ON UNIFORM RELATIONSHIPS BETWEEN COMBINATORIAL PROBLEMS

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ABSTRACT. The enterprise of comparing mathematical theorems according to
their logical strength is an active area in mathematical logic, with one of the
most common frameworks for doing so being reverse mathematics. In this set-
ting, one investigates which theorems provably imply which others in a weak
formal theory roughly corresponding to computable mathematics. Since the
proofs of such implications take place in classical logic, they may in principle
involve appeals to multiple applications of a particular theorem, or to non-
uniform decisions about how to proceed in a given construction. In practice,
however, if a theorem Q implies a theorem P, it is usually because there is
a direct uniform translation of the problems represented by P into the prob-
lems represented by Q, in a precise sense formalized by Weihrauch reducibility.
We study this notion of uniform reducibility in the context of several natural
combinatorial problems, and compare and contrast it with the traditional no-
tion of implication in reverse mathematics. We show, for instance, that for all
$n, j, k \geq 1$, if $j < k$ then Ramsey's theorem for n-tuples and k many colors
is not uniformly, or Weihrauch, reducible to Ramsey's theorem for n-tuples
and j many colors. The two theorems are classically equivalent, so our anal-
ysis gives a genuinely finer metric by which to gauge the relative strength of
mathematical propositions. We also study Weak König's Lemma, the Thin
Set Theorem, and the Rainbow Ramsey's Theorem, along with a number of
their variants investigated in the literature. Weihrauch reducibility turns out
to be connected with sequential forms of mathematical principles, where one
wishes to solve infinitely many instances of a particular problem simultane-
ously. We exploit this connection to uncover new points of difference between
combinatorial problems previously thought to be more closely related.

1. INTRODUCTION

The idea of reducing, or translating, one mathematical problem to another, with
the aim of using solutions to the latter to obtain solutions to the former, is a
basic and natural one in all areas of mathematics. For instance, the convolution
of two functions can be reduced to a pointwise product via the Fourier transform;

the study of a linear operator over a complex vector space can be reduced to the study of a matrix in Jordan normal form, via a change of basis; etc. In general, the precise forms of such reductions vary greatly with the particular problems, but they tend to be most useful when they are constructive or uniform in some appropriate sense. Typically, such reductions preserve various fundamental properties and yield more information, and they are usually easier to implement. These ideas have materialized in many areas such as category theory, complexity theory, proof theory, and set theory (see [3]). In this article, we investigate similar uniform reductions between various combinatorial problems in the setting of computability theory, reverse mathematics and computable analysis.

The program of reverse mathematics provides a unified and elegant way to compare the strengths of many mathematical theorems. Its setting is second-order arithmetic, which is a system strong enough to encompass most of classical mathematics. The formalism permits talking about natural numbers and about sets of natural numbers, and hence readily accommodates countable analogues of mathematical propositions. The fundamental idea is to calibrate the proof-theoretical strength of such propositions by classifying which set-existence axioms are needed to establish the structures needed in their proofs. In practice, we work with fragments, or subsystems, of second-order arithmetic, first finding the weakest one that suffices to prove a given theorem, and then obtaining sharpness by showing that the theorem is in fact equivalent to it. Each of the subsystems corresponds to a natural closure point under logical, and more specifically, computability-theoretic, operations. Thus, the base system, Recursive Comprehension Axiom (RCA_0), roughly corresponds to computable or constructive mathematics; the system Weak König's Lemma (WKL_0) corresponds to closure under taking infinite paths through infinite binary trees; and the Arithmetical Comprehension Axiom (ACA_0) corresponds to closure under arithmetical definability, or equivalently, under applications of the Turing jump. Other common subsystems, ATR_0 and $\Pi_1^1\text{-}\mathsf{CA}_0$, which we shall not consider in this article, admit similar characterizations. The point is that there is a rich interaction between proof systems on the one hand, and computability on the other.

We refer the reader to Simpson [29] for background on reverse mathematics, to Soare [30] for background on computability theory, and to Weihrauch [35] for background in computable analysis. For background on algorithmic randomness, to which some of our results in Sections 4 and 6 will pertain, we refer to Downey and Hirschfeldt [13].

In the context of reverse mathematics, we can say that a theorem P "reduces" to a theorem Q if there is a proof of P assuming Q over RCA_0. Since these proofs are carried out in a formal system, such a proof of P from Q may use Q several times to obtain P, or may involve non-uniform decisions about which sets to use in a construction. However, in many natural cases, a proof of P from Q uses direct, computable, and uniform translations between problems represented by P into problems represented by Q.

To describe these types of arguments more precisely, we restrict our focus to Π_2^1 statements in the language of second-order arithmetic, i.e., statements of the form

$$(\forall X)(\exists Y)\varphi(X, Y),$$

where φ is arithmetical. Each such principle has associated to it a natural class of *instances*, and for each instance, a natural class of *solutions* to that instance. The following are a few important examples.

Statement 1.1 (WKL)**.** Every infinite subtree of $2^{<\omega}$ has an infinite path.

Statement 1.2 (WWKL)**.** Every subtree T of $2^{<\omega}$ such that

$$\frac{|\{\sigma \in 2^n : \sigma \in T\}|}{2^n}$$

is uniformly bounded away from zero for all n has an infinite path.

Statement 1.3 (Ramsey's Theorem)**.** Fix $n, k \geq 1$. RT^n_k is the statement that for every $f\colon [\omega]^n \to k$, there exists an infinite set H (called *homogeneous* for f) such that f is constant on $[H]^n$.

Statement 1.4 (COH)**.** For every sequence of sets $\langle R_i : i \in \omega \rangle$, there exists an infinite set C such that for all i, either $C \cap R_i$ is finite or $C \cap \overline{R_i}$ is finite.

The idea of uniform direct translations alluded to above was made precise by Weihrauch [33, 34] in the realm of computable analysis and has been widely studied ever since (see [5, Section 1] for a partial bibliography). In this context, a Π^1_2 statement $(\forall X)(\exists Y)\varphi(X, Y)$ as above is viewed as a function specification; a partial realizer of such a specification is a function F such that $\varphi(X, F(X))$ holds for all X in the domain of F. For this reason, it is traditional in this context to understand the relation $\varphi(X, Y)$ as a partial multi-valued function where $\varphi(X, Y)$ holds exactly when Y is one of the possible values of the function at X. Weihrauch then introduced a notion of computable reducibility between partial multi-valued functions whereby there are computable processes that serve to uniformly translate realizers of one partial multi-valued function into realizers of another partial multi-valued function. We shall use here the following equivalent definition, which may appear more familiar from perspectives outside of computable analysis, particularly reverse mathematics. However, with a view towards encouraging more collaboration between these two similary-motivated but thus far largely separate approaches, we include an equivalence of the definitions in Appendix A.

Definition 1.5. Let P and Q be Π^1_2 statements of second-order arithmetic. We say that

(1) P *is Weihrauch reducible to* Q, and write $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, if there exist Turing reductions Φ and Ψ such that whenever A is an instance of P then $B = \Phi(A)$ is an instance of Q, and whenever T is a solution to B then $S = \Psi(A \oplus T)$ is a solution to A.

(2) P *is strongly Weihrauch reducible to* Q, and write $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$, if there exist Turing reductions Φ and Ψ such that whenever A is an instance of P then $B = \Phi(A)$ is an instance of Q, and whenever T is a solution to B then $S = \Psi(T)$ is a solution to A.

In other words, Weihrauch reducibility differs from strong Weihrauch reducibility only in that the "backwards" reduction Ψ takes as oracle not only the solution T to the instance $B = \Phi(A)$ of Q, but also the original instance, A, of P. The two notions thus agree on computable instances of problems, but not in general. (See also [14, Section 1] for a discussion of the distinction between these approaches in the non-uniform case; and [17, Section 2.2] for further discussion of (strong)

Weihrauch and related reducibilities in the context of computable combinatorics. We note that the notation in these sources differs from ours, with $\leq_u$ and $\leq_{su}$ being used in place of $\leq_W$ and $\leq_{sW}$, respectively.) For most of our results below (with the notable exception of Theorem 3.1) it will not matter which of the two reducibility notions we are working with, so to present the strongest possible results, we shall prove reductions for $\leq_{sW}$, and non-reductions for $\not\leq_W$.

It is straightforward to see each of these reducibilities is reflexive and transitive and thus defines a degree structure on Π_2^1 statements.

One simple example of a strong Weihrauch reduction is that $\mathsf{RT}_j^n \leq_{sW} \mathsf{RT}_k^n$ whenever $j \leq k$ because given $f\colon [\omega]^n \to j$, we may view f as a function $g\colon [\omega]^n \to k$ (by ignoring the additional colors) and then every set homogeneous for g is homogeneous for f. (Thus, here Φ and Ψ can both be taken to be the identity reduction.) A slightly more interesting example is that $\mathsf{RT}_k^m \leq_{sW} \mathsf{RT}_k^n$ whenever $m \leq n$. To see this, given $f\colon [\omega]^m \to k$, define $g\colon [\omega]^n \to k$ by letting $g(x_1, \ldots, x_m, \ldots, x_n) = f(x_1, \ldots, x_m)$ and notice that g is uniformly obtained from f via a Turing functional, and that every set homogenous for g is homogeneous for f.

There are also many examples of such reductions using more complicated Turing functionals. Friedman, Simpson, and Smith [16] showed that if P is the statement that every commutative ring with identity has a prime ideal, then $\mathsf{RCA}_0 \vdash \mathsf{WKL} \to \mathsf{P}$. Adapting the proof of this result, one can show that it is possible to uniformly computably convert a commutative ring R into an infinite tree T such that every path of T is a prime ideal of R, and hence that $\mathsf{P} \leq_{sW} \mathsf{WKL}$. For another example, Cholak, Jockusch, and Slaman [9, Theorem 12.5] exhibit a strong Weihrauch reduction of COH to RT_2^2 via a non-trivial Φ.[1]

Despite the fact that many natural implications in reverse mathematics correspond to Weihrauch reductions (even strong Weihrauch reductions), there are certainly examples where an implication holds in reverse mathematics but no Weihrauch reduction exists. For example, building on work of Jockusch in [21], it is known that $\mathsf{RCA}_0 \vdash \mathsf{RT}_k^n \leftrightarrow \mathsf{ACA}$ whenever $n \geq 3$ and $k \geq 2$, and in particular that $\mathsf{RCA}_0 \vdash \mathsf{RT}_2^3 \to \mathsf{RT}_2^4$. However, $\mathsf{RT}_2^4 \not\leq_W \mathsf{RT}_2^3$ because every computable instance of RT_2^3 has a $\emptyset'''$-computable solution, but there is a computable instance of RT_2^4 with no $\emptyset'''$-computable solution (see [21, Theorems 5.1 and 5.6]). The underlying reason why this implication holds in reverse mathematics is that $\emptyset'$ can be coded into a computable instance of RT_2^3, and by relativizing and iterating this result (i.e., by using multiple nested applications of RT_2^3), one can obtain the several jumps necessary to compute solutions to instances of RT_2^4.

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[1] The same is not true if RT_2^2 is replaced by the closely related principle D_2^2, introduced in [9, Statement 7.8]. This asserts that if $f : [\omega]^2 \to 2$ is *stable*, i.e., if for each x the limit of $f(x, y)$ as y tends to infinity exists, then there is an infinite set consisting either entirely of numbers for which this limit is 0, or entirely of numbers for which this limit is 1. A recent result by Chong, Slaman, and Yang [10, Theorem 2.7] resolves a longstanding open problem by showing that $\mathsf{RCA}_0 \nvdash \mathsf{D}_2^2 \to \mathsf{COH}$. However, the model they construct to witness the separation is a non-standard one, and so leaves open the question of whether every ω-model of $\mathsf{RCA}_0 + \mathsf{D}_2^2$ is also a model of COH. A typical reason for this being the case would be if COH were uniformly reducible to D_2^2. However, by results of Dzhafarov [14, Theorem 1.5 and Corollary 1.10] it follows that $\mathsf{COH} \not\leq_{sW} \mathsf{D}_2^2$, and more recently, Lerman, Solomon, and Towsner (unpublished) have shown even that $\mathsf{COH} \not\leq_W \mathsf{D}_2^2$.

There are also more subtle instances where no Weihrauch reduction exists despite the fact that degrees of solutions to the problems correspond. For example, Jockusch [22, Theorem 6] showed that for any $k \geq 2$, the degrees of DNR_k functions (i.e., functions $f \colon \omega \to k$ such that $f(e) \neq \Phi_e(e)$ for all $e \in \omega$) are the same as the degrees of DNR_2 functions, but there is no Weihrauch reduction witnessing this. More precisely, he showed that given $k \geq 2$, there is no Turing functional Φ such that $\Phi(g) \in \mathrm{DNR}_k$ for all $g \in \mathrm{DNR}_{k+1}$. If we let DNR_k be the Π_2^1 statement "for every X, there exists a DNR_k function relative to X", then Jockusch's theorem shows that $\mathsf{DNR}_k \not\leq_{\mathrm{W}} \mathsf{DNR}_{k+1}$.

