fuzzy data is defined over the domain of the associated data (which mostly will be a numerical domain). It is however not limited to fuzzy numeric data; classification (for instance land classification: forest, water, sand), can be modelled using fuzzy sets as well; as illustrated in 1.3.2 ([31]). As an example, fuzzy numbers are associated with points of the region.

Definition 71 (extended region for fuzzy real numbers)
An extended region associating fuzzy real numbers $A^{\tilde{\mathbb{R}}}$ is defined as:

$$A^{\tilde{\mathbb{R}}} = \{(p, f(p)\} \tag{2.74}$$

where

$$\begin{array}{rcl} f : U & \to & \tilde{\wp}(\mathbb{R}) \\ p & \mapsto & f(p) \end{array}$$

Here, U is the universe of all locations p and $\tilde{\wp}(\mathbb{R})$ is the set of all fuzzy sets defined over $\mathbb{R}$. The fuzzy real number associated with each point is obtained through measurements in the field, estimations, approximation or which ever data acquisition technique.

A region in which all points have a fuzzy set associated, will be called an *extended region*. The concept of extended regions allows for special cases of fuzzy regions to be defined.

Calculations on the associated data are the field of fuzzy arithmetic, and extend beyond the scope of this work.

Type-2 Fuzzy Regions

A type-2 fuzzy set is a fuzzy set in which the membership grades themselves are fuzzy sets over the domain $[0, 1]$. The philosophy behind type-2 fuzzy sets [34] is: "If information is uncertain or imprecise, it is not possible to assign a crisp membership grade". Consequently, it helps to refute the common criticism that membership grades are chosen arbitrary.

Consider a fuzzy region $\tilde{A}$. The region itself is interpreted as a fuzzy set, and values associated with each of the points are membership grades in the range $]0, 1]$. By defining the membership grades as fuzzy sets over this range, one obtains an *extended fuzzy region*, which can be denoted $\tilde{A}^{\tilde{[0,1]}}$ or $\tilde{\tilde{A}}$ for short. It supports the same operations as a fuzzy regions, but with the necessary modifications to deal with type-2 fuzzy sets.

Definition 72 (type-2 fuzzy region $\tilde{\tilde{A}}$)

$$\tilde{\tilde{A}} = \{(p, \mu_{\tilde{\tilde{A}}}(p))\} \tag{2.75}$$

where

$$\begin{array}{rcl} \mu_{\tilde{\tilde{A}}} : U & \to & \tilde{\wp}(]0, 1]) \\ p & \mapsto & \mu_{\tilde{\tilde{A}}}(p) \end{array}$$

Here, U is the universe of all locations p and $\tilde{\wp}(]0, 1])$ is the set of all fuzzy sets over the domain $]0, 1]$. The membership grade $\mu_{\tilde{\tilde{A}}}(p)$ expresses the extent to which p belongs to the fuzzy region using a fuzzy membership grade.

Similarly, type-2 fuzzy points can be defined. For the remainder of this work, this concept will not be considered further, but extending the results from this work to deal with type-2 fuzzy sets is part of the future work.

Possibilistic truth value regions

In 1.2.4, possibilistic truth values (PTV for short) were introduced. Representing truth values over a region is interesting for query purposes. A GIS capable of using such possibilistic truth value regions (or PTV regions), could answer a query by returning a PTV region: a region in which every location has a truth value, indicating how well each location matches with the given query. The possibilistic truth values use possibility distributions over $\{True, False\}$, but the same model as for fuzzy sets can be used; a possibilistic truth value can be considered as a new type of associated data.

The model therefore allows also for the modelling of a fuzzy region, where the associated values are possibility distributions over the boolean domain $\{True, False\}$. This model is quite analogous to the model where fuzzy numbers are associated with the locations; only the domain of the fuzzy set differs.

Definition 73 (PTV fuzzy region $A^{\widetilde{\{T,F\}}}$)

$$A^{\widetilde{\{T,F\}}} = \{(p, \mu_{A^{\widetilde{\{T,F\}}}}(p))\} \tag{2.76}$$

Where

$$\begin{aligned} \mu_{\tilde{\tilde{A}}} : U &\rightarrow \tilde{\wp}(\{True, False\}) \\ p &\mapsto \mu_{A^{\widetilde{\{T,F\}}}}(p) \end{aligned}$$

Here, U is the universe of all locations p and $\wp(]0,1])$ is the set of all fuzzy sets over the domain $]0,1]$. The membership grade $\mu_{\tilde{\tilde{A}}}(p)$ expresses the extent to which p belongs to the fuzzy region using a fuzzy membership grade.

Similarly, regions with extended possiblistic truth values, these are truth values defined over associated $\{True, False, \perp\}$, can be modelled as well.

2.6 Graphical representation

An important aspect of a GIS is the graphical representation of the data. In an entity based approach, the information is two dimensional, which makes it easy to represent on a graphical display. By moving to fuzzy regions, points in the two dimensional space have a membership grade associated with them. Consequently, there is more data to be shown than just the region itself. To display this additional data, a number of techniques are possible.

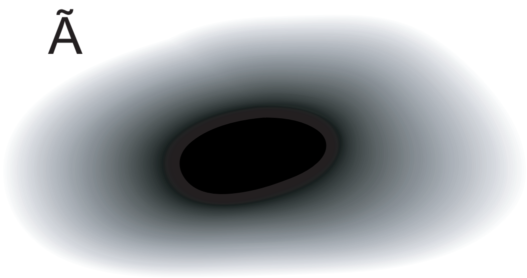


Figure 2.27: Examples of a fuzzy region represented using grey scales to indicate membership grades.

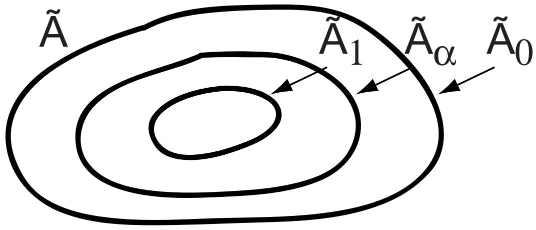


Figure 2.28: Examples of the a fuzzy region represented by means of contourlines to indicate membership grades.

2.6.1 Colours and grey scales

The first technique to draw fuzzy regions is to use colour to indicate the membership grades. Intuitively, one can consider a color gradient, where darker shades match higher membership grades; as illustrated on fig. 2.27. While this works for simple situations, it becomes a problem when different data needs to be displayed at the same location: making the colouring semi-transparent makes the membership grades (esp. the lower ones) harder to see and can become confusing.

2.6.2 Contour lines

Another technique to draw a fuzzy region is by making use of contour lines. This concept is similar to the altitude lines on topographical charts, or the representation of isobars on weather charts. While it has the advantage of less clutter in the drawing, care has to be taken to assure that the chosen iso-lines are representative for the gradual changes of the membership grades. An example of the representation with iso-lines can be seen on fig. 2.28.

The contour lines are used for example for altitude or air pressure, as only one type of data is considered at a time. When representing two regions or more using contour lines, the large amount of lines might become confusing.

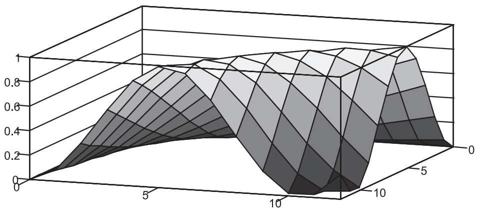


Figure 2.29: Examples of the a fuzzy region represented as a three dimensional structure, where the third dimensions is used to indicate membership grades.

2.6.3 Three dimensions

Finally, it is possible to consider the membership grade as a third dimension, and represent it as such. However, most GIS systems are not sufficiently equipped to represent three dimensional objects (some support 2.5D, meaning that they allow the notion of altitude to be modelled; full 3D systems are emerging); it would require many changes to the system and only serve limited applications. Furthermore, when having to draw a multitude of information, it can become difficult to discriminate the various objects. An example of a part of a fuzzy region represented in three dimensions is shown on fig. 2.29.

In this work, the different representation methods are used, depending on which one is more appropriate to illustrate the situation.

Chapter 3

Manageable fuzzy regions

3.1 Issues with the theoretical concept

The concept of fuzzy regions as developed and presented in chapter 2 is intended as a theoretical foundation to define and reason with fuzzy regions. As such, it is developed without taking into account limitations that may prevent a practical application of the concept. This results in two major issues, which make the conceptual model unsuited for direct implementation.

First, there is the fact that with each point of the two dimensional space, a membership grade is associated. However, as the membership grades of points are not required to share any mathematical relation between among them (one point can have a membership grade 1, another membership grade 0.35, etc.); which in practice would require an infinite amount of storage (the two dimensional space, even when limited to a region of interest, still contains an infinite number of points).

Secondly, there is the problem that most operations are defined by means of the extension principle. This principle provides for a definition of the membership grade for each element in the result of an extended operation, but does not provide for a means of computing the result (for example: in fuzzy arithmetic, the sum of two fuzzy real numbers yields a new fuzzy number; the extension principle will provide a way to compute the membership grade for each element of $\mathbb{R}$ with regard to the fuzzy sum, but does not provide for a way to compute which real numbers will contribute to the fuzzy sum). Consequently, while the extension principle can be used to uniquely and unambiguously define an operation mathematically, it cannot be used to compute the result in case the result has an infinite domain.

3.2 Practical approaches to fuzzy regions

To overcome these issues, traditional fuzzy models make use of limitations imposed on the models to allow results to be computed algorithmically. In the case of fuzzy arithmetic for instance, limiting the membership function of fuzzy numbers to piecewise linear functions in combination with the use of the Zadeh T-norm and T-conorm allows for the computation of sum and difference using interval calculus [32]. To make fuzzy regions manageable, limitations will be imposed on the distribution of membership grades over the fuzzy regions; these limitations will be chosen such that the computations can be performed algorithmically, while still yielding a result that is compliant with the conceptual model.

3.3 Fuzzy regions using a Contour-line model

3.3.1 Concept

The first technique uses multiple crisp boundaries (for which a representation method exists in current spatial databases) to group points that have the same membership grade. The membership grades for points that are not specified on a boundary are computed by means of a shape-function and the relative position of the points with regard to the boundaries. This causes the membership grades to be in relation with one another. The approach reverts to the use of contourlines as a building block, even though contourlines are normally used as a representation method for e.g. barometric pressure or altitudes. Different variants of this concept are presented: the first uses one crisp inner boundary, around which a second crisp outer boundary at a fixed distance is positioned (both this outer boundary and the membership grades for points inbetween the given inner and computed outer boundaries are defined using a shapefunction) The second variant allows for both the inner and outer boundaries to be specified (no longer requiring them to be equidistant).

The concept combines the broad boundary model (where two boundaries are used, 1.3.2), with the contourline representation (2.6.2) of a fuzzy region: instead of just using two boundaries (an inner and an outer boundary), intermediate lines can be included to assign membership grades. Inside the inner boundary all the points have membership grade 1 and outside the outer boundary the membership grades are 0. The difference between the aforementioned methods, is that membership grades are assigned to all locations. Points in between both boundaries will be assigned membership grades based on their distance from the boundaries: this resembles the use of an infinite number of boundaries within the broad boundary.

To assign membership grades for points between the inner and outer boundary, different approaches can be considered. The two presented methods both make use of a continuous, monotonic decreasing function, with both domain and range $[0, 1]$, which is called the *shape-function* S_f to define the gradual transition.

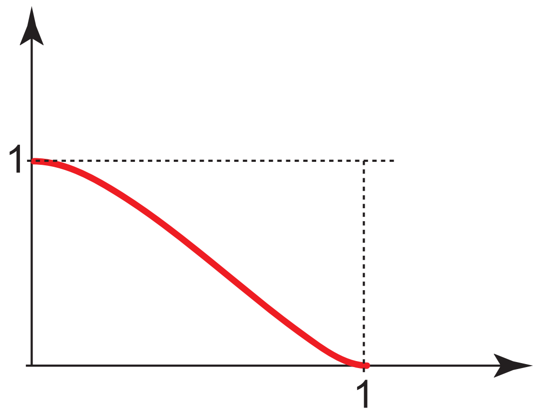


Figure 3.1: Example of a shape function.

Definition 74 (shape-function S_f)

$$\begin{array}{rcl} S_f : [0,1] & \rightarrow & [0,1] \\ x & \mapsto & S_f(x) \end{array}$$

This function must satisfy the following properties.

$$\begin{array}{l} S_f(0) = 1 \\ S_f(1) = 0 \\ \forall x \in [0,1] : S_f(x) \in [0,1] \\ \forall (x,y) \in [0,1]^2 : x \leq y \Rightarrow S_f(x) \geq S_f(y) \end{array} \tag{3.1}$$

The fourth property, claiming that $S_f(x)$ is a decreasing function, is not strictly necessary. This assumption is made to eliminate degenerate cases: we are modelling the property *belongs to* of the surrounding points with respect to a given crisp region, and therefore it is quite natural to assume that the degree of belonging to decreases with growing distances. It is possible to have a situation in which one would like to assign the center-point(s) the lowest membership-degree, while all the points outside of the maximum extent should get a value 1. In this case, a function with opposite properties should be used. An example of a shape function can be seen on fig. 3.1.

3.3.2 Definitions

Equidistant boundaries

The first technique is based on equidistant crisp boundaries. Because the two crisp boundaries are not independent of one another, it is sufficient to know only one boundary (we will consider the inner one). The inner boundary is noted B_1, the outer B_0. Each boundary outlines a crisp region, this region is denoted R_{B_1} for the boundary B_1 and R_{B_0} for B_0. The notation $p \in R_{B_1}$ is used to indicate that p is located inside the boundary B_1 or on the boundary B_1. The outer boundary B_0 can be defined as consisting of all the points at a given distance d_0 from B_1. Points located at a distance greater than d_0 will be considered to be outside the outer boundary. For vertices where there is an angle of less then π rad) B_0 will appearing round, whereas for the vertices with an angle greater than π rad, there is no change. It should be noted that - had we chosen the outer boundary as a reference - the resulting fuzzy area would have the exact opposite property (angles at vertices of B_1 would be round).

To determine the membership degree of a point p, its location relative to the two boundaries is determined. If $p \in R_{B_1}$, its membership degree will be 1. On the other hand, if $p \notin R_{B_0}$, its membership degree will be 0. For points in between both crisp boundaries ($p \in R_{B_0} \wedge p \notin R_{B_1}$), the distance $d_p = d(p, B_1)$ between the point and the chosen crisp boundary (in this example B_1) is calculated. This distance d_p is scaled to the interval $[0, 1]$ (based on the maximum distance d_0) and passed on as an argument to the shape function, which returns the membership degree for this point.

Definition 75 (membership function $\mu_{\tilde{A}}$)
(for a fuzzy region $\tilde{A}$ defined using equidistant boundaries)

$$\begin{array}{rcl} \mu_{\tilde{A}} : \mathbb{R}^2 & \rightarrow & [0,1] \\ p & \mapsto & \begin{cases} 1 & \text{if } p \in R_{B_1} \\ \\ 0 & \text{if } p \notin R_{B_1} \wedge d(p, B_1) > d_0 \\ S_f(\frac{d_p}{d_0}) & \text{if } p \notin R_{B_1} \wedge d(p, B_1) \leq d_0 \end{cases} \end{array}$$

Here, d_0 is the distance between the outer boundary B_0 and the inner boundary B_1.

This way, all points at the same distance from the crisp boundaries will be assigned the same membership degree. Because of the use of the shape-function, we can model any shape of transition, or add additional properties (i.e. it is possible to assign the points halfway in between both boundaries a membership degree anywhere in $]0, 1[$). The calculations needed are quite straightforward : the algorithms to determine whether or not a point is inside a polygon are known ([39]) and available in GIS-systems, so if the shape-function is not too complex, the required calculations can be executed very efficiently. A region with equidistant boundaries is shown on fig. 3.2.

Independent crisp boundaries

The second technique uses two independent (hence not necessarily equidistant) boundaries and a *shape-function* as defined above. Based on these two crisp boundaries, an infinite set of contours between both boundaries can be defined, each corresponding to a membership degree. All contours in the set are disjoint,

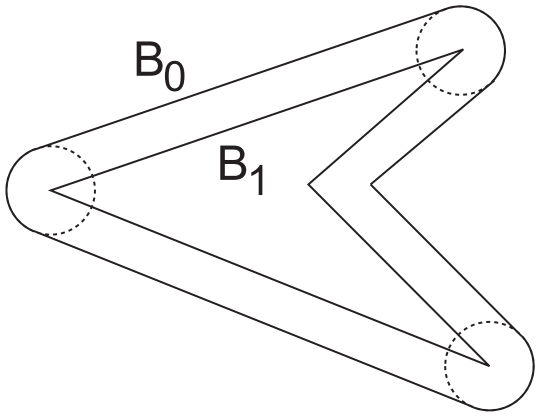


Figure 3.2: Example of two equidistant boundaries B_0 and B_1.

i.e. each point lies on exactly one contour corresponding with its membership degree. The effect of this technique can be visualized as a gradual deformation of the inner boundary towards the outer boundary. This is achieved by an appropriate scaling of the shape-function, taking into account the relative position of the point inside the broad boundary with respect to the two crisp boundaries. We will denote the inner boundary - within which the property is true - B_1, the outer one will be characterized B_0. The only requirement for both boundaries, is that $R_{B_1} \subset R_{B_0}$. Consider a point p: if $p \in R_{B_1}$, it is assigned a membership degree 1; if $p \notin R_{B_0}$, its membership grade will be 0. For the remaining option (between B_0 and B_1), the distances between the point p and the boundaries are calculated :

$$\begin{aligned} d_0 &= d(p, B_0) \\ d_1 &= d(p, B_1) \end{aligned}$$

The idea is to use these distances to provide information about the location of p. We will therefore scale the distance to a value in the range $[0, 1]$. The reference-distance to perform this scaling, can no longer be a constant value : the distance between both boundaries can vary. The *total distance used as a reference for this point* is denoted d_t and defined as

$$d_t = d_0 + d_1 \tag{3.2}$$

To determine the membership-grade for a given point p, we first need to check whether the point is inside B_1 or outside B_0. In the former case ($p \in B_1$), its membership-degree will be 1; in the latter ($p \notin B_0$) it will be 0. For points in between both boundaries, the shape-function is used.

Definition 76 (membership function $\mu_{\tilde{A}}$)
(for a fuzzy region $\tilde{A}$ defined using independent two crisp boundaries)

$$\mu_{\tilde{A}} : \mathbb{R}^2 \quad \rightarrow \quad [0, 1]$$

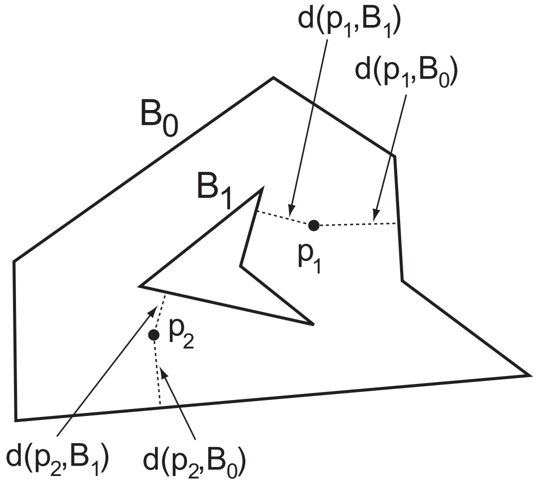


Figure 3.3: Example of a shape function.

$$p \quad \mapsto \quad \begin{cases} 1 & \textit{if } p \in R_{B_1} \\ 0 & \textit{if } p \notin R_{B_0} \\ S_f(\frac{d_1}{d_0+d_1}) & \textit{if } p \in R_{B_0} \backslash R_{B_1} \end{cases}$$

We have chosen to use d_1 as the numerator, though similar results can be acquired using d_0. Contrary to the previous technique, both choices will now lead to the same result. A region defined by two independent boundaries is illustrated on fig. 3.3

As an extension, it is possible for B_1 to be a disconnected region; the distance d_1 should then be interpreted as the shortest distance to any of the B_1. Even B_0 can be disconnected, provided that inside each separate part is a B_1 region. The distances d_0 and d_1 should be considered for the part of B_0 where p is located, and for the B_1 that is inside this B_0.

Multiple independent crisp boundaries

The above approach can be improved, by considering a number of crisp boundaries: apart from B_0 and B_1, a finite number of boundaries B_{i_j}, ($i_j \in]0,1[$), i_j is the membership grade for points on this boundary) can be considered (fig. 3.4). The interpolation between two successive boundaries is performed as described above. This allows for much more freedom in defining fuzzy regions. To define the gradual transition between the membership grades, using n boundaries, $n-1$ shape functions are needed. Each shape function has slightly different

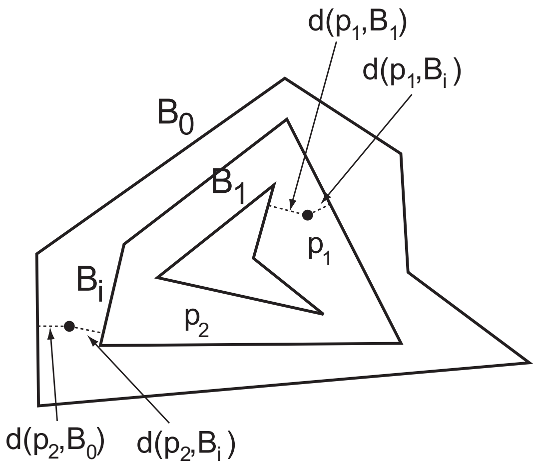


Figure 3.4: Example of the use of multiple independent crisp boundaries.

conditions than before though:

$$\begin{aligned}
&S_f^{i_j}(0) = i_{j+1} \\
&S_f^{i_j}(1) = i_j \\
&\forall x \in [0,1] : S_f^{i_j}(x) \in [i_j, i_{j+1}] \\
&\forall (x,y) \in [0,1]^2 : x \leq y \Rightarrow S_f^{i_j}(x) \geq S_f^{i_j}(y)
\end{aligned} \tag{3.3}$$

Consider the boundaries $B_{i_1}, B_{i_2}, \ldots, B_{i_n}$, where $B_{i_1} = B_0$ and $B_{i_n} = B_1$, the shape functions $S_f^{i_1}, S_f^{i_2}, \ldots, S_f^{i_n - 1}$ and a point $p(x, y)$. The notation d_{i_j} is used to represent the shortest distance between $p(x, y)$ and B_{i_j}. The membership grade for any point p is obtained from the membership function.

Definition 77 (membership function $\mu_{\tilde{A}}$)
(for a fuzzy region $\tilde{A}$ defined using multiple independent boundaries)

$$\begin{aligned}
\mu_{\tilde{A}} : \mathbb{R}^2 &\rightarrow [0,1] \\
p &\mapsto \begin{cases} 1 & \text{if } p \in B_1 \\ 0 & \text{if } p \notin B_0 \\ S_f^{i_j}\left(\frac{d_{i_{j+1}}}{d_{i_j} + d_{i_{j+1}}}\right) & \text{if } p \in R_{B_{i_j}} \backslash R_{B_{i_{j+1}}} \end{cases}
\end{aligned}$$

The membership function is a piecewise function, consisting of the $n - 1$ shape functions $S_F^{i_j}$, scaled appropriately.

3.3.3 Operations

Set operations

An obvious requirement to a model is that it is closed. For set operations, this means that the result of the operation should be of the same kind of object

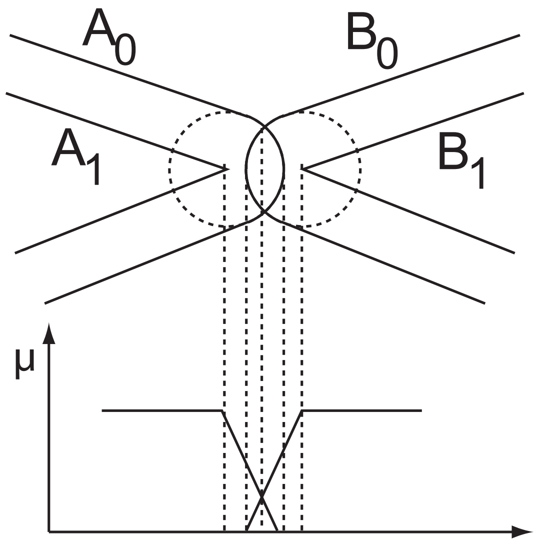


Figure 3.5: Example of the problems that occur for the union operator.

as both arguments. This allows for similar operations to be applied on the result. Consider two disjoint fuzzy regions $\tilde{A}$ and $\tilde{B}$. Both are defined using a number of contour lines and a number of shape functions. The model allows for disconnected regions, so it ought to be possible to consider the union as a new region $\tilde{C}$. While it is possible to define the union or the intersection, it is obvious that if both regions use different contour lines and/or different shape functions, this union can no longer be represented as a single region. This occurs both in the model with equidistant boundaries as well as in the model with independent boundaries.

The model could perhaps still be useful if all shape functions are required to be the same. Consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$, both defined using the same shape functions (and consequently the same contour lines), as shown of fig. 3.5. As the cross section illustrates, the new region would require a contour line to deal with with overlap. This contour line was not present initially, so again the closure property fails. In the above example, even the contour lines considered where the same, so even in such a limited model the property fails.

α-cut

Given the difficulties with the set operations, there is little point in developing the concept further. However, some lessons could be learned by seeing what difficulties arise in the calculation of the α-cut.

The definitions make it quite efficient to calculate the membership grade of any point of the region, however returning all the points with a given membership grade is much more difficult. When equidistant boundaries are used, the result is just a buffer line (1.1.4) for the original region, finding one point with the correct membership grade provides the distance d_1 from which this buffer needs to be defined.

In the case of independent boundaries, the shape of the contour line is quite difficult to compute, as both distances d_0 and d_1 need to be considered. In a system where the two-dimensional space is approximated using a finite number of elements (i.e. a realm, 1.3.2), one might consider to test each element between B_{i_j} and $B_{i_{j+1}}$ (which will be computationally intensive). The obtained set of points must then be simplified to a polygon-like structure, to minimize the points required for storage. This approach basically is a brute force algorithm, with little room for optimization.

Finally, if the boundaries are required to be equidistant, determining any α-cut is easily possible: for the straight line segments the lines of the α-cut are parallel (at the distance determined by the α-level), whereas in the corners the lines of the α-cut are represented by an arc (with a radius determined by the α-level. This kind of contour-line fuzzy region is however quite limited in the shape of regions it can represent.

3.3.4 Summary

This first concept obviously has too many drawbacks to be useful for use in implementations or even in theoretical considerations. It has however thought a number of lessons:

- The closure property should be satisfied for any model to be useful in real life.
- Limiting data structures with mathematical constraints may facilitate the calculation of membership grades, but makes the closure property harder to enforce. Consequently, care should be taken when imposing limitations on membership grades.
- It should be possible to (quickly) calculate the membership grade of any point.
- It should be possible to (quickly) return all the points that have a given membership grade.

The lessons learned in this first approach were taken into consideration, and led to both the bitmap-model (3.4) and the triangulated irregular network model (3.5).

Despite these shortcomings, the model is still quite appropriate as an extension of the traditional buffer concept. Conceptually, buffers can be considered to require membership grades that decrease from the core outward. The contourline model can also easily be queried when locations are given (queries of the from “how well is location x inside the fuzzy buffer” are easily answered), and requires only a little amount of data (more specifically: the object around which it is defined and the shape function).

3.4 Fuzzy regions using bitmaps

The second technique adopts the concept known in geographic systems as bitmaps. Bitmaps are a grid based structure, where the smallest considered element no longer is a point, but a cell (a set of points), and where only is a finite number of cells are required. The value associated with a cell is representative for all points within this cell, and will for fuzzy regions be a membership grade. By using this bitmap concept for fuzzy regions, the issues that occurred due to the fact that a fuzzy region contains an infinite number of points are resolved. In chapter 4, this approach is considered in further detail.

3.5 Fuzzy regions using TINs

The third technique uses the established spatial concept of a triangular network. In a triangular network, only a finite number of points (datapoints) are given an associated value. Using these datapoints, a triangular network is constructed, by means of this network the associated values for other points can be computed using linear interpolation on the associated values of the datapoints. When used for fuzzy regions, the associated values will concern membership grades. The operations on fuzzy regions are performed on the finite set of datapoints, after which appropriate modifications to the interpolation are required to obtain the desired result. This approach is dealt with in chapter 5.

Chapter 4

Fuzzy regions using bitmaps

4.1 Concept

In the bitmap approach, the infinite number of points that occur in the conceptual model (chapter 2), are grouped into a finite number of sets (which will be called cells). Membership grades are then assigned to these sets, and all points are considered to have the same membership grade as the set they belong to.

A bitmap is a known concept in GIS systems, as mentioned in 1.1.3. Traditionally, it is used to represent features that are present throughout the map, like population densities. A bitmap usually is an approximation of the real situation; where continuous values ought to exist but are either impossible or too difficult to obtain.

4.2 Definition

A fuzzy bitmap is in essence an extension of a regular, crisp bitmap. Unlike a regular bitmap in GIS, a fuzzy bitmap is considered to be limited to a certain, crisp region (the region of interest). In order to formally define a fuzzy bitmap, first the concepts of *cell* and *grid* will be defined. These definitions are similar to those in (1.1.3), with the difference that the universe is limited to a region of interest R, which is a bounded subset of the two dimensional space $U = \mathbb{R}^2$.

With the understanding that $U \subseteq \mathbb{R}^2$ is the universe of all the locations (points) considered in the GIS, a bounded subset $c \subseteq U$ is called a cell if it is convex, i.e.:

Definition 78 (cell)

$$\forall p_1, p_2 \in c,\ \exists p_3 \in c : \frac{\overrightarrow{p_1} + \overrightarrow{p_2}}{2} = \overrightarrow{p_3} \tag{4.1}$$

Definition 79 (grid)
A grid *— in GIS — is a finite collection $G \subseteq \wp(R)$, where $R \subset U$ is the region of interest, such that*

$$\begin{array}{ll} \textit{(i)} & (\forall c, c' \in G)(c \cap c' = \emptyset) \\ \textit{(ii)} & \bigcup_{c \in G} c = R \end{array}$$

In [50], the bitmap was considered a global structure; in subsequent publications it was altered by limiting it to a region of interest. This difference is resembled in this altered definition of a grid. In general, all cells have similar shapes and sizes, although the fuzzy bitmaps are not limited to this: in the examples here cells will be rectangular, but the length and width proportions of cells can differ. Each grid has a fixed number of (horizontal and vertical) cells. Other shapes of cells, i.e. hexagonal, are possible but not considered here. Similarly to traditional bitmaps, a value will be associated with every cell of the bitmap. In a fuzzy bitmap, these values are limited to the range $[0, 1]$ as they will represent membership grades. [1] The membership function associates every cell of a grid G with its membership grade for a given bitmap $\tilde{B}$:

Definition 80 (membership function $\mu_{\tilde{B}}$ of a fuzzy bitmap $\tilde{B}$)

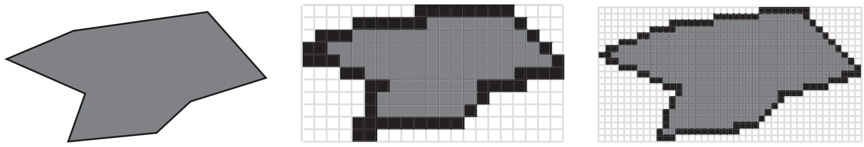
$$\begin{array}{rcl} \mu_{\tilde{B}} : G & \rightarrow & [0, 1] \\ c & \mapsto & \mu_{\tilde{B}}(c) \end{array}$$

Definition 81 (fuzzy bitmap $\tilde{B}$)
(using grid G and membership function $\mu_{\tilde{B}}$)

$$\tilde{B} = \{(c_j, \mu_{\tilde{B}}(c_j)) \mid c_j \in G\} \tag{4.2}$$

This definition differs from the original definition in [50] (later publications already use the above definition) in that the cells are no longer numbered using two indices (coordinates), but only using one index. The reason for this is twofold: first to accommodate the definition for both regular and irregular grids; second, this numbering also matches the numbering of cells in Esri MapObjects, which was used for a prototype implementation. The downside to this numbering is that given a cell, its relative position to other cells of the same bitmap is not immediately known.

[1]The membership grade 0 is included in a bitmap as this facilitates the implementation by allowing all bitmaps to bounded by a rectangular region of interest (the added cells then are associated membership grade 0). For any operation, they can simply be ignored; the definition includes 0 to allow for fuzzy regions with holes to be defined.



(a) (b) (c)

Figure 4.1: Use of a bitmap to approximate a crisp region: (a) crisp region, (b) approximation of the region in (a) using a coarse grid, (c) approximation of the region in (a) using a fine grid.

4.3 Using fuzzy bitmaps as regions

4.3.1 Representing crisp regions with fuzzy bitmaps

A bitmap can however be used to represent a crisp region, as is illustrated on fig. 4.1. Fig. 4.1a shows a crisp region, fig. 4.1b and 4.1c show this region in a bitmap representation. The accuracy of a bitmap is determined by its resolution[2], the larger this resolution, the more accurate the approximation is. In the example, 4.1c has a higher resolution than 4.1b. To obtain a bitmap from a vector shape, one of many rasterization techniques ([27]) can be used; these are common in the realm of computer graphics: normal computer displays can be considered to be bitmaps (they have a resolution), whereas computer graphics are vectorial in nature.

