ON DISTRIBUTIONAL POINT VALUES AND BOUNDARY VALUES OF ANALYTIC FUNCTIONS

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ABSTRACT. We give the following version of Fatou's theorem for distributions that are boundary values of analytic functions. We prove that if $f \in \mathcal{D}'(a, b)$ is the distributional limit of the analytic function F defined in a region of the form $(a, b) \times (0, R)$, if the one sided distributional limit exists, $f(x_0 + 0) = \gamma$, and if f is distributionally bounded at $x = x_0$, then the Łojasiewicz point value exists, $f(x_0) = \gamma$ distributionally, and in particular $F(z) \to \gamma$ as $z \to x_0$ in a non-tangential fashion.

1. INTRODUCTION

The study of boundary values of analytic functions is an important subject in mathematics. In particular, it plays a vital role in the understanding of generalized functions [1, 2, 4]. As well known, the behavior of an analytic function at the boundary points is intimately connected with the pointwise properties of the boundary generalized function [7, 9, 18, 19, 20] and the study of this interplay has often an Abelian-Tauberian character. There is a vast literature on Abelian and Tauberian theorems for distributions (see the monographs [8, 12, 14, 20] and references therein).

In this article we present sufficient conditions for the existence of Łojasiewicz point values [11] for distributions that are boundary values of analytic functions. The pointwise notions for distributions used in this paper are explained in Section 2. The following result by one of the authors is well known [7]:

Suppose that $f \in \mathcal{D}'(\mathbb{R})$ is the boundary value of a function F, analytic in the upper half-plane, that is, $f(x) = F(x + i0)$; if the distributional lateral limits $f(x_0 \pm 0) = \gamma_{\pm}$ both exist, then $\gamma_+ = \gamma_- = \gamma$, and the distributional limit $f(x_0)$ exists and equals γ.

On the other hand, the results of [5] imply that there are distributions $f(x) = F(x + i0)$ for which one distributional lateral limit exits but not the other. In Theorem 2 we show that the existence of one the distributional lateral limits may be removed from the previous statement if an additional Tauberian-type condition is assumed, namely, if the distribution is distributionally bounded at the point. We also show that when the distribution f is a bounded function near the point, then the distributional point value is of order 1. Furthermore, we give a general result of this kind for analytic functions that have distributional limits on a contour.

As an immediate consequence of our results, we shall obtain the following version of Fatou's theorem [13, 15] for distributions that are boundary values of analytic functions.

Corollary 1. *Let F be analytic in a rectangular region of the form $(a, b) \times (0, R)$. Suppose that $f(x) = \lim_{y \to 0^+} F(x + iy)$ in $\mathcal{D}'(a, b)$, that f is a bounded function near $x_0 \in (a, b)$, and that the following average lateral limit exists*

$$\lim_{x \to x_0^+} \frac{1}{(x - x_0)} \int_{x_0}^{x} f(t) \, \mathrm{d}t = \gamma .$$

Then,

$$\lim_{z \to x_0} F(z) = \gamma \quad (angularly).$$

Finally, we remark that Theorem 4 below generalizes some of our Tauberian results from [18].

2. PRELIMINARIES

We explain in this section several pointwise notions for distributions. There are several equivalent ways to introduce them. We start with the useful approach from [3]. Define the operator μ_a on locally integrable complex valued functions in $\mathbb{R}$ as

$$\mu_a \{f(t); x\} = \frac{1}{x - a} \int_a^x f(t) \, \mathrm{d}t, \quad x \neq a,$$

while the operator ∂_a is the inverse of μ_a,

$$\partial_a (g) = ((x - a) g(x))' .$$

Suppose first that $f_0 = f$ is *real*. Then if it is bounded near $x = a$, we can define

$$\overline{f_0}(a) = \limsup_{x \to a} f(x) , \qquad \underline{f_0}(a) = \liminf_{x \to a} f(x) .$$

Then $f_1 = \mu_a(f)$ will be likewise bounded near $x = a$ and actually

$$\underline{f_0}(a) \leq \underline{f_1}(a) \leq \overline{f_1}(a) \leq \overline{f_0}(a)$$

and, in particular, if $f(a) = f_0(a)$ exists, then $f_1(a)$ also exists and $f_1(a) = f_0(a)$.

Definition 1. A distribution $f \in \mathcal{D}'(\mathbb{R})$ is called distributionally bounded at $x = a$ if there exist $n \in \mathbb{N}$ and $f_n \in \mathcal{D}'(\mathbb{R})$, continuous and bounded in a pointed neighborhood $(a - \varepsilon, a) \cup (a, a + \varepsilon)$ of a, such that $f = \partial_a^n f_n$.

If f_0 is distributionally bounded at $x = a$, then there exists a *unique* distributionally bounded distribution near $x = a$, f_1, with $f_0 = \partial_a f_1$. Therefore, ∂_a and μ_a are isomorphisms of the space of distributionally bounded distributions near $x = a$. Given f_0 we can form a sequence of distributionally bounded distributions $\{f_n\}_{n=-\infty}^{\infty}$ with $f_n = \partial_a f_{n+1}$ for each $n \in \mathbb{Z}$.