A motivating question for this article is what happens when one varies the number of colors in Ramsey's Theorem. It is well known that if $n \geq 1$ and $j, k \geq 2$, then $\mathsf{RCA}_0 \vdash \mathsf{RT}_j^n \leftrightarrow \mathsf{RT}_k^n$. For example, to see that $\mathsf{RCA}_0 \vdash \mathsf{RT}_2^n \to \mathsf{RT}_3^n$, we can argue as follows. Suppose that $f \colon [\omega]^n \to 3$. Define $g \colon [\omega]^n \to 2$ by letting

$$g(\mathbf{x}) = \begin{cases} 0 & \text{if } f(\mathbf{x}) \in \{0, 1\}, \\ 1 & \text{if } f(\mathbf{x}) = 2. \end{cases}$$

By RT_2^n, we may fix a set H such that H is homogenous for g. Now if $g([H]^2) = \{1\}$, then H is homogeneous for f. Otherwise, the function $f \restriction [H]^2$ is a 2-coloring of $[H]^2$, so we may apply RT_2^n a second time to conclude that there is an infinite $I \subseteq H$ such that I is homogeneous for f. Notice that this proof requires two nested applications of RT_2^n to obtain a solution to RT_3^n. However, there are no known degree-theoretic differences between homogeneous sets of computable instances of RT_2^n and homogeneous sets of computable instances of RT_3^n, so it is unclear whether there is a proof of RT_3^n using one uniform application of RT_2^n. We prove below in Theorem 3.1 that $\mathsf{RT}_k^n \not\leq_{\mathrm{sW}} \mathsf{RT}_j^n$ when $j < k$.

Although the same basic idea of (strong) Weihrauch reducibility is used in the contexts of computable combinatorics, computable analysis, and reverse mathematics, there are important differences beyond terminology that the reader should keep in mind when translating back and forth.

- In [5], (strong) Weihrauch reduction is defined not only for partial multi-valued functions but also for abstract collections of partial functions on Baire space. This more general idea has no equivalent formulation in second-order arithmetic and, moreover, only definable relations make sense in the latter context. Thus, reverse mathematics has a limited view of the (strong) Weihrauch degrees considered in computable analysis. In practice, this limitation only surfaces when considering the general structure of (strong) Weihrauch degrees since these degrees, when considered for their own sake, generally correspond to definable relations. In this paper, we will only consider arithmetically-definable relations, which therefore make sense in all contexts.

- Computable combinatorics and computable analysis work exclusively with the standard natural numbers whereas reverse mathematics also considers non-standard models. Since the base system RCA_0 only postulates induction for Σ_1^0 formulas, issues related to induction often occur in translation and it is not the case that every reduction $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$ translates into a proof that $\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$. For example, a direct analysis of the reduction of Cholak, Jockusch, and Slaman showing that $\mathsf{COH} \leq_{\mathrm{sW}} \mathsf{RT}_2^2$ alluded to above appears to use Σ_2^0-induction in order to verify that homogenous sets

for the transformed coloring are indeed cohesive for the given instance, and
some additional work is required to verify that the proof goes through RCA_0
(see Mileti [27, Appendix A]).

- The typical use of oracles varies in the three contexts. In computable com-
binatorics, results are usually stated without any use of oracles and issues
of relativization are discussed where necessary. In computable analysis,
both the (strong) Weihrauch reduction above and its continuous analogue,
which permits the use of any oracle, are considered. In reverse mathe-
matics, it is customary to allow any oracle that exists in the model under
consideration. So the case of (strong) Weihrauch reduction corresponds
to the minimal standard model of RCA_0 where the only sets are the com-
putable ones, and the continuous analogue corresponds to the case of the
full standard model of second-order arithmetic where all sets are present.

For these reasons, most of our results will be stated in a way that includes all
relevant translations, though our proofs will generally focus only on one point of
view, with the others left to the reader.

We use standard notations and conventions from computability theory and re-
verse mathematics. We identify subsets of ω with their characteristic functions,
and we identify each $n \in \omega$ with its set of predecessors. Lower-case letters such as
$i, j, k, \ell, m, n, x, y, \ldots$ denote elements of ω. Given a set $A \subseteq \omega$, we let $[A]^n$ denote
the set of all subsets of A of size n. We use $\mathbf{x}, \mathbf{y}, \ldots$ to denote finite subsets of ω,
which we identity with the corresponding tuple listing the elements in increasing
order. We write $\mathbf{x} < \mathbf{y}$ if $\max \mathbf{x} < \min \mathbf{y}$. Given a Turing functional Φ, we assume
that if $\Phi(A)(x) \downarrow$, then $\Phi(A)(y) \downarrow$ for all $y \leq x$. We say that a Turing functional Φ
is *total* if $\Phi(A)$ is a total function for every $A \in 2^\omega$. Given sets A and B, we write
$\Phi(A, B)$ in place of $\Phi(A \oplus B)$.

2. The Squashing Theorem and sequential forms

We can naturally combine two Π_2^1 principles P and Q into one as follows. We
define the *parallel product* $\langle \mathsf{P}, \mathsf{Q} \rangle$ to be the Π_2^1 principle whose instances are pairs
$\langle A, B \rangle$ such that A is an instance of P and B is an instance of Q, and the solutions
to this instance are pairs $\langle S, T \rangle$ such that S is a solution to A and T is a solution
to B. Obviously, this can be generalized to combine any number of Π_2^1 principles,
even an infinite number. In particular, one of our interests will be in cases when P
and Q are the same principle. For $\alpha \in \omega \cup \{\omega\}$, we let α *applications of* P, or P^α,
refer to the Π_2^1 principle whose instances are sequences $\langle A_i : i < \alpha \rangle$ such that each
A_i is an instance of P, and the solutions to this instance are sequences $\langle S_i : i < \alpha \rangle$
such that each S_i is a solution to A_i. The infinite case P^ω is sometimes known as
the *parallelization* of P and is also denoted $\widehat{\mathsf{P}}$.

Notice that we trivially have $\mathsf{COH}^2 \leq_{\mathrm{sW}} \mathsf{COH}$ because given two sequences of
sets $\langle R_i : i \in \omega \rangle$ and $\langle S_i : i \in \omega \rangle$, we can uniformly computably interleave them
to form the sequence $\langle T_i : i \in \omega \rangle$ where $T_{2i} = R_i$ and $T_{2i+1} = S_i$, so that any set
cohesive for $\langle T_i : i \in \omega \rangle$ is cohesive for each of $\langle R_i : i \in \omega \rangle$ and $\langle S_i : i \in \omega \rangle$. In
fact, using a pairing function, it is easy to see that $\mathsf{COH}^\omega \leq_{\mathrm{sW}} \mathsf{COH}$. For another
example, we have that $\mathsf{WKL}^2 \leq_{\mathrm{sW}} \mathsf{WKL}$ as follows. Given two infinite trees $\langle T_0, T_1 \rangle$,
form a new tree S by letting $\sigma \in S$ if the sequence of even bits from σ is an element
of T_0 and the sequence of odd bits from σ is an element of T_1. It is straightforward
to check that S is an infinite tree uniformly computably obtained from $\langle T_0, T_1 \rangle$,

and that given a path B through S, the even bits form a path through T_0, and the odd bits form a path through T_1. Moreover, using a pairing function again, we can interleave a sequence $\langle T_i : i \in \omega \rangle$ of infinite trees together to form one infinite tree such that from any path we can uniformly computably obtain paths through each of the original trees, and hence $\mathsf{WKL}^\omega \leq_{\mathrm{sW}} \mathsf{WKL}$. (This fact is also a consequence of Theorem 8.2 of [5], which shows that WKL is strong Weihrauch equivalent to LLPO^ω; see also Lemma 5 of Hirst [20] for a formalized version in reverse mathematics.)

We have the following important example using distinct principles.

Proposition 2.1. *If $n, j, k \geq 1$, then $\langle \mathsf{RT}_j^n, \mathsf{RT}_k^n \rangle \leq_{\mathrm{sW}} \mathsf{RT}_{jk}^n$.*

Proof. Given $\langle f, g \rangle$ where $f \colon [\omega]^n \to j$ and $g \colon [\omega]^n \to k$, define $h \colon [\omega]^n \to jk$ by $h(\mathbf{x}) = \langle f(\mathbf{x}), g(\mathbf{x}) \rangle$ for all $\mathbf{x} \in [\omega]^n$. Then h is uniformly computable from $\langle f, g \rangle$, and any infinite homogeneous set for h is also homogeneous for both f and g. $\square$

Given a Π_2^1 principle P, if $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$, then it is straightforward to see (by repeatedly applying the reduction procedures) that $\mathsf{P}^n \leq_{\mathrm{sW}} \mathsf{P}$ for each fixed $n \in \omega$. For example, if $n = 4$ and we are given $\langle A_0, A_1, A_2, A_3 \rangle$ where each A_i is an instance of P, then

$$\Phi(A_0, \Phi(A_1, \Phi(A_2, A_3)))$$

is an instance of P uniformly obtained from $\langle A_0, A_1, A_2, A_3 \rangle$, and from any solution to this instance we can repeatedly apply Ψ to uniformly obtain a sequence $\langle S_0, S_1, S_2, S_3 \rangle$ such that each S_i is a solution to A_i. (The same is true if $\leq_{\mathrm{sW}}$ is replaced by $\leq_{\mathrm{W}}$.) It is not at all clear, however, whether this process can be continued into the infinite, i.e., does $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$ necessarily imply that $\mathsf{P}^\omega \leq_{\mathrm{sW}} \mathsf{P}$? Given a sequence $\langle A_i : i \in \omega \rangle$ where each A_i is an instance of P, the natural idea is to consider

$$\Phi(A_0, \Phi(A_1, \Phi(A_2, \Phi(A_3, \dots)))).$$

Of course, this process clearly fails to converge and so does not actually define an instance of P. In fact, we will see later that $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$ does not always imply that $\mathsf{P}^\omega \leq_{\mathrm{sW}} \mathsf{P}$.

However, if $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$ and P is reasonably well-behaved, we will prove that such a "squashing" of infinitely many applications of P into one application of P is indeed possible. For example, consider $\mathsf{P} = \mathsf{RT}_2^2$. The idea is to force some convergence in the above computation by approximating the second coordinate of Φ as follows. When attempting to simulate $\Phi(A_0, \Phi(A_1, \Phi(A_2, \dots)))$, we approximate the unknown result of $\Phi(A_1, \Phi(A_2, \dots))$ by guessing that it starts as the all zero coloring. By assuming this and hence that the second argument looks like a string of zeros, we eventually force convergence of $\Phi(A_0, 0^n)$ on 0, at the cost of introducing some finite initial error in the true "computation" of $\Phi(A_1, \Phi(A_2, \dots))$. Since removing finitely many elements from an infinite homogenous set results in an infinite homogeneous set, these finitely many errors we have introduced into the coloring will not be a problem.

More precisely, we will define a sequence $\langle B_i : i \in \omega \rangle$ of instances of P (where intuitively $B_i = \Phi(A_i, \Phi(A_{i+1}, \cdots))$) beyond some finite error introduced to force convergence), along with a uniformly computable sequence of numbers $\langle m_i : i \in \omega \rangle$, such that

$$B_i(x) = \Phi(A_i, B_{i+1})(x) \text{ for all } x \geq m_i.$$

Now since we no longer have $B_i = \Phi(A_i, B_{i+1})$ (due to the finite error), the Turing functional Ψ may not convert a solution of B_i into a pair of solutions to A_i and B_{i+1}. In order to deal effectively with these finite errors, to ensure that our B_i are actually instances of P, and to ensure that sequence $\langle m_i : i \in \omega \rangle$ is uniformly computable (and hence can be used as markers for cut-off points), we need to make some assumptions about P.

Definition 2.2. Let P be a Π_2^1 principle (or, more generally, any multi-valued function with domain 2^ω).

   (1) P is *total* if every element of 2^ω is (or codes) an instance of P.
   (2) P has *finite tolerance* if there exists a Turing functional Θ such that whenever B_1 and B_2 are instances of P with $B_1(x) = B_2(x)$ for all $x \geq m$, and S_1 is a solution to B_1, then $\Theta(S_1, m)$ is a solution to B_2.

Proposition 2.3. *For each $n, k \geq 1$, the principle RT_k^n is total and has finite tolerance.*

Proof. We can view every element of 2^ω as a valid k-coloring through simple coding. Define Θ as follows. Given $m \in \omega$, compute the largest element ℓ of any tuple of $[\omega]^n$ coded by a natural number less than m, and let $\Theta(S, m) = \{a \in S : a > \ell\}$. Now if B_1 and B_2 are colorings of $[\omega]^n$ using k colors such that $B_1(x) = B_2(x)$ for all $x \geq m$, and S_1 is an infinite set homogeneous for B_1, then $\Theta(S_1, m)$ is also an infinite set and it is homogeneous for B_2. $\square$

Another simple example of a total principle with finite tolerance is COH, where in fact we may take $\Theta(S, m) = S$ (because anything cohesive for a given family of sets is also cohesive for any finite modification of that family).