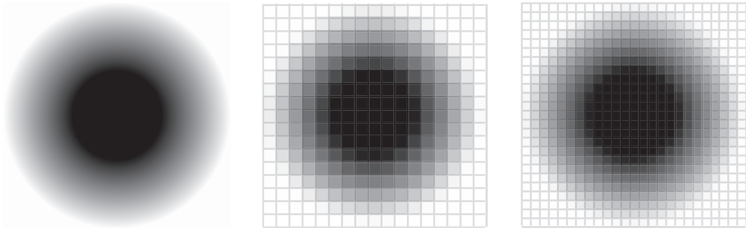
4.3.2 Representing fuzzy regions with Fuzzy bitmaps

To represent fuzzy regions, the cells - which contain an infinite number of points - are considered to be the smallest unit in the bitmap. Membership grades are associated with the cells, which basically means that all points in the cell have the same membership grade.

In figure 4.2a, a simple fuzzy region is shown. For representation purposes, greyscales are used: black equals membership grade 1, the lower the membership grade of a point (or in the case of the bitmap: a cell) is, the lighter its shade of grey will be. Figure 4.2a and figure 4.2b show a representation of this fuzzy region in the bitmap approach. The grid used in figure 4.2c is more refined than the grid used in figure 4.2b, which - as already was shown in the crisp case - yields a more accurate model.

Several bitmaps can be defined using the same grid. This means that they cover the same region of interest and that their cells are exactly the same size; the associated values of the cells can differ though.

[2]Strictly speaking, the resolution is expressed in dpi, while the size of a bitmap is expressed in a number of pixels (or cells). However, as bitmaps representing a same object (e.g. in fig. 4.1b and fig. 4.1c) cover roughly the same area, dpi and size expressed in pixels are linked: the more pixels, the higher the dpi.



(a) (b) (c)

Figure 4.2: Use of a bitmap to approximate a fuzzy region: (a) fuzzy region, (b) approximation of the region in (a) using a coarse grid, (c) approximation of the region in (a) using a fine grid.

4.4 Operations

4.4.1 Set-operations

The set operations make use of two fuzzy bitmaps. As no limitation is imposed on the grids, it can occur that the union or intersection of two fuzzy bitmaps with different grids is required. To accomplish this, the bitmaps will be redefined, so that they both use the same grid. A bitmap can be redefined, as long as every point maintains the same membership grade [3]. As matching the grids can be required for the set operations, this grid-problem will be considered first.

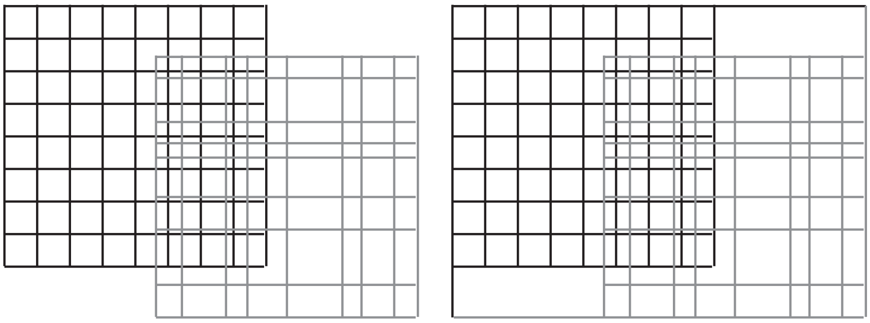
Matching the grid

The technique presented is one way of matching the grids of two unrelated bitmaps in order to combine them to a new bitmap. It creates a new grid, which can be seen as the combination of both grids, thus allowing the associated data of both bitmaps to be represented accurately in this common grid. This technique was chosen because it yields a mathematically accurate result. In true implementations, this approach can cause a fuzzy bitmap region to become too complex. Alternative methods can keep the complexity down, for instance by making use of the fact that the bitmaps are already representing imperfect information, and by exploiting this to match the grid of one bitmap to the grid of the other one by resampling the values to provide for new values for this new grid. This approach would keep the complexity down, but my require making changes to the original data. Other techniques in matching the grids, or additional limitations imposed on the allowed grids in the model might be more appropriate in some situations.

[3]This might be contradictory to the traditional use of bitmaps, as a finer grid is sometimes considered to imply more accurate information. Later on in this chapter, we will return to this issue.

Region of interest Consider two grids as shown on fig 4.3a. The first step in defining the grid is determining the new region of interest. The region of interest can be enlarged; as long as any points added will have a membership grade 0 associated.

The new region of interest is basically the union of the regions as considered by the bitmap arguments. However, as a bitmap is considered to have a rectangular outline, the region is extended in such a way that its outline is a rectangle, see fig 4.4b. This temporary structure is not yet a bitmap, the grid still needs to be adapted.



(a) (b)

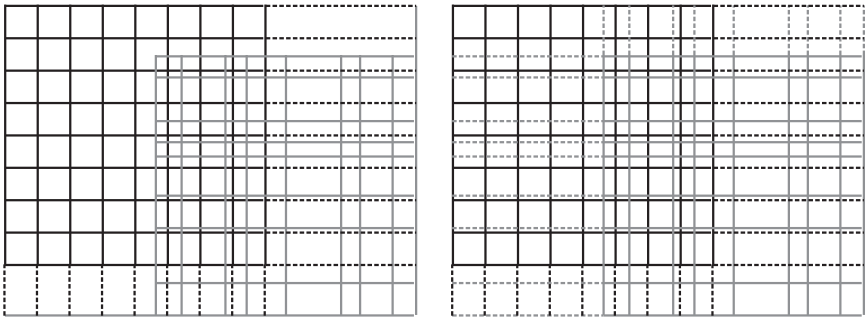
Figure 4.3: Defining the region of interest.

Grid lines In a second step, the region of interest is partitioned using the grid of the first bitmap. The original gridlines are maintained, but lengthened beyond its original region of interest to reach the outline of the newly defined region of interest. This action possibly divides cells of the other grid (for illustration purposes, the lengthened gridlines are drawn using dashed lines on fig 4.4a).

The third step is symmetrical to the second step: the grid lines of the second bitmap are lengthened to reach the outline of the new region of interest, thus possibly partitioning cells of the other bitmap further (4.4b).

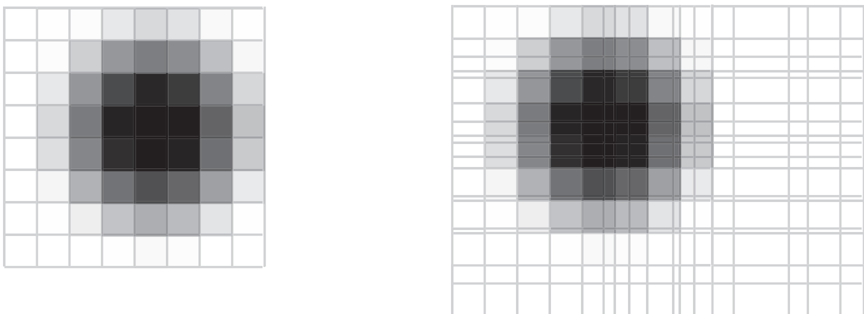
Associated values The above construction has provided a new grid for both bitmaps. For both bitmaps, their original grid can still be found in this new grid, the main difference is that the individual cells are possibly partitioned further. Each new cell is now assigned the membership grade of the original cell it is a part of. As the original cells are still present, but partitioned further, this operation yields no ambiguous assignments. With new cells that fall outside the original region of interest, the membership grade 0 is associated.

The result of this construction is that every cell that was present in one of the arguments is present in the new region of interest, either as a whole, or partitioned in a number of smaller cells. Now, the original grids are discarded, and every bitmap that is an argument is now using these grids for its



(a) (b)

Figure 4.4: Defining the gridlines.



(a) (b)

Figure 4.5: Mapping the bitmap from its original grid to the newly constructed grid.

cell definitions (cell coordinates used below are relative to this new grid), as illustrated on fig 4.5. While this action potentially changes the resolution of a bitmap (more cells defining the information, which might give the impression there is more accuracy), its overall appearance isn't altered by this: cells either inherit their membership grade from the original bitmap (so the points in the cell maintain the same value), or are assigned 0 if they cover regions not covered by the original bitmap.

Intersection

One way of combining data of multiple bitmaps is by considering their intersection. If the fuzzy bitmaps $\tilde{A}$ and $\tilde{B}$ model features 1 and 2 respectively, the intersection of both bitmaps will model the regions where both features 1 and 2 are present. The first step is to match the grids, so that both $\tilde{A}$ and $\tilde{B}$ make use of the same grid G. This will also be the grid used by the result.

In the theoretical approach, definition 54 states that each point p in the intersection is assigned a new membership grade: $\mu_{\tilde{A}\tilde{\cap}\tilde{B}}(p) = T(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p))$.

The intersection on fuzzy bitmaps should yield this result. As before, the intersection is performed by a T-norm operator (e.g. the minimum). The intersection on bitmaps is defined on a per cell basis.

Definition 82 (intersection of two fuzzy bitmaps)

$$\tilde{A}\tilde{\cap}\tilde{B} = \{(c, \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(c)) \mid c \in G\} \tag{4.3}$$

where

$$\begin{array}{rcl} \mu_{\tilde{A}\tilde{\cap}\tilde{B}} \to G & \to & [0,1] \\ c & \mapsto & T(\mu_{\tilde{A}}(c), \mu_{\tilde{B}}(c)) \end{array}$$

Verification Consider a point p, by definition $\exists c \in G : p \in c$. By definition, the membership grade $\mu_{\tilde{A}}(p) = \mu_{\tilde{A}}(c)$ and $\mu_{\tilde{B}}(p) = \mu_{\tilde{B}}(c)$.

$$\begin{array}{rcl} \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(p) & = & T(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p)) \\ & = & T(\mu_{\tilde{A}}(c), \mu_{\tilde{B}}(c)) \\ & = & \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(c) \end{array}$$

□

Implementation For the pseudo code, some assumptions are made. First, it is assumed there is a structure `Fuzzy_Bitmap`, which has an attribute `grid`. It also has a method `mu` which takes one argument, a cell, and returns the value associated with this cell. The function `MatchGrids` takes two grids as argument, and returns the single grid that is obtained by combining the grids, as described in the previous section. The function `ChangeGrid` takes a bitmap and a grid as argument, and returns a bitmap that is defined on the given grid; but technically holds all the same membership grades. The limitation is that the given grid must be a partition of the original grid. The pseudo code in this work is also not optimal. The reason for this is that any implementation will most likely be made inside a database or GIS, in which many data structures and algorithms will be present and can be reused. Optimizations will be dependent on the framework in which the implementation is made.

The implementation in pseudo code is given below.

```
Fuzzy_Bitmap Intersection (Fuzzy_Bitmap A, Fuzzy_Bitmap B)
  result, A1, B1: Fuzzy_Bitmap
BEGIN
  result.grid = MatchGrids(A.grid, B.grid)
  A1 = ChangeGrid(A, result.grid)
  B1 = ChangeGrid(B, result.grid)

  for each cell c in result.grid
```

```
    result.mu(c) = T(A1.mu(c),B1.mu(c))
  end for

  return result
END
```

Union

The union of two bitmaps can be used to yield the regions where one of two features (each modelled by its own bitmap) occurs.

This operator is performed by a T-conorm (e.g. maximum), but again, as the operator is applied on a per cell basis, any T-conorm can be used.

Definition 83 (union of two fuzzy bitmaps)

$$\tilde{A}\tilde{\cup}\tilde{B} = \{(c, \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(c)) \mid c \in G\} \tag{4.4}$$

where

$$\begin{aligned} \mu_{\tilde{A}\tilde{\cup}\tilde{B}} : G &\rightarrow [0,1] \\ c &= S(\mu_{\tilde{A}}(c), \mu_{\tilde{B}}(c)) \end{aligned}$$

Verification This definition yields the desired behaviour, as presented in 55. Consider a point p, by definition $\exists c \in G \mid p \in c$. By definition, the membership grade $\mu_{\tilde{A}}(p) = \mu_{\tilde{A}}(c)$ and $\mu_{\tilde{B}}(p) = \mu_{\tilde{B}}(c)$.

$$\begin{aligned} \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(p) &= S(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p)) \\ &= S(\mu_{\tilde{A}}(c), \mu_{\tilde{B}}(c)) \\ &= \mu_{\tilde{A}\tilde{\cup}\tilde{B}}(c) \end{aligned}$$

□

Implementation The pseudo code is very similar to the implementation of the intersection.

```
Fuzzy_Bitmap Union(Fuzzy_Bitmap A, Fuzzy_Bitmap B)
  result, A1, B1: Fuzzy_Bitmap
BEGIN
  result.grid = MatchGrids(A.grid, B.grid)
  A1 = ChangeGrid(A, result.grid)
  B1 = ChangeGrid(B, result.grid)

  for each cell c in result.grid
    result.mu(c) = S(A1.mu(c),B1.mu(c))
  end for
```

```
  return result
END
```

Complement

The complement of a fuzzy region $\tilde{A}$ yields a new fuzzy region, representing the complement of the original region. If $\tilde{A}$ is limited to a region of interest R, the complementary region will cover the considered universe U to which the GIS was limited (4.2). A new grid G' needs to be defined, so that

- $\bigcup_{c' \in G'} c' = U$
- $\forall c \in G, \exists c' \in G' : c = c'$

Definition 84 (complement of a fuzzy bitmap)

$$co\tilde{A} = \{(c', \mu_{co\tilde{A}}) \mid c' \in G'\} \tag{4.5}$$

where

$$\begin{array}{rcl} \mu_{\tilde{A}\tilde{\cup}\tilde{B}} : G' & \rightarrow & [0,1] \\ c' & \mapsto & \left\{ \begin{array}{ll} 1 - \mu_{\tilde{A}}(c) & \textit{if } \exists c \in G \mid c = c' \\ 1 & \textit{if } \nexists c \in G \mid c = c' \end{array} \right. \end{array}$$

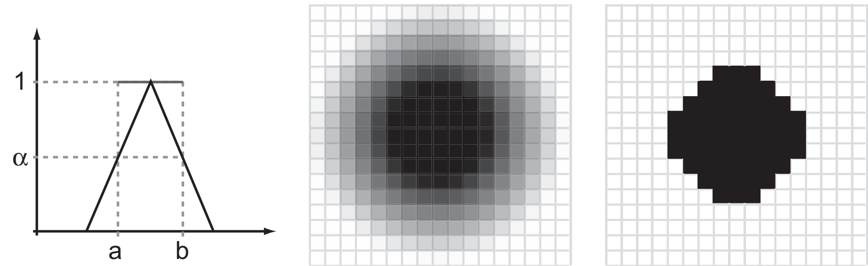
Verification This definition yields the desired behaviour, as presented in 56. Consider a point p, by definition $\exists c \in G \mid p \in c$. By definition, the membership grade $\mu_{\tilde{A}}(p) = \mu_{\tilde{A}}(c)$.

- For cells $c' \in G'$ for which $\exists c \in G : c = c'$

$$\begin{array}{rcl} \mu_{co\tilde{A}}(p) & = & \mu_{co\tilde{A}}(c) \\ & = & 1 - \mu_{\tilde{A}}(c) \\ & = & 1 - \mu_{\tilde{A}}(p) \end{array}$$

- For cells $c' \in G'$ for which $\nexists c \in G : c = c'$ Points not in the region of interest are considered to have a membership grade 0. By definition, the membership grade in the complement should be 1, which - by definition of the complement of a fuzzy bitmap - is the case.

□



(a) (b) (c)

Figure 4.6: Alphacuts for bitmaps.

Implementation The pseudo code is very similar to the implementation of the intersection and union. The new grid can easily be found by considering the universe as a bitmap consisting of one big cell (its membership grade is of no importance).

```
Fuzzy_Bitmap Complement(Fuzzy_Bitmap A)
  result, A1: Fuzzy_Bitmap
BEGIN
  result.grid = MatchGrids(A.grid, U)
  A1 = ChangeGrid(A, result.grid)

  for each cell c in result.grid
    result.mu(c) = 1-A1.mu(c)
  end for

  return result
END
```

4.4.2 Fuzzy operations

α-cut

When working with fuzzy structures, at some point there will be the need to *defuzzify* information, which implies there must be means to revert to crisp data. This can be needed for instance to display the results, but also to make it possible for a fuzzy model to be exported to a system that has no support for fuzzy models. As many extensions of geographic operators presented here make use of α-cuts, they are considered first.

Traditionally in fuzzy set theory, the α-cut is used for defuzzification: the α-cut of a fuzzy set is a regular set containing all the elements which have a membership grade greater than a given threshold. Elements whose membership grade is not greater than this threshold are not in the result set. On figure 4.6a is an illustration of this.

In the bitmap model, the first step of α-cut takes a fuzzy bitmap as argument (figure 4.6b) and results in a new fuzzy bitmap which only has associated values 0 or 1. This is illustrated on figure 4.6c. The resulting bitmap will share the same grid as the bitmap used as argument:

$$\tilde{G}_{result} = G_{orig} \tag{4.6}$$

In fuzzy set theory, a difference is made between a *strong α-cut* and *weak α-cut*; this difference is also reflected in our model.

Weak α-cut The *weak α-cut* of a fuzzy set returns the elements with a membership grade greater than or equal to a given threshold.

$$\tilde{B}_{\alpha} = \{(c_j, 1) \mid \mu_{B_{orig}}(c_j) \geq \alpha \wedge c_j \in G\} \tag{4.7}$$

core A special case of a weak α-cut is the *core*; this is the weak α-cut with threshold 1. This is an important alpha cut, as it results all the elements that fully belong to the fuzzy set.

$$\tilde{B}_{1} = \{(c_j, 1) \mid \mu_{B_{orig}}(c_j) \geq 1 \wedge c_j \in G\} \tag{4.8}$$

Strong α-cut The *strong α-cut* of a fuzzy set returns the elements with a membership grade strictly greater than a given threshold:

$$\tilde{B}_{\overline{\alpha}} = \{(c_j, 1) \mid \mu_{B_{orig}}(c_j) > \alpha \wedge c_j \in G\} \tag{4.9}$$

Support Similarly to the weak α-cut, the strong α-cut has a special case, now for a threshold equalling 0. This α-cut is called the *support*, and returns all the elements that belong to some extent to the given fuzzy set.

$$\tilde{B}_{\overline{0}} = \{(c_j, 1) \mid \mu_{B_{orig}}(c_j) > 0 \wedge c_j \in G\} \tag{4.10}$$

Verification To reflect the α-cut as defined in 2.4.1, the above definitions should have the same result on all points. The verification will be performed on the strong α-cut, the weak α-cut can be verified similarly. For fuzzy bitmaps, a cell c is in the α-cut if and only if its associated membership grade is greater than a given α; similarly, for fuzzy regions, p is in the α-cut if and only if its associated membership grade is greater than a given α. All points p in a cell c are considered to have the same membership grade.

$$\begin{aligned} c \in \tilde{A}_{\alpha} &\Leftrightarrow \mu_{\tilde{A}}(c) > \alpha \\ &\Leftrightarrow \forall p \in c : \mu_{\tilde{A}}(p) > \alpha \\ &\Leftrightarrow \forall p \in c : p \in \tilde{A}_{\alpha} \end{aligned}$$

□

Implementation The practical computation for the α-level of a fuzzy region is very straightforward.

```
Fuzzy_Bitmap BMP_weak_alpha (Fuzzy_Bitmap A, real Alpha)
  result: Fuzzy_Bitmap;

  result.grid = A.grid; // same grid => same number of cells

  for each cell c in fuzzy bitmap A
    if A.mu(c)c >= alpha
      result.mu(c) = 1
    else
      result.mu(c) = 0
  end for

  return result
END
```

For a strong α cut, it suffices to change the $\geq$ to $>$ in the `if`-test. For the kernel and the support, it suffices to replace `alpha` by respectively 1 or 0 in the respective algorithm for the weak or strong α-cut.

Finding the outline Traditionally, crisp regions are not represented using bitmaps, but using lines, polygons, etc. (1.1.2). While the result obtained above only holds crisp information, it is still not represented as a traditional crisp region. To achieve this, the α-level should be represented by polygons (with holes if necessary). This representation can be obtained by considering the outline of each cell of the bitmap representing the α level, that has an associated value 1. Each outline is a polygon (by definition, most likely a rectangle); the union of these polygons will yield a more traditional representation for the α-level.

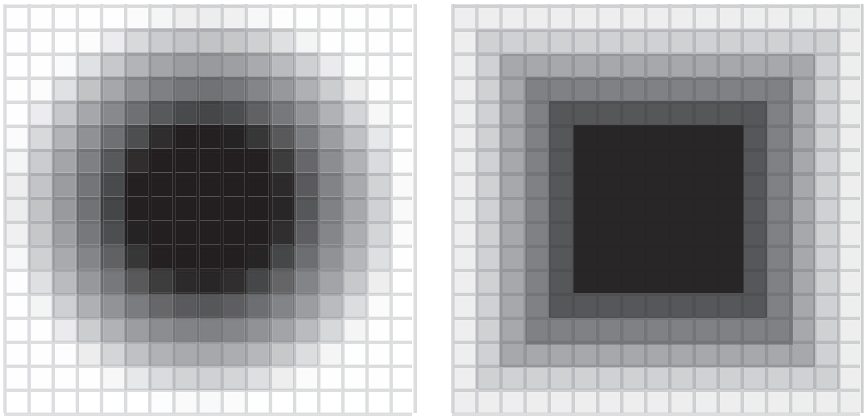
Verification This definition yields the desired behaviour, as presented in (2.4.1). Consider a point p, by definition $\exists c \in G \mid p \in c$. By definition, the membership grade $\mu_{\tilde{A}}(p) = \mu_{\tilde{A}}(c)$.

$\square$

4.4.3 Geo-spatial

Minimum bounding rectangle

In traditional GIS, a minimum bounding rectangle of a polygon is the smallest rectangle that can contain the polygon, and of which the sides are parallel to the axes used ([39]). This concept can be used for a number of purposes, ranging from determining the relative position of two features to optimizing operators (i.e. if the MBRs of two regions don't overlap, the regions don't overlap). For



(a) (b)

Figure 4.7: The fuzzy MBR of a bitmap.

a fuzzy region, two variants of the concept of an MBR are considered. The first is a fuzzy minimum bounding rectangle, which results in a fuzzy defined rectangle (i.e. another bitmap structure); the second requires an alpha level, and results in a crisp rectangle bounding this alpha level.

The concept of the fuzzy minimum bounding rectangle as introduced here should not be confused with the fuzzy minimum bounding rectangle defined in [45], where the authors define both a minimum bounding rectangle and the inscribed rectangle of a fuzzy region, along with a number of intermediate rectangles, in order to approximate this region.

Fuzzy minimum bounding rectangle A fuzzy MBR will yield a fuzzy bitmap with the same grid as the original fuzzy bitmap. The fuzzy MBR will be a new fuzzy bitmap where every α-cut is rectangular; it is defined such that these rectangular α-cuts are MBRs for the same α-cuts of the original fuzzy bitmap. This is illustrated on figure 4.7: figure 4.7a shows the original fuzzy bitmap, figure 4.7b shows its fuzzy MBR.

Definition 85 (fuzzy minimum bounding rectangle $\tilde{mbr}(\tilde{A})$)
(of a fuzzy bitmap region $\tilde{A}$)

$$\tilde{mbr}(\tilde{A}) = \{(c, \mu_{\tilde{mbr}(\tilde{A})})\} \tag{4.11}$$

where

$$\begin{aligned} \mu_{\tilde{mbr}(\tilde{A})} : U &\rightarrow \ ]0,1] \\ c &\mapsto \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid c \in mbr(\tilde{A}_{\alpha_i})\} \end{aligned}$$

Here, the $mbr(\tilde{A}_{\alpha_i})$ represents the minimum bounding rectangle - made up from cells - that bounds all the cells with membership grade 1 *(the α level of a region only has cells with values in* $\{0, 1\}$.

Verification To verify that this definition matches the definition for fuzzy regions 2.4.2, it suffices to verify that all points using the above definition have the same membership grades as when using the theoretical definition.

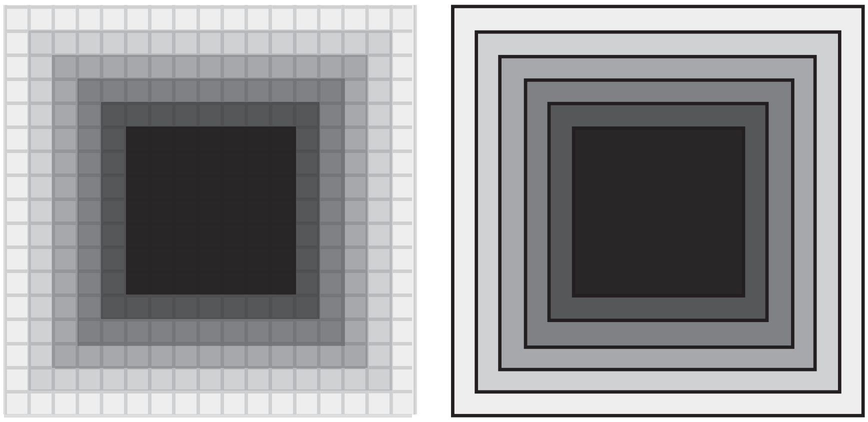
$$\begin{aligned}
\mu_{\tilde{mbr}(\tilde{A})}(c) = x \quad &\Leftrightarrow \quad \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid c \in mbr(\tilde{A}_{\alpha_i})\} = x \\
&\Leftrightarrow \quad \forall p \in c : \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid p \in mbr(\tilde{A}_{\alpha_i})\} = x \\
&\Leftrightarrow \quad \forall p \in c : \mu_{\tilde{mbr}(\tilde{A})}(p) = x
\end{aligned}$$

□

Implementation As a bitmap holds a finite number of cells, it also holds a finite number of membership grades. Consequently, only the grades present in the bitmap need to be considered when determining α-cuts to define the fuzzy MBR: other values will not add any more information.

Consider all these α-cuts. For each α-cut, a bitmap-MBR can be defined: a rectangle represented by a bitmap made of cells such that all cells belonging to this α-cut are inside the rectangle and no smaller rectangle can be defined. Such a rectangle can be considered for each α-level, and each rectangle can be considered as a bitmap. The union of all these (overlapping) bitmaps - as explained in section 4.4.1 - yields a new bitmap. The construction of this bitmap-MBR is explained below in pseudo-code.

```
Fuzzy_Bitmap Fuzzy_MBR (Fuzzy_Bitmap A)
  result, A_alpha: fuzzy_bitmaps, same grid as A, all grades=0
BEGIN
  determine available alpha levels in A

  for each alpha level alpha in A
    determine A_alpha

    find the cells with grade = 1 in A_alpha that are closest to
      the left/right/top/bottom side of the grid
    use these cells to define a rectangle of cells in A_alpha:
      leftmost cell determines lefthand side of the rectangle
      rightmost cell determines righthand side of the rectangle
      topmost cell determines tophand side of the rectangle
      bottommost cell determines bottom side of the rectangle

    for all cells c in A_alpha
      if c in rectangle
```



(a) (b)

Figure 4.8: The fuzzy MBR and the polygons derived from it: (a) fuzzy MBR bitmap, (b) polygons derived from the fuzzy MBR bitmap.

```
        A_alpha.mu(c) = 1
      else
        A_alpha.mu(c) = 0
      end if

    for each cell c with A_alpha.mu(c) = 1
      if result.mu(c) < alpha
        result.mu(c) = alpha
      end if
    end for
  end for
  return result
END
```

This new bitmap has the property that its α-cuts are MBRs for the same α levels in the original bitmap, this new bitmap is considered as the fuzzy MBR. Note that membership grades in the fuzzy MBR will always be simple (53): higher grades towards the middle, lower grades towards the edge of the fuzzy bitmap.

Fuzzy minimum bounding rectangle as polygons The fuzzy MBR has the disadvantage that it still is a bitmap-structure; making it impossible for existing systems to use this information without modifications. To overcome this, the crisp MBRs for the different α levels can be deduced from the fuzzy MBR. In addition to a bitmap, the calculation of such a crisp MBR also requires an α level: this level determines the cells of the bitmap around which the MBR is considered.

The crisp MBR is easily calculated from the fuzzy MBR. First, the fuzzy MBR is determined. Next, the α-cut at the given level is considered. This will yield a bitmap with cells having an associated membership grade 1 and cells with an associated grade 0. The outline of the cells with an associated grade 1 can now be represented as a polygon (by construction it will be a rectangle), resulting in a traditional MBR. On fig. 4.8, both the fuzzy MBR bitmap and all the derived polygons are shown (the shade of the polygons is used to give an indication of the α-cut from which the polygons are derived). The pseudo code is provided below.

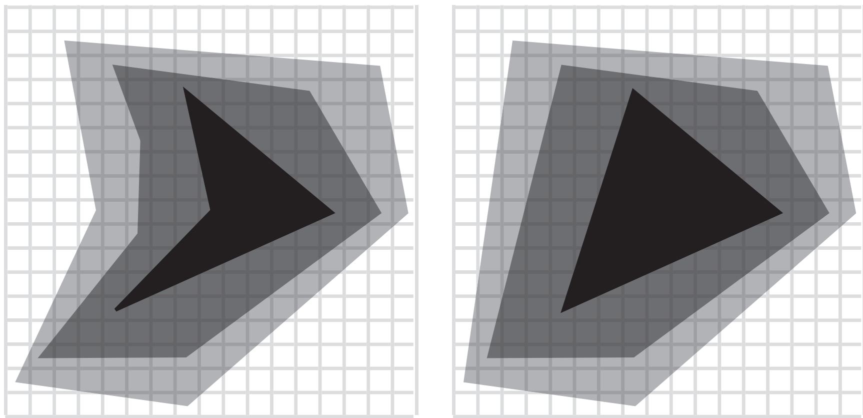
```
Polygon Fuzzy_MBR_polygon (Fuzzy_Bitmap A, real Alpha)
  A_MBR: fuzzy_bitmap, same grid as A, all grades=0
  result: Polygon, initially emtpy
BEGIN
  A_MBR = Fuzzy_MBR(A)

  A_alpha = BMP_weak_alpha(A_MBR, Alpha)

  For each cell c of A_alpha
    if A_alpha.mu(c) = 1 then
      result = result union outline(c)
  end for

  return result
END
```

Convex hull

The convex hull of a polygon ([39]) is an interesting operator in traditional GIS systems. It is commonly used to optimize other operators and tests, i.e. if the convex hulls of two polygons don't intersect, the polygons themselves don't intersect. It can also be used for indexing, similar to bounding rectangles. Even in this usage, defuzzification makes sense: the same fuzzy region can have index entries for different α-levels.

Traditionally, the convex hull of a polygon results in a new polygon; for fuzzy regions, the convex hull of a fuzzy region will result in a new fuzzy region. The approach is quite similar to the calculation of the fuzzy MBR: for every α-cut, the convex hull is considered. By recombining these results using the union operator, a new bitmap containing the fuzzy convex hull is obtained. On figure 4.9 is a simplified example (the cells aren't considered) to illustrate the concept: figure 4.9a shows a fuzzy region, its fuzzy convex hull is shown on figure 4.9b.

In order to determine the convex hull, the bitmap has to be interpreted as an approximation of a polygon. In order to find this polygon, each cell will need to be represented by a single point (such that the bitmap is the raster-representation of this polygon). This point is called the centerpoint of a cell in



(a) (b)

Figure 4.9: Simplified illustration of the concept of the fuzzy convex hull.

the algorithm; the most obvious choice for a centerpoint of a cell is the center of gravity.

Definition 86 (convex hull $\tilde{ch}(\tilde{A})$ of a fuzzy bitmap region $\tilde{A}$)

$$\tilde{ch}(\tilde{A}) = \{(c, \mu_{\tilde{ch}(\tilde{A})})\} \tag{4.12}$$

where

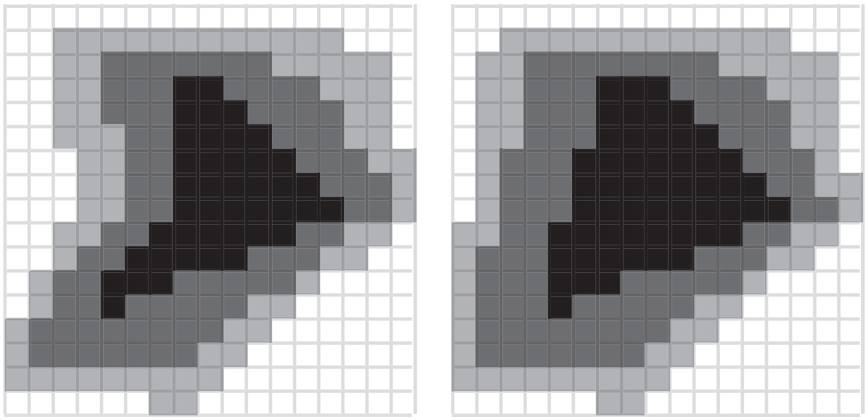
$$\begin{aligned} \mu_{\tilde{ch}\tilde{A}} : U &\rightarrow\ ]0,1] \\ c &\mapsto \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid c \in ch(\tilde{A}_{\alpha_i})\} \end{aligned}$$

Here, $ch(\tilde{A}_{\alpha_i})$ represents the convex hull - made up from cells - that bounds all the cells with membership grade 1 (the α level of a region only has cells with values in $\{0,1\}$).