We say that f has the distributional point value γ in the sense of Łojasiewicz [11, 10] and write

$$f(a) = \gamma \quad (\mathrm{L})\,,$$

if there exists $n \in \mathbb{N}$, the order of the point value, such that f_n is continuous near $x = a$ and $f_n(a) = \gamma$.

It can be shown [3, 8, 11, 14] that $f(a) = \gamma$ (L) if and only if

$$\lim_{\varepsilon \to 0} f(a + \varepsilon x) = \gamma\,,$$

distributionally, that is, if and only if

$$(2.1) \qquad \lim_{\varepsilon \to 0^+} \langle f(a + \varepsilon x), \phi(x) \rangle = \gamma \int_{-\infty}^{\infty} \phi(x)\,\mathrm{d}x\,,$$

for each $\phi \in \mathcal{D}(\mathbb{R})$. On the other hand, if f is distributionally bounded at $x = a$ then $\langle f(a + \varepsilon x), \phi(x) \rangle$ is bounded as $\varepsilon \to 0$.

We can also consider distributional lateral limits [11, 17]. We say that the distributional lateral limit $f(a + 0)$ (L) as $x \to a$ from the right exists and equals γ, and write

$$f(a + 0) = \gamma \quad (\mathrm{L})\,,$$

if (2.1) holds for all $\phi \in \mathcal{D}(\mathbb{R})$ with support contained in $(0, \infty)$. The distributional lateral limit from the left $f(a - 0)$ (L) is defined in a similar fashion.

Observe also that if $f = \partial_a^n f_n$, and f_n is bounded near $x = a$, then $f(a + 0)$ (L) exists, and equals γ, if and only if $f_n(a + 0) = \gamma$ (L).

These notions have straightforward extensions to distributions defined in a smooth contour of the complex plane. A natural extension of this pointwise notions for distributions is the so called quasiasymptotic behavior of distributions, explained, e.g., in [14, 16, 20].

3. Boundary values and distributional point values

We shall need the following well known fact [1]. We shall use the notation $\mathbb{H}$ for the half plane $\{z \in \mathbb{C} : \Im m\, z > 0\}$.

Lemma 1. *Let F be analytic in the half plane $\mathbb{H}$, and suppose that the distributional limit $f(x) = F(x + i0)$ exists in $\mathcal{D}'(\mathbb{R})$. Suppose that there exists an open, non-empty interval I such that f is equal to the constant γ in I. Then $f = \gamma$ and $F = \gamma$.*

Actually using the theorem of Privalov [15, Cor 6.14] it is easy to see that if F is analytic in the half plane $\mathbb{H}$, $f(x) = F(x + i0)$ exists in $\mathcal{D}'(\mathbb{R})$, and there exists a subset $X \subset \mathbb{R}$ of non-zero measure such that the distributional point value $f(x_0)$ exists and equals γ if $x_0 \in X$, then $f = \gamma$ and $F = \gamma$.

Our first result is for *bounded* analytic functions.

Theorem 1. *Let F be analytic and bounded in a rectangular region of the form $(a, b) \times (0, R)$. Set $f(x) = \lim_{y \to 0^+} F(x + iy)$ in $\mathcal{D}'(a, b)$, so that $f \in L^\infty(a, b)$. Let $x_0 \in (a, b)$ be such that*

$$(3.1) \qquad\qquad f(x_0 + 0) = \gamma \quad \text{(L)}$$

exists. Then the distributional point value

$$(3.2) \qquad\qquad f(x_0) = \gamma \quad \text{(L)}$$

also exists. In fact, the point value is of the first order, and thus

$$(3.3) \qquad\qquad \lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\, \mathrm{d}t = \gamma\,.$$

Proof. We shall first show that it is enough to prove the result if the rectangular region is the upper half-plane $\mathbb{H}$. Indeed, let C be a smooth simple closed curve contained in $(a, b) \times [0, R)$ such that $\mathsf{C} \cap (a, b) = [x_0 - \eta, x_0 + \eta]$, and which is symmetric with respect to the line $\Re e\, z = x_0$. Let φ be a conformal bijection from $\mathbb{H}$ to the region enclosed by C such that the image of the line $\Re e\, z = x_0$ is contained in $\Re e\, z = x_0$, so that, in particular, $\varphi(x_0) = x_0$. Then (3.1)–(3.3) hold if and only if the corresponding equations hold for $f \circ \varphi$.

Therefore we may assume that $a = -\infty$, and $b = R = \infty$. In this case, f belongs to the Hardy space H^∞, the closed subspace of $L^\infty(\mathbb{R})$ consisting of the boundary values of bounded analytic functions on $\mathbb{H}$.