Although we are certainly interested in the case where $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$, i.e., when $\langle \mathsf{P}, \mathsf{P} \rangle \leq_{\mathrm{sW}} \mathsf{P}$, we will need a slightly more general formulation below. As above, when $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathrm{sW}} \mathsf{P}$, it is straightforward to see that $\langle \mathsf{Q}^n, \mathsf{P} \rangle \leq_{\mathrm{sW}} \mathsf{P}$ for each fixed $n \in \omega$. When passing to the infinite case, however, our "squashing" never reaches the initial instance of P, but in good cases we can conclude that $\mathsf{Q}^\omega \leq_{\mathrm{sW}} \mathsf{P}$. Notice that if $\mathsf{Q} = \mathsf{P}$, this reduces to the case discussed above.

Remark 2.4. As a rule, all results in this section about Π_2^1 principles could be formulated more generally for any multi-valued function with domain 2^ω, as in Definition 2.2. For brevity, we shall omit repeatedly stating this.

Theorem 2.5 (Squashing Theorem). *Let P and Q be Π_2^1 statements, and assume that both are total and that P has finite tolerance.*

   (1) *If $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathrm{sW}} \mathsf{P}$ then $\mathsf{Q}^\omega \leq_{\mathrm{sW}} \mathsf{P}$.*
   (2) *If $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathrm{W}} \mathsf{P}$ then $\mathsf{Q}^\omega \leq_{\mathrm{W}} \mathsf{P}$.*

Proof. We prove (1), the proof of (2) being virtually the same (in fact, the argument can be made somewhat simpler because the oracle has access to the original problem). Throughout, if $\sigma, \tau \in 2^{<\omega}$, we write $\sigma\tau$ for the concatenation of σ by τ, and $\sigma^\frown\tau$ for the continuation of σ by τ, meaning

$$\sigma^\frown\tau(i) = \begin{cases} \sigma(i) & \text{if } i < |\sigma|, \\ \tau(i) & \text{if } |\sigma| \leq i < |\tau|, \end{cases}$$

for all $i < \max\{|\sigma|, |\tau|\}$. For $A \in 2^\omega$, we similarly define $\sigma^\frown A$.

Fix functionals Φ and Ψ witnessing the fact that $\langle \mathsf{Q}, \mathsf{P} \rangle \leq_{\mathrm{sW}} \mathsf{P}$. Since P is total, we may fix a computable instance C of P (one could take C to be the sequence of all 0s, but for some particular problems it might be more convenient or natural to use a different C). Given a sequence $\langle A_i : i \in \omega \rangle$ of instances of Q, we uniformly define a sequence $\langle B_i : i \in \omega \rangle$ of instances of P together with a uniformly computable sequence $\langle m_i : i \in \omega \rangle$ of numbers so that

$$B_i = (C \restriction m_i)^\frown \Phi(A_i, B_{i+1})$$

for all i. In other words, we will have $B_i(x) = C(x)$ for all $x < m_i$, and $B_i(x) = \Phi(A_i, B_{i+1})(x)$ for all $x \geq m_i$. We will then use the instance B_0 of P as our transformed version of $\langle A_i : i \in \omega \rangle$ and show how given a solution T_0 of B_0, we can uniformly transform T_0 into a sequence $\langle S_i : i \in \omega \rangle$ of solutions to $\langle A_i : i \in \omega \rangle$. One subtle but very important point here is that our sequence $\langle m_i : i \in \omega \rangle$ of cut-off positions will need to be uniformly computable independent of the instances $\langle A_i : i \in \omega \rangle$, so that we can use them to unravel a solution T_0 of B_0 without knowledge of the initial instance.

Thus, our first goal is to define the uniformly computable sequence $\langle m_i : i \in \omega \rangle$. We proceed in stages, initially letting $m_0 = 0$. At stage s, we define m_{s+1}. The goal is to choose m_{s+1} large enough to ensure that all potential B_i for $i \leq s$ will be defined on s. Intuitively, by placing enough of C down in column $s+1$ (i.e., at the beginning of B_{s+1}), we must eventually see convergence on previous columns through the cascade effect of the nested Φ. Since we do not have access to the sequence $\langle A_i : i \in \omega \rangle$, we make essential use of compactness and the fact that Q is total to handle all potential inputs at once.

To this end, assume m_t has been defined for each $t \leq s$. First we claim there exists an $n \in \omega$ such that for all $\sigma_0, \ldots, \sigma_s \in 2^n$,

$$\Phi(\sigma_s, C \restriction n)(s) \downarrow,$$
$$\Phi(\sigma_{s-1}, (C \restriction m_s)^\frown \Phi(\sigma_s, C \restriction n))(s) \downarrow,$$
$$\Phi(\sigma_{s-2}, (C \restriction m_{s-1})^\frown \Phi(\sigma_{s-1}, (C \restriction m_s)^\frown \Phi(\sigma_s, C \restriction n)))(s) \downarrow,$$

and for general $i \leq s$,

$$(1) \qquad \Phi(\sigma_i, (C \restriction m_{i+1})^\frown \cdots ^\frown \Phi(\sigma_{s-1}, (C \restriction m_s)^\frown \Phi(\sigma_s, C \restriction n)) \cdots)(s) \downarrow.$$

Observe that the set of all such n is closed under successor. Thus, once the claim is proved, we can define m_{s+1} to be the least such n that is greater than m_t for all $t \leq s$ and also greater than s (to ensure that B_{s+1} will be defined on $0, 1, \ldots, s$ as well). This observation also implies that to prove the claim, it suffices to fix $i \leq s$, and prove that we can effectively find an n such that (1) holds for all $\sigma_i, \ldots, \sigma_s \in 2^n$.

To this end, let T be the set of all tuples $\langle \sigma_i, \ldots, \sigma_s \rangle$ of binary strings with $|\sigma_i| = \cdots = |\sigma_s|$ such that

$$\Phi(\sigma_i, (C \restriction m_{i+1})^\frown \cdots ^\frown \Phi(\sigma_{s-1}, (C \restriction m_s)^\frown \Phi(\sigma_s, C \restriction |\sigma_i|)) \cdots)(s) \uparrow.$$

Since each of the computations here has a finite string as an oracle, T is a computable set. Furthermore, if $\langle \tau_i, \ldots, \tau_s \rangle$ is an initial segment of $\langle \sigma_i, \ldots, \sigma_s \rangle$ under component-wise extension, that is if $\tau_i \preceq \sigma_i, \ldots, \tau_s \preceq \sigma_s$, then $\langle \tau_i, \ldots, \tau_s \rangle$ belongs to T if $\langle \sigma_i, \ldots, \sigma_s \rangle$ does. Thus, T is a subtree in $(2^{<\omega})^s$ under component-wise extension.

Now if T is infinite, then it must have an infinite path $\langle U_i, \ldots, U_s \rangle$, where $U_i, \ldots, U_s \in 2^\omega$ and $\langle U_i \upharpoonright k, \ldots, U_s \upharpoonright k \rangle \in T$ for all k. Then by definition of T,

$$\Phi(U_i, (C \upharpoonright m_{i+1})^\frown \cdots ^\frown \Phi(U_{s-1}, (C \upharpoonright m_s)^\frown \Phi(U_s, C)) \cdots)(s) \uparrow .$$

As P and Q are both total, each of $U_i, \ldots, U_s$ are instances of Q, and each of the second components of any Φ above are instances of P. In particular,

$$(C \upharpoonright m_{i+1})^\frown \cdots ^\frown \Phi(U_{s-1}, (C \upharpoonright m_s)^\frown \Phi(U_s, C))$$

is an instance V of P, as is $\Phi(U_i, V)$. But then $\Phi(U_i, V)(s)$ cannot be undefined. We conclude that T is finite, whence its height can clearly serve as the desired n. To complete the proof, we note that an index for T as a computable tree can be found uniformly computably from i and $m_0, \ldots, m_s$, and therefore so can n.

We now define our reduction procedures witnessing that $\mathsf{Q}^\omega \leq_{\mathrm{sW}} \mathsf{P}$. Let $\langle A_i : i \in \omega \rangle$ be an instance of Q^ω. From this sequence, we uniformly computably define a sequence $\langle B_i : i \in \omega \rangle$ of instances of P as follows. Again, we proceed by stages, doing nothing at stage 0. At stage $s + 1$, we define $B_i(s)$ for each $i \leq s$ and define B_{s+1} on $0, 1, \ldots, s$. If $s < m_i$, we let $B_i(s) = C(s)$. Otherwise, we let

$$B_i(s) = \Phi(A_i, (C \upharpoonright m_{i+1})^\frown \cdots ^\frown \Phi(A_{s-1}, (C \upharpoonright m_s)^\frown \Phi(A_s, C \upharpoonright m_{s+1})) \cdots)(s),$$

the right-hand of which we know to be convergent by definition of m_{s+1}. That is, we have defined

$$
\begin{array}{rcl}
B_s(s) & = & \Phi(A_s, C \upharpoonright m_{s+1})(s), \\
B_{s-1}(s) & = & \Phi(A_{s-1}, (C \upharpoonright m_s)^\frown \Phi(A_s, C \upharpoonright m_{s+1}))(s), \\
B_{s-2}(s) & = & \Phi(A_{s-2}, (C \upharpoonright m_{s-1})^\frown \Phi(A_{s-1}, (C \upharpoonright m_s)^\frown \Phi(A_s, C \upharpoonright m_{s+1})))(s),
\end{array}
$$

and so forth. (Each of the A_t in the computations above could also be replaced by $A_t \upharpoonright m_{s+1}$.) We also define $B_{s+1}(j) = C(j)$ for all $j \leq s$. Since, from the next stage on, B_{s+1} will be defined so that $B_{s+1} \upharpoonright m_{s+1} = C \upharpoonright m_{s+1}$, it is not difficult to see that we do indeed succeed in arranging $B_i = (C \upharpoonright m_i)^\frown \Phi(A_i, B_{i+1})$, as desired. Furthermore, $\langle B_i : i \in \omega \rangle$ is defined uniformly computably from $\langle A_i : i \in \omega \rangle$, and each B_i is an instance of P because P is total. In particular, and there is a Turing functional that produces B_0 from $\langle A_i : i \in \omega \rangle$.

Let Θ be a Turing functional witnessing that P has finite tolerance. We claim that from any solution to the instance B_0 of P, we can uniformly computably obtain a sequence of solutions to $\langle A_i : i \in \omega \rangle$. So suppose T_0 is any such solution to B_0. The idea is to repeatedly apply the reduction Θ to deal with the finite errors, followed up by Ψ to convert individual solutions to pairs of solutions. Indeed, since $B_0(x) = \Phi(A_0, B_1)(x)$ for all $x \geq m_0$, we have that $\Theta(T_0, m_0)$ is a solution to $\Phi(A_0, B_1)$. Thus, $\Psi(\Theta(T_0, m_0)) = \langle S_0, T_1 \rangle$ is such that S_0 is a solution to A_0, and T_1 is a solution to B_1. The first of these, S_0, can serve as the first member of our sequence of solutions. Since $B_1(x) = \Phi(A_1, B_2)(x)$ for all $x \geq m_1$, we have that $\Theta(T_1, m_1)$ is a solution to $\Phi(A_1, B_2)$. Thus, $\Psi(\Theta(T_1, m_1)) = \langle S_1, T_2 \rangle$ is such that S_1 is a solution to A_1, and T_2 is a solution to B_2. Continuing in this way, we build an entire sequence $\langle S_i : i \in \omega \rangle$ of solutions to $\langle A_i : i \in \omega \rangle$, and since $\langle m_i : i \in \omega \rangle$ is uniformly computable, we do this uniformly computably from T_0 alone. The proof is complete. $\qquad \square$

The utility of the Squashing Theorem for our purposes, as we shall see in subsequent sections, is that in many cases it allows us to deduce that multiple applications of a given principle cannot be uniformly reduced to one. This is because there is

no (strong) Weihrauch reduction of ω instances of that principle to one, and in general, showing this tends to be easier.

Corollary 2.6. *Let* P *be a* Π^1_2 *principle that is total and has finite tolerance.*

    (1) *If* $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$, *then* $\mathsf{P}^\omega \leq_{\mathrm{sW}} \mathsf{P}$.
    (2) *If* $\mathsf{P}^2 \leq_{\mathrm{W}} \mathsf{P}$, *then* $\mathsf{P}^\omega \leq_{\mathrm{W}} \mathsf{P}$.