Verification The proof that the above definition matches the theoretical definition 2.4.2 is analogous to the proof for the minimum bounding rectangle.

$$\begin{aligned} \mu_{\tilde{ch}(\tilde{A})}(c) = x &\Leftrightarrow \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid c \in ch(\tilde{A}_{\alpha_i})\} = x \\ &\Leftrightarrow \forall p \in c : \sup_{\alpha_i \in]0,1]} \{\alpha_i \mid p \in ch(\tilde{A}_{\alpha_i})\} = x \\ &\Leftrightarrow \forall p \in c : \mu_{\tilde{ch}(\tilde{A})}(p) = x \end{aligned}$$

□



(a) (b)

Figure 4.10: Example of a fuzzy convex hull of an fuzzy bitmap.

Implementation Similar to the calculation of the fuzzy MBR, only the α-levels at membership grades that occur in $\tilde{B}$ need to be considered.

```
Fuzzy_Bitmap Fuzzy_Convex_Hull (Fuzzy_Bitmap A)
result, temp: fuzzy_bitmaps, same grid as A, all grades=0
BEGIN
  determine available alpha levels in A

  for each alpha level x
    determine A_x

    consider the centerpoints of cells with grade=1 in A_x that
      neighbour cells with grade=0 or
      neighbour the edge of the bitmap

generate convex hull of polygon defined by these centerpoints

rasterize the polygon (using the current grid as raster)
  if a cell c belongs to the edge or the inside of the polygon
    mu_temp(c) = 1

    for each cell c with temp.mu(c) = 1
      if the result.mu(c) < x
        result.mu(c) = x
      end for
  end for
  return result
END
```

In the algorithm, the center points of cells are needed. The center point of a cell is a point that is representative for the entire cell; the center of gravity makes for a nice representative point (due to the fact that the cell is defined as a convex subset, the center of gravity will always belong to the cell).

Also in the algorithm, a rasterization-method is required. These methods are common in the realm of computer graphics, for a description of different rasterization-techniques we refer to [27],[1]. As this extends beyond the scope of this work, it will not be considered in further detail.

On figure 4.10a fuzzy bitmap is considered (it is the rasterized example of figure 4.9). The fuzzy bitmap is shown on figure 4.10b. This result might not appear to be convex, but a bitmap representation is limited in that it can only consider cells as its smallest unit. The bitmap usually is an approximation of a polygon, the convex hull of a bitmap will also be an approximated polygon. The fuzzy bitmap as constructed above has the property that at every α-level it holds the convex hull for the original bitmap (at that same α-level).

Surface area

As mentioned before, for the calculation of the surface area of a fuzzy region, there are two possible interpretations. The first is when the surface area yields a fuzzy number. The second interpretation is when the surface area is considered to be an extension of fuzzy cardinality [32]; in this case the surface area of a fuzzy region will be a crisp number. Both interpretations are considered below.

Interpretation 1: fuzzy result The calculation of the fuzzy surface area $\tilde{S}^f$ of a fuzzy bitmap $\tilde{A}$ makes use of the previously defined α-cut. Conceptually, the surface area of each weak α-cut will be used to determine the fuzzy number that represents the surface area. Similar to the calculation of the distance, first the available α-cuts are considered. In practice, only the α-cuts at membership grades present in $\tilde{A}$ will need to be considered:

$$0 < \alpha_0 < \alpha_1 < ... < \alpha_n \leq 1 \tag{4.13}$$

Each α-cut of the fuzzy region yields a crisp region; consequently, the surface area can be calculated easily.

Definition 87 (surface area $\tilde{S}^f(\tilde{A})$ of a fuzzy bitmap region $\tilde{A}$) *(yielding a fuzzy result)*

$$\tilde{S}^f(\tilde{A}) = \{(x, \mu_{\tilde{S}^f(\tilde{A})}(x))\} \tag{4.14}$$

where

$$\begin{aligned} \mu_{\tilde{S}^f(\tilde{A})}(x) : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \begin{cases} \alpha_i & \textit{if } S(\tilde{A}_{\alpha_i+1}) < x \leq S(\tilde{A}_{\alpha_i}), \forall i = 0..n-1 \\ \alpha_n & \textit{if } S(\tilde{A}_{\alpha_n}) = x \\ 0 & \textit{elsewhere} \end{cases} \end{aligned}$$

where S represents the surface area of a crisp region.

Verification To prove that this definition has the same meaning as the definition 2.13, consider a fuzzy bitmap region $\tilde{A}$ where the occurring membership grades are $0 < \alpha_0 < \alpha_1 < ... < \alpha_n \leq 1$ (by definition, the number of membership grades in a fuzzy bitmap are finite). A value s is assigned membership grade $\mu_{\tilde{S}f(\tilde{A})}(s) = \alpha_i$ if s is the surface area of $s = \tilde{A}_{\mu_{\tilde{S}f(\tilde{A})}(s)} = \tilde{A}_\alpha$ and $\alpha_{i+1} < \alpha \leq \alpha_i$.

As there are a finite number of membership grades occurring in the fuzzy bitmap, and no grades occur between α_i and α_{i+1} it stands that $S(A_{\alpha_i}) = S(A_x), \alpha_i < x \leq \alpha_{i+1}$.

$\square$

Implementation By definition there are only a finite number of cells in a bitmap, so the above definitions are suited for implementation. The implementation of the fuzzy surface area assumes there is a mechanism to work with fuzzy numbers, using a data structure called `fuzzy_real`. It also makes use of the surface area of a cell, which is computed as the crisp surface area of the polygon (usually rectangle) limiting the cell. The algorithm in pseudo code is given below.

```
real S(Fuzzy_Bitmap A)
  result: real
BEGIN
  for each cell c in A
    result = result + S(c)
  end for
  return result
END

fuzzy_real Fuzzy_surface(Fuzzy_Bitmap A)
  A_x: Fuzzy_Bitmap, same grid, initially all cells 0
  result: fuzzy_real
BEGIN
  find all alpha levels x_0, x_i, ... x_n in A

  x_0 = lowest alphalevel in A
  A_x_0 = BMP_strong_alpha(A, x_0)
  add (S(B_x_0), x_0) to result

  for each alpha level x_i>x_0
    B_x_i = BMP_strong_alpha(A, x_i)
    B_x_(i-1) = BMP_strong_alpha(A, x_(i-1))
    add y=x_i for x in]S(B_x_i), S(B_x_(i-1))] to result
  end for
  return result
END
```

Interpretation 2: crisp result As mentioned before, the fuzziness of a region can also be considered to be the result of intrinsic fuzziness. In this concept, the surface area of each cell is considered and its associated membership grade will be used to determine to which extent this area contributes to the total area.

Definition 88 (surface area $\tilde{S}^c(\tilde{A})$ of a fuzzy bitmap region $\tilde{A}$) *(yielding a crisp result)*

$$\tilde{S}^c(\tilde{A}) = \sum_{c \in G} (S(c)\mu_{\tilde{A}}(c)) \tag{4.15}$$

where G is the grid used to define $\tilde{A}$.

Verification For a fuzzy bitmap, this definition matches the theoretical definition given in 2.4.2.

$$\begin{aligned}
\tilde{S}^c(\tilde{A}) &= \sum_{c \in \tilde{A}} (S(c)\ \mu_{\tilde{A}}(c)) \\
&= \sum_{c \in \tilde{A}} \left(\int_c d(x,y)\ \mu_{\tilde{A}}(c) \right) \\
&= \sum_{c \in \tilde{A}} \left(\int_c \mu_{\tilde{A}}(c)\ d(x,y) \right) \\
&= \sum_{c \in \tilde{A}} \left(\int_c \mu_{\tilde{A}}(p)\ d(x,y) \right) \\
&= \int_{\bigcup_i c_i} \mu_{\tilde{A}}(p)\ d(x,y) \\
&= \int_U \mu_{\tilde{A}}(p)\ d(x,y)
\end{aligned}$$

This is due to the fact that the cells c_i form partition of $\tilde{A}$, meaning that:

$$\bigcup_i c_i = \tilde{A}$$

$$\bigcap_i c_i = \emptyset$$

and due to the fact that the contribution is 0 for points $p \in U \backslash \bigcup_i c_i$.

Note that a cell c is defined as a convex, bounded subset of $\mathbb{R}^2$ (Definition 78); its surface area $S(c)$ is therefore a finite number which can be denoted as $\int_c d(x,y)$.

$\square$

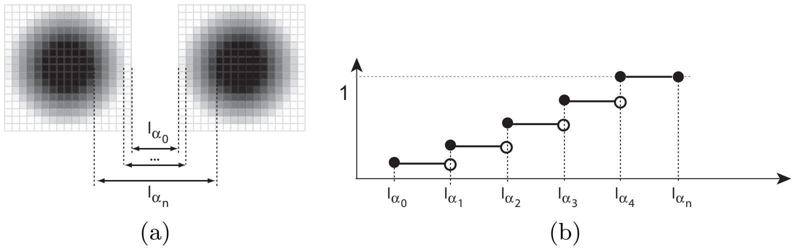


Figure 4.11: Concept of the distance between fuzzy regions.

Implementation The computation matching the above definition is given below in pseudo code.

```
real Fuzzy_surface_card(Fuzzy_Bitmap A)
  result: real
BEGIN
  for each cell c in A
    result = result + S(c)*mu_A(c)
  end for
  return result
END
```

As above, the surface area of a cell is computed as the area of the polygon defining it.

Distance calculation

As in the conceptual model (2.4.2), the distance can be interpreted differently, the two interpretations considered before, are both considered here.

α-level approach In the α-level approach, the distance between two regions is obtained by considering the possible distances between the cells of the α-cuts for both regions, and this for all α. This is illustrated on fig 4.11.

In order to extend the distance operator, first all the membership grades occurring in both arguments must be considered:

$$0 < \alpha_0 < \alpha_1 < ... < \alpha_n \leq 1 \tag{4.16}$$

where $\forall \alpha \in \{\alpha_0, \alpha_1, ...\alpha_n\} \exists c \in \tilde{A} \cup \tilde{B} : \mu_{\tilde{A}}(c) = \alpha \vee \mu_{\tilde{B}}(c) = \alpha$. Along with each of these α-values, l_α can be defined; l_α is the shortest distance between the α-levels of the bitmaps:

$$l_\alpha = min\left(d(p_1, p_2), \forall p_1 \in \tilde{A}_\alpha \wedge p_2 \in \tilde{B}_\alpha\right) \tag{4.17}$$

The distance is considered between all cells that belong to this α-level in each of the bitmaps; l_α is defined for all $\alpha \in \{\alpha_0, \alpha_1, ...\alpha_n\}$. The distance l_{α_0} is the shortest distance that occurs; the distance l_{α_n} is the longest.

The definition for the distance between two fuzzy regions is then defined using the l_α values as:

Definition 89 (distance $\tilde{d}(\tilde{A},\tilde{B})$ between two fuzzy bitmaps)
(in an α-level approach)

$$\tilde{d}(\tilde{A},\tilde{B}) = \{(x, \mu_{\tilde{d}(\tilde{A},\tilde{B})}(x))\} \tag{4.18}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \begin{cases} \alpha_i & \textit{if } \ l_{\alpha_i} \leq x < l_{\alpha_{i+1}}, \forall i = 0,1,...,n-1 \\ \alpha_n & \textit{if } \ x = l_{\alpha_n} \\ 0 & \textit{elsewhere} \end{cases} \end{aligned}$$

To calculate the distance to a crisp object (a point, line or polygon), it suffices to consider only the closest point from the fuzzy bitmap to this object. The crisp object can be treated as a fuzzy object; as all points are then assigned membership grade 1, it suffices to only consider the different α levels that occur in the fuzzy bitmap.

Verification To verify that the above definition matches the definition given in 2.4.2, consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$, and the distance represented by the fuzzy real number d. It is now necessary to prove that $\forall x \in \mathbb{R} : \mu_{\tilde{d}}(x) = sup\{\alpha : d(\tilde{A}_\alpha, \tilde{B}_\alpha) = x\}$.

In the conceptual definition (definition 61), x is assigned the largest α for which there are two points $p_1 \in \tilde{A}_\alpha$, $p_2 \in \tilde{B}_\alpha$ that have a distance $d(p_1,p_2) = x$. As in the bitmap only α-levels $\alpha_i, i = 0,1,...,n$ occur, only these levels need to be considered; suppose this largest α is α_i. Consequently, $p_1 \in \tilde{A}_{\alpha_i}$, but $p_1 \notin \tilde{A}_{\alpha_{i+1}}$; similarly $p_2 \in \tilde{B}_{\alpha_i}$, but $p_2 \notin \tilde{B}_{\alpha_{i+1}}$.

- is $d(p_1,p_2) \geq l_{\alpha_i}$?
  l_{α_i} is defined as the minimal distance between any two points in $\tilde{A}_{\alpha_i}$ and $\tilde{B}_{\alpha_i}$, this implies that $d(p_1,p_2) \geq l_{\alpha_i}$.

- is $d(p_1,p_2) < l_{\alpha_{i+1}}$?
  Suppose $d(p_1,p_2) \geq l_{\alpha_{i+1}}$. By definition of l_α, this would imply $p_1 \in \tilde{A}_{\alpha_{i+1}}$ and $p_2 \in \tilde{B}_{\alpha_{i+1}}$. This is false.

Consequently, $l_{\alpha_i} \leq d(p_1,p_2) < l_{\alpha_{i+1}}$ and $\mu_{\tilde{d}}(d(p_1,p_2)) = \alpha_i$.

$\square$

Implementation The pseudo code to implement the above definition is given below

```
fuzzy real Fuzzy_Distance (fuzzy_bitmap A, fuzzy_bitmap B)
  fuzzy real result
begin
  determine all available alpha levels in A and B
  n = number of alpha levels

  for each available alpha alpha_i
    l_alpha_i = crisp d(A_alpha_i, B_alpha_i)
  end for

  for all i = 0 to n-1
    add y = alpha_i for x in [l_alpha_i, l_alpha_(i+1)[ to result
  end for
  add y = alpha_n for x = l_alpha_n in result

  return result
end
```

Topological approach The topological approach, as described as a concept in 2.4.2, makes use of the notion of the fuzzy boundary (for bitmaps, see 4.4.4). The distance to a fuzzy region is considered to be the distance (in the α-level approach) to its fuzzy boundary.

Definition 90 (distance $\tilde{d}^{\triangle}(\tilde{A}, \tilde{B})$ between two fuzzy bitmaps)
(in a topological approach)

$$\tilde{d}^{\triangle}(\tilde{A}, \tilde{B}) = \{(x, \mu_{\tilde{d}^{\triangle}(\tilde{A},\tilde{B})}(x))\} \tag{4.19}$$

where

$$\begin{aligned} \mu_{\tilde{d}^{\triangle}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] && (4.20)\\ x &\mapsto \mu_{\tilde{d}(\Delta\tilde{A},\Delta\tilde{B})}(x) && (4.21)\end{aligned}$$

where $\Delta(\tilde{A})$ is the notation for the boundary of $\tilde{A}$, as defined conceptually in 2.4.3 and specifically for bitmaps in 4.4.4:

$$\Delta\tilde{A} = \bigcup_{\alpha\in]0,1]} \{(c, 2(0.5 - |0.5 - \alpha|))|c \in \partial A_\alpha\}$$

For some applications, the topology approach feels more natural: for points that have membership grade greater than 0.5, the distance starts decreasing. This makes sense, as these points actually belong more to the region than not.

4.4.4 Topology

The topology concepts of a fuzzy bitmap are defined analogous to the theoretical definition 62. Each of the concepts boundary, interior and exterior of a fuzzy bitmap yields a new fuzzy bitmap which uses the same grid as the original bitmap.

Boundary

Definition 91 (boundary $\Delta\tilde{A}$ of a fuzzy region $\tilde{A}$)

$$\Delta\tilde{A} = \bigcup_{\alpha\in]0,1]} \{(c, 2(0.5 - |0.5 - \alpha|))|c \in \partial A_\alpha\} \tag{4.22}$$

implementation

```
fuzzy_bitmap Fuzzy_Boundary(fuzzy_bitmap A)
  result: fuzzy_bitmap, same grid as A, all grades=0
BEGIN
  for each cell c in A
    add c to result
    result.mu(c) = 2*(0.5-abs(0.5-A.mu(c)))
  end for

  return result
END
```

Interior

Definition 92 (interior $\tilde{A}^\circ$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^\circ = \{(c, \mu_{\tilde{A}^\circ}(c))\} \tag{4.23}$$

where

$$\begin{array}{rcl} \mu_{\tilde{A}^\circ} : G & \rightarrow & [0,1] \\ c & \mapsto & \begin{cases} 0 & \mu_{\tilde{A}}(c) \leq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(c) & \textit{elsewhere} \end{cases} \end{array}$$

implementation

```
fuzzy_bitmap Fuzzy_Interior(fuzzy_bitmap A)
  result: fuzzy_bitmap, same grid as A, all grades=0
BEGIN
  for each cell c in A
    add c to result
    if A.mu(c) <= 0.5
      result.mu(c) = 0
    else
      result.mu(c) = 1-2*(0.5-abs(0.5-A.mu(c)))
    end if
```

```
  end for

  return result
END
```

Exterior

The exterior can be defined similarly to the interior. Remark that if $\tilde{A}$ is limited to a region of interest R, the exterior will cover the universe U as considered by the GIS 4.2 (this is similar to the remark for the complement in 4.4.1). A new grid G' needs to be defined, so that

- $\bigcup_{c' \in G'} c' = U$
- $\forall c \in G, \exists c' \in G' : c = c'$

The exterior of fuzzy bitmap $\tilde{A}$ can now be considered on this new grid:

Definition 93 (exterior $\tilde{A}^-$ of a fuzzy region $\tilde{A}$)

$$\tilde{A}^- = \{(c', \mu_{\tilde{A}^-}(c'))\} \tag{4.24}$$

where

$$\begin{aligned} \mu_{\tilde{A}^-} : G' &\rightarrow [0,1] \\ c' &\mapsto \begin{cases} 0 & \textit{if } \exists c \in G : c = c' \wedge \mu_{\tilde{A}}(c) \geq 0.5 \\ 1 - \mu_{\Delta\tilde{A}}(c) & \textit{if } \exists c \in G : c = c' \wedge \mu_{\tilde{A}}(c) < 0.5 \\ 1 & \forall c \in G : c \neq c' \end{cases} \end{aligned}$$

implementation

```
fuzzy_bitmap Fuzzy_Exterior(fuzzy_bitmap A)
  result: fuzzy_bitmap, same grid as A, all grades=0
BEGIN
  result.grid=MatchGrids(Fuzzy_Bitmap A, U)
  result.grid = A.grid

  for each cell c in result
    if A.mu(c) >= 0.5
      result.mu(c) = 0
    else
      result.mu(c) = 1-2*(0.5-abs(0.5-A.mu(c)))
    end if
  end for

  return result
END
```

Using the above concepts, the topology can be determined completely analogous to the methodology described in 2.4.3: for any two regions, an intersection matrix can be constructed. The interpretation of the different intersection matrices, has been dealt with in 2.4.3.

4.5 Extensions

As the bitmap model serves as an implementation-model for fuzzy regions, the extensions considered in 2.5 also need to be considered here.

4.5.1 Fuzzy locations

As mentioned in 2.5.1, fuzzy locations can be represented by fuzzy points, which in turn are fuzzy regions with a possibilistic interpretation instead of the veristic interpretation. The two dimensional fuzzy set is then interpreted as a set of "possible locations for the imprecise or uncertain point". The definition is completely similar to the one before.

Definition 94 (fuzzy point $\tilde{p}$ using bitmap B and grid G)

$$\tilde{p}^B = \{(c_j, \mu_{\tilde{p}^B}(c_j))\} \tag{4.25}$$

where

$$\begin{aligned} \mu_{\tilde{p}^B} : G &\rightarrow [0,1] \\ c &\mapsto \mu_{\tilde{p}^B}(c) \end{aligned}$$

Operations

The overview of operations on fuzzy regions that remain meaningful on fuzzy points was provided in the conceptual model. For completeness, this overview is also repeated here.

Set-operations The set operations can be used to combine information from different sources regarding the modelled location, for instance one bitmap could model a source indicating that the location is close to a city; another bitmap could model a different source indicating that the location is close to a given road. The intersection of both will yield the possible locations that satisfy both criteria.

α-cut operation Determination of the α-cut is also still meaningful, and can be used to determine the crisp region in which the locations have at least a given membership grade.

Minimum bounding rectangle and convex hull Both the minimum bounding rectangle and the convex hull can be used to approximate the outline of the region of possible locations.

Surface area calculation By definition, the surface area of a single point is 0. The definition of the surface area for fuzzy regions can still be used to serve as an indication over which area the imprecisely known or inaccurately known point is located. It does not indicate the surface area of the point.

Distance calculation The definition of the distance between fuzzy points differs from the definition of the distance between fuzzy regions. The reason for this is that for fuzzy regions the distance could not exceed the distance between both cores (the definition between regions is the minimum definition between them). The imprecisely known or inaccurately known point can however be positioned in any point of the region. This will also be reflected in the definition of the distance between two fuzzy points. The distance is considered to be a possibility distribution, where for each possible distance between points of both regions, a possibility is assigned. This possibility is the largest α-value such that the distance is still present it the α-levels of both bitmaps.

Consider cells c_A used in point $\tilde{p}^A$, and c_B used in point $\tilde{p}^B$. For every combination of cells c_A and c_B, the following values can be defined:

$$d_{min}(c_A, c_B) = \inf\{d(p_1, p_2) : p_1 \in c_A, p_2 \in c_B\} \tag{4.26}$$

$$d_{max}(c_A, c_B) = \sup\{d(p_1, p_2) : p_1 \in c_A, p_2 \in c_B\} \tag{4.27}$$

The distance between two fuzzy points is then defined as:

Definition 95 (distance $\tilde{d}(\tilde{A}, \tilde{B})$ between two fuzzy bitmaps)
(for fuzzy bitmaps that represent fuzzy points)

$$\tilde{d}(\tilde{p}^A, \tilde{p}^B) = \{(x, \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)}(x))\} \tag{4.28}$$

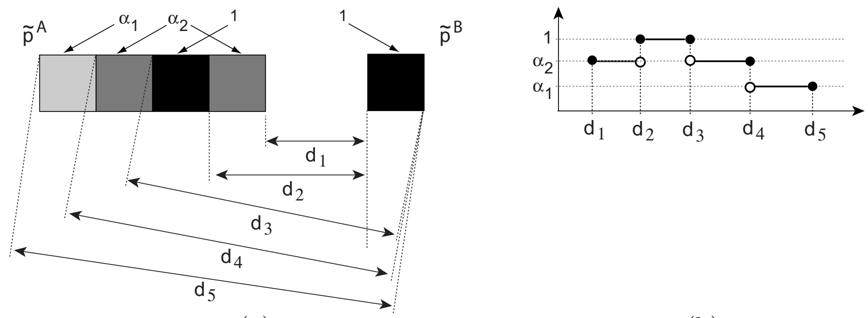
where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{p}^A, \tilde{p}^B)} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \sup_{\forall c_A, c_B} \begin{array}{l} \{\alpha \mid \alpha = \min(\mu_{\tilde{p}^A}(c_A), \mu_{\tilde{p}^B}(c_B)) : \\ \quad d_{min}((c_A, c_B)) \leq x \leq d_{max}((c_A, c_B))\} \end{array} \end{aligned}$$

where c_A and c_B represent cells in respectively $\tilde{p}^A$ and $\tilde{p}^B$, and where the notation $\mu_{\tilde{p}^X}(c_X)$ is used to represent the membership grade associated with cell c_X in bitmap X.

The above definition is illustrated on fig. 4.12. For fuzzy points represented by complex fuzzy bitmaps, the remark is similar as for the conceptual fuzzy points (2.5.1).

A downside to representing fuzzy points by using fuzzy bitmaps, is that it is not possible to pinpoint a single location or even a one-dimensional structure (e.g. along a line): the smallest unit known to the bitmap is a cell. Within a cell, no further details can be modelled regarding the fuzzy point.



(a) (b)

Figure 4.12: Illustration of the fuzzy distance between fuzzy points: (a) fuzzy points $\tilde{p}^A$ and $\tilde{p}^B$ (illustrated using grey scales, (b) the fuzzy distance between $\tilde{p}^A$ and $\tilde{p}^B$.

Topology The topology relations for fuzzy points can be used similarly to the topology relations for fuzzy regions. The result should not be interpreted as a topologyical relation on the points as such, but on the regions delimiting the points.

4.5.2 Regions at fuzzy positions

In the current approach for fuzzy regions, fuzzy regions are considered to be at crisp positions: the membership grade indicates to which each point belongs to the region. It can however be interesting to represent a region (either crisp or fuzzy) for which its position is unknown or uncertain: a region at a fuzzy position. This could for instance be used to model possible locations for building a new airport, where the uncertainty of the position stems from environmental limitations (noise pollution, wildlife areas, safety zones, ...).

The concept of modelling this, using a first fuzzy region to model the position of an anchor point, and a second fuzzy region to model the region itself, has been mentioned in 2.5.2. The concept is similar here, but with the above remark for fuzzy points using a fuzzy bitmap (4.5.1) that it is impossible to limit anchor point represented by a fuzzy bitmap to either a single location or even to a one-dimensional line.

As mentioned in 2.5.2, the combination of both the uncertainty concerning the location, and the imprecision concerning the region will yield level-2 fuzzy sets.

4.5.3 Fuzzy associated data

Similar to the conceptual fuzzy regions, fuzzy regions represented by bitmaps can be used also for fuzzy associated data. In this case, the region is no longer considered to be fuzzy, but the data associated with each cell is comprised of a fuzzy set. Bitmaps that are adopted for this use, are referred to a *extended* *bitmaps*. The associated data can be a fuzzy set over a number of domains (e.g. soil types, rock composition, vegetation, real numbers, etc.). The example for real numbers is given below.

Definition 96 (extended bitmap with fuzzy real numbers $B^{\tilde{\mathbb{R}}}$) *(using grid G)*

$$B^{\tilde{\mathbb{R}}} = \{(c_j, f_(c_j))\} \tag{4.29}$$

where

$$\begin{array}{rcl} f : G & \to & \tilde{\wp}(\mathbb{R}) \\ c & \mapsto & f(c) \end{array}$$

Here, G is the grid containing the cells c and $\tilde{\wp}(\mathbb{R})$ the set of all fuzzy sets over the domain $\mathbb{R}$.

With each cell, the associated fuzzy set is obtained through measurements, estimations or approximations. Traditionally, bitmaps are an approximation; the data associated with each cell is usually computed from a number of sample points in this cell. When using extended bitmaps, and thus allowing for fuzzy sets to be modelled, the associated data for each cell becomes a fuzzy set. This fuzzy set can be used to take into account the fact that the data stems from an approximation.

Type-2 fuzzy bitmaps

The above extension can be modified to accommodate type-2 fuzzy sets, as mentioned in the conceptual model.

Definition 97 (type-2 fuzzy bitmap region region $\tilde{\tilde{B}}$) *(using grid G)*

$$\tilde{\tilde{B}} = \{(c, \mu_{\tilde{\tilde{B}}}(c))\} \tag{4.30}$$

where

$$\begin{array}{rcl} \mu_{\tilde{\tilde{B}}} : G & \to & \tilde{\wp}([0,1]) \\ c & \mapsto & \mu_{\tilde{\tilde{B}}}(p) \end{array}$$

Here, G is the grid containing the cells c and $\tilde{\wp}([0,1])$ the set of all fuzzy sets over the domain $[0,1]$. The membership grade $\mu_{\tilde{\tilde{B}}}(c)$ expresses the extent to which p belongs to the fuzzy region using a fuzzy membership grade.

Similarly, type-2 fuzzy points can be defined.

Membership grades are assigned with cells based on sample values, estimations, etc. By means of type-2 fuzzy sets, the fact that the single associated value for the cell (a fuzzy set in this case) usually stems from multiple different fuzzy sets for various sample points inside the cell, can be modelled more accurately: fuzzy aggregation on the membership grades of the sample points can yield a more representative fuzzy set for the value associated with the considered cell.

Possibilistic truth value regions

The bitmap model for a region representing possiblisitic truth values, is defined similarly to the type-2 fuzzy bitmaps.

Definition 98 (PTV extended bitmap $B^{\widetilde{\{T,F\}}}$)
(using grid G)

$$B^{\widetilde{\{T,F\}}} = \{(c_j, f_(c_j))\} \tag{4.31}$$

where

$$\begin{aligned} f : G &\rightarrow \tilde{\wp}(\{True, False\}) \\ c &\mapsto f(c) \end{aligned}$$

Here, G is the grid containing the cells c and $\tilde{\wp}(\{True, False\})$ the set of all fuzzy sets over the domain $\{True, False\}$.

Chapter 5

Fuzzy regions using Triangulated Irregular Networks

5.1 Concept

A fuzzy region is defined as a fuzzy set over the two dimensional domain. To make this concept workable, the model needs to be simplified. In fuzzy bitmaps (chapter 4), this simplification is done by limiting the domain to a finite set. In this chapter, a simplification using triangulated irregular networks is presented. Unlike the bitmaps, this approach - like the conceptual model in chapter 2- still uses an infinite two dimensional domain. The triangulated irregular network (or TIN for short) method simplifies on the conceptual model by changing the way information is defined: only a limited number of points (the vertices) are defined; information for other points is calculated from this limited number of points.

As mentioned in 1.1.3, triangulated irregular networks are a known structure in GIS. With a number of locations (no limitation is imposed on either the number of the locations or on the actual position of each location), data is associated. Using this set of locations (referred to as *data points*), a triangular network is constructed (this usually is a Delaunay triangulation, see 1.1.3). For fuzzy regions, the associated data with the triangular network are membership grades, with a veristic interpretation (all points belong to the set, but some to a greater degree than others).

5.2 Definition

Given a set of data points, a triangular network can be constructed. This implies that both the edges and triangles need to be determined. A commonly used triangulation method is the Delaunay triangulation [43] which yields a uniquely defined network [1]. A Delaunay triangulation has the property that for all triangles, the circumscribing circle of a triangle does not contain data points of the TIN other than the corner points of this triangle. This property avoids long, narrow triangles in the TIN. Several algorithms exist that implement the Delaunay triangulation, as mentioned in [42] and illustrated in A.1. Further properties of triangular networks extends beyond the research of this work.

As will become clear later on, some operations will yield a result in which the presence of some edges must be enforced. This can be done by means of a constrained Delaunay triangulation: the triangulation is not performed merely on a set of data points, but allows for specific edges that must be part of the TIN network to be specified. Consequently a constrained Delaunay triangulation is no longer a Delaunay triangulation (the definition of a Delaunay triangulation is no longer satisfied: there can be data points other than the corner points inside the circumscribing circle of a triangle). For many operations, the edges and triangles themselves are referred to, hence they are included in characterizing a TIN:

Definition 99 (fuzzy TIN region)

$$\textit{fuzzy TIN} = [(P, E, T), f] \tag{5.1}$$

where P is a set of data points on which the TIN is constructed, E is a set of edges (including both the edges obtained through a Delaunay triangulation, and the edges required to be in the result in the case of a constrained Delaunay triangulation, see 1.1.3), and T is a set of triangles that make up the TIN. The function f is a mapping function defined as:

$$\begin{array}{rcl} f : P & \rightarrow & [0,1] \\ p(x,y) & \mapsto & f(p(x,y)) \end{array}$$

This function associates each data point with a value; which will be the membership grade for this data point.

A triangular network is considered in two dimensions, the notation $p(x, y)$ refers to points used in the triangulation process. For some operations and calculations, it is interesting to consider the associated value with each of the point as a third dimension, for which the notation $p(x, y, z)$ will be used, which is a shorthand for $p(x, y, f(p(x, y)))$.

In addition to a traditional TIN, TINs used for the representation of fuzzy regions are limited to a region of interest. The region of interest is similar in interpretation to the outline of a crisp region (the polygon) and is immediately deduced from the TIN as the (crisp) convex hull of a number of points in a two dimensional space (the triangulation will automatically yield this outline for any given set of points and edges).

[1] As mentioned in 1.1.3, there are trivial cases where there would be multiple solutions, but these can be circumvented by adding specific, global rules.

Based on the linear interpolation as is applied on a TIN and the mapping function f, the membership function for a fuzzy region $\tilde{A}$ can be defined as

Definition 100 (membership function $\mu_{\tilde{A}}$ of a fuzzy TIN $\tilde{A}$)

$$\begin{array}{rcl} \mu_{\tilde{A}} : U & \rightarrow & [0,1] \\ p(x,y) & \mapsto & \left\{ \begin{array}{ll} f(p(x,y)) & \textit{if } \ p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \textit{if } \ p(x,y) \in R \backslash P \\ 0 & \textit{if } \ p(x,y) \notin R \end{array} \right. \end{array}$$

where R represents the region of interest of the TIN and A, B, C and D are the parameters of the equation $Ax + By + Cz + D = 0$ of the plane containing the three points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ and $p_3(x_3, y_3, z_3)$ (with the understanding that $z_j = f(x_j, y_j)$, $j = 1, 2, 3$), such that the triangle $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ is a triangle of the TIN and $p(x, y, 0)$ is inside or on an edge of this triangle.

$$A = y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2) \quad (5.2)$$

$$B = z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2) \quad (5.3)$$

$$C = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \quad (5.4)$$

$$D = -Ax_1 - By_1 - Cz_1 \quad (5.5)$$

The points $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ in the XY-plane should not be co-linear, which is guaranteed by the fact that no Delaunay triangulation (or even a constrained Delaunay triangulation) would result in a triangulation containing such a degenerate case. For the remainder of the work, *TIN* will be used to refer to both a TIN obtained through a Delaunay or through a constrained Delaunay triangulation.