Let $f_\varepsilon(x) = f(x_0 + \varepsilon x)$. Clearly, the set $\{f_\varepsilon : \varepsilon > 0\}$ is weak* bounded (as a subset of the dual space $(L^1(\mathbb{R}))' = L^\infty(\mathbb{R})$) and, consequently, a relatively weak* compact set. If $\{\varepsilon_n\}_{n=0}^\infty$ is a sequence of positive numbers with $\varepsilon_n \to 0$ such that the sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ is weak* convergent to $g \in L^\infty(\mathbb{R})$, then $g \equiv \gamma$, since $g \in H^\infty$, and $g(x) = \gamma$ for $x > 0$. In fact, the condition (3.1) means that

$$\int_0^\infty g(x)\psi(x)\mathrm{d}x = \lim_{n\to\infty} \int_0^\infty f_{\varepsilon_n}(x)\psi(x)\mathrm{d}x = \gamma \int_0^\infty \psi(x)\mathrm{d}x\ ,$$

for all $\psi \in \mathcal{D}(0,\infty)$, which yields the claim. Since any sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ with $\varepsilon_n \to 0$ has a weak* convergent subsequence, and since that subsequence converges to the constant function γ, we conclude that $f_\varepsilon \to \gamma$ in the weak* topology of $L^\infty(\mathbb{R})$. Furthermore, (3.3) follows by taking $x = x_0 + \varepsilon$ and $\phi(t) = \chi_{[0,1]}(t)$, the characteristic function of the unit interval, in the limit $\lim_{\varepsilon\to 0} \langle f_\varepsilon(t), \phi(t)\rangle = \gamma \int_{-\infty}^\infty \phi(t)\,\mathrm{d}t.$ $\square$

We can now prove our main result, a distributional extension of Theorem 1.

Theorem 2. *Let F be analytic in a rectangular region of the form $(a,b)\times(0,R)$. Suppose $f(x) = \lim_{y\to 0^+} F(x+iy)$ in the space $\mathcal{D}'(a,b)$. Let $x_0 \in (a,b)$ such that $f(x_0 + 0) = \gamma$ (L). If f is distributionally bounded at $x = x_0$ then $f(x_0) = \gamma$ (L). Furthermore, $F(z) \to \gamma$ as $z \to x_0$ in an angular fashion.*

Proof. There exists $n \in \mathbb{N}$ and a function f_n bounded in a neighborhood of x_0 such that $f = \partial_{x_0}^n f_n$; notice that $f(x_0) = \gamma$ (L) if and only if $f_n(x_0) = \gamma$ (L). But $f_n(x) = F_n(x + i0)$ distributionally, where F_n is analytic in $(a,b) \times (0,R)$; here F_n is the only angularly bounded solution of $F(z) = \partial_{x_0}^n F_n(z)$ (derivatives with respect to z). Clearly, $f_n(x) = F_n(x+i0)$. Since f_n is bounded near $x = x_0$, F_n is also bounded in a rectangular region of the form $(a_1,b_1)\times(0,R_1)$, where $x_0 \in (a_1,b_1)$. Clearly $f_n(x_0 + 0) = \gamma$ (L), so the Theorem 1 yields $f_n(x_0) = \gamma$ (L), as required. Finally, the fact that $F(z) \to \gamma$ as $z \to x_0$, angularly, is a consequence of the existence of the distributional point value, as shown in [6, 16]. $\square$

Observe that in general the result (3.3) does not follow if f is not bounded but just distributionally bounded near x_0.

We may use a conformal map to obtain the following general form of the Theorem 2.

Theorem 3. *Let C be a smooth part of the boundary $\partial\Omega$ of a region Ω of the complex plane. Let F be analytic in Ω, and suppose that*

$f \in \mathcal{D}'(\mathsf{C})$ *is the distributional boundary limit of F. Let* $\xi_0 \in \mathsf{C}$ *and suppose that the distributional lateral limit* $f(\xi_0 + 0) = \gamma$ (L) *exists and f is distributionally bounded at* $\xi = \xi_0,$ *then* $f(\xi_0) = \gamma$ (L) *and* $F(z)$ *has non-tangential limit* γ *at the boundary point* $\xi_0.$

We also immediately obtain the following Tauberian theorem. As mentioned at the Introduction, it generalizes some Tauberian results by the authors from [18].

Theorem 4. *Let F be analytic in a rectangular region of the form* $(a, b) \times (0, R)$. *Suppose* $f(x) = \lim_{y \to 0^+} F(x + iy)$ *in the space* $\mathcal{D}'(a, b)$. *Let* $x_0 \in (a, b)$ *such that the distributional limit* $\lim_{y \to 0^+} F(x_0 + iy) = \gamma$ (L) *exists. If f is distributionally bounded at* $x = x_0$ *then* $f(x_0) = \gamma$ (L) *and the angular (ordinary) limit exists:* $\lim_{z \to x_0} F(z) = \gamma.$

Proof. If we consider the curve C to be the union of the segments $(a, x_0]$ and $[x_0, iR)$, then the distributional lateral limit of the boundary value of F on C exists and equals γ as we approach x_0 from the right along C and so the Theorem 3 yields that the distributional limit from the left, which is nothing but $f(x_0 - 0)$ (L), also exists and equals γ. Then the Theorem 2 gives us that $f(x_0) = \gamma$ (L). The existence of the angular limit of $F(z)$ as $z \to x_0$ then follows. $\square$

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