Proof. Apply Theorem 2.5 with $\mathsf{Q} = \mathsf{P}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.7. *Let* P *and* Q *be* Π^1_2 *principles.*

    • *If both* P *and* Q *are total, then* $\langle \mathsf{P}, \mathsf{Q} \rangle$ *is total.*
    • *If both* P *and* Q *have finite tolerance, then* $\langle \mathsf{P}, \mathsf{Q} \rangle$ *has finite tolerance.*

Proof. Immediate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.8. *Let* P *and* Q *be* Π^1_2 *statements, assume that both are total and that* P *has finite tolerance, and let* $m \geq 1$ *be given.*

    (1) *If* $\langle \mathsf{Q}, \mathsf{P}^m \rangle \leq_{\mathrm{sW}} \mathsf{P}^m$ *then* $\mathsf{Q}^\omega \leq_{\mathrm{sW}} \mathsf{P}^m$.
    (2) *If* $\langle \mathsf{Q}, \mathsf{P}^m \rangle \leq_{\mathrm{W}} \mathsf{P}^m$ *then* $\mathsf{Q}^\omega \leq_{\mathrm{W}} \mathsf{P}^m$.

Proof. Repeatedly applying Lemma 2.7, we see that P^m is total and has finite tolerance. The result follows from the Squashing Theorem. $\qquad\qquad\qquad\square$

Corollary 2.9. *Let* P *be a* Π^1_2 *principle that is total and has finite tolerance, and let* $m \geq 1$ *be given.*

    (1) *If* $\mathsf{P}^{m+1} \leq_{\mathrm{sW}} \mathsf{P}^m$, *then* $\mathsf{P}^\omega \leq_{\mathrm{sW}} \mathsf{P}^m$.
    (2) *If* $\mathsf{P}^{m+1} \leq_{\mathrm{W}} \mathsf{P}^m$, *then* $\mathsf{P}^\omega \leq_{\mathrm{W}} \mathsf{P}^m$.

Proof. Since $\mathsf{P}^{m+1} \leq_{\mathrm{sW}} \mathsf{P}^m$, we know that $\langle \mathsf{P}, \mathsf{P}^m \rangle \leq_{\mathrm{sW}} \mathsf{P}^m$, so the result follows from the previous corollary. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the remainder of this article, we employ the following short-hand to avoid excessive exponents and to give P^ω a more evocative name.

Statement 2.10. For any Π^1_2 principle P, we denote ω applications of P, or P^ω, by SeqP. We call SeqP the *sequential version* of P.

So, for instance, Corollary 2.6 says that that if P is total and has finite tolerance, then $\mathsf{P}^2 \leq_{\mathrm{sW}} \mathsf{P}$ implies that $\mathsf{SeqP} \leq_{\mathrm{sW}} \mathsf{P}$. With this terminology, we have the following simple result.

Proposition 2.11. *Let* P *and* Q *be* Π^1_2 *principles.*

    (1) *If* $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$, *then* $\mathsf{SeqP} \leq_{\mathrm{sW}} \mathsf{SeqQ}$.
    (2) *If* $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$, *then* $\mathsf{SeqP} \leq_{\mathrm{W}} \mathsf{SeqQ}$.

Proof. For (1), fix Φ and Ψ witnessing the reduction $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$. Given an instance $\langle A_i : i \in \omega \rangle$ of SeqP, we have that $\langle \Phi(A_i) : i \in \omega \rangle$ is an instance of SeqQ uniformly computably obtained from it. Also, if $\langle T_i : i \in \omega \rangle$ is a solution to $\langle \Phi(A_i) : i \in \omega \rangle$, then $\langle \Psi(T_i) : i \in \omega \rangle$ is a solution to $\langle A_i : i \in \omega \rangle$. For (2), the proof is the same, except we must take $\langle \Psi(A_i, T_i) : i \in \omega \rangle$ as the solution. $\qquad\square$

3. Ramsey's theorem for different numbers of colors

Throughout this section, let $n \geq 1$ be fixed. Our goal is to work up towards a proof of the following theorem.

Theorem 3.1. *For all $j, k \geq 2$ with $j < k$, we have $\mathsf{RT}_k^n \not\leq_{\mathrm{sW}} \mathsf{RT}_j^n$.*

As pointed out above, we have that $\mathsf{RCA}_0 \vdash \mathsf{RT}_j^n \to \mathsf{RT}_k^n$, but the obvious proof uses multiple nested applications of RT_j^n. Theorem 3.1 says that it is impossible to give a uniform proof of this implication using just one application of RT_j^n.

The key ingredients of the proof are Proposition 2.1, the Squashing Theorem, and the fact that it is possible to code more into SeqRT_k^n than into RT_k^n alone. To illustrate the last of these, consider RT_2^1. Notice that every computable instance of RT_2^1 trivially has a computable solution because either there are infinitely many 0s or there are infinitely many 1s (and each of these sets is computable), but there is one non-uniform bit of information used to determine which of these two statements is true. However, it is a straightforward matter to build a computable instance of SeqRT_2^1 such that every solution computes $\emptyset'$. The idea is to use each column to code one bit of $\emptyset'$ by exploiting this one non-uniform decision. In fact, for higher exponents this result can be made sharper, as we now prove. (See also [24, Proposition 47] for a related result in the context of proof mining and program extraction.)

Lemma 3.2. *There is a computable instance of SeqRT_2^n every solution to which computes $\emptyset^{(n)}$.*

Proof. We prove the result for n being odd; the case where n is even is analogous. Fix a computable predicate φ such that

$$\emptyset^{(n)} = \{i \in \omega : (\exists x_0)(\forall x_1) \cdots (\exists x_{n-1}) \ \varphi(i, x_0, x_1, \ldots, x_{n-1})\}.$$

Define a computable sequence of colorings $\langle f_i : i \in \omega \rangle$ by

$$f_i(\mathbf{y}) = \begin{cases} 1 & \text{if } (\exists x_0 < y_0)(\forall x_1 < y_1) \cdots (\exists x_{n-1} < y_{n-1}) \ \varphi(i, x_0, x_1, \ldots, x_{n-1}), \\ 0 & \text{otherwise}, \end{cases}$$

for all $\mathbf{y} = \langle y_0, y_1, \ldots, y_{n-1} \rangle \in [\omega]^n$.

Let $\langle H_i : i \in \omega \rangle$ be any sequence of infinite homogeneous sets for the f_i. We claim that $\emptyset^{(n)}(i) = f_i([H_i]^n)$ for all i, and hence that $\emptyset^{(n)} \leq_T \langle H_i : i \in \omega \rangle$. To see this, suppose first that $i \in \emptyset^{(n)}$. Let $\langle w_{2j} : 2j < n \rangle$ be Skolem functions for membership in $\emptyset^{(n)}$, so that

$$(\forall x_1)(\forall x_3) \cdots (\forall x_{n-2}) \ \varphi(i, w_0(i), x_1, w_2(i, x_1), x_3, \ldots, w_{n-1}(i, x_1, x_3, \ldots, x_{n-2})).$$

Now define an increasing sequence $z_0 < z_1 < \cdots < z_{n-1}$ of elements H_i as follows. Start by letting z_0 be the least $z \in H_i$ that is greater than $w_0(i)$. Then, given j with $1 \leq j \leq n-1$, suppose we have defined z_k for all $k < j$. If j is odd, let z_j be the least $z \in H_i$ that is greater than z_{j-1}. If j is even, let z_j be the least $z \in H_i$ that is greater than z_{j-1}, and also greater than $w_j(i, x_1, x_3, \ldots, x_{j-1})$ for all sequences $x_1, x_3, \ldots, x_{j-1}$ with $x_k < z_k$ for each odd $k < j$.

The sequence of z_j so constructed now clearly satisfies

$$(2) \qquad (\exists x_0 < z_0)(\forall x_1 < z_1) \cdots (\exists x_{n-1} < z_{n-1}) \ \varphi(i, x_0, x_1, \ldots, x_{n-1}).$$

So by definition of f_i, we have that $f_i(z_0, \ldots, z_{n-1}) = 1$. And since the z_j all belong to H_i, it follows that $f([H_i]^n) = 1$, as desired.

Now suppose that $i \notin \emptyset^{(n)}$. We can similarly construct a sequence $z_0 < \cdots < z_{n-1}$ of elements of H_i witnessing that $f([H_i]^n) = 0$. Let $\langle w_{2j+1} : 2j + 1 < n \rangle$ be Skolem functions for non-membership in $\emptyset^{(n)}$, so that

$$(\forall x_0)(\forall x_2) \cdots (\forall x_{n-1}) \, \neg\varphi(i, x_0, w_1(i, x_0), x_2, \ldots, w_{n-2}(i, x_0, x_2, \ldots, x_{n-3}), x_{n-1}).$$

Let z_0 be the least element of H_i, and suppose we are given a j with $1 \leq j \leq n-1$ such that z_k has been defined for all $k < j$. If j is even, let z_j be the least $z \in H_i$ that is greater than z_{j-1}. If j is odd, let z_j be the least $z \in H_i$ that is greater than z_{j-1}, and also greater than $w_j(i, x_0, x_2, \ldots, x_{j-1})$ for all sequences $x_0, x_2, \ldots, x_{j-1}$ with $x_k < z_k$ for each even $k < j$.

This sequence of z_j satisfies the negation of (2) above, so $f_i(z_0, \ldots, z_{n-1}) = 0$ by definition. Since all the z_j belong to H_i, the claim follows. $\square$

After relativization and translation into the language of strong Weihrauch reductions, we obtain from the above that $\mathsf{TJ}^n \leq_{\mathrm{sW}} \mathsf{SeqRT}_2^n$. (See the discussion following Corollary 5.21 for a definition of the iterated Turing jump, TJ^n.)

Lemma 3.3. *For all $n \geq 1$ and $k \geq 2$, we have $\langle \mathsf{RT}_2^n, \mathsf{RT}_k^n \rangle \not\leq_{\mathrm{W}} \mathsf{RT}_k^n$.*

Proof. Suppose instead that $\langle \mathsf{RT}_2^n, \mathsf{RT}_k^n \rangle \leq_{\mathrm{W}} \mathsf{RT}_k^n$. Since RT_2^n and RT_k^n are both total and have finite tolerance by 2.3, we may use the Squashing Theorem 2.5 to conclude that $\mathsf{SeqRT}_2^n \leq_{\mathrm{W}} \mathsf{RT}_k^n$. Fix Φ and Ψ witnessing the reduction, and let $f = \langle f_i : i \in \omega \rangle$ be any computable instance of SeqRT_2^n. Apply Φ to this sequence to obtain an instance g of RT_k^n and notice that g is computable. By Theorem 5.6 of Jockusch [21], we can find an infinite set H homogeneous for g such that $H' \leq_T \emptyset^{(n)}$ (since RT_k^1 is computably true, Jockusch's Theorem 5.6 holds also when $n = 1$). We then have that $S = \langle S_i : i \in \omega \rangle = \Psi(f, H)$ is a solution to $f = \langle f_i : i \in \omega \rangle$ with $S' \leq_T \emptyset^{(n)}$.

But as the sequence $f = \langle f_i : i \in \omega \rangle$ was chosen as an arbitrary computable instance of SeqRT_2^n, this would imply that every computable instance of SeqRT_2^n has a solution with jump computable in $\emptyset^{(n)}$. This contradicts Lemma 3.2, since no such set can compute $\emptyset^{(n)}$. Therefore, we must have $\langle \mathsf{RT}_2^n, \mathsf{RT}_k^n \rangle \not\leq_{\mathrm{W}} \mathsf{RT}_k^n$. $\square$

We shall prove Theorem 3.1 by means of the following weaker version of the theorem, which now follows easily.

Corollary 3.4. *For all $n \geq 1$ and $k \geq 2$, we have $\mathsf{RT}_{2k}^n \not\leq_{\mathrm{W}} \mathsf{RT}_k^n$.*

Proof. Suppose instead that $\mathsf{RT}_{2k}^n \leq_{\mathrm{W}} \mathsf{RT}_k^n$. We know from Proposition 2.1 that $\langle \mathsf{RT}_2^n, \mathsf{RT}_k^n \rangle \leq_{\mathrm{W}} \mathsf{RT}_{2k}^n$. Hence, using transitivity of $\leq_{\mathrm{W}}$, we have $\langle \mathsf{RT}_2^n, \mathsf{RT}_k^n \rangle \leq_{\mathrm{W}} \mathsf{RT}_k^n$, contrary to Lemma 3.3. $\square$

In order to use this corollary to handle all cases of Theorem 3.1, we use the following result saying that we can fan out a strong Weihrauch reduction $\mathsf{RT}_k^n \leq_{\mathrm{sW}} \mathsf{RT}_j^n$ to obtain a strong Weihrauch reduction with a larger spread between the number of colors used.