In its current form, the TIN does not support discontinuities, apart from possible discontinuities at its outline (between points inside the region of interest R and points not in the region of interest). Normally, points outside the region of interest do not belong to the region (and are assigned membership grade 0). Some applications may require regions that encompass the considered universe (which, is a genuine subset of $\mathbb{R}^2$) and may require a membership grade other than 0 for points outside the region of interest (examples are the complement and the exterior). Therefore, the membership function can be generalized as:

$$\begin{array}{rcl} \mu_{\tilde{A}} : U & \rightarrow & [0,1] \\ p(x,y) & \mapsto & \left\{ \begin{array}{ll} f(p(x,y)) & \text{if } \ p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } \ p(x,y) \in R \backslash P \\ a & \text{if } \ p(x,y) \notin R \end{array} \right. \end{array}$$

where a is any value in the range $[0, 1]$.

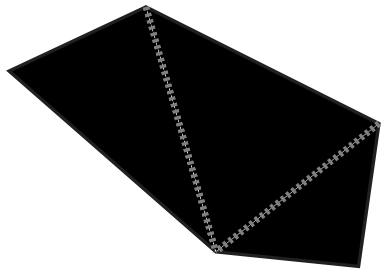


Figure 5.1: Representing crisp objects with TINs.

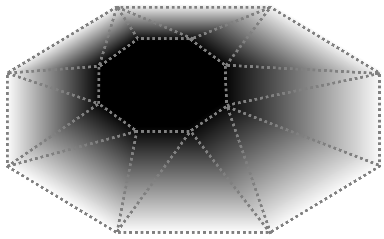


Figure 5.2: Representing fuzzy objects using the TIN model.

It is important to note that essentially, a (fuzzy) TIN structure is a two dimensional model: the data points exist in two dimensions, the triangulation is performed in two dimensions. However, for the interpolation, and also for some operators it is useful to consider the TIN in three dimensions, with third dimension representing the associated values. To explain how fuzzy objects are modelled using fuzzy TINs, it will first be illustrated how a crisp object might be represented by a fuzzy TIN.

5.3 Using fuzzy TINs as regions

5.3.1 Representing crisp regions with fuzzy TINs

A fuzzy TIN for a crisp region can be defined by considering the points of the polygons as datapoints associating a membership grade 1 with each of them. A Delaunay triangulation using these points will yield a TIN for which all points inside the polygon have a membership grade 1. This is essentially what a crisp region is; it is therefore perfectly possible to model this crisp region using a TIN with a limited region of interest, as shown in fig. 5.1. This approach of course yields little added value over the polygon, but it facilitates the concept that will be used further on in this work.

5.3.2 Representing fuzzy regions with fuzzy TINs

The concept of adapting TINs - a field based model - to model features, or in this section, regions, is quite analogous to the way the field based bitmap model was adapted to model fuzzy regions in the fuzzy bitmap model. The data associated with the TIN are membership grades: for each point this grade represents the extent to which this point is part of the region.

Unlike the commonly used field based models that extend over the entire map (or over the entire region of interest), this extended TIN model has an outline. This outline is a closed polyline and basically divides the space in two[2] distinct sets: the outside of the polyline consists of all the points that do not belong to the fuzzy region, whereas the inside holds all the points that belong to the region to some degree which differs from 0. It is important to note that *all* the points that have a degree strictly greater than 0 belong to the region, but some more than others. In fig.5.2 a fuzzy TIN region is representing, here points closer to the center are assigned higher membership grades (darker colour) than points closer to the outside (lighter colour). There is however no reason to limit the fuzzy TIN to convex regions, nor is there a reason to have the membership grades decrease from the inside (where they are 1) towards the outside.

Inside this outline, each of the points is assigned a membership grade in the range $[0, 1]$. For this purpose, a TIN structure is used to model the membership grades for each of the points; the data points will have a known grade associated (coming from observations, calculations, estimates or even personal opinions); the other points will have a grade calculated through linear interpolation in each of the triangles as explained above.

5.3.3 Remark

It is important to note that it should be possible for points and edges to be specified in the TIN, but having membership grade 0. While these elements do not belong to the fuzzy region, they can be required to define the region.

As shall become clear in some operations (more specifically the strong α-cut and the related support), it can be interesting to *tag* points and edges to indicate that they are not part of the fuzzy set.

[2]It is possible for the fuzzy region to consist of multiple disjunct closed polylines. For clarity reasons, this is not considered in the explanation, but the concept is completely analogous: the fuzzy region then consists of multiple polylines.

5.4 Operations

5.4.1 Set-operations

Intersection

As fuzzy TIN is an extension of a TIN, no assumption on the location of its data-points is made, nor do two (or more) TINs that are intended to be combined need to have the same number of data points or have their points at the same locations.

Consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$, by definition their intersection is given using a t-norm:

$$\tilde{A}\tilde{\cap}\tilde{B} = \{(x, \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x)) \mid x \in U \wedge \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(x) = T((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)))\} \quad (5.6)$$

When $\tilde{A}$ and $\tilde{B}$ are represented as fuzzy TINs, they are characterized by respectively $[(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ and $[(P_{\tilde{B}}, E_{\tilde{B}}, T_{\tilde{B}}), f_{\tilde{B}}]$. The intersection of both fuzzy regions will be a new fuzzy region $\tilde{A}\tilde{\cap}\tilde{B}$, which will be characterized by $[(P_{\tilde{A}\tilde{\cap}\tilde{B}}, E_{\tilde{A}\tilde{\cap}\tilde{B}}, T_{\tilde{A}\tilde{\cap}\tilde{B}}), f_{\tilde{A}\tilde{\cap}\tilde{B}}]$, such that

$$\forall p \in U : \mu_{\tilde{A}\tilde{\cap}\tilde{B}}(p) = T(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p)) \quad (5.7)$$

The intersection of fuzzy sets is performed using a t-norm; commonly used is the minimum. It will also be used here. The reason to choose a particular t-norm, is that an algorithm to compute it efficiently will be constructed. The minimum is interesting, as it maintains linearity, a property which will allow for the resulting TIN to be mathematically correct rather than an approximation. Informally, when considering the fuzzy TINs as three dimensional structures, the desired result of the minimum-operation would be that all the "lowest" points of the two fuzzy TINs are retained. These include the data-points, but also the points on edges and inside triangles.

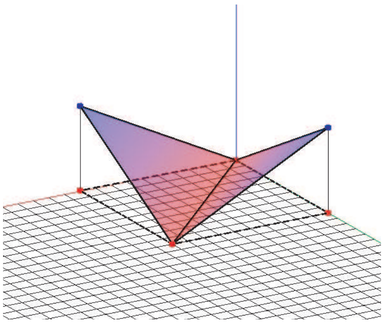
It is not sufficient to only consider the minimum of the data points, as is illustrated by a simple example. Suppose that we have two fuzzy TIN structures $\widetilde{Tin_1}$ and $\widetilde{Tin_2}$, defined as shown in resp. Fig. 5.3 and Fig. 5.4. The data points of $\widetilde{Tin_i}$ will be denoted as

$$p_{j_i}(x_{j_i}, y_{j_i}, f_i(x_{j_i}, y_{j_i})), i = 1, 2; j = 1, 2, ...n_i \quad (5.8)$$

where n_i stands for the number of elements in P_i (cardinality). An edge e_{k_i} connecting the points p_{l_i} and p_{m_i} is denoted as $e_{k_i}(p_{l_i}, p_{m_i})$. As a shorthand notation, the indices i will be omitted if no confusion is possible. Although a Delaunay triangulation algorithm would — in this simple case — have yielded the same networks for these same sets of four points, this artificial example best illustrates the problem that can occur in more elaborate (genuine) Delaunay triangulations as well.

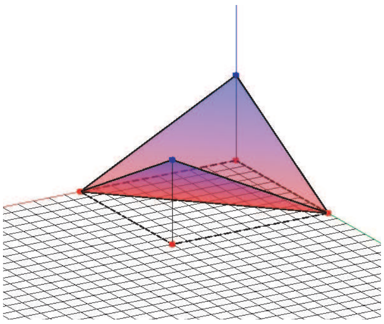
Obviously, all the data points overlap (which facilitates the operations). For all data points

$$\mu_{\widetilde{Tin_1}\tilde{\cap}\widetilde{Tin_2}}(p_i) = \min(\mu_{\widetilde{Tin_1}}(p_i), \mu_{\widetilde{Tin_2}}(p_i)) \quad (5.9)$$



P_1	E_1
$p_1(0,0,0)$	$e_1(p_1,p_3)$
$p_2(0,100,1)$	$e_2(p_3,p_2)$
$p_3(100,100,0)$	$e_3(p_1,p_2)$
$p_4(100,0,1)$	$e_4(p_3,p_4)$
	$e_5(p_1,p_4)$

Figure 5.3: $\widetilde{Tin_1}$



P_2	E_2
$p_1(0,0,1)$	$e_1(p_2,p_1)$
$p_2(0,100,0)$	$e_2(p_1,p_4)$
$p_3(100,100,1)$	$e_3(p_4,p_2)$
$p_4(100,0,0)$	$e_4(p_4,p_3)$
	$e_5(p_3,p_2)$

Figure 5.4: $\widetilde{Tin_2}$

This yields a membership grade 0 for all data points , and consequently the interpolation will yield 0 for all points. However, the point $p(50,0)$ has membership grade $\mu_{\widetilde{Tin_1}}(p) = \mu_{\widetilde{Tin_2}}(p) = 0.5$, the minimum of these two values should be 0.5. This also illustrates why in equation 5.7, the function f would have been insufficient, and μ needs to be used.

The intersection of the fuzzy regions defined by the TINs

$$\begin{aligned}\widetilde{Tin_1} &= [(P_1,E_1,T_1),f_1]\\ \widetilde{Tin_2} &= [(P_2,E_2,T_2),f_2]\end{aligned}$$

with derived membership functions respectively

$$\begin{aligned}&\mu_{\widetilde{Tin_1}}\\ &\mu_{\widetilde{Tin_2}}\end{aligned}$$

is by definition obtained by considering the

$$\min(\mu_{\widetilde{Tin_1}}(p(x,y)),\mu_{\widetilde{Tin_2}}(p(x,y)))$$

of the membership grades associated with each point $p(x,y)$ in the considered region of interest. A computable definition of the minimum denoted $\widetilde{Tin_3}$ of two TINs $\widetilde{Tin_1}$ and $\widetilde{Tin_2}$ can be derived by using the actual definitions of both these arguments. To "build" the resultant network, first, the set P_3 containing the points that will define $\widetilde{Tin_3}$ has to be determined. Calculating the points of P_3 is an incremental process, starting from an empty set.

1. First, the points of P_1 that are located "below" $\tilde{Tin_2}$ are added to P_3. These are points $p \in P_1$ for which $\mu_{\widetilde{Tin_1}}(p) = \mu_{\tilde{Tin_2}}(p)$. As $p \in P_1$, $\mu_{\widetilde{Tin_1}}(p) = f_1(p)$. There is no requirement regarding the relative locations of the data points of both networks; it is possible that values associated with points $p \in P_1$ will be compared either with values associated with points $p \in P_2$ or with values computed for points $p \notin P_2$. In either case, if $f_1(p) \leq \mu_{\widetilde{Tin_2}}(p)$, the point p is contained in P_3, with associated value $f_1(p)$.
$$p_{t_1} = \{p \mid p \in P_1 \wedge f_1(p) \leq \mu_{\widetilde{Tin_2}}(p)\} \tag{5.10}$$

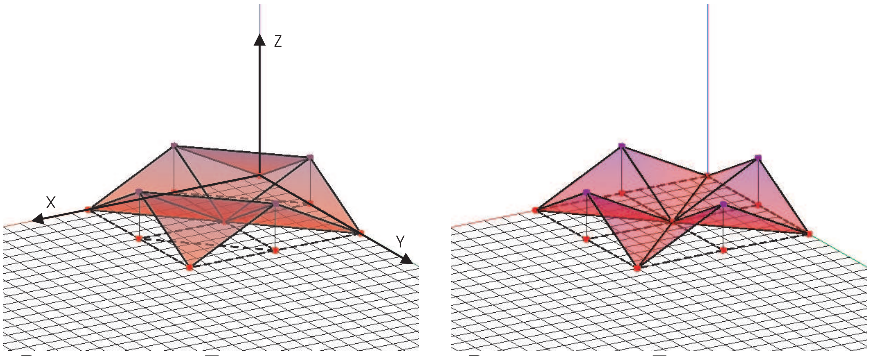
2. Completely analogue, the points of P_2 that are "below" $\widetilde{Tin_1}$ are added to P_3: for points $p \in P_2$, if $f_2(p) \leq \mu_{\widetilde{Tin_1}}(p)$, the point p is contained in P_3, with associated value $f_2(p)$.
$$p_{t_2} = \{p \mid p \in P_2 \wedge f_2(p) \leq \mu_{\widetilde{Tin_1}}(p)\} \tag{5.11}$$

3. Finally, P_{t_3} is defined as the set of points that result from the intersection of the triangles in $\widetilde{Tin_1}$ and the edges in $\widetilde{Tin_2}$ (and vice versa); these points were not necessarily present in any of the original TINs. The points in this new set P_{t_3} are also added to P_3.
$$\begin{aligned} P_{t_3} \quad = \quad & \{p \mid p \notin (P_1 \cup P_2) \wedge \exists e \in E_1, \exists t \in T_2 : \\ & \quad p \in e \wedge p \in t \wedge \mu_{\widetilde{Tin_1}}(p) = \mu_{\widetilde{Tin_2}}(p)\} \\ & \cup \\ & \{p \mid p \notin (P_1 \cup P_2) \wedge \exists e \in E_2, \exists t \in T_1 : \\ & \quad p \in e \wedge p \in t \wedge \mu_{\widetilde{Tin_1}}(p) = \mu_{\widetilde{Tin_2}}(p)\} \end{aligned} \tag{5.12}$$
These points are needed as they determine where an edge of one network "stops" being located "below" the other network. As they are intersection points between edges of one TIN and triangles of the other TIN, each point has the same membership grade in both TINs. This value is associated with this point in the resulting network.

The minimum will then be a new TIN $\widetilde{Tin_3} = [(P_3, E_3, T_3), f_3]$, defined by the points as explained above:
$$P_3 = P_{t_1} \cup P_{t_2} \cup P_{t_3} \tag{5.13}$$

Using the set P_3 and considering the points in the XY-plane, as the input for a Delaunay triangulation will yield a unique triangulated irregular network.

The TIN that is obtained by applying a Delaunay triangulation algorithm on the set P_3 is shown in Fig. 5.5a. It can easily be verified that the result is incorrect, considering the expected geometrical minimum in Fig. 5.5b. Consider for instance the edges $e_2(p_2, p_3)$ and $e_{12}(p_5, p_4)$ of $\widetilde{Tin_3}$ (Fig. 5.5a) that are generated by the triangulation algorithm. The point $p(25, 25)$, which is a point with an interpolated associated value in each of the three networks ($\tilde{Tin_1}$, $\widetilde{Tin_2}$ and $\widetilde{Tin_3}$) is located on $e_2(p_2, p_3)$. As can be clearly seen in Fig. 5.5a, this point has a associated, calculated membership grade $\mu_{\tilde{Tin_3}}(p(25, 25)) = 0.5$ in $\tilde{Tin_{3'}}$. However, membership grades for this point in both $\widetilde{Tin_1}$ and $\widetilde{Tin_2}$ are $\mu_{\widetilde{Tin_1}}(p(25, 25)) = \mu_{\widetilde{Tin_2}}(p(25, 25)) = 0$; the minimum value should therefore also equal to 0. This difference is due to the fact that the Delaunay triangulation generates new edges, which are not part of any of the original TINs, and do not satisfy the minimum criterion. Especially when higher accuracy is desired, this result is most likely to be insufficient, which calls for a more accurate approach.



P_3	E_3
$p_1(50, 50, 0)$	$e_1(p_6, p_2)$
$p_2(50, 0, 0.5)$	$e_2(p_2, p_3)$
$p_3(0, 50, 0.5)$	$e_3(p_3, p_6)$
$p_4(100, 50, 0.5)$	$e_4(p_1, p_8)$
$p_5(50, 100, 0.5)$	$e_5(p_8, p_3)$
$p_6(0, 0, 0)$	$e_6(p_3, p_1)$
$p_7(100, 100, 0)$	$e_7(p_2, p_1)$
$p_8(0, 100, 0)$	$e_8(p_1, p_9)$
$p_9(100, 0, 0)$	$e_9(p_9, p_4)$
	$e_{10}(p_4, p_1)$
	$e_{11}(p_2, p_9)$
	$e_{12}(p_5, p_4)$
	$e_{13}(p_4, p_7)$
	$e_{14}(p_7, p_5)$
	$e_{15}(p_5, p_1)$
	$e_{16}(p_5, p_8)$

(a)

P_3	E_3
$p_1(50, 50, 0)$	$e_1(p_2, p_1)$
$p_2(50, 0, 0.5)$	$e_2(p_1, p_6)$
$p_3(0, 50, 0.5)$	$e_3(p_6, p_2)$
$p_4(100, 50, 0.5)$	$e_4(p_1, p_8)$
$p_5(50, 100, 0.5)$	$e_5(p_8, p_3)$
$p_6(0, 0, 0)$	$e_6(p_3, p_1)$
$p_7(100, 100, 0)$	$e_7(p_3, p_6)$
$p_8(0, 100, 0)$	$e_8(p_1, p_9)$
$p_9(100, 0, 0)$	$e_9(p_9, p_4)$
	$e_{10}(p_4, p_1)$
	$e_{11}(p_2, p_9)$
	$e_{12}(p_1, p_7)$
	$e_{13}(p_7, p_5)$
	$e_{14}(p_5, p_1)$
	$e_{15}(p_4, p_7)$
	$e_{16}(p_5, p_8)$

(b)

Figure 5.5: Graphical representation of the minimum: (a) incorrect result obtained by only considering the minimum of the datapoints and performing a Delaunay triangulation, (b) correct result by taking into account the intersections of edges and performing a constrained Delaunay triangulation.

In order to overcome this problem, a set of predefined edges E_3 will be included. These edges will be used to force the triangulation algorithm to maintain edges that are needed in the resulting TIN; the constrained Delaunay triangulation can then be used to calculate the correct minimum.

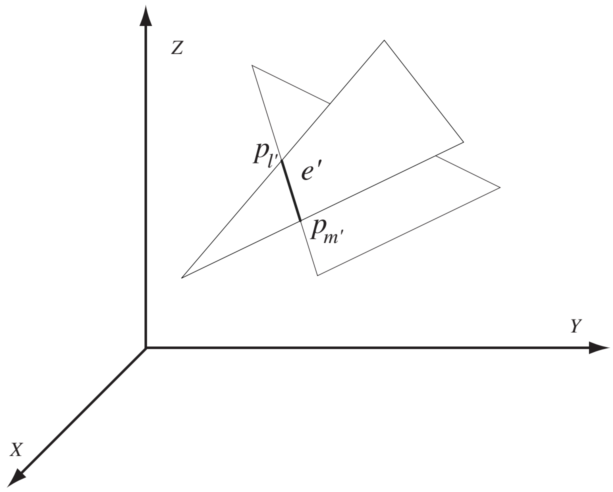


Figure 5.6: Illustration of E_{t_1}

Defining E_3 is also an incremental process.

1. In the first step, line segments that are part of an existing edge in either $\widetilde{Tin}_1$ or $\widetilde{Tin}_2$, and that connect two newly added points (obtained through the intersection of the triangles of both TINs $\widetilde{Tin}_1$ and $\widetilde{Tin}_2$) are added. This is illustrated in figure 5.6. The set of these edges is:

$$E_{t_1} \quad = \quad \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \wedge p_{b'} \in P_{t_3} \wedge \exists e \in E_1 \cup E_2 : e' \subset e\}$$

2. Next, the intersection of a triangle in $T\tilde{i}n_1$ and a triangle in $T\tilde{i}n_2$ can yield an edge that did not belong to either TIN. The endpoints $p_{a'}$ and $p_{b'}$ of such edges are contained in P_t. These new edges also need to be added to E_3, but they must form a closed segment in which no other points of P_t are contained; as illustrated in figure 5.7. This is expressed by

$$\begin{aligned} E_{t_2} \quad = \quad & \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_{t_3} \ \wedge \\ & \forall p \in e' : p \notin P_{t_3} \wedge \mu_{\widetilde{Tin_3}}(p) = min(\mu_{\widetilde{Tin_1}}(p), \mu_{\widetilde{Tin_2}}(p))\} \end{aligned}$$

3. As a next step, all the segments $e'(p_{a'}, p_{b'})$ of existing edges $e(p_a, p_b)$ in E_1 or E_2 that connect a point $p_{a'} \in P_t$ (i.e. a point obtained through the intersection of a triangle and the edge e of which e' is a segment) with a point $p_{b'} \in P_3 \setminus P_t$ (i.e. a point that is definitely part of the minimum) are added. This is illustrated on figure 5.8. These segments are in

$$\begin{aligned} E_{t_3} \quad = \quad & \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\ & (\exists\, e(p_a, p_b) \in E_1 \cup E_2 \ : \ e'(p_{a'}, p_{b'}) \subseteq e(p_a, p_b))\} \end{aligned}$$

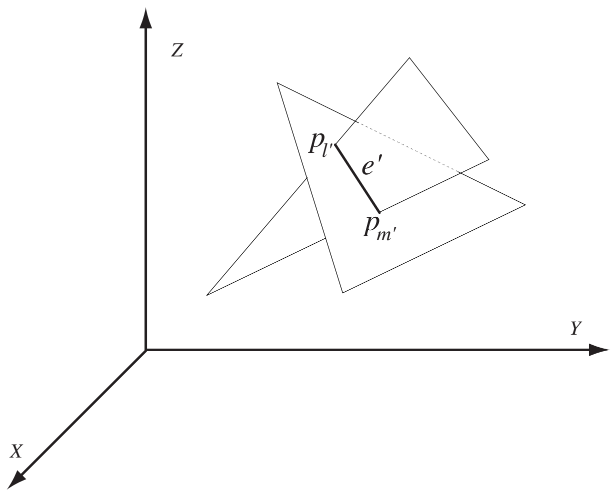


Figure 5.7: Illustration of E_{t_2}

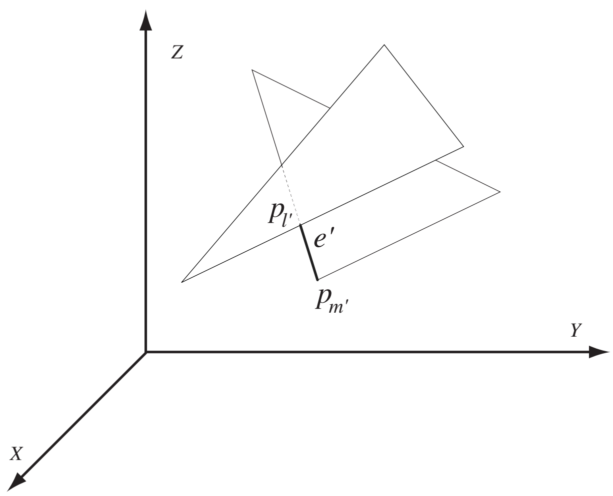


Figure 5.8: Illustration of E_{t_3}

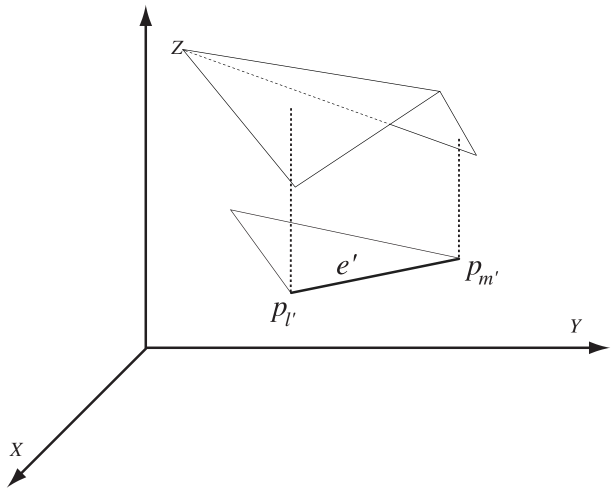


Figure 5.9: Illustration of E_{t_4}

4. Finally, the edges that interconnect two points; which are definitely part of the minimum but are not intersection points (i.e. $p_{a'} \in P_3 \setminus P_t$ and $p_{b'} \in P_3 \setminus P_{t_3}$), and that form an existing edge in either TIN ($e(p_{a'}, p_{b'}) \in E_1 \cup E_2$) are also added to E_3. This is illustrated in figure 5.9. These edges are contained in

$$\begin{aligned} E_{t_4} \quad = \quad & \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_3 \setminus P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\ & (\exists \ e(p_a, p_b) \in E_1 \cup E_2 \ \wedge \ \forall p \in e' : \\ & \quad p \notin P_{t_3} \wedge \mu_{\widetilde{Tin_3}}(p) = min(\mu_{\widetilde{Tin_1}}(p), \mu_{\widetilde{Tin_2}}(p)))\} \end{aligned}$$

Combining these sets results in E_3 being defined as

$$\begin{aligned}
E_3 &= E_{t_1} \cup E_{t_2} \cup E_{t_3} \cup E_{t_4} \\
&= \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3}, p_{b'} \in P_{t_3} \wedge \exists e \in E_1 \cup E_2 : e' \subset e\} \\
&\quad \cup \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_{t_3} \ \wedge \\
&\qquad \forall p \in e' : p \notin P_{t_3} \wedge \mu_{\widetilde{Tin_3}}(p) = min(\mu_{\widetilde{Tin_1}}(p), \mu_{\widetilde{Tin_2}}(p))\} \\
&\quad \cup \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\
&\qquad (\exists\, e(p_a, p_b) \in E_1 \cup E_2 \ : \ e'(p_{a'}, p_{b'}) \subseteq e(p_a, p_b))\} \\
&\quad \cup \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_3 \setminus P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\
&\qquad \exists\, e(p_a, p_b) \in E_1 \cup E_2 \ \wedge \ \forall p \in e' : \\
&\qquad\quad p \notin P_{t_3} \wedge \mu_{\widetilde{Tin_3}}(p) = min(\mu_{\widetilde{Tin_1}}(p), \mu_{\widetilde{Tin_2}}(p))\}
\end{aligned}$$

In the simplified example (Fig. 5.3 and Fig. 5.4), the set E_3 will contain all the edges defining the triangulated irregular network. With more complex networks, edges in E_3 yield a set of non-overlapping planar polygons. In general, when a Delaunay (or constrained Delaunay) triangulation is applied to a planar polygon [3], it results in a planar triangulation ([42]). In this case triangulating the planar polygons (which is in fact a constrained Delaunay triangulation with the edges of the polygons specified to be part of the result), results in a triangulation that does not exhibit the problems caused by the regular Delaunay triangulation on the non-planar set of points P_3. As can be seen in Fig. 5.5b, this definition of the minimum is exactly the same as the minimum that should be obtained.

Even for simple examples, it is necessary to perform the triangulation algorithm: while all the edges may be found, the triangles have not been determined yet. For characterizing a TIN, all three sets P, E and T are needed; the contrained Delaunay triangulation will create the appropriate set of triangles T_3 from the sets P_3 and E_3.

Verification To verify that the above construction is conform with the theoretical definition given in 2.4.1, consider two fuzzy regions $\tilde{A}$ and $\tilde{B}$ in a fuzzy TIN representation: $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ and $\tilde{B} = [(P_{\tilde{B}}, E_{\tilde{B}}, T_{\tilde{B}}), f_{\tilde{B}}]$. For every point p of the universe, its membership grade $\mu_{\tilde{A}\cap\tilde{B}}(p)$ should be equal to $min(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p))$.

For datapoints $p \in P_{\tilde{A}\cap\tilde{B}}$, there are three possible reasons as to why they are in $P_{\tilde{A}\cap\tilde{B}}$: they were in P_{t_1}, P_{t_2} or P_{t_3}. In other words:

$$\begin{aligned}
p \in P_{\tilde{A}\cap\tilde{B}} \quad \Leftrightarrow \quad & p \in P_{\tilde{A}} \wedge f_{\tilde{A}}(p) < \mu_{\tilde{B}}(p) && (p \in P_{t_1}, \text{ see } 5.10) \\
& \vee p \in P_{\tilde{B}} \wedge f_{\tilde{B}}(p) < \mu_{\tilde{A}}(p) && (p \in P_{t_2}, \text{ see } 5.11) \\
& \vee \mu_{\tilde{A}}(p) = \mu_{\tilde{B}}(p) && (p \in P_{t_3}, \text{ see } 5.12)
\end{aligned} \tag{5.14}$$

In each of these three cases, $\mu_{\tilde{A}\cap\tilde{B}} = min(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p))$.

[3] A planar polygon is a polygon in a three dimensional space, but whose vertices are all located in the same plane.

For points $p \notin P_{\tilde{A}\cap\tilde{B}}$

$$\mu_{\tilde{A}\cap\tilde{B}}(p(x,y)) = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} \quad \text{(see definition 100)}$$

Here, A, B, C and D are computed from the three vertices of the triangle in which p is located (see 100). It now suffices to show that this plane $-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}$ is part of a triangle in either T_A or T_B, and that this triangle is located below the other TIN.

The corner points p_1, p_2, p_3 of this triangle are in $P_{\tilde{A}\cap\tilde{B}}$, and these all have the an associated value $f_{\tilde{A}\cap\tilde{B}}(p_i) = min(\mu_{\tilde{A}}(p_i), \mu_{\tilde{B}}(p_i))$ (by construction). The three points are definitely the lowest of both TINs.

It now suffices to show that the triangle formed by these three points is a triangle (or part of a triangle) in one of the two original TINs. The triangle consists of the edges $e(p_1,p_2)$, $e(p_2,p_3)$ and $e(p_1,p_3)$. As the intersection of two triangles (of a TIN) is either a single point or an edge, at most two of these points are intersection points. This yields the following cases

- Suppose that there is one intersection point p_1; this implies that p_2 and p_3 are points that belong to the lowest TIN. By construction, the edges $e(p_1,p_2)$ and $e(p_1,p_3)$ were added to E_{t_3} ; the edge $e(p_2,p_3)$ was added to E_{t_2}; they are consequently in $E_{\tilde{A}\cap\tilde{B}}$. These three edges form a triangle, constraining a Delaunay with these three edges, will cause this triangle to be in the result.

- Suppose that there are two intersection points p_1 and p_2. By construction, the edges $e(p_1,p_3)$ and $e(p_2,p_3)$ will have been added to E_{t_3}. The edge $e(p_1,p_2)$ will be added to E_{t_1} (if $e(p_1,p_2)$ was part of an edge in $E_{\tilde{A}}$ or $E_{\tilde{B}}$), or to E_{t_2} (if $e(p_1,p_2)$ was not part of an edge in $E_{\tilde{A}}$ or $E_{\tilde{B}}$). In either case, the triangle $t(p_1,p_2,p_3)$ is a portion of a triangle in either $T_{\tilde{A}}$ or $T_{\tilde{B}}$.

- Suppose that there are no intersection points. All three edges $e(p_1,p_2)$, $e(p_2,p_3)$, $e(p_1,p_3)$ will have been added to E_{t_4}.

In each of the above cases, the three points are in $P_{\tilde{A}\cap\tilde{B}}$, and the three matching edges are in $E_{\tilde{A}\cap\tilde{B}}$. The constrained Delaunay triangulation is only performed to define the set $T_{\tilde{A}\cap\tilde{B}}$, but already all edges are known. By construction, the triangles in the result are either triangles or triangular parts of triangles for which all the cornerpoints p satisfy $\mu_{\tilde{A}\cap\tilde{B}}(p) = min(\mu_{\tilde{A}}(p), \mu_{\tilde{B}}(p))$. As the three points defining the triangle (and thus also the plane), the equations will match, and the property will be fulfilled for all points inside the triangles. □

Implementation

```
fuzzy_tin Intersect_min(fuzzy_tin A, fuzzy_tin B)
  pointset Pt,P , edgeset E
BEGIN
```

```
for all p in P_A
  if mu_A(p) <= mu_B(p)
    add p to P
    f(p) = mu_A(p)
end for

for all p in P_B
  if mu_A(p) >= mu_B(p)
    add p to P
    f(p) = mu_B(p)
end for

 for all t in T_A
  for all e in E_B
    if e is not in the plane of t
          and intersection(t,e) is not empty
      add point = intersection(t,e) to Pt
      f(p) = mu_A(p)
     end if
   end for
 end for

for all t in T_B
  for all e in E_A
    if e is not in the plane of t
          and intersection(t,e) is not empty
      add point = intersection(t,e) to Pt
      f(p) = mu_B(p)
     end if
   end for
 end for

for all p1,p2 in Pt
  if there is an edge e in E_A or E_B for which
        e(p1,p2) is subset of e
    add e(p1,p2) to E
  end if
  if e(p1,p2) has no other points from Pt
    add e(p1,p2) to E
  end if
end for

for all p1 in Pt and all p2 in P
  if there is an edge e in E_A or E_B for which
        e(p1,p2) is subset of e
```

```
      add e(p1,p2) to E
    end if
  end for

  for all p1,p2 in P
    if there is an edge e in E_A or E_B for which
          e(p1,p2) is subset of e
      add e(p1,p2) to E
    end if
  end for

  return Constrained_Delaunay(union(P,Pt),E)
END
```

Union

The definition for the union (using the maximum) is obtained in a completely analogue way. The resulting TIN $\tilde{Tin}_3 = [(P_3, E_3, T_3), f_3]$, determined by performing a constained Delaunay triangulation using the sets P_3 and E_3. Determining the set P_3 is also done in three steps:

1. The condition $\leq$ is now replaced by $\geq$, in order to find the points "above" the $\tilde{Tin}_2$.
$$p_{t_1} = \{p \mid p \in P_1 \wedge f_1(p) \geq \mu_{\tilde{Tin}_2}(p)\} \tag{5.15}$$
2. Similarly, the points "above" $\tilde{Tin}_1$ need to be found.
$$p_{t_2} = \{p \mid p \in P_2 \wedge f_2(p) \geq \mu_{\tilde{Tin}_1}(p)\} \tag{5.16}$$
3. The points obtained through an intersection are the same, so P_{t_3} is exactly the same as for the intersection.
$$\begin{aligned} P_{t_3} \quad = \quad & \{p \mid p \notin (P_1 \cup P_2) \wedge \exists e \in E_1, \exists t \in T_2 : \\ & \quad p \in e \wedge p \in t \wedge \mu_{\tilde{Tin}_1}(p) = \mu_{\tilde{Tin}_2}(p)\} \\ & \cup \\ & \{p \mid p \notin (P_1 \cup P_2) \wedge \exists e \in E_2, \exists t \in T_1 : \\ & \quad p \in e \wedge p \in t \wedge \mu_{\tilde{Tin}_1}(p) = \mu_{\tilde{Tin}_2}(p)\} \end{aligned} \tag{5.17}$$

This results in the set P_3 defined as:

$$P_3 = P_{t_1} \cup P_{t_2} \cup P_{t_3} \tag{5.18}$$

The set of edges E_3 is constructed in four steps:

1. First, the line segments that are part of an existing edge, and that connect two newly added points are considered. This is the same as for the intersection.

$$E_{t_1} \quad = \quad \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \wedge p_{b'} \in P_{t_3} \wedge \exists e \in E_1 \cup E_2 : e' \subset e\}$$

2. Next, the line segments that are the result of an intersection between two triangles are added. This is analogues as before, but now using the *max* instead of the *min*.

$$\begin{aligned} E_{t_2} &= \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_{t_3} \ \wedge \\ &\quad \forall p \in e' : p \notin P_{t_3} \wedge \mu_{\tilde{Tin_3}}(p) = max(\mu_{\tilde{Tin_1}}(p), \mu_{\tilde{Tin_2}}(p))\} \end{aligned}$$

3. As a third step, segments of existing edges that connect a newly added point with a point that definitely belongs to the union is added. This definition is the same as for the intersection.

$$\begin{aligned} E_{t_3} &= \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\ &\quad (\exists\, e(p_a, p_b) \in E_1 \cup E_2 \ : \ e'(p_{a'}, p_{b'}) \subseteq e(p_a, p_b))\} \end{aligned}$$

4. Finally, points that match existing edges which should be present are added, this is similar to the last step in the intersection, but again the *min* is replaced with *max*.

$$\begin{aligned} E_{t_4} &= \{e'(p_{a'}, p_{b'}) \mid p_{a'} \in P_3 \setminus P_{t_3} \ \wedge \ p_{b'} \in P_3 \setminus P_{t_3} \ \wedge \\ &\quad (\exists\, e(p_a, p_b) \in E_1 \cup E_2 \ \wedge \\ &\qquad \forall p \in e' : p \notin P_{t_3} \wedge \mu_{\tilde{Tin_3}}(p) = max(\mu_{\tilde{Tin_1}}(p), \mu_{\tilde{Tin_2}}(p)))\} \end{aligned}$$

The set E_3 is the union of the above sets:

$$E_3 = E_{t_1} \cup E_{t_2} \cup E_{t_3} \cup E_{t_4} \tag{5.19}$$

The TIN that represents the union of both sets is obtained by considering a constrained Delaunay triangulation, using the sets P_3 and E_3.

The verification and implementation are analogous to the verification and implementation of the intersection.

Complement

The complement of a fuzzy region $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ yields a new fuzzy region $co\tilde{A} = [(P_{co\tilde{A}}, E_{co\tilde{A}}, T_{co\tilde{A}}), f_{co\tilde{A}}]$, representing the complement of the original region. If $\tilde{A}$ is limited to a region of interest R, the complementary region will cover the entire universe U.

To define the complement, the characterizing sets and the mapping function f need to be constructed. These three sets are all the same as the sets $(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}})$ as in the original region $\tilde{A}$. The only difference is in the mapping function and the resulting membership function:

$$\begin{aligned} f : P &\rightarrow [0, 1] \\ p &\mapsto 1 - \mu_{\tilde{A}}(p) \end{aligned}$$

The membership function differs slightly from the theoretical definition; points not in the region of interest now need to be assigned a membership grade 1.

$$\begin{array}{rcl} \mu_{co\tilde{A}} : U & \rightarrow & [0,1] \\ p(x,y) & \mapsto & \begin{cases} f(p(x,y)) & \text{if } p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } p(x,y) \in R \backslash P \\ 1 & \text{if } p(x,y) \notin R \end{cases} \end{array}$$

where A,B,C and D are defined respectively as in 5.2, 5.3, 5.4 and 5.5.

Verification To verify that this definition matches the theoretical definition, it suffice to show that for all points $\mu_{co\tilde{A}}(p) = 1 - \mu_{\tilde{A}}(p)$. Consider a fuzzy tin $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ and its complement $co\tilde{A} = [(P_{co\tilde{A}}, E_{co\tilde{A}}, T_{co\tilde{A}}), f_{co\tilde{A}}]$

1. $p \in P_{co\tilde{A}}$ For points in $P_{co\tilde{A}}$, this requirement is immediately fulfilled, due to the definition of both $P_{co\tilde{A}}$ and the definition of the mapping function $f_{co\tilde{A}}$.

2. For $p \notin P_{co\tilde{A}}$, it is necessary to prove that $\mu_{co\tilde{A}} = 1 - \mu_{\tilde{A}}(p)$. A triangle in $T_{co\tilde{A}}$ is a triangle of $T_{\tilde{A}}$ (remember that the triangulation process happens in two dimensions): only the associated values differ. The coefficients of the equation of a plane in the original TIN are denoted A, B, C and D; those of a plane in the complement are denoted A', B', C' and D'; the associated membership grade for a point p_i is $\mu_{co\tilde{A}}(p_i) = z'_i$.

   Consider the coefficient A':

$$\begin{aligned} A' &= y_1(z'_2 - z'_3) + y_2(z'_3 - z'_1) + y_3(z'_1 - z'_2) \\ &= y_1(1 - z_2 - 1 - z_3) + y_2(1 - z_3 - 1 - z_1) + y_3(1 - z_1 - 1 - z_2) \\ &= -y_1(z_2 - z_3) - y_2(z_3 - z_1) - y_3(z_1 - z_2) \\ &= -A \end{aligned}$$

   Consider B':

$$\begin{aligned} B' &= z'_1(x_2 - x_3) + z'_2(x_3 - x_1) + z'_3(x_1 - x_2) \\ &= (1 - z_1)(x_2 - x_3) + (1 - z_2)(x_3 - x_1) + (1 - z_3)(x_1 - x_2) \\ &= (x_2 - x_3) - z_1(x_2 - x_3) + (x_3 - x_1) - z_2(x_3 - x_1) \\ &\qquad + (x_1 - x_2) - z_3(x_1 - x_2) \\ &= (x_2 - x_3) + (x_3 - x_1) + (x_1 - x_2) - z_1(x_2 - x_3) \\ &\qquad - z_2(x_3 - x_1) - z_3(x_1 - x_2) \\ &= -(z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\ &= -B \end{aligned}$$

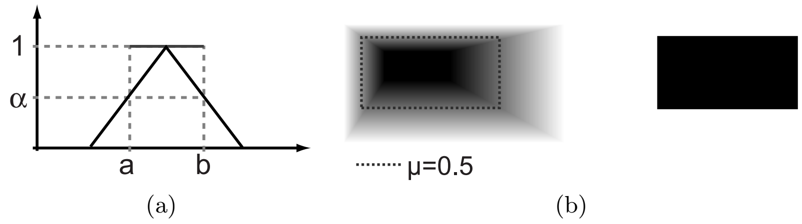


Figure 5.10: Meaning of the α-cut of fuzzy sets (a) and fuzzy TINs (b).

Consider C':

$$\begin{aligned} C' &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= C \end{aligned}$$

Consider D':

$$\begin{aligned} D' &= -A'x_1 - B'y_1 - C'z'_1 \\ &= Ax_1 + By_1 - C(1 - z_1) \\ &= -(-Ax_1 - By_1 + C(1 - z_1)) \\ &= -(-Ax_1 - By_1 - Cz_1 + C) \\ &= -(-Ax_1 - By_1 - Cz_1) - C \\ &= -D - C \end{aligned}$$

This yields

$$\begin{aligned} \mu_{co\tilde{A}}(p(x, y)) &= -\frac{A'}{C'}x - \frac{B'}{C'}y - \frac{D'}{C'} \\ &= -\frac{A'}{C}x - \frac{B'}{C}y - \frac{D'}{C} \\ &= -\frac{-A}{C}x - \frac{-B}{C}y - \frac{-D-C}{C} \\ &= -(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) + \frac{C}{C} \\ &= 1 - (-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) \\ &= 1 - \mu_{\tilde{A}}(p) \end{aligned}$$

□

5.4.2 Fuzzy operations

α-cut

An important concept is the *α-cut*, which stems from the fuzzy realm. Commonly, the α-cut of a fuzzy set is used to defuzzify the fuzzy set: it results

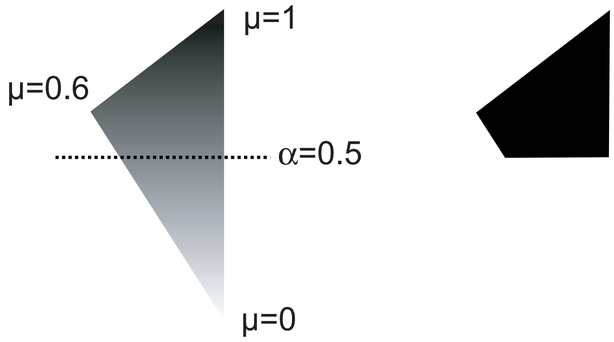


Figure 5.11: Illustration of the calculation of the α-cut of fuzzy TINs.

a crisp set containing all elements with a membership grade greater than (or equal to, if a weak α-cut is considered) a given value α. An illustration of this is shown in fig.5.10a. While working with fuzzy information provides a richer model, at some point the information will have to be defuzzified; most likely to exchange data with a non-fuzzy system, but also to just present the data. The α-cut of a fuzzy TIN is a new fuzzy TIN in which all membership grades equal 1, as shown in fig.5.10b; this can easily be converted to a more traditional model for a crisp region, by considering its outline.

Weak α-cut Determination of the *weak α-cut* of a fuzzy region represented by a fuzzy TIN $[(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ is performed by considering the TIN in three dimensions and a plane parallel to the XY plane, with $z = \alpha$. These three dimensional structures will intersect (the TIN has z coordinates in the range $[0, 1]$, and $\alpha \in]0, 1]$). The result of this intersection will now be considered for each triangle, as illustrated on fig. 5.12. The intersection between a triangle in three dimensional space, and a (horizontal) plane will always be either a straight line segment, a point, or will be empty. If the triangle is horizontal, and located in the α plane, the intersection will yield the entire triangle; in this case only its points and edges need considered.

To define the α-level, the sets P and E of the result will be determined.

1. The first step is to retain all the datapoints that are "above" or on the given α-level.

$$P_{t_1} = \{p \mid p \in P_{\tilde{A}} \wedge \mu_{\tilde{A}}(p) \geq \alpha\} \tag{5.20}$$

2. Next, all the points that define the intersection between an edge and the α-plane are considered. If there is an intersection, it consists of either one point (which was added before) or of an edge, defined by two points that are located on the edges of the triangle where it intersects the horizontal plane.

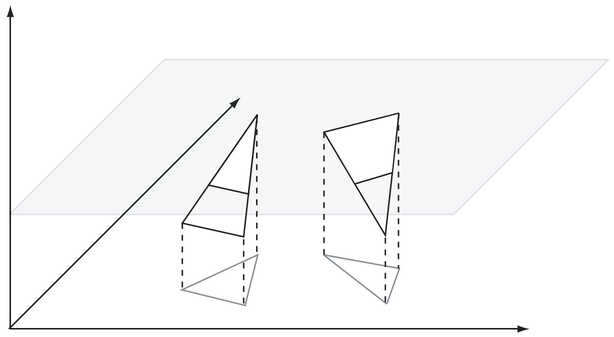


Figure 5.12: Illustration of the intersection between triangles and a horizontal plane.

$$P_{t_2} = \{p \mid \exists e(p_1,p_2) \in E_{\tilde{A}} : p \in e \wedge \mu_{\tilde{A}}(p_1) > \mu_{\tilde{A}}(p) \quad (5.21)$$
$$\mu_{\tilde{A}}(p) = \alpha \wedge > \alpha \mu_{\tilde{A}}(p_2)\} \quad (5.22)$$

Determining the set E is also done in a number of steps.

1. First, the edges that completely belong to the α-level are determined.

$$E_{t_1} = \{e_{t_1}(p_1,p_2) \mid p_i \in P_{\tilde{A}} \wedge \mu_{\tilde{A}}(p_i) \geq \alpha, \forall i\} \quad (5.23)$$

   Note that for horizontal triangles located in the α-plane, this definition adds the three edges defining this triangle.

2. Next, the intersections between the triangles and the α-plane are considered. Basically, there are four possibilities: the triangle is completely above or inside the α-plane (dealt with above), the triangle is completely below the α-plane (needs not to be included), the triangle has two points above the α-plane (and one below) and the triangle has one point above the α-plane (and two below). In either of the last two cases, two edges need to be added:

$$E_{t_2} = \{e_{t_2}(p_1,p_2) \mid p_1 \in P_{t_1} \wedge p_2 \in P_{t_2} \wedge \exists e \in E_{\tilde{A}} : e_{t_2} \subset e\} \quad (5.24)$$

3. Finally, edges that are the result of the intersection of a triangle and the horizontal α plane need to be considered.

$$E_{t_3} = \{e_{t_3}(p_1,p_2)\} \mid p_1 \in P_{t_2} \wedge p_2 \in P_{t_2} \wedge \quad (5.25)$$
$$\exists t \in T_{\tilde{A}} : \forall p \in e_{t_3}, p \in t\} \quad (5.26)$$

   where $p \in t$ means that the point is inside or on the circumference of the triangle.

The sets P and E by means of which the α-cut can be calculated are now obtained by

$$\begin{aligned} P &= P_{t_1} \cup P_{t_2} \\ E &= E_{t_1} \cup E_{t_2} \cup E_{t_3} \end{aligned}$$

The function f, that associates membership grades to each of the datapoints is defined as

$$\begin{aligned} f : P &\rightarrow [0, 1] \\ p(x, y) &\mapsto 1 \end{aligned}$$

At this point, it is possible to either perform a constrained Delaunay triangulation on P and E (taking into account the fact that there may be holes), which yields a triangular network as result. However, as the α-cut will be a crisp region, it makes sense to represent it as a crisp region, i.e. a polygon. For this it is only required to determine the outline (including outlines of existing holes, if any), and consider this polygon as the α-cut.

Verification To verify the definition for fuzzy TINs, it is necessary to show that the points p for which $\mu_{\tilde{A}}(p) \geq \alpha$ are contained in the result, and that there are no points for which $\mu_{\tilde{A}}(p) < \alpha$ in the result. For any triangle, there are 4 possibilities: no datapoint p has $\mu_{\tilde{A}}(p) \geq \alpha$, one datapoint p has $\mu_{\tilde{A}}(p) \geq \alpha$, two datapoints p have $\mu_{\tilde{A}}(p) \geq \alpha$ or all datapoints p have $\mu_{\tilde{A}}(p) \geq \alpha$.

- Suppose all three datapoints p have $\mu_{\tilde{A}}(p) \geq \alpha$. All three datapoints are added (see 5.20), with membership grade 1 (see 5.27). By definition of the membership function, all points inside this triangle will also have membership grade 1.

- Suppose two datapoints have $\mu_{\tilde{A}}(p) \geq \alpha$, consider them p_1 and p_2. Both points p_1 and p_2 have been added to the P (5.20); the edge $e(p_1, p_2)$ has been added to E (5.23). For the third point p_3, $\mu_{\tilde{A}}(p) < \alpha$. Considering the triangle in three dimensions, there will be an intersection with the plane $z = \alpha$. This plane holds all the points with the lowest values that should still be in the result. The intersection will be a straight line, delimited by the points $p'_1 \in e(p_1, p_3) \wedge \mu_{tildeA}(p'_1) = \alpha$ and $p'_2 \in e(p_2, p_3) \wedge \mu_{tildeA}(p'_2) = \alpha$. Both points are added to the result (5.22), as are the edges $e(p'_1, p_1)$ and $e(p'_2, p_2)$ (5.24). The polygon (p_1, p_2, p'_1, p'_2) is planar (by construction it is part of a triangle in the original TIN) and all points are either above or in the plane $z = \alpha$.

- Suppose only one datapoint p has $\mu_{\tilde{A}}(p) \geq \alpha$; consider this to be p_3. This is case is similar to the above case: p_3 is added to the set P (5.20), the intersection between the triangle (p_1, p_2, p_3) is an edge $e(p'_1, p'_2)$ and the points p'_1 and p'_2 are added to P (5.22). Also, the edges $e(p_1, p'_1)$ and $e(p_2, p'_2)$ are added to E (5.24), and the edge $e(p'_1, p'_2)$ is added to E (5.26).

- Suppose no datapoint p has $\mu_{\tilde{A}}(p) \geq \alpha$. In this case, no point has been added to P; consequently, no point inside this triangle will be added.

In each of the above cases, only datapoints p for which $\mu_{\tilde{A}}(p) \geq \alpha$ were added. Also, edges were added such that planar polygons that were part of the original TIN were formed. Consequently, only those points that should be part of the α level, are contained. □

Core The core is determined in exactly the same way as the weak α-cut, for the value $\alpha = 1$.

Strong α-cut The strong α-cut is similar, but now the points and edges that have membership-grade α should not be part of the α-level. As the TINs are considered over a two dimensional real domain, it is not possible to simply consider the next available α-level. Consequently, if this result is represented as a TIN, we need to tag the edges and points as they are not part of the α-cut. This is not an issue when representing the α-level as a polygon: the strong α-level is represented by the interior of polygon obtained from the weak α-cut.

Support The support is determined in exactly the same way as the strong α-cut, for the value $\alpha = 0$.

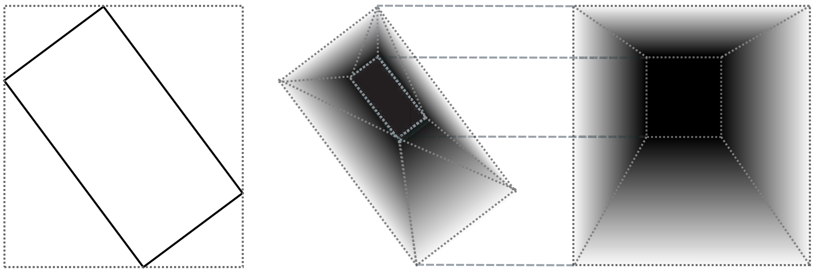
Algorithm for Fuzzy TINs For illustration purposes, the algorithm to calculate the weak α-cut is explained, the strong α-cut and both derived core and support, are similar. In order to calculate the α-cut, the fuzzy TIN is considered in three dimensions; together with a plane at the required α-level ($z = \alpha$). The points that are above this plane and the intersection points between edges of triangles and this plane are considered (fig.5.11). The intersection points are used to define additional edges: the line segments that are the result of an intersection of a triangle and the horizontal plane, as well as the remainder of the edges that intersected with the plane are retained. To get the resulting TIN, a constrained Delaunay triangulation is applied on the created sets of points and edges. The algorithm is shown below in pseudo code.

```
polygon WeakAlpha(Fuzzy_TIN A, alpha)
VAR pointset,edgeset

BEGIN
For all datapoints p in A
if mu_A(p) >= alpha add p in ResultP
endfor

For all edges e(p1,p2) in A
if mu_A(p1)>=alpha and mu_A(p2)>=alpha
  add e in edgeset
```

```
endfor

For all triangles t
  if mu_A(two points of t) > alpha then
    name these two points p1, p2
ip1 = intersection point between edge e(p1,p3) and alphaplane
ip2 = intersection point between edge e(p2,p3) and alphaplane
add ip1, ip2 to pointset
    add edge e(p1,p2) to edgeset
    add edge e(p1,ip1) to edgeset
    add edge e(p2,ip2) to edgeset
    add edge e(ip1,ip2) to edgeset
    endif
  else
  if mu_A(one point of t) > alpha then
    name this point p1
ip2 = intersection point between e(p1,p2) and alphaplane
ip3 = intersection point between e(p1,p3) and alphaplane
add ip2, ip3 to pointset
    add edge e(p1,ip2) to edgeset
    add edge e(p1,ip3) to edgeset
    add edge e(ip2,ip3) to edgeset
    endif
endfor

For all points p in pointset
  mu_WeakAlpha(p) = 1

alpha_tin = Constrained_Delaunay(pointset,edgeset)

return Outline(alpha_tin)

END
```

In the above code, the crisp result is represented as a polygon. Should the end result be represented as a fuzzy TIN (in which all membership grades are 1), it suffices in the last steps to `return alpha_tin` instead of the outline.

The algorithm for strong α-cuts is similar, apart from the fact that instead of returning the outline, it is necessary to return the interior of the α-cut: `return Interior(Outline(alpha_tin))`. Should the result be desired in a fuzzy TIN representation of a crisp region (a TIN in which all membership grades equal to 1), then points for which $\mu_{\tilde{A}}(p) = \alpha$ and edges $e(p_1, p_2)$ for which $\mu_{\tilde{A}}(p_1) = \mu_{\tilde{A}}(p_2) = \alpha$ should be tagged not to belong to the fuzzy region; prior to `return(alpha_tin)`.



(a) (b)

Figure 5.13: Minimum bounding rectangle of a crisp region (a) and a fuzzy TIN-region (b).

5.4.3 Geo-spatial

Minimum bounding rectangle

A minimum bounding rectangle has a number of uses, mostly in indexing. The fuzzy minimum bounding rectangle for a fuzzy TIN, is similar in concept to the fuzzy minimim bounding rectangle for fuzzy bitmaps 4.7: the fuzzy minimum bounding rectangle is a structure of which each α-cut is a minimum bounding rectangle for the same α-cut of the original fuzzy region. Consequently, the fuzzy MBR of a fuzzy TIN will be a new fuzzy region, also represented by a TIN. This operator returns the fuzzy MBR of a fuzzy TIN and has no relation with the *Fuzzy MBR* model used to approximate regions as presented in [45]. In fig.5.13a, the minimum bounding rectangle of a crisp region is shown (this yields a rectangle oriented parallel to the reference axes); fig. 5.13b shows a fuzzy MBR for a fuzzy region (also rectangular and oriented parallel to the reference axes).

Consider a fuzzy TIN $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$. The set $P_{\tilde{A}}$ contains a finite number of datapoints, consequently, the image $f_{\tilde{A}}(P_{\tilde{A}})$ also contains a finite number of elements. These elements will be considered as different α-levels α_i by means of which the fuzzy minimum bounding rectangle is determined. For each $\alpha_i \in f_{\tilde{A}}(P_{\tilde{A}})$ a bounding rectangle can be found, but some of the points and edges might overlap. Points in which this happens need to be assigned the highest α value for which they still are a corner point.

The set P of points is defined as

$$P = \{p \mid \exists \alpha_i \in f_{\tilde{A}}(P_{\tilde{A}}) : p \text{ is corner of } MBR(\tilde{A}_{\alpha_i})\} \tag{5.27}$$

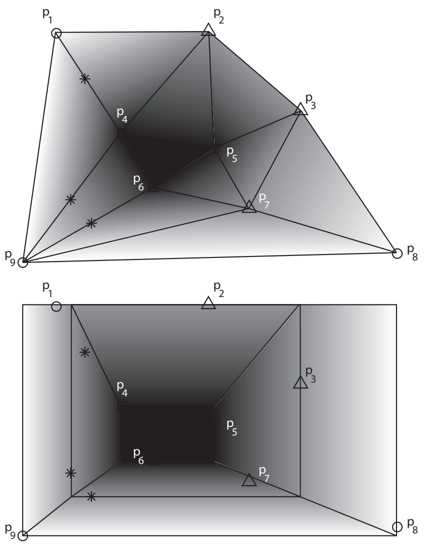
This set contains points that may or may not belong to $P_{\tilde{A}}$, as illustrated in fig. 5.14.

The mapping function of the fuzzy MBR is then defined as:

$$\begin{aligned} f : P &\rightarrow [0,1] \\ p &\mapsto \sup\{\alpha_i \mid p \text{ is corner of } MBR(\tilde{A}_{\alpha_i})\} \end{aligned} \tag{5.28}$$

(a)

(b)



$\mu_{\tilde{A}}(p_1) = 0$
$\mu_{\tilde{A}}(p_2) = 0.5$
$\mu_{\tilde{A}}(p_3) = 0.5$
$\mu_{\tilde{A}}(p_4) = 1$
$\mu_{\tilde{A}}(p_5) = 1$
$\mu_{\tilde{A}}(p_6) = 1$
$\mu_{\tilde{A}}(p_7) = 0.5$
$\mu_{\tilde{A}}(p_8) = 0$
$\mu_{\tilde{A}}(p_9) = 0$

(c)

Figure 5.14: Illustration of the algorithm for determining the fuzzy minimum bounding rectangle: (a) fuzzy TIN $\tilde{A}$, (b) MBR of $\tilde{A}$ prior to triangulation, (c) membership grades of the datapoints of $\tilde{A}$.

The set E which will constrain the triangulation of the MBR is defined in two steps, and makes use of the above mapping function.

1. First, all the sides that connect matching corners of subsequent α-levels (e.g. the top left corner of an α-level with the top left corner of the next α-level) are added.

$$\begin{aligned} E_{t_1} \quad = \quad & \{e(p_1,p_2) \mid \exists \alpha_i, \alpha_j \in f(P) : f(p_1) = \alpha_i \wedge f(p_2) = \alpha_j \wedge \\ & \nexists \alpha \in f(P) : \alpha_i > \alpha > \alpha_j \wedge p_1 \text{ is like } p_2\} \end{aligned} \tag{5.29}$$

   Where *is like* means that both points are top left corners, both points are top right corners, both points are bottom left corners or both points are bottom right corners and where $f(P)$ represents the image of P under f.

2. Next, all the sides of the MBRs at different α-levels are added to E, provided no segment of such a side has been added above.

$$\begin{aligned} E_{t_2} \quad = \quad & \{e(p_1,p_2) \mid \exists \alpha_i \in f(P) : e(p_1,p_2) \text{ is a side of } MBR(\tilde{A}_{\alpha_i}) \wedge \\ & \nexists e_1 \in E_{t_1} : e_1 \subseteq e(p_1,p_2)\} \end{aligned} \tag{5.30}$$

   Note that the sides of the MBR are parallel in space (and both parallel to the X-axis or both parallel to the Y-axis); the sides of the MBR of two subsequent α-levels consequently define a plane.

The sets P holds a number of datapoints, the set E now holds a number of edges, which all form planar polygons with four corners. The fuzzy MBR for

$\tilde{A}$ is now obtained by performing a constrained Delaunay triangulation using the set P as pointset and the set E as edgeset.

Verification To verify the above definition of the MBR for fuzzy regions in a TIN representation, consider a fuzzy TIN $\tilde{A}$ and its fuzzy MBR $\tilde{mbr}(\tilde{A})$:

$$\begin{aligned}\tilde{A} &= [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]\\ \tilde{mbr}(\tilde{A}) &= [(P_{\tilde{mbr}(\tilde{A})}, E_{\tilde{mbr}(\tilde{A})}, T_{\tilde{mbr}(\tilde{A})}), f_{\tilde{mbr}(\tilde{A})}]\end{aligned}$$

By definition (57), conceptually, $\mu_{\tilde{mbr}(\tilde{A})}(p) = \sup\{\alpha_i : p \in MBR(\tilde{A}_{\alpha_i}), \forall \alpha_i \in]0,1]\}$. It will now be verified for both datapoints and non-datapoints (inside the region of interest) that this also is the case for the fuzzy TIN model.

1. $p \in P_{\tilde{A}}$

$$\begin{aligned}\mu_{\tilde{mbr}(\tilde{A})}(p) &= f_{\tilde{mbr}(\tilde{A})}(p)\\ &= \sup\{\alpha_i : p \text{ is corner of } MBR(\tilde{A}_{\alpha_i})\}\\ &= \sup\{\alpha_i : p \in MBR(\tilde{A}_{\alpha_i})\}\end{aligned}$$

2. $p \notin P_{\tilde{A}}$

$$\begin{aligned}\mu_{\tilde{mbr}(\tilde{A})}(p) &= -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}\\ &\quad \text{with A,B,C,D defined as in 5.2, 5.3, 5.4, 5.5}\end{aligned}$$

Each of the above triangles is part of a polygon in three dimensional space. These polygons are planar, as illustrated in 5.30 and satisfy (by construction) the MBR requirement. Consequently, the property will be satisfied for each point inside the triangle. □

Implementation The construction of the fuzzy MBR starts with considering all the α-levels occurring in each of the data points. For the α-cut on each of these α-levels a crisp MBR is considered. Assigning each point of each MBR the maximum of the α-levels that were considered and applying linear interpolation between the different α-levels yields the fuzzy MBR. The algorithm is supplied in pseudo code.

```
Fuzzy_TIN FuzzyMBR(Fuzzy_TIN A)
VAR alphaset,pointset,edgeset
BEGIN
For all datapoints p in A
  add mu_A(p) to alphaset
endfor

For all alpha in alphaset, in increasing order
```

```
  polygon B = WeakAlpha(A,alpha)
  consider MBR(B), this yields 4 points and 4 edges

  For each of the 4 points of MBR(B)
    if point p already in pointset
      mu_FuzzyMBR(p) = alpha
    else
      add point p to pointset
      mu_FuzzyMBR(p) = alpha
    endif

    if alpha <> lowest alpha-value in alphaset
      find cornerpoints of previous mbr-alpha-level
      if current top left <> previous top left
        add edge(current top left, previous level top left)
         to edgeset
      if current top right <> previous top right
        add edge(current top right, previous level top rigt)
         to edgeset
      if current bottom left <> previous bottom left
        add edge(current bottom left, previous level bottom left)
         to edgeset
      if current top right <> previous top right
        add edge(current bottom left, previous level bottom left)
         to edgeset
    endif
  endfor


  For each of the 4 new edges e of MBR(B)
  if edge e(p1,p2) is part of
        an existing edge e2(p1,p3) in edgeset
    remove e2(p1,p3) from edgeset
     add edge e(p2,p3) to edgeset
     add edge e(p1,p2) to edgeset
    else
      add edge e(p1,p2) to edgeset
    endif
  endfor

return Constrained_Delaunay(pointset,edgeset)
END
```

Note that the fuzzy MBR will always be simple (53): the center will hold the highest membership grades, the outline will have the lowest membership grades and membership grades will decrease from the centre outwards. In general, the fuzzy MBR yields an infinite number of crisp MBRs for a given fuzzy region. Consequently, the fuzzy MBR as such might not be as applicable for indexing as a crisp MBR would in a traditional geographic database. However, one can consider only using a limited number of α-cuts of MBRs (i.e. α-cuts at the membership grades of points in P).

Convex hull

The concept of the fuzzy convex hull of a fuzzy TIN $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ is that it yields a new fuzzy TIN which holds the convex hull of the fuzzy TIN $\tilde{A}$ at each α-level. To define this, the convex hull at each α-level for which there is a data point is considered; the convex hull at each α-level will have a (convex) polygon as its outline.

The first step in finding the fuzzy TIN that represents the convex hull of a given fuzzy TIN, is similar to finding the fuzzy MBR: determining the set P, the set E and the mapping function. This is done by considering a horizontal plane at each α-level for which there is a datapoint with $\mu_A(p) = f_A(p) = \alpha$. Considering the TIN in three dimensions, each of these α-planes will intersect the fuzzy TIN at the datapoint p (most likely other points will be in the plane also), but also at a number of line segments; thus forming a polygon in the horizontal plane. In each plane; the convex hull of this polygon is constructed.