Lemma 3.5. *Let $n, j, k, s \geq 1$. If $\mathsf{RT}_k^n \leq_{\mathrm{sW}} \mathsf{RT}_j^n$, then $\mathsf{RT}_{k^s}^n \leq_{\mathrm{sW}} \mathsf{RT}_{j^s}^n$.*

Proof. Fix Φ and Ψ witnessing the fact that $\mathsf{RT}_k^n \leq_{\mathrm{sW}} \mathsf{RT}_j^n$. In what follows, define $e(b, a, i)$ for all $b, a \in \omega$ and all $i < \lfloor \log_b a \rfloor$ to be the ith digit in the base b expansion of a. Thus, for example, $e(10, 25, 0) = 5$ and $e(2, 25, 0) = 1$.

Fix an arbitrary $f\colon [\omega]^n \to k^s$. We now convert f into s many colorings $f_0, \ldots, f_{s-1}\colon [\omega]^n \to k$ by setting

$$f_i(\mathbf{x}) = e(k, f(\mathbf{x}), i)$$

for all $i < s$ and all $\mathbf{x} \in [\omega]^n$. Then for any $\mathbf{x}$, the expansion of $f(\mathbf{x})$ in base k is precisely $f_0(\mathbf{x}) \cdots f_{s-1}(\mathbf{x})$. Hence, any set that is simultaneously homogeneous for each of the f_i is also homogeneous for f.

Now apply the reduction Φ to each f_i to obtain colorings $g_i\colon [\omega]^n \to j$ for each $i < s$. We merge these m many colorings into one coloring $g\colon [\omega]^n \to j^s$ defined by

$$g = \sum_{i=0}^{s-1} j^i g_i.$$

Notice that any infinite set H homogeneous for g is simultaneously homogeneous for each of the g_i. Hence, $\Psi(H)$ is simultaneously homogeneous for each of the f_i. But then by the observation above, it follows that $\Psi(H)$ is an infinite homogeneous set for f. Since the reduction from f to g was uniformly computable, the lemma is proved. $\qquad\qquad\square$

We can now prove our main result.

Proof of Theorem 3.1. Seeking a contradiction, fix $j < k$ and assume $\mathsf{RT}_k^n \leq_{\mathrm{sW}} \mathsf{RT}_j^n$. Since $\frac{k}{j} > 1$, we may fix $s \in \omega$ with $(\frac{k}{j})^s > 4$, so that $4j^s < k^s$. Let $m \in \omega$ be least such that $j^s \leq 2^m$. We then have $2^{m-1} < j^s$, so $2^{m+1} < 4j^s < k^s$, and hence

$$j^s \leq 2^m < 2^{m+1} < k^s.$$

Since we are assuming $\mathsf{RT}_k^n \leq_{\mathrm{sW}} \mathsf{RT}_j^n$, we can use Lemma 3.5 to conclude that $\mathsf{RT}_{k^s}^n \leq_{\mathrm{sW}} \mathsf{RT}_{j^s}^n$. We therefore have

$$\mathsf{RT}_{2^{m+1}}^n \leq_{\mathrm{sW}} \mathsf{RT}_{k^s}^n \leq_{\mathrm{sW}} \mathsf{RT}_{j^s}^n \leq_{\mathrm{sW}} \mathsf{RT}_{2^m}^n$$

Since $\leq_{\mathrm{sW}}$ is transitive, it follows that $\mathsf{RT}_{2^{m+1}}^n \leq_{\mathrm{sW}} \mathsf{RT}_{2^m}^n$, contradicting Corollary 3.4. $\qquad\qquad\square$

It is worth pointing out that, in proving of Theorem 3.1, the proof of Lemma 3.5 was the only moment where it mattered that we were working with the *strong* form of Weihrauch reducibility. Specifically, since Ψ there took solutions to g_i to solutions to f_i for each i, in finding a simultaneous solution H for all the g_i we found a simultaneous solution $\Psi(H)$ for all the f_i. This would no longer be the case if joining with original instances was permitted, since then we could not guarantee that $\Psi(g_i, H_i)$ would be the same set for all i. We do not know how to overcome this difficulty, and hence leave open the question of whether Lemma 3.5 and Theorem 3.1 also holds with $\leq_{\mathrm{sW}}$ replaced by $\leq_{\mathrm{W}}$.

4. Weak Weak König's Lemma

As discussed in Section 2, it is straightforward to see that $\mathsf{SeqWKL} \leq_{\mathrm{sW}} \mathsf{WKL}$ (and the reverse direction is obvious). However, the situation of WWKL is more interesting. By performing the same interleaving process to show that $\mathsf{WKL}^2 \leq_{\mathrm{sW}} \mathsf{WKL}$, one checks that the resulting tree has positive measure if each of the two input trees do (in fact, the measure of the interleaved tree is the product of the measures of the original trees), and hence it follows that $\mathsf{WWKL}^2 \leq_{\mathrm{sW}} \mathsf{WWKL}$. By iterating this, it follows that given any *finite* sequence $\langle T_i : i < n \rangle$ of trees with positive

measure many paths, one can interleave them to obtain a tree S whose measure will be the product of the T_i (and hence also positive) such that from any path through S, one can uniformly compute paths through the T_i. However, this idea does not carry over to the case of an infinite sequence of trees of positive measure, since then the interleaving process can produce a tree of measure 0. Indeed, this can happen even if the measures of the trees in the sequences are uniformly bounded away from 0.

Notice that we trivially have WWKL $\leq_{\mathrm{sW}}$ WKL, so SeqWWKL $\leq_{\mathrm{sW}}$ SeqWKL by Proposition 2.11. As explained in Section 2, we have SeqWKL $\leq_{\mathrm{sW}}$ WKL, and hence SeqWWKL $\leq_{\mathrm{sW}}$ WKL by transitivity of $\leq_{\mathrm{sW}}$. One can also show that this can be formalized to give RCA$_0 \vdash$ WKL $\rightarrow$ SeqWKL $\rightarrow$ SeqWWKL. The next theorem shows that the converses are also true, and hence SeqWWKL, even in this weaker form, is in fact strictly stronger than WWKL.

Theorem 4.1.
   (1) WKL $\leq_{\mathrm{sW}}$ SeqWWKL.
   (2) RCA$_0 \vdash$ SeqWWKL $\rightarrow$ WKL.
In fact, both of these statements hold even if we restrict SeqWWKL *to infinite sequences of subtrees of* $2^{<\omega}$ *of measure uniformly bounded away from* 0.

Proof. We prove (2) in the stronger form in order to handle the formalized version carefully, but our construction is completely uniform and hence can be turned into a proof of (1).

Let S be an arbitrary infinite subtree of $2^{<\omega}$. We define a sequence of trees $\langle T_\sigma : \sigma \in 2^{<\omega}\rangle$ indexed by finite binary strings $\sigma \in 2^{<\omega}$ (which of course can be put in bijection with ω). Intuitively, T_σ is constructed as follows. Put the empty string $\emptyset$, 0, and 1 in T_σ. Keep building above both 0 and 1 putting in all possible extensions as long as $\sigma 0$ and $\sigma 1$ both look extendible in S. If we discover that one of $\sigma 0$ or $\sigma 1$ is not extendible in S, then stop building above 0 or 1 in T_σ accordingly, and forever build above the other side (even if the other also ends up not extendible in S). In this way, T_σ will always have measure either $\frac{1}{2}$ or 1.

More formally, we define our sequence as follows. Given $\rho \in 2^{<\omega}$ and $k \in \omega$, let $Ext_S(\rho, k)$ be the Δ_0 predicate saying that either $k \leq |\rho|$, or there exists an element of S extending ρ of length k. Given $\sigma \in 2^{<\omega}$, define T_σ to be $\emptyset$ together with the set of $\tau \in 2^{<\omega} \setminus \{\emptyset\}$ satisfying one of the following:

   • $\tau(0) = 0$ and $Ext_S(\sigma 0, |\tau|)$.
   • $\tau(0) = 1$ and $Ext_S(\sigma 1, |\tau|)$.
   • $\tau(0) = 0$ and $(\exists k < |\tau|)[Ext_S(\sigma 0, k) \wedge \neg Ext_S(\sigma 1, k)]$.
   • $\tau(0) = 1$ and $(\exists k < |\tau|)[Ext_S(\sigma 1, k) \wedge \neg Ext_S(\sigma 0, k)]$.
   • $(\exists k < |\tau|)[Ext_S(\sigma 0, k) \wedge Ext_S(\sigma 1, k) \wedge \neg Ext_S(\sigma 0, k+1) \wedge \neg Ext_S(\sigma 1, k+1)]$.

Note that the last condition handles the case when both sides die at the same level, and in this situation we (arbitrarily) build the full tree.

Since S is tree, if $k < m$ and $Ext_S(\rho, m)$, then $Ext_S(\rho, k)$. By Σ_1^0-induction and the fact that $Ext_S(\rho, 0)$ holds by definition, if $\neg Ext_S(\rho, m)$ then there exists a unique $k \in \omega$ with $k < m$ such that $Ext_S(\rho, k)$ and $\neg Ext_S(\rho, k+1)$. Using these facts, it is straightforward to check that each T_σ is a tree, and that for each $m \in \omega$, either every element of 2^m is in T_σ or exactly half of the elements of 2^m are in T_σ.

Applying SeqWWKL to the sequence $\langle T_\sigma : \sigma \in 2^{<\omega}\rangle$, we obtain a sequence $\langle B_\sigma : \sigma \in 2^{<\omega}\rangle$ of paths through the trees $\langle T_\sigma : \sigma \in 2^{<\omega}\rangle$. We now define a function

$C \colon \omega \to \{0, 1\}$ recursively by letting $C(n) = B_{C \restriction n}(0)$, where $C \restriction n$ is the finite sequence $C(0)C(1)\cdots C(n-1)$. We claim that C is a path through S. To show this, we prove the stronger fact that for each $n \in \omega$, we have $(\forall m)Ext_S(C \restriction n, m)$. The proof is by induction on n (using Π^0_1-induction, which follows from Σ^0_1-induction). For $n = 0$, note that $C \restriction n = \emptyset$, and we know that $(\forall m)Ext_S(\emptyset, m)$ because S is an infinite tree by assumption. Suppose that we have a given $n \in \omega$ for which $(\forall m)Ext_S(C \restriction n, m)$. In this case, at least one of $(\forall m)Ext_S((C \restriction n)0, m)$ or $(\forall m)Ext_S((C \restriction n)1, m)$ must hold. Now if $i \in \{0, 1\}$ is such that $\neg Ext_S((C \restriction n)i, m)$, then $T_{C \restriction n}$ has no node extending i of length m (by definition of the T_σ), so it must be the case that $B_{C \restriction n}(0) = 1 - i$. Therefore, we must have $(\forall m)Ext_S(C \restriction (n+1), m)$. This completes the induction, and the proof. $\qquad\square$

Fact (1) above can also be derived from the result of Brattka and Gherardi [5, Theorem 8.2] that $\mathsf{WKL} \leq_{\mathrm{sW}} \mathsf{SeqLLPO}$ and the observation of Brattka and Pauly [7, Figure 1] that that $\mathsf{LLPO} \leq_{\mathrm{sW}} \mathsf{WWKL}$. (See [5, Section 1] for a definition of LLPO.)

On the other hand, we have the following fact, which follows in this form from more general results of Brattka and Pauly [7, Proposition 22], and also essentially by the proof of Simpson and Yu [36] that $\mathsf{WWKL} \nrightarrow \mathsf{WKL}$ over RCA_0. We include a proof for completeness.

Proposition 4.2. $\mathsf{WKL} \nleq_{\mathrm{W}} \mathsf{WWKL}$.

Proof. By results of Jockusch and Soare [23, Theorem 5.3], there is a computable instance of WKL for which only measure 0 many elements of 2^ω compute a solution. However, every 1-random computes an infinite path through every infinite computable instance of WWKL. (See, e.g., [1, Lemma 1.3].) $\qquad\square$

Thus while $\mathsf{WWKL}^n \leq_{\mathrm{sW}} \mathsf{WWKL}$ for each $n \in \omega$, we have that $\mathsf{SeqWWKL} \nleq_{\mathrm{W}}$ WWKL. Notice that the Squashing Theorem does not apply to WWKL because it is not total (there is no clear way to view every real as coding an instance of WWKL).

We now turn to questions about uniformly passing back and forth between trees of positive measure. Consider any such tree T of $2^{<\omega}$. A question that seems natural is whether from a positive rational $q < 1$, it is possible to build a tree S of measure at least q, each path through which computes a path through T. Intuitively, is it possible to blow up the measure of T without losing information about its paths? It is not difficult to see that the answer is affirmative, and in fact, that such an S can be obtained uniformly from q and an index for T. Indeed, fix a universal Martin-Löf test $\{U_i : i \in \omega\}$ and let $S = 2^{<\omega} - U_i$ for the least i with $q \leq 1 - 2^{-i}$. Every path through S is 1-random, and hence computes a path through T, but not uniformly. The following lemma and proposition show that if we allow S to be defined non-uniformly from T and q, then we can arrange for the computations from paths to paths to be uniform.

Lemma 4.3. *Given a tree $T \subseteq 2^{<\omega}$ of positive measure p, and given $\varepsilon > 0$, there is a tree S, each path of which uniformly computes a path through T, such that the measure of the complement of S is at most $(1 + \varepsilon)(1 - p)^2$.*

Proof. We may assume $p < 1$, since otherwise we can just take $S = T$. Fix a positive $\delta < 1$ such that $1 - \delta p \leq (1 + \varepsilon)(1 - p)$. Choose minimal, hence incompatible, strings $\sigma_0, \ldots, \sigma_{n-1} \notin T$ such that

$$\sum_{i < n} 2^{-|\sigma_i|} \geq \delta(1 - p),$$

and let

$$S = T \cup (\bigcup_{i < n} \sigma_i T),$$

where $\sigma_i T = \{\sigma_i \tau : \tau \in T\}$. Then the measure of the complement of S is

$$(1 - p) - \sum_{i < n} 2^{-|\sigma_i|} p \leq (1 - p) - \delta p (1 - p) = (1 - p)(1 - \delta p) \leq (1 + \varepsilon)(1 - p)^2.$$

Now let Φ be the functional that sends $A \in 2^\omega$ to $A(|\sigma_i|)A(|\sigma_i| + 1) \cdots$ if $\sigma_i \preceq A$ for some $i < n$, and to A otherwise. Clearly, $\Phi(A)$ is a path through T whenever A is a path through S. $\qquad\square$

Proposition 4.4. *Given a tree $T \subseteq 2^{<\omega}$ of positive measure p, and given a positive rational $q < 1$, there is a tree $S \subseteq 2^{<\omega}$ of measure at least q, each path of which uniformly computes a path through T.*

Proof. Given T, p, and q, choose $\varepsilon_0, \ldots, \varepsilon_{n-1}$ so that

$$(3) \qquad (1 + \varepsilon_{n-1})(1 + \varepsilon_{n-2})^2 \cdots (1 + \varepsilon_0)^{2(n-1)}(1 - p)^{2n} < 1 - q.$$

Now iterate the lemma. Let $S_{-1} = T$, and given S_{i-1} obtain S_i with complement of measure at most $(1 + \varepsilon_i)(1 - \mu(S_{i-1}))^2$ such that each path through S_i computes a path through S_{i-1}. By induction, the complement of S_{n-1} has measure bounded by (3), and each path through it computes a path through T. $\qquad\square$

Thus, we can either uniformly blow up the measure of a given tree T, and have paths through the new tree non-uniformly compute paths through the old; or we can non-uniformly blow up the measure of T, and have paths through the new tree uniformly compute paths through the old. The following proposition, which is a direct corollary of Theorem 4.1, shows that we cannot achieve both types of uniformity simultaneously.

Proposition 4.5. *There is no effective procedure that, given (an index for) a computable subtree T of $2^{<\omega}$ of positive measure, and a positive rational q, produces (an index for) a computable subtree S of $2^{<\omega}$ of measure at least q and an $e \in \omega$ such that Φ_e^A is a path through T for every path A through S.*

Proof. Suppose otherwise and fix any computable sequence $\langle T_i : i \in \omega \rangle$ of (indices for) subtrees of $2^{<\omega}$ of positive measure. We build a single tree S of positive measure, every path through which computes a sequence of sets $\langle A_i : i \in \omega \rangle$ such that each A_i is a path through T_i. In particular, every 1-random set computes such a sequence. Of course, this contradicts the proof of Theorem 4.1, as it follows from what is shown there that there exists a sequence of trees for which only sets of PA degree can compute a sequence of paths, but not every 1-random computes a set of PA degree.

We obtain S by interleaving the members of a new sequence $\langle S_i : i \in \omega \rangle$ of subtrees of $2^{<\omega}$, constructed inductively as follows. By adding a tree to $\langle T_i : i \in \omega \rangle$ if necessary, we may assume $\mu(T_0) < 1$, and fix a positive rational number r with $\mu(T_0) < r < 1$. Define $S_0 = T_0$, choose $q_0 < 1$ with $r < q_0$, and let e_0 be an index for the identity reduction. Now suppose we have defined S_i, q_i, and e_i. Choose a rational $q_{i+1} < 1$ such that $\prod_{j \leq i+1} q_j \geq r$, which we may assume exists by induction. Let S_{i+1} and e_{i+1} be as given by the hypothesized effective procedure in the statement, with $T = T_{i+1}$ and $q = q_{i+1}$.

Clearly, the resulting sequences $\langle S_i : i \in \omega \rangle$ and $\langle e_i : i \in \omega \rangle$ are computable. It follows that S is computable, and by construction, $\mu(S) = \prod_{i \in \omega} \mu(S_i) \geq r > 0$. Now suppose B is any path through S. By undoing the interleaving process along B, we computably define a sequence $\langle B_i : i \in \omega \rangle$ such that each B_i is a path through S_i. Setting $A_i = \Phi_{e_i}^{B_i}$ for each i, it follows that $\langle A_i : i \in \omega \rangle$ is the desired sequence of paths through the T_i. $\qquad\square$

The preceding results inspire the following restriction of WWKL. Let $q < 1$ be a positive rational.

Statement 4.6 (q-WWKL). Every subtree T of $2^{<\omega}$ such that

$$\frac{|\{\sigma \in 2^n : \sigma \in T\}|}{2^n} \geq q$$

for all n has an infinite path.

Note that Proposition 4.4 can be formalized to show that $\mathsf{RCA}_0 \vdash \mathsf{WWKL} \leftrightarrow q\text{-}\mathsf{WWKL}$, for each q. We conclude this section with the following contrasting result.

Proposition 4.7. *For all positive rationals $p < q < 1$, p-WWKL $\not\leq_{\mathrm{W}}$ q-WWKL.*

Proof. Suppose not, and let Φ and Ψ witness a Weihrauch reduction from p-WWKL to q-WWKL. We build a computable tree T of measure at least p such that $\Phi(T)$ has measure less than q, and thus obtain the desired contradiction. Intuitively, we use the fact that Ψ must take paths through $\Phi(T)$ to paths through T to successively cut down larger and larger portions of $\Phi(T)$ by cutting down larger and larger portions of T. Although this results in the measures of both trees becoming smaller, we will only cut down each tree finitely many times, and we will be able to control for how much of the measure of T is left.

We shall regard each partial computable function as defining an initial segment of a computable subtree of $2^{<\omega}$, with each new convergence giving an entire new level of the tree, and only strings of maximal length at the previous level being extended. Then Φ in the construction can be viewed as a monotone map between such initial segments. This will ensure that the construction of T will be uniform, and so by the recursion theorem, we can fix an index for it ahead of time. This permits us the convenience of not needing to consider T in the oracle for Ψ, by replacing that functional, if necessary, by $\widehat{\Psi}(X) = \Psi(T, X)$.

Construction. Fix a positive number a such that $2^{-a} < q - p$. At stage s of the construction we shall define $T_s = T \cap 2^{\leq s}$, starting with $T_0 = \{\emptyset\}$. That is, T_s will have height s. Let n_s be the height of $\Phi(T_s)$, and assume without loss of generality that $n_s \leq s$ for all s and that $\Phi(T_0) = \{\emptyset\}$.

At stage $s + 1$, choose the least a many numbers $x_0 < \cdots < x_{a-1}$ that we have not yet acted for, as defined below. Assume inductively that for each $\alpha \in 2^a$ there is a string $\sigma \in T_s$ of length s with $\sigma(x_j) = \alpha(j)$ for all j with $x_j < s$. We consider two cases.

Case 1. If any of the following apply:

- $\Phi(T_s)$ contains fewer than $2^{n_s} q$ many strings of length n_s;
- $x_{a-1} \geq s$;
- $x_{a-1} < s$ but $\Psi(\tau)(x_j) \uparrow$ for some $\tau \in \Phi(T_s)$ of length n_s and some $j < a$;

then we obtain T_{s+1} from T_s by adding $\sigma 0$ and $\sigma 1$ for each $\sigma \in T_s$ of length s.

Case 2. Otherwise, choose $\alpha \in 2^a$ so that $\Psi(\tau)(x_j) \downarrow = \alpha(j)$ for all $j < a$ for at least 2^{-a} many strings $\tau \in \Phi(T_s)$ of length n_s. Then, we obtain T_{s+1} from T_s by adding $\sigma 0$ and $\sigma 1$ for each $\sigma \in T_s$ of length s with $\sigma(x_j) \neq \alpha(j)$ for some $j < a$. Say we have *acted* for $x_0, \ldots, x_{a-1}$.

Verification. Clearly, T is a computable subtree of $2^{<\omega}$. Note that the measure of T is cut down only when the construction enters Case 2, at which point it is cut down by a factor of precisely 2^{-a}. Likewise, whenever the construction enters Case 2, the measure of $\Phi(T)$ is cut down by at least a factor of 2^{-a}. We claim there is a stage s such that $\Phi(T_s)$ contains fewer than $2^{n_s} q$ many strings of length s, so that the measure of $\Phi(T)$ is less than q. Fix the least such s. Then as $2^{-a} < q - p$, it follows that T_s contains at least $2^s p$ many strings of length s. But the construction can never enter Case 2 at any stage after s, so the measure of T is at least p.

It thus remains only to prove the claim. To this end, let t be any stage such that $\Phi(T_t)$ contains at least $2^{n_t} q$ many strings of length n_t. Fix the least $x_0 < \cdots < x_{a-1}$ not yet acted for prior to stage $t + 1$. For each path B through $\Phi(T)$, we have that $\Psi(B)(x_j) \downarrow$ for all $j < a$, so by compactness, there is an $s > \max\{t, x_{a-1}\}$ such that $\Psi(\tau)(x_j) \downarrow$ for all $j < a$ and all $\tau \in \Phi(T_s)$ of length n_s. Fix the least such s. Then the construction never enters Case 2 strictly between stages t and s, so T_s contains at least $2^{n_s} q$ many strings of length n_s, so Case 2 applies at stage s. Hence, we have shown that the construction continues to enter Case 2 until the measure of $\Phi(T)$ has been sufficiently cut down, from which the claim follows. $\square$

5. The Thin Set Theorem

For all $n \geq 1$ and $k \in \{2, 3, 4, \ldots, \omega\}$, say that a subset S of ω is *thin* for a coloring $f: [\omega]^n \to k$ if there exists a $c < k$ such that $f(\mathbf{x}) \neq c$ for all $\mathbf{x} \in [S]^n$. In this section, we shall concentrate on the following combinatorial principle, known as the Thin Set Theorem.

Statement 5.1 (TS_k^n)**.** Let $n \geq 1$ and let $k \in \{2, 3, 4, \ldots, \omega\}$. Every $f: [\mathbb{N}]^n \to k$ admits an infinite thin set.

The statement TS_ω^n is the usual Thin Set Theorem as studied in [8].[2] Note that TS_2^n is logically equivalent to RT_2^n, i.e., the thin sets for 2-colorings are precisely the homogeneous sets. Likewise, observe that whereas RT_1^n is plainly true, TS_1^n is not even defined above, as it would be plainly false.

Implications between versions of the Thin Set Theorem for different numbers of colors go opposite the way they do for Ramsey's theorem. For the purpose of viewing TS_k^n as a multi-valued function, it is important that a solution to an instance of TS_k^n includes which color is omitted by the thin set since there is no uniformly computable way to recover that information from the thin set alone.

Proposition 5.2. *Let $n \geq 1$.*

  (1) *If $j, k \geq 2$ with $j < k$, then $\mathsf{TS}_k^n \leq_{\mathrm{sW}} \mathsf{TS}_j^n$.*

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[2]This should not be confused with the principle $\mathsf{TS}_{<\infty}^n$, which, by analogy with Ramsey's theorem, should be defined as $(\forall k \geq 2)\, \mathsf{TS}_k^n$. By contrast, TS_ω^n is the statement of the Thin Set Theorem for colorings $f: [\mathbb{N}]^n \to \omega$, i.e., colorings employing infinitely many colors. Using Proposition 5.2, is not difficult to see that $\mathsf{TS}_{<\infty}^n$ is equivalent to TS_2^n under strong Weihrauch reducibility.