1. As a first step, only the datapoints that will be part of the convex hull of some α-level are added to P

$$\begin{aligned} P_{t_1} = \{ & p \mid p \in P_{\tilde{A}} \wedge \\ & p \text{ is corner of the convex hull in the plane } z = f_{\tilde{A}}(p)\} \end{aligned}$$

2. Next, all the points that are an intersection point between an edge of the TIN and the horizontal plane, and that are corner points of the polygon in the horizontal plane are added to P.

$$\begin{aligned} P_{t_2} = \{ & p \mid p \notin P_{\tilde{A}}, \exists e(p_1, p_2) : p \in e, \exists p_3 \in P_{\tilde{A}} : \\ & p \text{ corner of the convex hull in the plane } z = f(p_3)\} \end{aligned}$$

The set P is the union of P_{t_1} and P_{t_2}: $P = P_{t_1} \cup P_{t_2}$. The mapping function f is defined as:

$$\begin{aligned} f : P & \rightarrow [0,1] \\ p(x,y) & \mapsto \mu_{\tilde{A}}(p(x,y)) \end{aligned}$$

The set E is also defined in two steps.

1. First, all the edges of the (convex) polygons in the different α-planes are considered.

$$\begin{aligned} E_{t_1} = \{ & e(p_1, p_2) \mid p_1, p_2 \in P, f(p_1) = f(p_2) \wedge \\ & e(p_1, p_2) \text{ is a side of the convex polygon} \\ & \text{in the plane } z = f(p_1)\} \end{aligned}$$

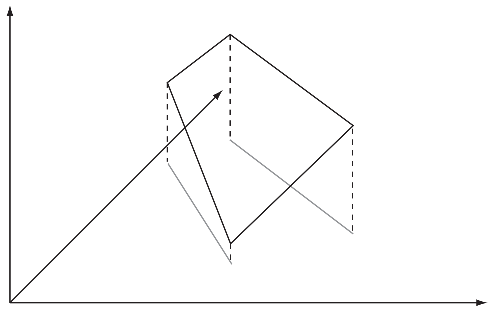


Figure 5.15: The case yielding a hyperbolic paraboloid.

2. Next, edges between the polygons at different α-levels need to be defined.

$$
\begin{aligned}
E_{t_2} \quad = \quad & \{e(p_1, p_2) \mid p_1, p_2 \in P, f(p_1) \neq f(p_2) \wedge \\
& \exists e_{\tilde{A}} \in E_{\tilde{A}} : \; e(p_1, p_2) \subseteq e_{\tilde{A}}\}
\end{aligned}
$$

The set E is the union of E_{t_1} and E_{t_2}: $E = E_{t_1} \cup E_{t_2}$. The sets P and E, and the mapping function f are now used as arguments for a constrained Delaunay triangulation, which will yield the fuzzy convex hull of the given fuzzy TIN $\tilde{A}$.

Non planar results The above construction yields polygons consisting of four vertices. However, unlike in the case of the fuzzy minimum bounding rectangle, the situation here can yield non-planar sections. This occurs when no plane can be found in which all four vertices are situated. The result in such situations is a curved plane, called a hyperbolic paraboloid and is illustrated on fig. 5.15. This curved plane basically can be defined as a ruled surface, by moving one line along two other non equidistant lines (that are not situated in the same plane) in a three dimensional space. While it is possible to define such a plane mathematically, incorporating this in the model would make the TIN model no longer closed for convex hull. To overcome this, the convex hull will be approximated. The accuracy of the approximation depends on the requirements of the user and should be chosen accordingly. The above definitions provide for the most coarse approximation (the hyperbolic paraboloid will be approximated by two triangles), but by adding points (and edges), the approximation can be refined.

As it makes sense for the convex hull to be exact for all the α-levels that occur in the datapoints of the original TIN, this case is added as an illustration. To achieve this, points which on the edges connecting the polygons at different levels (edges in E_{t_2}) are added.

$$
\begin{aligned}
P_{t_3} \quad = \quad & \{p \mid \exists p' \in P_{\tilde{A}}, \exists \alpha \in f_{\tilde{A}}(p') : p \in H_\alpha \wedge \\
& \exists e(p_1, p_2), e(p_3, p_4), e(p_1, p_3), e(p_2, p_4) \in E : p \in e(p_1, p_2) \wedge \\
& p_1, p_2, p_3, p_4 \text{ are not planar } \}
\end{aligned}
$$

Where H_α is the notation for the horizontal plane $z = \alpha$.

Originally, The set P held all the points that are corner-points of a convex hull at each considered α-level. The set P_{t_3} results in more α-levels being considered, yielding α an level for every associated value of the datapoints.

This is just one illustration, one is free to add more points (any intersection with a horizontal plane yields a convex polygon at this level), should a more accurate approximation be required.

The set P now is the union of P_{t_1}, P_{t_2} and P_{t_3}: $P = P_{t_1} \cup P_{t_2} \cup P_{t_2}$. The mapping function f is defined using the newly obtained set P as domain:

$$\begin{array}{rcl} f : P & \rightarrow & [0,1] \\ p(x,y) & \mapsto & \mu_{\tilde{A}}(p(x,y)) \end{array}$$

Using the new set P, set set of edges is defined similarly as before:

1. First, all the edges of the (convex) polygons in the different α-planes are considered.

$$\begin{array}{rcl} E_{t_1} & = & \{e(p_1,p_2) \mid p_1,p_2 \in P, f(p_1) = f(p_2) \wedge \\ & & e(p_1,p_2) \text{ is a side of the convex polygon} \\ & & \text{in the plane } z = f(p_1)\} \end{array}$$

2. Next, edges between the polygons at different α-levels need to be defined.

$$\begin{array}{rcl} E_{t_2} & = & \{e(p_1,p_2) \mid p_1,p_2 \in P, f(p_1) \neq f(p_2) \wedge \\ & & \exists e_{\tilde{A}} \in E_{\tilde{A}} : \ e(p_1,p_2) \subseteq e_{\tilde{A}}\} \end{array}$$

Similar as before, the set E_{t_1} contains the edges of the convex polygons at every level, whereas the set E_{t_2} contains the edges that connect polygons at subsequent levels. Again, the triangular network representing the approximated convex hull is obtained by performing a constraint Delaunay triangulation on the newly obtained sets P and E.

Implementation The construction of the fuzzy convex hull starts with considering all the α-levels occurring in each of the data points. For the α-cut on each of these α-levels a crisp MBR is considered. Assigning each point of each MBR the maximum of the α-levels that were considered and applying linear interpolation between the different α-levels yields the fuzzy MBR. The algorithm for the refined definition is supplied in pseudo code.

```
Fuzzy_TIN FuzzyConvexHull(Fuzzy_TIN A)
VAR alphaset,pointset,edgeset,pointtemp
BEGIN
For all datapoints p in P_A
  add mu_A(p) to alphaset
end for
```

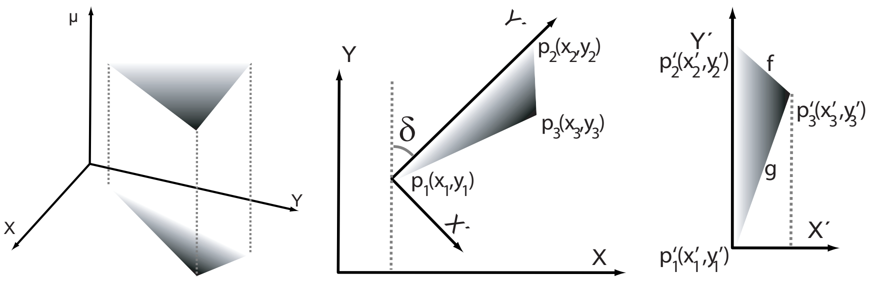
```
For alpha in alphaset, in increasing order
For all points p in P_A
   if alpha = mu_A(p) add p to pointtemp
   end for

For all edges e in E_A
  if e intersects z=mu_A(p)
    find intersection point p with z=alpha
    add intersection point p to pointtemp
  end if
end for

find the convex hull of the points in pointtemp

For all points p in pointtemp
  if p is a corner of the convex hull add p to pointset
  if (p,p1) is an edge of the convex hull add e(p,p1) to edgeset
end for
clear pointtemp
end for

  For all points p1 in pointset
    mu_FuzzyConvexHull(p1) = mu_A(p1)
    if e(p1,p2) part of an edge in E_A and there is
          no point p3 on e
      add e(p1,p2) to edgeset
    end if
  end for

return Constrained_Delaunay(pointset,edgeset)
END
```

To add additional points for refinement, it suffices to add the α levels at which the convex hull should be accurate to `alphaset`.

Surface area

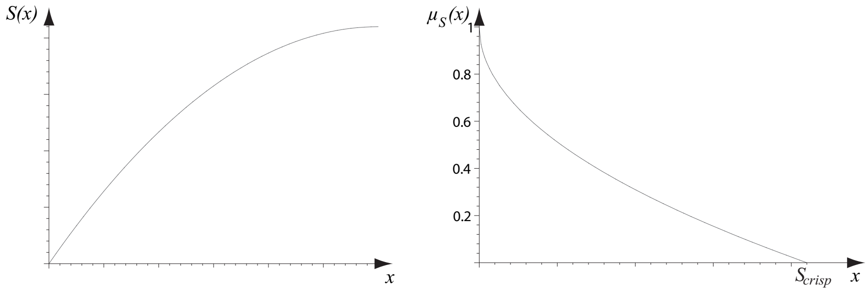
As in the theoretical model, the surface area has two possible interpretations: as a fuzzy number, or as a reflection of fuzzy cardinality.

Similar to the theoretical model (2.4.2) and the bitmap model (4.4.3), two interpretations for the calculation of the surface area are considered. In the first interpretation, the surface is interpreted as a measurement for the surface area; for a fuzzy region, this will yield a fuzzy number. In the second interpretation, the surface area is considered as an extension of fuzzy cardinality; the surface area of a fuzzy region related to the cardinality of a fuzzy set and will yield a



(a) (b) (c)

Figure 5.16: Transforming a triangle to calculate its fuzzy surface.



(a) (b)

Figure 5.17: Calculation of the fuzzy surface area. (a) result of $\int(f) - \int(g)$, (b) transformation of the result in (a) to a fuzzy number.

crisp number [32].

Fuzzy result The fuzzy surface area of a fuzzy region $\tilde{A}$ represented by a TIN, will be a fuzzy number. This number is a summation for all the fuzzy surfaces of each triangle; the calculation will be given for a single triangle. Using fuzzy arithmetic [32], the different surface areas can be added up.

To explain the calculation of the fuzzy area in the first interpretation, the algorithm is first performed on a triangle, as illustrated in fig. 5.16a. For every triangle of the TIN, there are two possibilities: the triangle (considered in three dimensional space with the associated membership grades along the Z-axis) is parallel to the XY-plane, or the triangle is not parallel to the XY plane. For the sake of argumentation, consider a triangle defined by three corner points $p_1(x_1, y_1)$, $p_2(x_2, y_2)$ and $p_3(x_3, y_3)$, with associated membership grades respectively $\mu_1 = f_{\tilde{A}}(p_1)$, $\mu_2 = f_{\tilde{A}}(p_2)$ and $\mu_3 = f_{\tilde{A}}(p_3)$. The first step in calculating the area is to determine the smallest membership grade that occurs in the triangle; this level is found by considering the minimum of the membership grades of the three corner points.

$$\alpha_{min} = \min(\mu_1, \mu_2, \mu_3) \tag{5.31}$$

Next, the greatest membership grade that occurs in the triangle must be calculated; this level is found by considering the maximum of all the α levels at the three corner points.

$$\alpha_{max} = \max(\mu_1, \mu_2, \mu_3) \tag{5.32}$$

If $\alpha_{min} = \alpha_{max}$, then the triangle is parallel to the XY-plane. The fuzzy surface area of this triangle is then the crisp surface area, but with membership grade equal to α_{min}.

If $\alpha_{min} \neq \alpha_{max}$, then suppose that $\mu_1 = \alpha_{min}$ and $\mu_3 = \alpha_{max}$. For μ_2 there are three possibilities: $\mu_2 = \mu_1$, $\mu_2 = \mu_3$ and $\mu_1 < \mu_2 < \mu_3$. Only the first case will be explained in further detail, but the other cases are analogous.

If $\mu_2 = \mu_1$, this means that the triangle has the edge e(p_1,p_2) parallel to the XY plane, and points on this edge have the lowest membership grades (equalling α_{min}) that occur in this triangle. This knowledge yields us some indication of the fuzzy surface area: at membership grade α_{min}, the surface equals the surface area of the triangle in the XY plane. At membership grade α_{max}, there is only the point p_3; the area at this α-level equals 0.

To calculate the surface, consider the triangle formed by the datapoints $p_1(x_1, y_1)$, $p_2(x_2, y_2)$, $p_3(x_3, y_3)$ in the XY plane. For ease of calculation, a coordinate transformation - as shown in fig. 5.16b - will be applied: the X- and Y-axes are chosen such that edge e(p'_1, p'_2) is on the Y-axis ($x'_1 = x'_2 = 0$) and the X-axis is perpendicular to it and passes through $\min(y'_1, y'_2, y'_3) = 0$. The transformation implies a rotation with an angle δ

$$\delta = \begin{cases} 0 & x_1 = x_2 \\ -\frac{\pi}{2} - \frac{y_2 - y_1}{x_2 - x_1} & x_1 \neq x_2 \end{cases} \tag{5.33}$$

This rotation is followed by a translation along X-axis over $-x_1$ such that $x'_1 = x'_2 = 0$ and a translation along the Y-axis over $\min(y_1, y_2, y_3)$, such that $\min(y'_1, y'_2, y'_3) = 0$ (in this case $y'_1 = 0$). The transformed points of the triangle are:

$$p'_1(0, x_1 \cos\delta - y_1 \sin\delta - \min(y_1, y_2, y_3)) \tag{5.34}$$
$$p'_2(0, x_2 \cos\delta - y_2 \sin\delta - \min(y_1, y_2, y_3)) \tag{5.35}$$
$$p'_3(x_3 \cos\delta - y_3 \sin\delta - x_1, x_3 \cos\delta - y_3 \sin\delta - \min(y_1, y_2, y_3)) \tag{5.36}$$

Next, the equations for the remaining edges e(p'_2, p'_3) and e(p'_1, p'_3)of the triangle are constructed in this new coordinate system, fig. 5.16c :

$$\begin{aligned} f(x) &:= \frac{y'_3 - y'_2}{x'_3 - x'_2}(x - x'_2) + y'_2 \\ &= \frac{y'_3 - y'_2}{x'_3} x + y'_2 \quad (\text{as } x'_2 = 0) \end{aligned}$$

$$g(x) := \frac{y'_3 - y'_1}{x'_3 - x'_1}(x - x'_1) + y'_1$$

$$= \frac{y'_3}{x'_3}x \quad (\text{as } x'_2 = 0, y'_1 = 0)$$

As p'_1, p'_2 and p'_3 form a triangle, and both p'_1 and p'_2 are on the Y-axis, neither p'_2, p'_3 nor p'_1, p'_3 can be parallel to the Y-axis. The next step to calculate the surface area of the triangle of interest, is determining the surface area below f and g.

$$\begin{aligned}
\int_x^{x'_3} f(x)dx &= \int_x^{x'_3} \frac{y'_3 - y'_2}{x'_3}x + y'_2 \\
&= \frac{(y'_3 - y'_2)(x'^2_3 - x^2)}{2x'_3} + y'_2(x'_3 - x) \\
\int_x^{x'_3} g(x)dx &= \int_x^{x'_3} \frac{y'_3}{x'_3}xdx \\
&= \frac{y'_3(x'^2_3 - x^2)}{2x'_3}
\end{aligned}$$

The above functions represent the surface area below them as a function of the distance to the edge $e(p1, p2)$. Subtracting both functions will yield the surface in between, which is plotted in fig.5.17a.

$$\int_x^{x'_3} f(x)dx - \int_x^{x'_3} g(x)dx = S(x) = \frac{x'_3 y'_2}{2} + \frac{y'_2 x^2}{2x'_3} - y'_2 x \tag{5.37}$$

As a next step, this function - which basically represents the surface information along the Y-axis - needs to be inverted: the fuzzy number representing the surface should have surface information along the X-axis (and membership information on the Y-axis). To accomplish this, the quadratic equation needs to be solved. It yields two solutions:

$$S_1(x) = \frac{x'_3 y'_2 + \sqrt{2y'_2 x'_3 x}}{y'_2} \tag{5.38}$$

$$S_2(x) = \frac{x'_3 y'_2 - \sqrt{2y'_2 x'_3 x}}{y'_2} \tag{5.39}$$

Only one of the solutions is valid for our purpose, more specifically the one for which $S_i(S_{crisp}) = 0$: the largest surface area occurs when the triangle in fig. 5.16c is considered entirely: from $x' = 0$. For this particular case, it is $S_2(x)$. Currently, it represents the x-values that match changing surface areas. But it is also known how the membership values in the triangle evolve with changing x-values (as it is a linear interpolation), this is modeled by this equation (for this triangle):

$$membership(x) := \frac{\alpha_{max} - \alpha_{min}}{x'_3} x + \alpha_{min} \tag{5.40}$$

So applying this on top of the original equation, gives us the membership for each matching surface area: the fuzzy surface area of this triangle. The fuzzy surface for this triangle therefore is:

$$\begin{aligned}
\tilde{S} &= membership(S_2(x)) && (5.41)\\
&= \frac{\alpha_{max} - \alpha_{min}}{x'_3} S_2(x) + \alpha_{min} && (5.42)\\
&= \frac{\alpha_{max} - \alpha_{min}}{x'_3} \frac{x'_3 y'_2 - \sqrt{2y'_2 x'_3 x}}{y'_2} + \alpha_{min} && (5.43)
\end{aligned}$$

The fuzzy number is shown in fig. 5.17b.

As mentioned before, there are other possibilities for the value of μ_2. If $\mu_2 = \mu_3$ than the triangle has the edge p_2-p_3 parallel to the XY plane, and points on this edge have the highest membership grades (equalling α_{max}) that occur in this triangle. The calculation is completely analogous to the one made above.

If $\mu_1 < \mu_2 < \mu_3$, the triangle p_1, p_2, p_3 can be divided on the α-level μ_2: the edge $p_1 - p_3$ must contain a point with level μ_2, consider this point p_4. The calculation of the surface is now done for both triangles p_1, p_2, p_4 and p_2, p_3, p_4 using the above algorithms and the results are combined (either as a piecewise function or by treating them as two separate triangles).

Obviously, the calculation of the surface is a very computationally intensive operation. The above formula only yields the fuzzy surface area for a single triangle. In practice, if this calculation is often required, the calculation can be optimized by for instance storing the fuzzy surface of each TIN - or even of each triangle - in the database.

Implementation

```
fuzzy real Triangle_surface1(p_1, p_2, p_3)
BEGIN
   transformation of p_1, p_2, p_3 as in 5.11, 5.12 and 5.13
   determination of the correct equation (5.18 or 5.19)
   calculate as in 5.20
END

fuzzy real Triangle_surface2(p_1, p_2, p_3)
BEGIN
   transformation of p_1, p_2, p_3 as in 5.11, 5.12 and 5.13
   determination of the correct equation similar to 5.18 or 5.19
   calculate similarly to the formula in 5.20
```

```
END


fuzzy real Fuzzy_surface(fuzzy_tin A)
  fuzzy real S_triangle, S_total
BEGIN
  for each triangle t(p_1(x_1,y_1),p_2(x_2,y_2),p_3(x_3,y_3))
        of A
    if mu_A(p_1) = mu_A(p_2) = mu_A(p_3)
      S_triangle = { (crisp_surface(p_1,p_2,p_3), grade) }
    end if

    p'_1 = point with lowest membership grade
    mu_A(p'_1) = min(mu_A(p_1),mu_A(p_2),mu_A(p_3))
    p'_3 = point with highest membership grade
    mu_A(p'_3) = max(mu_A(p_1),mu_A(p_2),mu_A(p_3))
    p'_2 = point with mu_A(p'_1) < mu_A(p'_2) < mu_A(p'_3)


    if mu_A(p'_1) = mu_A(p'_2) and mu_A(p'_3) > mu_A(p'_1)
      triangle_surface1(p'_1,p'_2,p'_3)
    end if

    if mu_A(p'_1) = mu_A(p'_2) and mu_A(p'_3) < mu_A(p'_1)
      triangle_surface2(p'_1,p'_2,p'_3)
    end if

    if mu_A(p'_1)<>mu_A(p'_2) and mu_A(p'_1)<>mu_A(p'_3) and
          mu_A(p'_2)<>mu_A(p'_3)
      find p_4 on e(p'_1,p'_3) such that mu_A(p_4)=mu_A(p'_2)
      S_triangle = triangle_surface1(p'_1,p'_2,p'_4)
                     + triangle_surface2(p'_3,p'_2,p'4)
    end if

  S_total = S_total + S_triangle
  end for

  return S_total
END
```

Crisp result As mentioned before, the surface area of a region can also be considered to be an extension of cardinality (in a sense, one can consider this operation to count the number of points in the region). For a fuzzy region, the surface area for each membership grade is considered, and this area counts for a given portion (indicated by the membership grade). This interpretation for the surface calculation is related to the notion of fuzzy cardinality [32], which yields a crisp number as result. In this interpretation, one can say that each point contributes to some extent to the total surface area; its membership grade determines the extent to which a point contributes.

This interpretation is easily illustrated when considering triangles whose corner points have the same membership grade. If they all are equal to 1, the entire surface area of the triangle counts. If they all are equal to 0.5, only half of the surface area of the triangle counts (the points only belong to the extent 0.5 to the triangle). In general, this single surface value is obtained by considering the total area below the fuzzy number representing the surface area in the previous interpretation; this can be calculated by integrating the membership function which represents the fuzzy number as obtained in the previous interpretation between the values 0 and the total surface area of the triangle.

$$\int_0^{S_{max}} \frac{-b+\sqrt{b^2-4ac}}{2a} dS \tag{5.44}$$

where

$$\begin{aligned}
a &= -\frac{(y_2-y_1)}{2x_3} \\
b &= y_2-y_1 \\
c &= \frac{\alpha_{min}(-2x_3+\alpha_{min})(y_2-y_1)}{2x_3} - S \\
S_{max} &= \frac{y_2' x_3'}{2}
\end{aligned}$$

(S_{max} is the crisp surface area of the triangle p_1, p_2, p_3)

Implementation In the above equation, all variables are numeric values (apart from S). It suffices to perform the integration, after which S can be computed easily. The surface for the different triangles are then added up, an operation for which we refer to fuzzy arithmetic.

Distance calculation

As mentioned in the theoretical concepts (see 2.4.2), the distance between fuzzy regions can be interpreted in different ways.

α-level approach The distance between fuzzy regions will be represented by a fuzzy number. This number will be constructed from the distances between the different α-levels.

Consider two fuzzy TINs

$$\begin{aligned}
\tilde{A} &= [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}] \\
\tilde{B} &= [(P_{\tilde{B}}, E_{\tilde{B}}, T_{\tilde{B}}), f_{\tilde{B}}]
\end{aligned}$$

Next, consider all the α-levels for which there are datapoints in either TIN:

$$L = \{f_{\tilde{A}}(p) \mid \exists p \in P_{\tilde{A}}\} \cup \{f_{\tilde{B}}(p) \mid \exists p \in P_{\tilde{B}}\} \tag{5.45}$$

Name them α_0, α_i, α_{i+1}, ... αn, such that $\forall i : \alpha_i < \alpha_{i+1}$.

For every α-level $\alpha_i \in L$, determine $d_i = d(\tilde{A}_{\alpha_i}, \tilde{B}_{\alpha_i})$ and $d_{\bar{i}} = d(\tilde{A}_{\overline{\alpha_i}}, \tilde{B}_{\overline{\alpha_i}})$; as the α-cut of a fuzzy region is crisp region, the calculation of this distance is known. The $d_{\bar{i}}$ is required to provide a correct result should one of the TINs contain a triangle for which all three datapoints have the same membership grade.

The distance is now represented by:

Definition 101 (distance between two fuzzy TINs in the α-level approach)

$$\tilde{d}(\tilde{A}, \tilde{B}) = \{(x, \mu_{\tilde{d}(\tilde{A},\tilde{B})}(x))\} \tag{5.46}$$

where

$$\begin{aligned} \mu_{\tilde{d}(\tilde{A},\tilde{B})} : \mathbb{R} &\rightarrow [0,1] \\ x &\mapsto \begin{cases} \alpha_i & \text{if } d_i \le x \le d_{\bar{i}} \\ \frac{\alpha_{i+1}-\alpha_i}{d_{i+1}-d_{\bar{i}}}(x - d_{\bar{i}}) + \alpha_i & \text{if } d_{\bar{i}} < x < d_{i+1}, \forall i = 0..n-1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

The distance is an approximation for α-levels that do not occur as the associated membership grade for a datapoint in any of the two arguments. The approximation can be made accurate for any desired α-level, by considering it in the set L prior to determining the distances.

Verification It can easily be seen that the above definition matches the theoretical definition for distances that are the shortest distance between datapoints of one region and datapoints of the other region.

Implementation

```
fuzzy real Distance(fuzzy_tin A, fuzzy_tin B)
VAR alphaset
    fuzzy real: result
BEGIN
For all datapoints p in P_A
  add mu_A(p) to alphaset
end for
For all datapoints p in P_B
  add mu_B(p) to alphaset
end for
```

```
For alpha in alphaset, in increasing order
  add (d(weakalpha(A,alpha), weakalpha(B,alpha)),alpha) to result
  determine sd_i(strongalpha(A,alpha), strongalpha(B,alpha))

end for

  For successive d_i,d_j in result
    for x in [sd_i,d_j[
      add (y=(result(d_j) - result(sd_i))/(d_j-sd_i) (x-sd_{i})
              +result(sd_i)) to result
    end for
  end for

  return result;

END
```

Topological approach

Definition 102 (distance between two fuzzy TINs)
(in the topological approach)

$$\tilde{d}^{\Delta}(\tilde{A}, \tilde{B}) = \{(x, \mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})}(x))\} \tag{5.47}$$

where

$$\mu_{\tilde{d}^{\Delta}(\tilde{A},\tilde{B})} : \mathbb{R} \rightarrow [0,1] \tag{5.48}$$
$$x \mapsto \mu_{\tilde{d}(\tilde{A}^{\circ},\tilde{B}^{\circ})}(x) \tag{5.49}$$

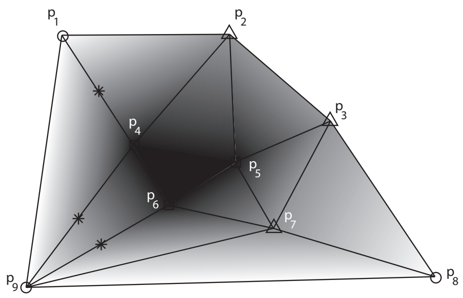
5.4.4 Topology

To determine the topology for fuzzy regions, first the concepts interior, exterior and boundary of a fuzzy TIN regions need to be defined. To illustrate, consider the fuzzy region $\tilde{A}$ as shown in fig. 5.18

Boundary Determining the fuzzy boundary $\Delta\tilde{A}$ of a fuzzy region $\tilde{A}$, represented by a TIN $[(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ is similar to finding the fuzzy MBR: first the set of datapoints P of the result is determined, then the set of edges E, after which a constrained Delaunay triangulation is applied.

1. First, the datapoints points $p \in P_{\tilde{A}}$ are considered.

$$P_{t_1} = \{p \mid p \in P_{\tilde{A}}\}$$



(a)

$$\begin{aligned}
\mu_{\tilde{A}}(p_1) &= 0\\
\mu_{\tilde{A}}(p_2) &= 0.5\\
\mu_{\tilde{A}}(p_3) &= 0.5\\
\mu_{\tilde{A}}(p_4) &= 1\\
\mu_{\tilde{A}}(p_5) &= 1\\
\mu_{\tilde{A}}(p_6) &= 1\\
\mu_{\tilde{A}}(p_7) &= 0.5\\
\mu_{\tilde{A}}(p_8) &= 0\\
\mu_{\tilde{A}}(p_9) &= 0
\end{aligned}$$

(b)

Figure 5.18: A fuzzy TIN (a) and a list of the datapoints (b).

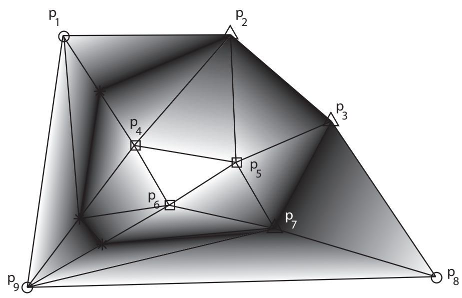


Figure 5.19: The fuzzy boundary of the fuzzy TIN in 5.18.

2. The α-level for $\alpha = 0.5$ is crucial: points with membership grades less than 0.5 are treated differently from points with membership grades greater than 0.5. For this, outline (defined by both datapoints and intersection points) of all the points with membership grade 0.5 need to be determined, so the intersection points between edges of the TIN and a horizontal plane at level $z = 0.5$ are added (these are indicated in fig. 5.18 by $*$). The datapoints for edges that are completely located in the plane $z = 0.5$, have been considered in the previous step.

$$P_{t_2} \quad = \quad \{p \mid \exists e(p_1, p_2) \in E_{\tilde{A}} : p \in e(p_1, p_2) \wedge \mu_{\tilde{A}}(p) = 0.5\}$$

The set of datapoints $P_{\Delta\tilde{A}}$ for the fuzzy boundary of $\tilde{A}$ is the union of the above sets:

$$P_{\Delta\tilde{A}} = P_{t_1} \cup P_{t_2} \tag{5.50}$$

The mapping function $f_{\Delta\tilde{A}}$ should map the membership grades in accordance with the definition of the fuzzy boundary (2.4.3) and is therefore defined as

$$\begin{aligned}
f_{\Delta\tilde{A}} : P_{\Delta\tilde{A}} &\rightarrow [0, 1]\\
p &\mapsto 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|)
\end{aligned}$$

Next, the set of edges E must be determined.

1. First, all the edges that existed in the original network, for which no newly added point is on the edge are added. Bear in mind that the TIN is defined in two dimensions; the altered associated value for the datapoints does not play a part here.

$$\begin{aligned} E_{t_1} &= \{e(p_1, p_2)|p_1, p_2 \in P \wedge e(p_1, p_2) \in E_{\tilde{A}} \wedge \forall p \in P : \\ &\quad p \neq p_1 \wedge p \neq p_2 \wedge p \notin e(p_1, p_2)\} \end{aligned}$$

2. Next, all the edges that are part of an existing edge (in E_A) and that connect an intersection point (in P_{t_2}) with a datapoint from $P_{\tilde{A}} = P_{t_1}$ are added.

$$E_{t_2} = \{e(p_1, p_2)|p_1 \in P_{t_1}, p_2 in P_{t_2} \wedge \exists e \in E_{\tilde{A}} : e(p_1, p_2) \subseteq e\}$$

The set of edges E that will be used to constrain the triangulation is $E_{\Delta\tilde{A}} = E_{t_1} \cup E_{t_2}$.

Verification To verify that the above definitions yield the fuzzy boundary of a fuzzy TIN, while matching the theoretical definition, consider a fuzzy TIN $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$. It is now necessary to prove that $\mu_{\Delta\tilde{A}}(p) = 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|)$, for all points in the TIN.