(2) *If $j, k \geq 2$ with $j < k$, then $\mathsf{RCA}_0 \vdash \mathsf{TS}_j^n \to \mathsf{TS}_k^n$.*

(3) *If $j \geq 2$, then $\mathsf{TS}_\omega^n \leq_{\mathrm{sW}} \mathsf{TS}_j^n$.*

(4) *If $j \geq 2$, then $\mathsf{RCA}_0 \vdash \mathsf{TS}_j^n \to \mathsf{TS}_\omega^n$.*

Proof. We prove (1) and (2) (the argument is uniform and can easily be formalized in RCA_0). Let $j, k \geq 2$ with $j < k$. Fix $f \colon [\omega]^n \to k$. Define $g \colon [\omega]^n \to j$ by letting

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) < j - 1, \\ j - 1 & \text{otherwise} \end{cases}$$

for all $\mathbf{x} \in [\omega]^n$. Now suppose $S \subseteq \omega$ is an infinite thin set for g, say with $c < j$ such that $g(\mathbf{x}) \neq c$ for all $\mathbf{x} \in [S]^n$. If $c < j - 1$, then $f(\mathbf{x}) \neq c$ for all $\mathbf{x} \in S$, while if $c = j - 1$ then $f(\mathbf{x}) < j - 1$ for all such $\mathbf{x}$, so in particular $f(\mathbf{x}) \neq j - 1 = c$. Either way, c witnesses that S is an infinite thin set for f. The proof of (3) and (4) similarly proceeds by collapsing all colors greater then $j - 1$ to be $j - 1$. □

Thus, we have the following chain for any n:

$$\mathsf{TS}_\omega^n \leq_{\mathrm{sW}} \cdots \leq_{\mathrm{sW}} \mathsf{TS}_4^n \leq_{\mathrm{sW}} \mathsf{TS}_3^n \leq_{\mathrm{sW}} \mathsf{TS}_2^n = \mathsf{RT}_2^n \leq_{\mathrm{sW}} \mathsf{RT}_3^n \leq_{\mathrm{sW}} \mathsf{RT}_4^n \leq_{\mathrm{sW}} \cdots$$

By Theorem 3.1, none of the reductions to the right of the equals sign reverse. We shall see in Theorem 5.27 that the same is true of the left side when $n = 1$.

5.1. **General reverse mathematics results.** Before discussing uniform implications and sequential forms, we prove several results about the principles TS_k^n. General questions about the strength of TS_k^n were asked by J. Miller at the *Reverse Mathematics: Foundations and Applications Workshop* in Chicago in November, 2009. Another recent investigation of these principles appears in Wang [32].

Proposition 5.3. *For each $m, n, k \geq 1$, we have $\mathsf{RCA}_0 \vdash \mathsf{TS}_{k^n}^{mn+1} \to \mathsf{TS}_k^{m+1}$.*

Proof. The result is trivial for $n = 1$, so we may assume $n \geq 2$. Let $f \colon [\mathbb{N}]^{m+1} \to k$ be a coloring. Define $g \colon [\mathbb{N}]^{mn+1} \to k^n$ by

$$g(x, \mathbf{y}_0, \ldots, \mathbf{y}_{n-1}) = \langle f(x, \mathbf{y}_0), \ldots, f(x, \mathbf{y}_{n-1}) \rangle$$

for all $x \in \omega$ and $\mathbf{y}_0, \ldots, \mathbf{y}_{n-1} \in [\omega]^m$ with $x < \mathbf{y}_0 < \cdots < \mathbf{y}_{n-1}$.

Suppose H is an infinite set that avoids the color $\langle a_0, \ldots, a_{n-1} \rangle < k^n$ for the coloring g. Choose the greatest $i < n$ for which there are infinitely many $x \in H$ such that

$$(4) \qquad\qquad f(x, \mathbf{y}_0) = a_0, \ldots, f(x, \mathbf{y}_i) = a_i$$

for some $\mathbf{y}_0, \ldots, \mathbf{y}_i \in [H]^m$ with $x < \mathbf{y}_0 < \cdots < \mathbf{y}_i$. By assumption on the color avoided by H, it must be that $i < n - 1$.

By choice of i, we can remove finitely many elements from H if necessary to ensure that if $x < \mathbf{y}_0 < \cdots < \mathbf{y}_i$ satisfy (4) above, then there is no $\mathbf{y} > \mathbf{y}_i$ such that $f(x, \mathbf{y}) = a_{i+1}$. Let H' be H with these finitely many elements deleted.

Now using Δ_1^0 comprehension, we can define a sequence $\langle x_j : j \in \omega \rangle$ of elements of H' so that for each j, (4) holds for some $\mathbf{y}_0, \ldots, \mathbf{y}_i \in [H']^m$ with

$$(5) \qquad\qquad x_j < \mathbf{y}_0 < \cdots < \mathbf{y}_i < x_{j+1}.$$

Let $R \subseteq H'$ be the range of this sequence, which exists because the sequence is increasing. We claim that R avoids the color a_{i+1} for f. Indeed, suppose $f(x, \mathbf{y}) = a_{i+1}$ for some $x \in R$ and $\mathbf{y} \in [R]^m$ with $x < \mathbf{y}$. Let $\mathbf{y}_0, \ldots, \mathbf{y}_i \in [H']^m$ be the

witnesses for having chosen x to belong to our sequence. Then by (5), it follows that $\mathbf{y}_i < \mathbf{y}$, which contradicts the definition of H'. $\square$

Setting $k = 2$, we obtain:

Corollary 5.4. *For each $m, n \geq 1$, we have* $\mathsf{RCA}_0 \vdash \mathsf{TS}_{2^n}^{mn+1} \to \mathsf{RT}_2^{m+1}$.

The proof of Proposition 5.3 is not entirely uniform. Indeed, one assumes Σ_2^0-induction in order to prove that $\forall n(\mathsf{TS}_{k^n}^{mn+1} \to \mathsf{TS}_k^{m+1})$. Similarly, the proof does not give a Weihrauch reduction of TS_k^{m+1} to $\mathsf{TS}_{k^n}^{mn+1}$.

Since RT_2^3 is equivalent to arithmetic comprehension, we also get that $\mathsf{TS}_{2^n}^{2n+1}$ implies arithmetic comprehension for each $n \in \omega$. However, we can do better by carefully choosing the coloring.

Proposition 5.5. *For each $n \geq 1$, we have* $\mathsf{RCA}_0 \vdash \mathsf{TS}_{2^n}^{n+2} \to \mathsf{ACA}$.

Proof. We will show how to reduce finding the range of an injection $f : \mathbb{N} \to \mathbb{N}$ to an instance $g : [\mathbb{N}]^{n+2} \to 2^n$ of $\mathsf{TS}_{2^n}^{n+2}$. Namely, for each $i < n$, define the coloring $g_i : [\mathbb{N}]^{n+2} \to 2$ by

$$g_i(x_0, \ldots, x_{n+1}) = \begin{cases} 1 & (\exists z)[x_i < z < x_{i+1} \wedge f(z) < x_0], \\ 0 & (\forall z)[x_i < z < x_{i+1} \to f(z) \geq x_0], \end{cases}$$

and let $g = \langle g_0, \ldots, g_{n-1} \rangle$. Let H be an infinite set that avoids at least one color $b = \langle b_0, \ldots, b_{n-1} \rangle < 2^n$. Let $m < n$ be the largest index for which there are $x_0 < \cdots < x_{n+1}$ in H with $g_i(x_0, \ldots, x_{n+1}) = b_i$ for all $i < m$, and assume without loss of generality that such $x_0, \ldots, x_{n+1}$ can be found in every tail of H.

To determine whether some number y is in the range of f, choose some elements $x_0 < \cdots < x_n$ of H with $y < x_0$ and $g_i(x_0, \ldots, x_n) = b_i$ for $i < m$. Note that $f(z) \geq x_0$ for all $z > x_m$, otherwise we could pick $y_0 = x_0 < \cdots < y_m = x_m < y_m < \cdots < y_{n+1}$ in H to realize at least $m + 1$ bits of b, contradicting the choice of m. Therefore, y is in the range of f if and only if $y \in \{f(0), \ldots, f(x_m)\}$. $\square$

By contrast, Wang [32, Theorem 3.1] has shown that for every n, there is a k such that TS_k^n does not imply ACA over RCA_0. Thus, the number of colors above is important.

Proposition 5.6 (RCA_0). *For all $n \geq 1$, we have* $\mathsf{RCA}_0 \vdash \mathsf{TS}_{3^n}^{n+1} \to \mathsf{RT}_{<\infty}^1$.

Proof. By Proposition 5.3, it suffices to show that TS_3^2 implies $\mathsf{RT}_{<\infty}^1$. Given $f : \mathbb{N} \to k$, define $g : [\mathbb{N}]^2 \to 3$ by

$$g(x, y) = \begin{cases} 0 & \text{if } f(x) = f(y), \\ 1 & \text{if } f(x) > f(y), \\ 2 & \text{if } f(x) < f(y), \end{cases}$$

for all $x < y$. Suppose that H is an infinite set that avoids one of the three colors.

Note that H cannot avoid the color 0, since otherwise the restriction of f to H would be an injection, which is impossible since H is infinite. So suppose H avoids the color 1, so that the restriction of f to H is then non-decreasing. Any bounded non-decreasing function on an infinite set eventually stabilizes to a maximal value m. Then $f^{-1}(m)$ is an infinite homogeneous set for f. The case when H avoids color 2 is symmetric. $\square$

5.2. **Results for triples.** In order to properly state the strong Weihrauch reduction form of the next few results, we introduce an operation on Π_2^1 statements that essentially corresponds to composition of partial multi-valued functions in the context of computable analysis. Given Π_2^1 statements P and Q and a Turing functional Θ, we define $\mathsf{Q} \bullet \mathsf{P}$ to be the Π_2^1 principle whose instances are instances A of P and whose solutions are pairs $\langle B, C \rangle$ where B is a P-solution for A and C is a Q-solution for the instance $\Theta(A, B)$ of Q. In other words, $\mathsf{Q} \bullet \mathsf{P}$ first takes an instance A of P and seeks a solution B, then uses A and B to construct an instance $\Theta(A, B)$ of Q and seeks a solution C. The Turing functional Θ is only a matter of convenience in order to translate the output of P into an input for Q and it is usually obvious from the context what Θ needs to be. (The notion here essentially corresponds to function composition from the context of computable analysis, although because of the use of the functional Θ, and because solutions here are pairs, the two are not formally the same. Compare this with the compositional product in the Weihrauch lattice, as defined in [6, Definition 4.1].)

To illustrate this definition, consider for example the implication of RT_2^2 by $\mathsf{SRT}_2^2 + \mathsf{COH}$ from Cholak, Jockush and Slaman [9, Theorem 12.5]. Their proof that $\mathsf{SRT}_2^2 + \mathsf{COH}$ implies RT_2^2 breaks into three steps, as follows. A given coloring $f : [\omega]^2 \to 2$ is first transformed into the instance $\langle R_x : x \in \omega \rangle$ of COH given by $R_x = \{y : y > x \wedge f(x, y) = 0\}$; a solution $C = \{c_0 < c_1 < \ldots\}$ to this instance is then transformed, along with f, into the stable coloring $g : [\omega]^2 \to 2$ given by $g(x, y) = f(c_x, c_y)$; and finally, a homogeneous set G for g is transformed into the homogeneous set $H = \{c_x : x \in G\}$ for f. Thus, by letting Θ be the functional that defines g from C and f as here, we see that this argument corresponds to a reduction $\mathsf{RT}_2^2 \leq_{\mathrm{sW}} \mathsf{SRT}_2^2 \bullet \mathsf{COH}$. The key property used here is that the restriction of a computable coloring to a cohesive set is stable, and thus keeping the output C of COH in the composition is essential.

We know that TS_2^3 implies arithmetic comprehension, and Corollary 5.4 shows that TS_4^3 implies RT_2^2. This leaves a gap around TS_3^3. Wang [32, Corollary 3.2] has shown that TS_3^3 does not imply ACA. The next result gives a little more information.

Proposition 5.7. $\mathsf{RCA}_0 \vdash \mathsf{TS}_3^3 \to \mathsf{RT}_{<\infty}^2$.

Proof. Let $f \colon [\mathbb{N}]^2 \to k$ be any finite coloring. Let $g \colon [\mathbb{N}]^3 \to 3$ be defined by

$$g(x, y, z) = |\{f(x, y), f(x, z), f(y, z)\}| - 1.$$

By TS_3^3 there is an infinite set H that omits one of the three possible colors.

Since every infinite set contains at least one homogeneous triangle for f, the set H cannot omit color 0 for g. If H omits color 1 for g, then pick $x \in H$ and consider the sets $H_i = \{y \in H : y > x \wedge f(x, y) = i\}$ for $i < k$. Since H omits triples which take exactly two f-colors, each H_i is homogeneous with color i. By $\mathsf{B}\Pi_1^0$, which follows from RT_2^2 and hence from TS_3^3, one of these sets must be infinite. Thus we have an infinite homogeneous set for f.

The only remaining case is when H omits color 2 for g. In that case, consider the coloring

$$h(x, y, z) = \begin{cases} 0 & \text{if } f(y, z) = f(x, y) = f(x, z), \\ 1 & \text{if } f(y, z) \neq f(x, y) = f(x, z), \\ 2 & \text{if } f(x, z) \neq f(x, y) = f(y, z), \\ 3 & \text{if } f(x, y) \neq f(x, z) = f(y, z), \end{cases}$$

where $x < y < z$ are elements of H. Since H omits color 2 for g, these four cases are exhaustive. By TS_3^3, there is an infinite set $G \subseteq H$ that omits two colors for h.