1. For $p \in P_{\Delta\tilde{A}}$, this requirement is immediately fulfilled, due to the definition of both $P_{\Delta\tilde{A}}$ and the definition of the mapping function $f_{\Delta\tilde{A}}$.
2. For $p \notin P_{\Delta\tilde{A}}$, it is necessary to prove that $\mu_{\Delta\tilde{A}} = 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|)$. A triangle in $\Delta\tilde{A}$ is part of a triangle of $\tilde{A}$ (bear in mind that the triangulation process happens in two dimensions): only the associated values differ. The coefficients of the equation of a plane in the boundary are denoted A^Δ, B^Δ, C^Δ and D^Δ (100); the associated membership grade for a datapoint p_i is $f_{\Delta\tilde{A}}(p_i) = \mu_{\Delta\tilde{A}}(p_i) = z_i^\Delta$.
   - Suppose $\mu_{\tilde{A}}(p) \geq 0.5$, then $0.5 - \mu_{\tilde{A}}(p) \leq 0$ and $2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) = 2(0.5 + (0.5 - \mu_{\tilde{A}}(p))) = 2 - 2\mu_{\tilde{A}}(p)$
     Consider the coefficient A^Δ:

$$\begin{aligned} A^\Delta &= y_1(z_2^\Delta - z_3^\Delta) + y_2(z_3^\Delta - z_1^\Delta) + y_3(z_1^\Delta - z_2^\Delta) \\ &= y_1(2 - 2z_2 - (2 - 2z_3)) + y_2(2 - 2z_3 - (2 - 2z_1)) \\ &\quad + y_3(2 - 2z_1 - (2 - 2z_2)) \\ &= y_1(2 - 2z_2 - 2 + 2z_3) + y_2(2 - 2z_3 - 2 + 2z_1) \\ &\quad + y_3(2 - 2z_1 - 2 + 2z_2) \\ &= y_1(-2z_2 + 2z_3) + y_2(-2z_3 + 2z_1) + y_3(-2z_1 + 2z_2) \\ &= -2(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)) \\ &= -2A \end{aligned}$$

Consider B^Δ:

$$\begin{aligned}
B^\Delta &= z_1^\Delta(x_2 - x_3) + z_2^\Delta(x_3 - x_1) + z_3^\Delta(x_1 - x_2) \\
&= (2 - 2z_1)(x_2 - x_3) + (2 - 2z_2)(x_3 - x_1) \\
&\quad + (2 - 2z_3)(x_1 - x_2) \\
&= 2((1 - z_1)(x_2 - x_3) + (1 - z_2)(x_3 - x_1) \\
&\quad + (1 - z_3)(x_1 - x_2)) \\
&= 2((x_2 - x_3) - z_1(x_2 - x_3) + (x_3 - x_1) - z_2(x_3 - x_1) \\
&\quad + (x_1 - x_2) - z_3(x_1 - x_2)) \\
&= 2((x_2 - x_3) + (x_3 - x_1) + (x_1 - x_2) - z_1(x_2 - x_3) \\
&\quad - z_2(x_3 - x_1) - z_3(x_1 - x_2)) \\
&= -2(z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\
&= -2B
\end{aligned}$$

Consider C^Δ.

$$\begin{aligned}
C^\Delta &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\
&= C
\end{aligned}$$

Consider D^Δ:

$$\begin{aligned}
D^\Delta &= -A^\Delta x_1 - B^\Delta y_1 - C^\Delta z_1^\Delta \\
&= 2Ax_1 + 2By_1 - C(2 - 2z_1) \\
&= -2(-Ax_1 - By_1 + C(1 - z_1)) \\
&= -2(-Ax_1 - By_1 - Cz_1 + C) \\
&= -2(-Ax_1 - By_1 - Cz_1) - 2C \\
&= -2D - 2C
\end{aligned}$$

$$\begin{aligned}
\mu_{\Delta\tilde{A}}(p(x,y)) &= -\frac{A^\Delta}{C^\Delta}x - \frac{B^\Delta}{C^\Delta}y - \frac{D^\Delta}{C^\Delta} \\
&= -\frac{A^\Delta}{C}x - \frac{B^\Delta}{C}y - \frac{D^\Delta}{C} \\
&= -\frac{-2A}{C}x - \frac{-2B}{C}y - \frac{-2D - 2C}{C} \\
&= -2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) + \frac{2C}{C} \\
&= 2 - 2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) \\
&= 2 - 2\mu_{\tilde{A}}(p)
\end{aligned}$$

- Suppose $\mu_{\tilde{A}}(p) < 0.5$, then $0.5 - \mu_{\tilde{A}}(p) > 0$ and $2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) = 2(0.5 - (0.5 - \mu_{\tilde{A}}(p))) = 2\mu_{\tilde{A}}(p)$
Consider A^{Δ}:

$$\begin{aligned} A^{\Delta} &= y_1(z_2^{\Delta} - z_3^{\Delta}) + y_2(z_3^{\Delta} - z_1^{\Delta}) + y_3(z_1^{\Delta} - z_2^{\Delta}) \\ &= y_1(2z_2 - 2z_3) + y_2(2z_3 - 2z_1) + y_3(2z_1 - 2z_2) \\ &= 2(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)) \\ &= 2A \end{aligned}$$

Consider B^{Δ}:

$$\begin{aligned} B^{\Delta} &= z_1^{\Delta}(x_2 - x_3) + z_2^{\Delta}(x_3 - x_1) + z_3^{\Delta}(x_1 - x_2) \\ &= 2z_1(x_2 - x_3) + 2z_2(x_3 - x_1) + 2z_3(x_1 - x_2) \\ &= 2(z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\ &= 2B \end{aligned}$$

Consider C^{Δ}:

$$\begin{aligned} C^{\Delta} &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= C \end{aligned}$$

Consider D_{Δ}:

$$\begin{aligned} D^{\Delta} &= -A^{\Delta}x_1 - B^{\Delta}y_1 - C^{\Delta}z_1^{\Delta} \\ &= -2Ax_1 - 2By_1 - C(2z_1) \\ &= 2(-Ax_1 - By_1 - Cz_1) \\ &= -2D \end{aligned}$$

$$\begin{aligned} \mu_{\Delta\tilde{A}}(p(x,y)) &= -\frac{A^{\Delta}}{C^{\Delta}}x - \frac{B^{\Delta}}{C^{\Delta}}y - \frac{D^{\Delta}}{C^{\Delta}} \\ &= -\frac{A^{\Delta}}{C}x - \frac{B^{\Delta}}{C}y - \frac{D^{\Delta}}{C} \\ &= -\frac{2A}{C}x - \frac{2B}{C}y - \frac{2D}{C} \\ &= 2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) \\ &= 2\mu_{\tilde{A}}(p) \end{aligned}$$

This implies that for all points, their membership grade in the boundary matches the membership grade as defined in the conceptual model (Chapter 2). $\square$

Implementation The fuzzy region (in TIN representation) of the boundary of $\tilde{A}$, is obtained by performing a constrained Delaunay triangulation on P and E. The pseudo code to find the fuzzy boundary of a fuzzy region is given below.

```
Fuzzy_TIN FuzzyBoundary(Fuzzy_TIN A)
VAR pointset,edgeset
BEGIN
For all datapoints p in P_A
  add p to pointset
  mu_DA(p)=2(0.5-abs(0.5-mu_A(p)))
end for

  For all edges e(p_1,p_2) in E_A
    if (mu_A(p_1) < 0.5 AND mu_A(p_2) > 0.5) OR
       (mu_A(p_1) > 0.5 AND mu_A(p_2) < 0.5)
      find point p on e with mu_A(p) = 0.5
      add p to pointset
      mu_DA(p) = 1
      add e(p_1,p) to edgeset
      add e(p_2,p) to edgeset
    else
      add e(p_1,p_2) to edgeset
    endif

  end for

  return constraint_Delaunay(pointset,edgeset)

END
```

Interior The interior of a fuzzy region $\tilde{A}$ is a new fuzzy region. It is determined similarly as the boundary: by first finding all the datapoints, all the edges, and defining the mapping function. The interior is illustrated on fig. 5.20.

1. First, the datapoints points p for which $f_{\tilde{A}}(p) \geq 0.5$ are considered.

$$P_{t_1} = \{p \mid p \in P_{\tilde{A}} \wedge f_{\tilde{A}}(p) \geq 0.5\}$$

2. Next, the intersection points between edges of the TIN and a horizontal plane at level $z = 0.5$ are added. The datapoints for edges that are completely located in the plane $z = 0.5$, have been considered in the previous step.

$$\begin{aligned} P_{t_2} = \{p \mid \exists e(p_1, p_2) \in E_{\tilde{A}} : p_1, p_2 \notin P_{t_1} \wedge \\ p \in e(p_1, p_2) \wedge \mu_{\tilde{A}}(p) = 0.5\} \end{aligned}$$

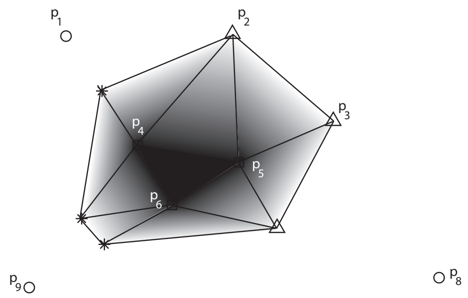


Figure 5.20: The fuzzy interior of the fuzzy TIN in 5.18; for illustration purposes all the datapoints of the original TIN are shown.

The union of these two sets yields the set of datapoints for the interior:

$$P_{\tilde{A}^\circ} = P_{t_1} \cup P_{t_2} \tag{5.51}$$

Next, the set of edges E must be determined.

1. First, all the edges that existed in the original network, and whose points are in P, are considered. Bear in mind that the TIN is defined in two dimensions; the altered associated value for the datapoints does not play a part here.

$$E_{t_1} = \{e(p_1, p_2) \mid p_1, p_2 \in P \wedge e(p_1, p_2) \in E_{\tilde{A}}\}$$

2. Next, all the edges that are part of an existing edge (in $E_{\tilde{A}}$) and that connect an intersection point (in P_{t_2}) with a datapoint from P_{t_1} are added.

$$E_{t_2} = \{e(p_1, p_2) \mid p_1 \in P_{t_1}, p_2 \in P_{t_2} \wedge \exists e \in E_{\tilde{A}} : e(p_1, p_2) \subseteq e\}$$

The union of E_{t_1} and E_{t_2} yields the set $E_{\tilde{A}^\circ}$ which will constrain the Delaunay triangulation.

The mapping function is defined to match the theoretical definition (2.4.3):

$$\begin{aligned} f_{\tilde{A}^\circ} : P_{\tilde{A}^\circ} &\rightarrow [0, 1] \\ p &\mapsto 1 - \mu_{\Delta\tilde{A}}(p) = 1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) \end{aligned}$$

As the set of points and edges for the interior is a subset of the set of points and edges for the boundary, it is possible to deduce the sets P and E from the TIN that represents the boundary:

$$P_{\tilde{A}^\circ} = \{p | p \in P_{\Delta\tilde{A}} \wedge \mu_{\tilde{A}}(p) \geq 0.5\} \tag{5.52}$$

$$E_{\tilde{A}^\circ} = \{e(p_1, p_2) | e(p_1, p_2) \in E_{\Delta\tilde{A}} \wedge p_1, p_2 \in P\} \tag{5.53}$$

Verification The verification of the interior is completely similar to the verification of the boundary, as discussed above; consider a fuzzy TIN $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$. It is now necessary to prove that $\mu_{\tilde{A}^\circ}(p) = 1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|)$, for all points in the TIN.

1. $p \in P_{\tilde{A}^\circ}$ For points in $P_{\tilde{A}^\circ}$, this requirement is immediately fulfilled, due to the definition of both $P_{\tilde{A}^\circ}$ and the definition of the mapping function $f_{\tilde{A}^\circ}$.

2. $p \notin P_{\tilde{A}^\circ}$, but still inside the outline of the interior (i.e. $\mu_{\Delta\tilde{A}}(p) \geq 0.5$), it is necessary to prove that $\mu_{\tilde{A}^\circ}(p) = 1 - 2(0.5 - |0.5 - \mu_{\tilde{A}^\circ}(p)|)$; for points $p \notin P_{\tilde{A}^\circ}$ but outside the interior (i.e. $\mu_{\Delta\tilde{A}}(p) < 0.5$), $\mu_{\tilde{A}^\circ}(p) = 0$.

   A triangle in $\tilde{A}^\circ$ is part of a triangle of $\tilde{A}$ (bear in mind that the triangulation process happens in two dimensions): only the associated values differ. The coefficients of the equation of a plane in the boundary are denoted A°, B°, C° and D° (100); the associated membership grade for a point p_i is $\mu_{\tilde{A}^\circ}(p_i) = z_i^\Delta$.

   - Suppose $\mu_{\tilde{A}}(p) \geq 0.5$, then $0.5 - \mu_{\tilde{A}}(p) < 0$ and $1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) = 1 - 2(0.5 + (0.5 - \mu_{\tilde{A}}(p))) = 1 - 2 - 2\mu_{\tilde{A}}(p) = -1 - 2\mu_{\tilde{A}}(p)$
     Consider the coefficient A°:

$$
\begin{aligned}
A^\circ &= y_1(z_2^\circ - z_3^\circ) + y_2(z_3^\circ - z_1^\circ) + y_3(z_1^\circ - z_2^\circ) \\
&= y_1(-1 - 2z_2 - (-1 - 2z_3)) + y_2(-1 - 2z_3 - (-1 - 2z_1)) \\
&\quad + y_3(-1 - 2z_1 - (-1 - 2z_2)) \\
&= y_1(-1 - 2z_2 + 1 + 2z_3) + y_2(-1 - 2z_3 + 1 + 2z_1) \\
&\quad + y_3(-1 - 2z_1 + 1 + 2z_2) \\
&= y_1(-2z_2 + 2z_3) + y_2(-2z_3 + 2z_1) + y_3(-2z_1 + 2z_2) \\
&= -2(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)) \\
&= -2A
\end{aligned}
$$

     Consider B°:

$$
\begin{aligned}
B^\circ &= z_1^\circ(x_2 - x_3) + z_2^\circ(x_3 - x_1) + z_3^\circ(x_1 - x_2) \\
&= (-1 - 2z_1)(x_2 - x_3) + (-1 - 2z_2)(x_3 - x_1) \\
&\quad + (-1 - 2z_3)(x_1 - x_2) \\
&= -(x_2 - x_3) - 2z_1(x_2 - x_3) - (x_3 - x_1) - 2z_2(x_3 - x_1) \\
&\quad - (x_1 - x_2) - 2z_3(x_1 - x_2) \\
&= -x_2 + x_3 - x_3 + x_1 - x_1 + x_2 - 2(z_1(x_2 - x_3) \\
&\quad + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\
&= -2B
\end{aligned}
$$

Consider C°.

$$\begin{aligned} C^\circ &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= C \end{aligned}$$

Consider D°:

$$\begin{aligned} D^\circ &= -A^\circ x_1 - B^\circ y_1 - C^\circ z_1^\circ \\ &= 2Ax_1 + 2By_1 - C(-1 - 2z_1) \\ &= 2Ax_1 + 2By_1 + 2Cz_1 + C \\ &= -2(-Ax_1 - By_1 - Cz_1) + C \\ &= -2D + C \end{aligned}$$

The equation then yields:

$$\begin{aligned} \mu_{\tilde{A}^\circ}(p(x,y)) &= -\frac{A^\circ}{C^\circ}x - \frac{B^\circ}{C^\circ}y - \frac{D^\circ}{C^\circ} \\ &= -\frac{A^\circ}{C}x - \frac{B^\circ}{C}y - \frac{D^\circ}{C} \\ &= -\frac{-2A}{C}x - \frac{-2B}{C}y - \frac{-2D+C}{C} \\ &= -2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) - \frac{C}{C} \\ &= -1 - 2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) \\ &= -1 - 2\mu_{\tilde{A}}(p) \end{aligned}$$

- Suppose $\mu_{\tilde{A}}(p) < 0.5$; by definition of the membership function (derived from the mapping function $f_{\tilde{A}^\circ}$), the associated membership grade for the interior will be 0.

This implies that for all points, their membership grade in the interior matches the membership grade as defined in the conceptual model (Chapter 2). □

Implementation This similarity can also be found in the pseudo code for the construction of the fuzzy interior.

```
Fuzzy_TIN FuzzyInterior(Fuzzy_TIN A)
VAR pointset,edgeset
BEGIN
For all datapoints p in P_A
  if mu_A(p) >= 0.5
    add p to pointset
    mu_A(p)=1-2(0.5-abs(0.5-mu_A(p)))
  end if
```

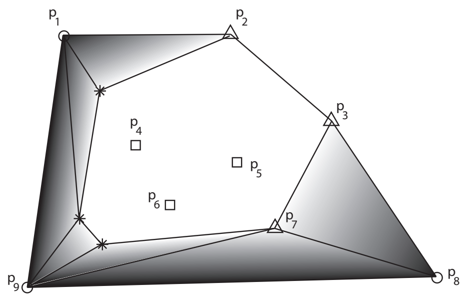


Figure 5.21: The fuzzy interior of the fuzzy TIN in 5.18. All points outside the polygon (p_1, p_2, p_8, p_9) are assigned a membership grade 1, all points inside the polygon defined by the 3 points indicated with $*$ and the points p_2, p_3, p_7 are assigned membership grade 0.

```
end for

  For all edges e(p_1,p_2) in E_A
    if (mu_A(p_1) > 0.5 AND mu_A(p_2) > 0.5)
      add e(p_1,p_2) to edgeset
    end if
    if (mu_A(p_1) > 0.5 AND mu_A(p_2) < 0.5)
      find point p on e with mu_A(p) = 0.5
      add p to pointset
      mu_A(p) = 0
      add e(p_1,p) to edgeset
    end if
    if (mu_A(p_2) > 0.5 AND mu_A(p_1) < 0.5)
      find point p on e with mu_A(p) = 0.5
      add p to pointset
      mu_A(p) = 0
      add e(p_2,p) to edgeset
    end if
  end for

  return constraint_Delaunay(pointset,edgeset)

END
```

Exterior The exterior of a fuzzy region $\tilde{A}$ also is a new fuzzy region. It is similar in definition to the interior. The exterior is illustrated on fig. 5.21; for illustration purposes all the datapoints of the original TIN are shown.

1. First, the datapoints points p for which $f_{\tilde{A}}(p) \leq 0.5$ are considered.

$$P_{t_1} = \{p \mid p \in P_{\tilde{A}} \wedge f_{\tilde{A}}(p) \leq 0.5\}$$

2. Next, the intersection points between edges of the TIN and a horizontal plane at level $z = 0.5$ are added. The datapoints for edges that are completely located in the plane $z = 0.5$, have been considered in the previous step.

$$\begin{aligned} P_{t_2} &= \{p \mid \exists e(p_1, p_2) \in E_{\tilde{A}} : p_1, p_2 \notin P_{t_1} \wedge \\ &\qquad p \in e(p_1, p_2) \wedge \mu_{\tilde{A}}(p) = 0.5\} \end{aligned}$$

Here, U represents the universe. The union of these two sets yields the set of datapoints for the interior: $P = P_{t_1} \cup P_{t_2}$. Next, the set of edges E must be determined.

1. First, all the edges that existed in the original network, and whose points are in P, are considered. Bear in mind that the TIN is defined in two dimensions; the altered associated value for the datapoints does not play a part here.

$$E_{t_1} = \{e(p_1, p_2) \mid p_1, p_2 \in P \wedge e(p_1, p_2) \in E_{\tilde{A}}\}$$

2. Next, all the edges that are part of an existing edge (in E_A) and that connect an intersection point (in P_{t_2}) with a datapoint from $P_{\tilde{A}} = P_{t_1}$ are added.

$$E_{t_2} = \{e(p_1, p_2) \mid p_1 \in P_{t_1}, p_2 \in P_{t_2} \wedge \exists e \in E_{\tilde{A}} : e(p_1, p_2) \subseteq e\}$$

The union of E_{t_1} and E_{t_2} yields us the set E which will constrain the Delaunay triangulation.

The mapping function is defined to match the theoretical definition (2.4.3):

$$\begin{aligned} f : P &\rightarrow [0, 1] \\ p &\mapsto 1 - \mu_{\Delta\tilde{A}}(p) = 1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) \end{aligned}$$

The membership function associates 1 with points outside the region of interest, which yields:

$$\begin{aligned} \mu_{\tilde{A}} : U &\rightarrow [0, 1] \\ p(x, y) &\mapsto \begin{cases} f(p(x, y)) & \text{if } p(x, y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } p(x, y) \in R \backslash P \\ 1 & \text{if } p(x, y) \notin R \end{cases} \end{aligned}$$

where A,B,C and D are defined respectively as in 5.2, 5.3, 5.4 and 5.5.

Verification The verification that the constructed TIN matches the theoretical definition of the exterior, is completely similar to the verification of the fuzzy interior; consider a fuzzy TIN $\tilde{A} = [(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$. It is now necessary to prove that $\mu_{\tilde{A}^-}(p) = 1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|)$, for all points in the TIN.

1. $p \in P_{\tilde{A}^-}$ For points in $P_{\tilde{A}^-}$, this requirement is immediately fulfilled, due to the definition of both $P_{\tilde{A}^-}$ and the definition of the mapping function $f_{\tilde{A}^-}$.
2. $p \notin P_{\tilde{A}^-}$, but still inside the outline of the interior (i.e. $mu_{\Delta\tilde{A}}(p) \geq 0.5$), it is necessary to prove that $\mu_{\tilde{A}^-}(p) = 1-2(0.5-|0.5-\mu_{\tilde{A}^-}(p)|)$; for points $p \notin P_{\tilde{A}^-}$ but outside the interior (i.e. $mu_{\Delta\tilde{A}}(p) < 0.5$), $mu_{\tilde{A}^-}(p) = 0$.

   A triangle in $\tilde{A}^-$ is part of a triangle of $\tilde{A}$ (bear in mind that the triangulation process happens in two dimensions): only the associated values differ. The coefficients of the equation of a plane in the boundary are denoted A^-, B^-, C^- and D^- (100); the associated membership grade for a point p_i is $\mu_{\tilde{A}^-}(p_i) = z_i^\Delta$.

   - Suppose $\mu_{\tilde{A}}(p) \leq 0.5$, then $0.5 - \mu_{\tilde{A}}(p) > 0$ and $1 - 2(0.5 - |0.5 - \mu_{\tilde{A}}(p)|) = 1 - 2(0.5 + (0.5 - \mu_{\tilde{A}}(p))) = 1 - 2 - 2\mu_{\tilde{A}}(p) = -1 - 2\mu_{\tilde{A}}(p)$ Consider the coefficient A^-:

$$
\begin{aligned}
A^- &= y_1(z_2^- - z_3^-) + y_2(z_3^- - z_1^-) \\
&\quad + y_3(z_1^- - z_2^-) \\
&= y_1(-1 - 2z_2 - (-1 - 2z_3)) + y_2(-1 - 2z_3 - (-1 - 2z_1)) \\
&\quad + y_3(-1 - 2z_1 - (-1 - 2z_2)) \\
&= y_1(-1 - 2z_2 + 1 + 2z_3) + y_2(-1 - 2z_3 + 1 + 2z_1) \\
&\quad + y_3(-1 - 2z_1 + 1 + 2z_2) \\
&= y_1(-2z_2 + 2z_3) + y_2(-2z_3 + 2z_1) + y_3(-2z_1 + 2z_2) \\
&= -2(y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2)) \\
&= -2A
\end{aligned}
$$

     Consider B^-:

$$
\begin{aligned}
B^- &= z_1^-(x_2 - x_3) + z_2^-(x_3 - x_1) + z_3^-(x_1 - x_2) \\
&= (-1 - 2z_1)(x_2 - x_3) + (-1 - 2z_2)(x_3 - x_1) \\
&\quad + (-1 - 2z_3)(x_1 - x_2) \\
&= -(x_2 - x_3) - 2z_1(x_2 - x_3) - (x_3 - x_1) - 2z_2(x_3 - x_1) \\
&\quad - (x_1 - x_2) - 2z_3(x_1 - x_2) \\
&= -x_2 + x_3 - x_3 + x_1 - x_1 + x_2 - 2(z_1(x_2 - x_3) \\
&\quad + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\
&= -2(z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2)) \\
&= -2B
\end{aligned}
$$

Consider C^-.

$$\begin{aligned} C^- &= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \\ &= C \end{aligned}$$

Consider D^-:

$$\begin{aligned} D^- &= -A^- x_1 - B^- y_1 - C^- z_1^- \\ &= 2Ax_1 + 2By_1 - C(-1 - 2z_1) \\ &= 2Ax_1 + 2By_1 + 2Cz_1 + C \\ &= -2(-Ax_1 - By_1 - Cz_1) + C \\ &= -2D + C \end{aligned}$$

The equation then yields:

$$\begin{aligned} \mu_{\tilde{A}^-}(p(x,y)) &= -\frac{A^-}{C^-}x - \frac{B^-}{C^-}y - \frac{D^-}{C^-} \\ &= -\frac{A^-}{C}x - \frac{B^-}{C}y - \frac{D^-}{C} \\ &= -\frac{-2A}{C}x - \frac{-2B}{C}y - \frac{-2D + C}{C} \\ &= -2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) - \frac{C}{C} \\ &= -1 - 2(-\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}) \\ &= -1 - 2\mu_{\tilde{A}}(p) \end{aligned}$$

- Suppose $\mu_{\tilde{A}}(p) \geq 0.5$; by definition of the membership function (derived from the mapping function $f_{\tilde{A}^-}$), the associated membership grade for the interior will be 0.

This implies that for all points, their membership grade in the exterior matches the membership grade as defined in the conceptual model (Chapter 2). □

Implementation In pseudo-code, this yields:

```
Fuzzy_TIN FuzzyExterior(Fuzzy_TIN A)
VAR pointset,edgeset
BEGIN
For all datapoints p in P_A
  if mu_A(p) <= 0.5
    add p to pointset
    mu_A-(p)=1-2(0.5-abs(0.5-mu_A(p)))
  end if
```

```
end for

  For all edges e(p_1,p_2) in E_A
    if (mu_A(p_1) < 0.5 AND mu_A(p_2) < 0.5)
      add e(p_1,p_2) to edgeset
    end if
    if (mu_A(p_1) < 0.5 AND mu_A(p_2) > 0.5)
      find point p on e with mu_A(p) = 0.5
      add p to pointset
      mu_A-(p) = 0
      add e(p_1,p) to edgeset
    end if
    if (mu_A(p_2) < 0.5 AND mu_A(p_1) > 0.5)
      find point p on e with mu_A(p) = 0.5
      add p to pointset
      mu_A(p) = 0
      add e(p_2,p) to edgeset
    end if
  end for

  return constraint_Delaunay(pointset,edgeset)

END
```

The concepts boundary, interior and exterior are closely linked; constructing them together in a single operation would no doubt be more efficient than considering them all three separately. Furthermore, as an additional optimization, it can be beneficial to store the boundary, interior and exterior for each fuzzy TIN known to the system.

Using the above concepts, it becomes possible to generate intersection matrices for fuzzy regions in a TIN representation. These intersection matrices can then be interpreted similarly to those in the theoretical model 2.4.3.

5.5 Extensions

5.5.1 Fuzzy locations

The first extension to be considered, is the extension in which fuzzy regions are used to represent fuzzy points. The definitions for a fuzzy TIN representing a fuzzy point are the same as those for a fuzzy TIN region. The difference is in the interpretation of the membership grades: for fuzzy regions, the membership grade was interpreted in a veristic way (indicating the extent to which a point belongs to the region), whereas for fuzzy points it is interpreted in a possibilistic way (indicating the extent to which a point is a possible location).

Definition 103 (fuzzy TIN point)

$$\textit{fuzzy TIN} = [(P, E, T), f] \tag{5.54}$$

where P is a set of data points, E a set of edges (both the edges obtained through a Delaunay triangulation, and the edges required to be in the result in the case of a constrained Delaunay triangulation, see 1.1.3), and T set of triangles that make up the TIN. The function f is a mapping function defined as:

$$\begin{array}{rcl} f : P & \rightarrow & [0,1] \\ p(x,y) & \mapsto & f(p(x,y)) \end{array}$$

This function associates each data point with a value; which will be the membership grade for this data point.

Based on the linear interpolation as applied on a TIN and the mapping function f, the membership function for a fuzzy region $\tilde{A}$ is then defined as

Definition 104 (membership function $\mu_{\tilde{A}}$ of a fuzzy point $p^{\tilde{A}}$) *(represented by a fuzzy TIN $\tilde{A}$)*

$$\begin{array}{rcl} \mu_{\tilde{A}} : U & \rightarrow & [0,1] \\ p(x,y) & \mapsto & \left\{ \begin{array}{ll} f(p(x,y)) & \textit{if } \; p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \textit{if } \; p(x,y) \in R \backslash P \\ 0 & \textit{if } \; p(x,y) \notin R \end{array} \right. \end{array}$$

where R represents the region of interest of the TIN and A, B, C and D are the parameters of the equation $Ax + By + Cz + D = 0$ of the plane containing the three points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ and $p_3(x_3, y_3, z_3)$ (with the understanding that $z_j = f(x_j, y_j)$, $j = 1, 2, 3$), such that the triangle $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ is a triangle of the TIN and $p(x, y, 0)$ is inside or on an edge of this triangle. For the calculation of these values, we refer to 5.2, 5.3, 5.4 and 5.5.

Operations

The overview of operations on fuzzy regions that remain meaningful on fuzzy points was provided in the conceptual model. For completeness, similarly as in the bitmap model, this overview is also repeated here.

Set-operations As before, information regarding a fuzzy location can originate from different constraints. Applying set operations on the fuzzy regions that hold the possible locations for the fuzzy locations, allows for combining the information from different sources, for instance a single fuzzy point can on one hand be said to be *close to a river* and on the other hand *near a water tower*. The union of both fuzzy regions will provide the possible locations for the fuzzy point for which one of the criteria needs to be fulfilled, the intersection will provide the possible locations for the fuzzy point if both criteria need to be fulfilled, and the complement provides the negation of a criterion.

α-cut operation The determination of the α-cut of a fuzzy point and can be used to determine a crisp region (for a given α) in which the locations have at least a given membership grade. This can be useful when needing to represent a region where the fuzzy point is located in a system that has no support for fuzzy geographic structures.

Minimum bounding rectangle and convex hull Both the minimum bounding rectangle and the convex hull can be used to approximate the outline of the fuzzy region of possible locations.

Surface area calculation By definition, the surface area of a single point is 0. The definition of the surface area for fuzzy regions can still be used to serve as an indication over which area the imprecisely known or inaccurately known point is located. It does not indicate the surface area of the point.

Distance calculation Calculating the distance between two fuzzy locations (or fuzzy points) in a TIN representation differs, just like it did in the bitmap model (chapter 4) and in the conceptual model (chapter 2), from calculating the distance between fuzzy regions. The reason again is that for fuzzy points, every possible distance needs to be taken into account, not only the possible distances between the regions.

To define the distance for fuzzy points, consider two fuzzy points that are defined using a single triangle; fuzzy point $p^{\tilde{A}}$ is represented by the triangle $p_1^{\tilde{A}}, p_2^{\tilde{A}}, p_3^{\tilde{A}}$; fuzzy point $p^{\tilde{B}}$ by the triangle $p_1^{\tilde{B}}, p_2^{\tilde{B}}, p_3^{\tilde{B}}$. The fuzzy points are characterized by respectively $[(P_{\tilde{A}}, E_{\tilde{A}}, T_{\tilde{A}}), f_{\tilde{A}}]$ and $[(P_{\tilde{B}}, E_{\tilde{B}}, T_{\tilde{B}}), f_{\tilde{B}}]$.

The shortest distance that can occur d_{min} for the distance between any point in $p^{\tilde{A}}$ and any point in $p^{\tilde{B}}$ is equal to the minimal distance between between the edges of both triangles.

$$d_{min} = \min_{i=1,2,3;j=1,2,3} \{d(e_i^{p^{\tilde{A}}}, e_j^{p^{\tilde{B}}})\} \tag{5.55}$$

where $e_i^{p^{\tilde{X}}}$ is the notation for an edge of the triangle representing $p^{\tilde{X}}$. The distance between two edges $e_1(p_1, p_2)$ and $e_2(p_3, p_4)$ is defined as the minimum distance between any endpoint of one edge and a point p on the other edge (which can be an endpoint):

$$\begin{aligned} d(e_1(p_1,p_2), e_2(p_3,p_4)) &= \min\{\inf\{d(p_i,p) \mid i \in \{1,2\}, p \in e_2\}, \\ &\quad \inf\{d(p,p_i) \mid i \in \{3,4\}, p \in e_1\}\} \end{aligned}$$

The membership grade that will be associated with the smallest possible distance between both triangles is

$$\begin{aligned}
\alpha_{d_{min}} &= \mu_{d(p^{\tilde{A}},p^{\tilde{B}})}(d_{min}) \\
&= \sup\{\min(\mu_{p^{\tilde{A}}}(p), \mu_{p^{\tilde{B}}}(p')) : \\
&\qquad \exists e \in E_{\tilde{A}}, p \in e \wedge \exists e' \in E_{\tilde{B}}, p' \in e' \wedge d(p,p') = d_{min}\}
\end{aligned}$$

Similarly, the maximum distance d_{max} that can occur between points of both triangles is

$$d_{max} = \max_{i=1,2,3;j=1,2,3}\{d(e_i^{p^{\tilde{A}}}, e_j^{p^{\tilde{B}}})\} \tag{5.56}$$

The membership grade that will be associated with this longest possible distance between both triangles is:

$$\begin{aligned}
\alpha_{d_{max}} &= \mu_{d(p^{\tilde{A}},p^{\tilde{B}})}(d_{max}) \\
&= \sup\{\min(\mu_{p^{\tilde{A}}}(p), \mu_{p^{\tilde{B}}}(p')) : \\
&\qquad \exists e \in E_{\tilde{A}}, p \in e \wedge \exists e' \in E_{\tilde{B}}, p' \in e' \wedge d(p,p') = d_{max}\}
\end{aligned}$$

Finally, also the distances that will be assigned the highest membership grade $d_{\mu_{max}}$ will be determined. This highest membership grade is given by:

$$\begin{aligned}
\alpha_{max} &= \min\{\max\{f_{p^{\tilde{A}}}(p_1^{\tilde{A}}), f_{p^{\tilde{A}}}(p_2^{\tilde{A}}), f_{p^{\tilde{A}}}(p_3^{\tilde{A}})\}, \\
&\qquad \max\{f_{p^{\tilde{B}}}(p_1^{\tilde{B}}), f_{p^{\tilde{B}}}(p_2^{\tilde{B}}), f_{p^{\tilde{B}}}(p_3^{\tilde{B}})\}\}
\end{aligned}$$

There can however be several distances that will match this membership grade (if one of the triangles has two or more points which share the highest membership grade in this triangle). Only the shortest and longest distances with this membership grade are needed.

$$d_{min}^{\alpha_{max}} = \min_{i=1,2,3;j=1,2,3}\{d(p_i^{\tilde{A}}, p_j^{\tilde{B}}) : \min(f_{p^{\tilde{A}}}(p_i^{\tilde{A}}), f_{p^{\tilde{B}}}(p_j^{\tilde{B}})) = \alpha_{max}\} \tag{5.57}$$

$$d_{max}^{\alpha_{max}} = \max_{i=1,2,3;j=1,2,3}\{d(p_i^{\tilde{A}}, p_j^{\tilde{B}}) : \min(f_{p^{\tilde{A}}}(p_i^{\tilde{A}}), f_{p^{\tilde{B}}}(p_j^{\tilde{B}})) = \alpha_{max}\} \tag{5.58}$$

This provides us with two more distances (and their membership grades), that definitely belong to the representation of the distance between both fuzzy points. As the triangles are convex structures, all distances in between these four distances are also valid.

The remaining question now is: how do the membership grades change for these distances? It can easily be shown that these changes are not linear. Consider the parametric equations of the straight lines in which the segments ab and de are contained. As ab is chosen to match the X-axis, the parametric equation for the straight line is:

$$\left\{\begin{array}{lcl} x &=& \lambda x_b \\ y &=& 0 \end{array}\right. \tag{5.59}$$

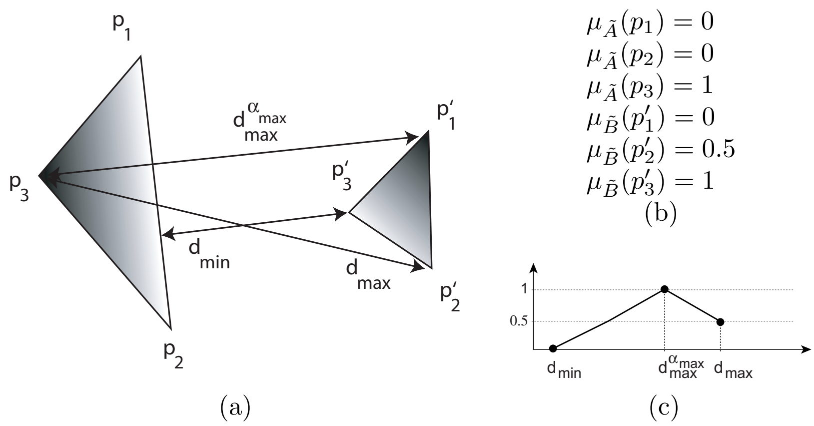


Figure 5.22: Illustration of the distance between two fuzzy regions represented by triangles: (a) sample triangles $\tilde{A}$ and $\tilde{B}$, (b) membership grades of the datapoints in $\tilde{A}$ and $\tilde{B}$, (c) the distance between the points represented by $\tilde{A}$ and $\tilde{B}$

For the straight line in which de is contained, the parametric equation is:

$$\left\{ \begin{array}{lcl} x & = & x_d + \lambda(x_e - x_d) \\ y & = & y_d + \lambda(y_e - y_d) \end{array} \right. \tag{5.60}$$

where a, b, c, d, e, f are defined by the coordinates (x_a, y_a), (x_b, y_b), (x_c, y_c), (x_d, y_d), (x_e, y_e) and (x_f, y_f) respectively and $\lambda \in [0, 1]$ (which limits the equations to the required line segments). The line segments ab and de could represent two sides of a triangle between which the evolution for the distance for different λ values (which could represent membership grades) needs to be determined. This is for instance the case on fig. 5.23, for finding distances between d_{min} and $d_{max}^{\alpha_{max}}$. These distances are given by:

$$d_\lambda(ab, de) \quad = \quad \sqrt{(y_d + \lambda(y_e - y_d))^2 + (x_d + \lambda(x_e - x_d) - \lambda x_b)^2}$$

Obviously, the distance for each λ-value does not vary in a linear way with the changing λ values (it can happen in particular cases, but not in general). Consequently, the distance between the points will merely be approximated by treating these changes as linear. To calculate the distance between two fuzzy points in a TIN representation, first the distance between two triangles is needed.

Definition 105 (distance between two fuzzy points)
(each represented by a single triangle)

$$\tilde{d}(p^{\tilde{A}}, p^{\tilde{B}}) = \{(x, \mu_{\tilde{d}(p^{\tilde{A}}, p^{\tilde{B}})}(x))\} \tag{5.61}$$

where

$$\mu_{\tilde{d}(p^{\tilde{A}}, p^{\tilde{B}})} : \mathbb{R} \quad \rightarrow \quad [0, 1]$$

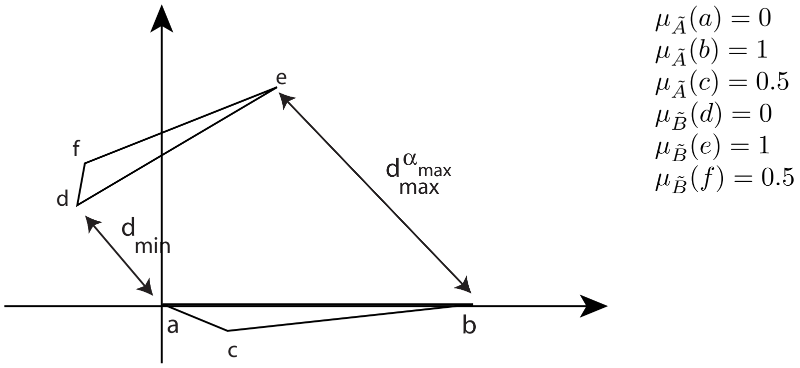


Figure 5.23: Example to calculate the changes to the distance depending on the changing λ-value.

$$x \mapsto \begin{cases} \alpha_{d_{min}} & \textit{if } x = d_{min} \\ \frac{\alpha_{max} - \alpha_{d_{min}}}{d^{\alpha_{max}}_{min} - d_{min}}(x - d_{min}) + \alpha_{d_{min}} & \textit{if } d_{min} < x < d^{\alpha_{max}}_{min} \\ \alpha_{max} & \textit{if } d^{\alpha_{max}}_{min} \leq x \leq d^{\alpha_{max}}_{max} \\ \frac{\alpha_{d_{max}} - \alpha_{max}}{d_{max} - d^{\alpha_{max}}_{max}}(x - d_{max}) + \alpha_{d_{max}} & \textit{if } d^{\alpha_{max}}_{max} < x < d_{max} \\ \alpha_{d_{max}} & \textit{if } x = d_{max} \end{cases}$$

with the values d_{min}, $d^{\alpha_{max}}_{min}$, $d^{\alpha_{max}}_{max}$, d_{max}, $\alpha_{d_{min}}$, α_{max} *and* $\alpha_{d_{max}}$ *defined as above.*

For two triangular networks, all the possible triangle-combinations need to be taken into account (as in the above definition), after which the union (using the maximum T-conorm, 1.2.3) of all obtained fuzzy sets will yield the distance between both fuzzy points in TIN representation.

Regions at fuzzy positions

So far, membership grades in fuzzy regions are interpreted as degrees of belonging to the region. As mentioned in earlier chapters, it can be interesting to consider a region located at a fuzzy position. As mentioned before, this can be accomplished by defining a region's position by means of an anchor point.

The TIN-structure is very well suited to be used for representing the anchor point: not only does it allow for a model in which a single point has membership grade 1, but it is also possible to model a single line where the membership grade will be 1. Furthermore, due to its arbitrary distribution of datapoints, it can be tailored to suit various constraints (e.g. a fuzzy point can consist of possible locations in streets, but not in houses along this street).

The region attached to the anchor point can be either a crisp region, a fuzzy region represented by a bitmap, or a fuzzy region represented by a TIN. In the latter two cases, the region can be considered to be represented by a level-2 fuzzy set.

5.5.2 Fuzzy associated data

In the above model and extensions, the uncertainty and/or imprecision concerned the region itself, not the associated data. It can however also be interesting to consider the associated data to be fuzzy. A region which is in itself crisp, but with which fuzzy data is associated is referred to as an extended region. The associated data can be numeric, but it can also concern other data (categories of land, types of land use, soil composotion, etc). In the example below, an extended region used to model fuzzy real numbers is given.

Definition 106 (extended TIN region)

$$\textit{extended TIN} = [(P, E, T), f] \tag{5.62}$$

where P is a set of data points on which the TIN is constructed, E is a set of edges (including both the edges obtained through a Delaunay triangulation, and the edges required to be in the result in the case of a constrained Delaunay triangulation, see 1.1.3), and T is a set of triangles that make up the TIN. The function f is a mapping function defined as:

$$\begin{aligned} f : P &\rightarrow \wp(\mathbb{R}) \\ p(x, y) &\mapsto f(p(x, y)) \end{aligned}$$

This function associates each data point with a value; which will be the membership grade for this data point.

Based on the linear interpolation as is applied on a TIN and the mapping function f, the full mapping function g (which also provides the information for interpolated points) for an extended region can be defined as

Definition 107 (full mapping function g of a fuzzy TIN $\tilde{A}$)

$$\begin{aligned} g : U &\rightarrow \wp(\mathbb{R}) \\ p(x, y) &\mapsto \begin{cases} f(p(x, y)) & \text{if } p(x, y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } p(x, y) \in R \backslash P \\ 0 & \text{if } p(x, y) \notin R \end{cases} \end{aligned}$$

where R represents the region of interest of the TIN and A, B, C and D are the parameters of the equation $Ax + By + Cz + D = 0$ of the plane containing the three points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ and $p_3(x_3, y_3, z_3)$ (with the understanding that $z_j = f(x_j, y_j)$, $j = 1, 2, 3$), such that the triangle $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ is a triangle of the TIN and $p(x, y, 0)$ is inside or on an edge of this triangle.

Notice that for associated fuzzy real number, this requires fuzzy arithmetic. For discrete data (for instance land classification, where there are only a finite number of classes), the extended TIN will appear to consist of multiple crisp TINs: each class will yield one TIN which models the membership grade for every location with regard to this class.

Type-2 fuzzy sets

The above extension can be combined with the model for fuzzy regions, to yield a representation for type-2 fuzzy regions.

Definition 108 (type-2 fuzzy TIN region)

$$\textit{type-2 fuzzy TIN} = [(P, E, T), f] \tag{5.63}$$

where P is a set of data points on which the TIN is constructed, E is a set of edges (including both the edges obtained through a Delaunay triangulation, and the edges required to be in the result in the case of a constrained Delaunay triangulation, see 1.1.3), and T is a set of triangles that make up the TIN. The function f is a mapping function defined as:

$$\begin{array}{rcl} f : P & \rightarrow & \wp([0,1]) \\ p(x,y) & \mapsto & f(p(x,y)) \end{array}$$

This function associates each data point with a value; which will be the membership grade for this data point.

Based on the linear interpolation as is applied on a TIN and the mapping function f, the full mapping function g (which also provides the information for interpolated points) for an extended region can be defined as

Definition 109 (type-2 membership function $\mu_{\tilde{\tilde{A}}}$)
(of a type-2 fuzzy TIN $\tilde{\tilde{A}}$)

$$\begin{array}{rcl} \mu_{\tilde{\tilde{A}}} : U & \rightarrow & \wp([0,1]) \\ p(x,y) & \mapsto & \begin{cases} f(p(x,y)) & \text{if } p(x,y) \in P \\ -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} & \text{if } p(x,y) \in R \backslash P \\ 0 & \text{if } p(x,y) \notin R \end{cases} \end{array}$$

where R represents the region of interest of the TIN and A, B, C and D are the parameters of the equation $Ax + By + Cz + D = 0$ of the plane containing the three points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$ and $p_3(x_3, y_3, z_3)$ (with the understanding that $z_j = f(x_j, y_j)$, $j = 1, 2, 3$), such that the triangle $p_1(x_1, y_1, 0)$, $p_2(x_2, y_2, 0)$ and $p_3(x_3, y_3, 0)$ is a triangle of the TIN and $p(x, y, 0)$ is inside or on an edge of this triangle.

Just like in the case of type-2 fuzzy bitmaps (4.5.3), the type-2 fuzzy region can be used to overcome criticism on crisp membership grades, but also to take into account the fact that the TIN representation is an approximation of reality. Where in the type-2 fuzzy bitmap, the uncertain membership grade could be obtained from several sample points, it is possible in a fuzzy region to consider an alternate interpolation method. This alternate interpolation method could yield fuzzy membership grades that are more imprecise for locations further away from the datapoints.

PTV-regions

It is possible to work with fuzzy associated data regardless of the domain. Consequently, this domain can also be $\{True, False\}$; fuzzy sets over this domain are possibilistic truth values, as mentioned in 1.2.4. The definition for such regions is similar as for regions with other fuzzy associated data, but this time with $\wp(\{True, False\})$ as the co-domain. If desired, extended possibilistic truthvalues (fuzzy sets over $\wp(\{True, False, \perp\}$, [19]) can also be defined similarly.

PTV regions can be useful for representing results from queries on fuzzy information, or queries that use fuzzy predicates (even if the information in the database itself is crisp).

Chapter 6

Conclusion

6.1 Application fields

6.1.1 Spatial databases

The first application field is of course in field from which the research originated: spatial databases (and consequently geographic information systems). In spatial databases, there can be applications for either the fuzzy regions, fuzzy points, or any of the other extensions considered in this work. For every concept, applications will be considered in further detail.

Fuzzy regions

The concept of fuzzy regions allows for the modelling of regions which have an indetermined boundary. The cause for this indetermination can either be uncertainty (there is no certain knowledge regarding the boundary), but can also be imprecision (the boundary is imprecise). The uncertainty or imprecision can be inherent to the region, but it can also have been introduced by limitations imposed when determining the region (it can be physically impossible or too expensive to determine the actual crisp boundary). Below is a list of some examples of such occurrences.

- Soil composition
  The first example considered is soil composition. Depending on the soil considered, the fuzziness is either inherent (where does a section of sandy soil ends, and a section of clay soil begins?) or virtually impossible to determine accurately (e.g. underground caverns). Traditionally, geologists place a fixed, crisp boundary at a position they deem representative for the boundary between the regions that have a different type of soil. While this is sufficient for many uses, inclusion of the fuzziness of the boundary can have advantages in more advanced analysis: it becomes possible to take into account a "partial belonging to"-relation whic in turn prevents a system of having to discriminate on a pure binary (yes/no) basis.

- Vegetation and wildlife
  Modelling which plants are located in which regions tends to be a fuzzy process. The presence of a single plant is expressed as a crisp number: for grasses or similar plants, this number is a percentage: 70% of a region is covered in grass, for large plants it is expressed as a an average number of plants per square metre; which in both cases is likely to be an estimate. Similar to the vegetation in a region, is the distribution of wildlife. Most animals have a certain territory, which, while often quite crisp for the animal, is difficult to find for researchers. Counting the number of animals in a given region is also prone to inherent uncertainty, as the animals tend to move around. Currently, observations are extrapolated on the entire region, which of course introduces uncertainty. As the territory of one species is often influenced by either the presence (or absence) of another species, or by the presence or absence of plants, modelling the regions as fuzzy regions would allow to take into account the changes to territories under the influence of vegetation changes or under the influence of migration of other species.

- Weather
  Any climate related information, from measurements (temperature, humidity, wind direction and wind velocity, and even rainfall) is inherently imprecise: the measurements are always performed on a fixed number of locations, and extrapolated to other locations. Forecasts regarding the weather are inherently uncertain. Also information that is related to weather, like areas affected by hurricanes are also imprecise or uncertain. The path of hurricanes themselves is also fuzzy, but can also considered as an application of a fuzzy point.

Fuzzy points

Fuzzy locations (and fuzzy points) can be used to model a person's whereabouts or an object's position, given a number of criteria. This can be a last known position, combined with terrain information and further combined with other (possibly non geographic) data. Quite often, there will be a spatio-temporal aspect to fuzzy points.

- Prediction of possible future locations
  An application for this could be in tracking persons and predicting possible locations where they are now: there often is a last known position is (e.g. the scene of a crime, witness report). Given the terrain information (roads, buildings, stretches of water, etc.) and information that the person is for instance moving by car, the system could calculate different possible locations for this person. Any added information (e.g. if the car was seen somewhere), can be taken into account as well, and provide additional restrictions on the possible locations. These principle could also be applied for location based services, where persons are given infor-

mation based on their location. Modelling the path of migrating animals or weather phenomena such as hurricanes is similar.

- Past location tracing
  Similar to the above application, the techniques can be used to determine where a person has been in the past. This can be useful in tracing steps from criminals, but also to obtain clues regarding found persons. The system could combine distance information with samples of soil, eyewitness reports, and other information.

- Identifying locations
  The fuzzy regions concept can also be used to identify locations given a number of properties: the properties “by a river, near a bridge, close to a water tower” can be combined and a list of all locations that satisfy these properties can be returned. This could be used for emergency services, to pinpoint the location where they have to go to if no accurate information (e.g. the name of the street) is provided.

- Modelling spread of pollutants / Tracing possible sources of pollutants
  In Belgium for instance, the air force is using unmanned aircraft to patrol the coast for illegal oil pollution caused by ships. However, when the pollution is not found within a short timeframe, it becomes much more difficult to find the source. By combining the information of the present location of the oil (which is quite crisp) with the information of ocean currents (prone to imprecision) and traffic information about the ships (prone to uncertainty), a fuzzy approach could be used to allow for a bigger timeframe in which the guilty party can still be traced. The reverse problem is that when a pollution happens (a leak in a tank, a truck accident, etc.), it is important to quickly assess what area is likely to be affected by it.

- Object matching
  The last geographical example is a field our department research is also investigating. The goal is to provide an automated system to match satellite images and aerial photographs with given cartographic data to detect changes to the road network. Ideally, the system should discover where there have been changes (as can be seen on the images) to the current representation of the road network (as stored in the geographic database). Detecting which roads and intersections match is prone to uncertainty or imprecision, due to perspective errors in the images; clouds, vehicles and bridges obscuring the road network or genuine changes to the network.

6.1.2 Image processing

Even though the presented techniques have been developed with geographic systems in mind, there are applications in other fields, mainly in image processing (as partly illustrated by the object matching example).

- Image segmentation
  Image segmentation is the process where an image is divided into different subsections, based on the content of the image. As the content of the image not always leads to a crisp boundary (e.g. detecting the outline of a tree on a picture), it stands to reason that allowing for fuzzy segments can have its advantages. A fuzzy segment on an image is conceptually the same as a fuzzy region; given the fact that images are bitmaps, the presented bitmap-based model becomes extremely well suited for this purpose.

- Object recognition
  Recognizing objects is one step further than image segmentation: it involves tagging the segments, e.g. stating that an object on the image is a tree (after the outline of the tree has been found and marked as a segment). Depending on the circumstances, an object can differ from the object one is trying to identify on the image, even if it concerns the same objects. Examples are a tree in summer and that same tree in winter, a person viewed from the front or in profile, etc. Given this wide variety, using a fuzzy matching technique to compare segments on the image with a set of predefined objects could yield better results (this application partly overlaps with the object matching mentioned previously).

6.2 Conclusion

6.2.1 General spatial datatype criteria

In [40], the author lists a number of properties, which are interesting for spatial data types at the implementation level. These properties are used to test the presented models. First, an overview of the properties is given.

- Generality
  It should be possible to model spatial objects as general as possible, a line object for example should be able to model the Nile delta, whereas a region object should be able to cope with holes and disconnected regions.

- Closure properties The domains of spatial datatypes should be closed with respect to union, intersection and difference. This property means that the result of an operation should be of the same type as the arguments of the operation, which allows for the result to be used as an argument in a subsequent operation.

- Rigorous definition
  The semantics should be defined in a clear and unique way as to avoid ambiguities for both user and implementor.

- Finite resolution, numerical robustness, topological correctness
  Formal definition should take into account the finite representation as

is available in computers. This is particularly the case with geometrical calculations (for example determining the exact location of intersection points, etc.), as errors can be propagated. Subsequent operations can cause an escalation of these errors, causing the end result to deviate quite a lot from the desired result. When left to the programmer, such calculation errors tend to lead to both numerical and topological problems.

- Geometric consistency
  Distinct spatial types may be related through geometric consistency constraints (e.g. adjacent regions have a common boundary). For fuzzy regions, this concept can become difficult, as boundaries are allowed to be uncertain or imprecise. Nevertheless, it is still possible to enforce different regions to use the same locations/points.

- Extensibility
  There may always be applications that require operations (and perhaps types) not considered in the original spatial data types. A type system should be extendible for new datatypes.

- Data model independence
  Spatial datatypes need to be integrated in a DBMS data model and in the query language that handles them. However, the spatial datatype itself should be independent of any particular DBMS.

The presented models will now be verified against the above properties.

Conceptual model

The conceptual model has been presented in chapter 2. This model is not intended to represent a spatial data type, but more as a theoretical basis on which spatial data types can be based.

- Generality: No assumption whatsoever is made regarding the distribution of locations (or points), making this model extremely general: regions made up of several disconnected regions, regions with holes and any combination of these two are possible.

- Closure: The model is closed with respect to the required operations (union, intersection and complement). Furthermore, the model is closed for the minimum bounding rectangle, convex hull and even α-cuts (while the α-cut of a region is a crisp region, it can still be represented as a fuzzy region with membership grades in $\{0, 1\}$). Any operation on two or more regions which yields a (finite) region can be represented in this conceptual model.

- Rigorous definition: The definitions are unambiguous, but due to the nature of the concept not suited for implementation (which is not the point of this model).

- Extensibility: The definitions of the regions are an application of fuzzy set theory. An extension principle can be used to extend operations that work on crisp regions, to operators that work on fuzzy regions as represented in the conceptual model.

Due to the fact that this model is not intended for direct implementation, but more as a theoretical foundation for other models, the properties that deal with practical implementation aspects (resolution/robustness, geometric consistency, data model independence) are of no real importance for this model.

Contourline model

The contourline model has been dealt with in 3.3. The model is a straight forward extension of the models that use an inner and an outer boundary to define a broad boundary.

- Generality: The model itself allows for regions that consist of several disconnected regions, and even for regions with holes. However, all the boundaries that occur in a single region (the boundaries for the different disjoint regions, as well as the boundaries that delimit holes) need to be defined in the same way (using the same shape function).

- Closure: The contourline model is generally not closed with respect to union and intersection. This is easily seen when considering the union or intersection of two regions that use a different shape function, but it is also the case for regions that use the same shape function (i.e. in the case they intersect only with their boundaries).

- Rigorous definition: The definitions are formally specified and unambiguous.

- Resolution/robustness: The contourline model is based on the two dimensional space. Within the model, the limitations imposed by finite number representation (in computers) are not taken into account, making the model not very robust. However, it is possible to alter the definitions to make use of a realm 1.3.2, thus making the model more robust.

- Geometric consistency: In the contourline model, some geometric consistency constraints may be quite difficult to enforce, for instance specifying that two adjacent regions share the same boundary for instance is not straight forward, as boundaries are defined for each region individually.

- Extensibility: While it is possible to add operations to the model, they are hard if not impossible to define if the closure property is required, as is illustrated in 3.3.

- Data model independence: The model was built from the theoretical concept, and is not related to any DBMS data model.

Despite the fact that the contourline model fails at a number of properties, the concept is still useful as an extension of the traditional buffer concept. The contourline model allows for a buffer to be defined in a fuzzy way, while providing a simple model (both conceptually as from an implementation point of view) with little data requirements and easy querying to provide information for given locations.

Fuzzy bitmap model

- Generality: The model makes the conceptual model manageable by limiting the domain to a finite space, but maintains the generality: regions made up of disconnected regions, regions with holes, etc. are all possible.

- Closure: The model is closed with respect to union, intersection and complement. Additionally, it is even closed with respect to minimum bounding rectangle, convex hull, and α-cuts (α-cuts are crisp regions, but the initial result of the α-cut is a crisp region in a bitmap representation, with membership grades in $\{0, 1\}$).

- Rigorous definition: The definitions of both the model and the various operations are unambiguously specified.

- Resolution/robustness: The model makes the conceptual model manageable by considering a finite space (where a grid is used to define the elements) instead of the two dimensional space. By using a finite number of elements as the basic structures, the numerical robustness for geometric properties can be guaranteed.

- Geometric consistency: The bitmap model makes it easy to enforce geometric consistency constraints (even though the concept for fuzzy regions is dubious). For a shared boundary between fuzzy regions for instances, it suffices to consider the same cells in each region as the cells that make up the overlapping boundary.

- Extensibility: A number of operations have been defined, but many others can be added by applying an extension principle. Due to the fact that the fuzzy regions are defined over a discrete domain, the extension principle can be applied in practice (although it won't yield an optimized algorithm for the required operation).

- Data model independence: The model was built upon the conceptual model, and is independent from any DBMS data type. Due to the fact however that the bitmap structure is a known structure to most geographic database systems, the fuzzy bitmap model should be relatively straightforward to add to such a database system.

Fuzzy TIN model

- Generality: The model simplifies on the conceptual model by limiting the number of points that define the region, but not at the expense of generality: regions made up of disconnected regions, regions with holes, etc. can all be modelled.

- Closure: The model is closed with respect to union, intersection and complement. Furthermore, it is also closed for minimum bounding rectangle, convex hull (which is an approximation, but accurate for every chosen α-cut) and α-cut (the initial result of the α-cut is a crisp region in TIN-region representation, with membership grades in $\{0, 1\}$).

- Rigorous definition: Both the structure and the operations are unambiguously defined.

- Resolution/robustness: Even though only a limited number of points are used to define a fuzzy TIN, the model may be prone to propagation of errors (stemming from the limited representation in computers) in some of the calculations. This is for example the case in any operation that requires the geometric intersection between lines and/or planes. However, it is also possible to define a fuzzy TIN by means of a realm (1.3.2), in which case the numerical robustness can be guaranteed. It may have an impact on the interpolation method though, depending on how the realm elements are defined and treated.

- Geometric consistency: While the concept of geometric consistency is unclear, regions can share common points and even triangles.

- Extensibility: In the fuzzy TIN model, a number of operations have been considered. Defining additional datatypes and operations is possible, but the linear interpolation requirement may cause some operations to be more difficult to define (e.g. the product norm for union).

- Data model independence: The fuzzy TIN model was based on the conceptual model. As such, it is independent from any DBMS data type. It makes use of triangular networks and Delaunay triangulation, both of which are concept already known to most geographic database systems.

6.2.2 Summary

The above results are summarized in the following table.

The conceptual model provides for a solid theoretical foundation on which different implementation models can be based. The contourline model is not really suited to provide a rich model for fuzzy regions, but still has its merits in providing for a better model for the representation of buffer regions, as it allows for fuzzy buffers. Both the fuzzy bitmap and fuzzy TIN models are well suited to represent fuzzy regions and while both are suitable for fuzzy

Property	Concept	Contour-line	Fuzzy Bitmap	Fuzzy TIN
Generality	+	0	+	+
Closure	+	-	+	+
Rigorous	+	+	+	+
Resolution robustness	/	0	+	0
Geometric consistency	/	-	+	+
Extensibility	+	0	+	+
Data model independence	+	+	+	+

Figure 6.1: Overview spatial datatype properties for the presented models; legend: +=satisfied, 0=neutral, -=not satisfied, /=not applicable

points, the fuzzy TIN model has the advantage to allow fuzzy regions with a 1-dimensional core to be represented.

6.3 Future work

The models presented in this work are to be considered a first step towards a geographic information system in which fuzziness is supported at various levels. Using the representations and operations presented, fuzziness at both the boundary of the regions as well as at the position of points can be achieved. For this purpose, a number of basic operations have been considered, defined and presented, but there still is a lot of future work, both research in the theoretical field, but research toward a practical system.

While a number of operations have been defined, there are additional operations that are also interesting and often required on fuzzy regions and points. Not only operations that work on single regions (e.g. center of gravity), but also more interaction between different objects (e.g. better support for geometric consistency constraints) need to be developed further. Particularly in the fuzzy TIN model, work is in progress on supporting discontinuities within a TIN network (currently, discontinuities are only possible at the edges of the region of interest). This will be achieved by allowing different associated values for datapoints, and then considering the appropriate value depending on the triangle at hand. Future research is also aimed at combining the concepts of regions and points: working with regions at fuzzy positions, and defining operations that provide adequate support for this. The other mentioned extensions (fuzzy associated values, type-2 fuzzy sets, etc.) also have many uses (e.g. to model fuzzy measurements) and need to be researched and developed further.

Apart from further theoretical work, a important next step is the development of a (prototype) implementation which supports the different models and operations. During the research, some small scale test implementations were made; either to verify the feasibility of the operations, or to clearly verify the impact in some degenerate cases. A full implementation would allow for a proper comparison against existing techniques, not only to compare the results of different analyses or query performed on data, but also to test for performance and possible limitations imposed by a computer system (speed, accuracy, etc.). It would also allow for an easier illustration on how models and queries could benefit from incorporating the fuzziness that occurs.

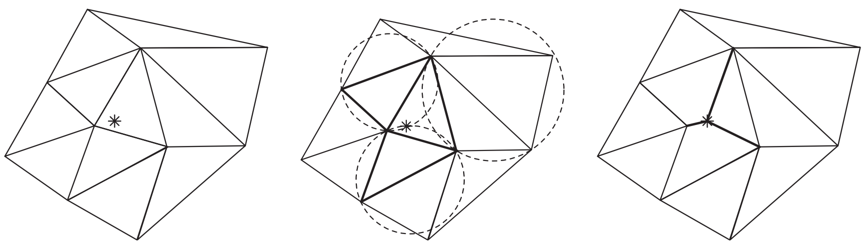
Appendix A

Appendix

A.1 Delaunay Triangulation

The concept of Delaunay triangulation is often referred to in this work (particularly in chapter 5); as mentioned, various algorithms to perform this triangulation exist. For the purpose of illustration, one algorithm will be considered in further detail. As indicated, a Delaunay triangulation is a triangulation in which for each triangle, no vertex points other than its own three vertex points are contained inside the circumscribing circle.

An incremental algorithm [5] is the most straightforward way of computing a triangulation: vertices are added one point at a time, and the parts of the triangulation affected by adding this vertex are corrected. This is illustrated on fig. A.1.



(a) (b) (c)

Figure A.1: Incremental algorithm to find the Delaunay triangulation: (a) existing network with newly added point, (b) circumcircles for the relevant triangles, (c) triangular network with the newly added point.

Whenever a vertex is added (fig. A.1a), the circumcircles of all triangles, in which this vertex is located are computed (fig. A.1b); those triangles are removed and this part of the graph is retriangulated (fig. A.1c). Without any optimization, this takes $O(n^2)$. By ordering the vertices on e.g. their first coordinate, and adding them in this order, the algorithm is more efficient with an average complexity of $O(n^{3/2})$. With other optimizations, the average complexity can be brought down to $O(n \log(n))$. Commonly, as a starting point, a triangle is constructed (called the *supertriangle*), in which all the points are contained. This supertriangle makes no use of the vertex points and is to be removed at the end of the algorithm. Notice how this algorithm immediately allows for the addition of points, and how the removal of vertex points is basically analogous.

The divide and conquer algorithm [9] starts with a set of vertices. This set is split in two sets, which are then merged after having computed the Delaunay triangulation of both sets (which is done recursively). The process of splitting takes $O(\log n)$, whereas the merging operation can be performed in $O(n)$; yielding a total complexity of $O(n \log n)$.

The sweepline algorithm *scans* the vertex points, adding multiple points and egdes at the same time. While this algorithm is more advanced, it still has an $O(n \log n)$ complexity.

The incremental algorithm [5] is given below in pseudocode:

```
subroutine triangulate
input : vertex list
output : triangle list
   initialize the triangle list
   determine the supertriangle
   add supertriangle vertices to the end of the vertex list
   add the supertriangle to the triangle list
   for each sample point in the vertex list
      initialize the edge buffer
      for each triangle currently in the triangle list
         calculate the triangle circumcircle center and radius
         if the point lies in the triangle circumcircle then
            add the three triangle edges to the edge buffer
            remove the triangle from the triangle list
         endif
      endfor
      delete all doubly specified edges from the edge buffer
         this leaves the edges of the enclosing polygon only
      add to the triangle list all triangles formed between
         the point and the edges of the enclosing polygon
   endfor
   remove any triangles from the triangle list that use the
     supertriangle vertices
   remove the supertriangle vertices from the vertex list
end
```

A.2 List of symbols

$\mathbb{R}$	set of real numbers
$\forall$	for all
$\exists$	exists
$\nexists$	not exists
$\exists!$	exists only one
$\wp(A)$	powerset of the set A: the set that contains all subsets of A
U	notation for the universe considered, in this work this is always a genuine, limited subset of $\mathbb{R}^2$
$f(A)$	where A is a set, is the notation for the set containing all the elements obtained by applying f to each element of A i.e. $\{y \mid y = f(x) \wedge x \in A\}$
$A \backslash B$	notation for set difference: A minus B
$[a, b]$	notation for the closed interval: $\{x \in \mathbb{R} \mid a \leq x \wedge x \leq b\}$
$]a, b]$	notation for the halfopen interval: $\{x \in \mathbb{R} \mid a < x \wedge x \leq b\}$
$[a, b[$	notation for the halfopen interval: $\{x \in \mathbb{R} \mid a \leq x \wedge x < b\}$
$]a, b[$	notation for the open interval: $\{x \in \mathbb{R} \mid a \leq x \wedge x \leq b\}$
$p(x, y)$	point with coordinates (x, y)
$p(x, y, z)$	point with coordinates (x, y, z), commonly the z-coordinate is a shorthand notation for the membership grade associated with $p(x, y)$
$\mu_A(x)$	membership grade of x in the fuzzy set A
$\tilde{A}_\alpha$	notation for the weak α-cut of A
$\tilde{A}_{\overline{\alpha}}$	notation for the strong α-cut of A
ΔA	boundary of the crisp region A
A°	interior of the crisp region A
A^-	exterior of the crisp region A
$\Delta\tilde{A}$	boundary of the fuzzy region $\tilde{A}$
$\tilde{A}^\circ$	interior of the fuzzy region $\tilde{A}$
$\tilde{A}^-$	exterior of the fuzzy region $\tilde{A}$

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Index