Since every infinite set contains a homogeneous triangle, color 0 cannot be among the colors omitted by G. If 1 is among the two colors omitted by G, then pick $x \in G$ and consider the sets $G_i = \{y \in G : y > x \land f(x,y) = i\}$ for $i < k$. Since G omits color 1 for h, each G_i is homogeneous of color i. By $\mathsf{B\Pi}_1^0$, one of these sets G_i must be infinite and thus we have an infinite homogeneous set for f.

The only remaining case is when G omits both colors 2 and 3 for h. In that case, G is min-homogeneous for f, i.e., $f(x,y) = f(x,z)$ for all $x < y < z$ in G. We may then unambiguously define the coloring $\bar{f} \colon \mathbb{N} \to k$ by $\bar{f}(x) = f(x,y)$ where $x < y$ in G. By $\mathsf{B\Pi}_1^0$, one of the sets $G_i = \{x \in G : \bar{f}(x) = i\}$ must be infinite, and this G_i is an infinite homogeneous set for f of color i. $\qquad\square$

Because this argument compounds multiple uses of TS_3^3 and its consequences, Proposition 5.7 does not lead a direct reduction of $\mathsf{RT}_{<\infty}^2$ to TS_3^3. Carefully going through the proof and using the $\bullet$ composition described above, we obtain the following cumbersome reduction:

$$\mathsf{RT}_k^2 \leq_{\mathrm{sW}} \mathsf{RT}_k^1 \bullet \mathsf{TS}_3^3 \bullet \mathsf{TS}_3^3 \bullet \mathsf{RT}_k^1 \bullet \mathsf{TS}_3^3.$$

It is also unclear how strong TS_k^3 is for $k \geq 4$. The next three results give some non-trivial lower bounds for $k = 6, 7, 8$.

For these results, we use the following related notions:

Definition 5.8. A coloring $f \colon [\mathbb{N}]^2 \to k$ is *transitive* if for all $i < k$ and all $x < y < z$, whenever $f(x,y) = f(y,z) = i$ then $f(x,z) = i$. A coloring is *semi-transitive* if this property holds for all but possibly one $i < k$.

A coloring $f \colon [\mathbb{N}]^2 \to k$ is *semi-hereditary* if for all $i < k$ except possibly one, whenever $x < y < z$ and $f(x,z) = f(y,z) = i$ then $f(x,y) = i$.

A coloring $f \colon [\mathbb{N}]^2 \to k$ is *semi-trivial* if for all $i < k$ except possibly one, the set $\{y \in \mathbb{N} : x < y \land f(x,y) = i\}$ is homogeneous for f for each x (in a possibly different color).

These are associated with restrictions of RT_2^2.

Statement 5.9 (ADS)**.** Every transitive coloring $f \colon [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

Statement 5.10 (CAC)**.** Every semi-transitive coloring $f \colon [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

Statement 5.11 (SHER)**.** Every semi-hereditary coloring $f \colon [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

Statement 5.12 (STRIV)**.** Every semi-trivial coloring $f \colon [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set.

The restrictions CAC and ADS were studied by Hirschfeldt and Shore [19], who showed that ADS is implied by CAC over RCA_0, and that both are strictly weaker than RT_2^2. Recently, Lerman, Solomon and Towsner [26, Section 2] have shown that ADS does not imply CAC over RCA_0. (The usual definitions of these principles, as given in Section 1 of [19], are equivalent to the ones above by Theorems 5.2 and 5.3 of [19], respectively.) The restriction SHER was studied by Dorais (unpublished), who showed that it follows from ADS. It is unknown whether SHER implies ADS

over RCA_0. The last restriction, STRIV, is equivalent to the infinite pigeonhole principle $\mathsf{RT}^1_{<\infty}$ over RCA_0. However, the proof of STRIV from $\mathsf{RT}^1_{<\infty}$ is not uniform and hence there does not appear to be a strong Weihrauch equivalence between STRIV and $\mathsf{RT}^1_{<\infty}$. CAC, ADS, SHER all imply STRIV over RCA_0 and none of those implications reverse.

We now prove a number of implications between Π^1_2 principles in RCA_0 that can also be presented as strong Weihrauch reductions between compositions of Π^1_2 principles. To properly state the relevant reductions, we need the *alternative product* $[\mathsf{P}, \mathsf{Q}]$ of Π^1_2 principles P and Q. An instance of $[\mathsf{P}, \mathsf{Q}]$ is a either a pair $\langle 0, A \rangle$ where A is an instance of P or a pair $\langle 1, B \rangle$ where B is an instance of Q; a corresponding solutions are, respectively, solutions to the instance A of P or solutions to the instance B of Q. This is indeed a product since $\mathsf{P}, \mathsf{Q} \leq_{\mathrm{sW}} [\mathsf{P}, \mathsf{Q}]$ but it is not always equivalent to the parallel product $\langle \mathsf{P}, \mathsf{Q} \rangle$. In fact, $[\mathsf{P}, \mathsf{Q}]$ is the least upper bound of P and Q in the strong Weihrauch preordering. In particular, $[\mathsf{P}, \mathsf{P}]$ is always strong Weihrauch equivalent to P. (See Blass [3] for a discussion of these two products in a broader setting. The alternative product is called the co-product in the Weihrauch lattice, and was originally introduced in this context by Pauly [28].)

Proposition 5.13. $\mathsf{RT}^2_2 \leq_{\mathrm{sW}} [\mathsf{STRIV}, \mathsf{CAC}, \mathsf{SHER}] \bullet \mathsf{TS}^3_8$ *and* $\mathsf{RCA}_0 \vdash (\mathsf{TS}^3_8 + \mathsf{CAC}) \to \mathsf{RT}^2_2$.

Proof. We argue in RCA_0. Let $f \colon [\mathbb{N}]^2 \to 2$ be a coloring. Define the coloring $g \colon [\mathbb{N}]^3 \to 8$ by
$$g(x, y, z) = \langle f(y, z), f(x, z), f(x, y) \rangle,$$
for all $x < y < z$. By TS^3_8, we know that there is an infinite set H that avoids one of the eight possible colors for g.

The proof now divides into cases according to which color is avoided. Call this color c.

Case 1. If $c = \langle 0, 0, 0 \rangle$ or $c = \langle 1, 0, 0 \rangle$, then for every $x \in H$, the set $H_x = \{y \in H : y > x \wedge f(x, y) = 0\}$ is homogeneous for f (with color 1 or 0, respectively). If $c = \langle 1, 1, 1 \rangle$ or $c = \langle 0, 1, 1 \rangle$, then for every $x \in H$, the set $H_x = \{y \in H : y > x \wedge f(x, y) = 1\}$ is homogeneous for f (with color 0 or 1, respectively). In all these cases, applying STRIV gives an infinite homogeneous set for f.

Case 2. If $c = \langle 0, 1, 0 \rangle$ or $c = \langle 1, 0, 1 \rangle$, then f is semi-transitive on H. Applying CAC gives an infinite homogeneous set for f.

Case 3. If $c = \langle 0, 0, 1 \rangle$ or $c = \langle 1, 1, 0 \rangle$, then f is semi-hereditary on H. Applying SHER gives an infinite homogeneous set for f. $\qquad\square$

Proposition 5.14. $\mathsf{RT}^2_2 \leq_{\mathrm{sW}} [\mathsf{STRIV}, \mathsf{ADS}, \mathsf{SHER}] \bullet \mathsf{TS}^3_7$ *and* $\mathsf{RCA}_0 \vdash (\mathsf{TS}^3_7 + \mathsf{ADS}) \to \mathsf{RT}^2_2$.

Proof. The construction is basically the same as that of Proposition 5.13, with the exception that the dual pair of colors $\langle 0, 1, 0 \rangle$ and $\langle 1, 0, 1 \rangle$ are merged into one. Any infinite set H that avoids both of these colors is transitive for f, so ADS suffices to give an infinite homogeneous set for c. $\qquad\square$

Proposition 5.15. $\mathsf{RT}^2_2 \leq_{\mathrm{sW}} [\mathsf{STRIV}, \mathsf{SHER}] \bullet \mathsf{TS}^3_6$ *and* $\mathsf{RCA}_0 \vdash (\mathsf{TS}^3_6 + \mathsf{SHER}) \to \mathsf{RT}^2_2$.

Proof. The construction is basically the same as that of Proposition 5.13, with the exception that the pairs of colors $\langle 0, 1, 0 \rangle$, $\langle 1, 1, 0 \rangle$ are merged into one, and similarly for the dual pair $\langle 1, 0, 1 \rangle$, $\langle 0, 0, 1 \rangle$. Then SHER suffices to give an infinite homogeneous set for f these two cases.   □

5.3. **Sequential forms.** From TS_k^n we can, in accordance with Statement 2.10, form the sequential version SeqTS_k^n. Surprisingly, the sequential forms of these weaker thin set principles can still code $\emptyset^{(n)}$ just as SeqRT_k^n did in Lemma 3.2. We need the following theorem of Kummer.

Theorem 5.16 (Kummer [25], p. 678). *Fix $k \geq 2$, and let $A, B \subseteq \omega$ be arbitrary. Suppose g is a computable function such that, for all $\mathbf{x} \in [\omega]^{k-1}$, the B-c.e. set $W_{g(\mathbf{x})}^B$ is a proper subset of k and contains $|\mathbf{x} \cap A|$. Then A is computable in B.*

Corollary 5.17. *For all $n \geq 1$ and all $k \geq 2$, there is a computable instance of SeqTS_k^n, every solution to which computes $\emptyset^{(n)}$.*

Proof. The proof is somewhat similar to that of Lemma 3.2, but our argument here is slightly more delicate on account of needing to fit the rather unique conditions of Kummer's theorem. We define a computable sequence $\langle f_{\mathbf{x}} : \mathbf{x} \in [\omega]^{k-1} \rangle$ of k-colorings of $[\omega]^n$ to serve as the desired instance of SeqTS_k^n. Let h be a $\{0, 1\}$-valued computable function such that

$$\emptyset^{(n)} = \{ i \in \omega : \lim_{y_0} \cdots \lim_{y_{n-1}} h(i, y_0, \ldots, y_{n-1}) = 1 \},$$

and for each $\mathbf{x} \in [\omega]^{k-1}$ set

$$f_{\mathbf{x}}(\mathbf{y}) = |\{ i \in \mathbf{x} : h(i, y_0, \ldots, y_{n-1}) = 1 \}|$$

for all $\mathbf{y} = \langle y_0, \ldots, y_{n-1} \rangle \in [\omega]^n$.

Suppose we are given $\vec{H} = \langle H_{\mathbf{x}} : \mathbf{x} \in [\omega]^{k-1} \rangle$ such that each $H_{\mathbf{x}}$ is an infinite thin set for $f_{\mathbf{x}}$. That is, $\vec{H}$ is a solution to the instance $\langle f_{\mathbf{x}} : \mathbf{x} \in [\omega]^{k-1} \rangle$. Let g be a computable function such that for all $\mathbf{x} \in [\omega]^{k-1}$,

$$W_{g(\mathbf{x})}^{\vec{H}} = \{ c < k : (\exists \mathbf{y} \in [H_{\mathbf{x}}]^n)[f_{\mathbf{x}}(\mathbf{y}) = c] \}.$$

Since it is thin, $H_{\mathbf{x}}$ necessarily avoids some $c < k$, so $W_{g(\mathbf{x})}^{\vec{H}}$ is a proper subset of k. We claim that $|\mathbf{x} \cap \emptyset^{(n)}| \in W_{g(\mathbf{x})}^{\vec{H}}$, whence it will follow by Theorem 5.16 that $\emptyset^{(n)} \leq_T \vec{H}$, as desired.

To prove the claim, fix $\mathbf{x} \in [\omega]^{k-1}$. For each $i \in \mathbf{x}$, we have that

$$(\exists s_0)(\forall y_0 > s_0) \cdots (\exists s_{n-1})(\forall y_{n-1} > s_{n-1}) \, [h(i, y_0, \ldots, y_{n-1}) = \emptyset^{(n)}(i)]$$

by definition of the limit. Let $w_0, \ldots, w_{n-1}$ be Skolem functions for this definition, so that for each i,

$$(\forall y_0 > w_0(i)) \cdots (\forall y_{n-1} > w_{n-1}(i, y_0, \ldots, y_{n-2})) \, [h(i, y_0, \ldots, y_{n-1}) = \emptyset^{(n)}(i)].$$

We define a sequence $s_0 < y_0 < s_1 < y_1 \cdots < s_{n-1} < y_{n-1}$ with each $s_j \in \omega$ and each $y_j \in H_{\mathbf{x}}$, as follows. Let $s_0 = \max_{i \in \mathbf{x}} \{w_0(i)\}$, and suppose s_j has been defined for some $j < n$. Let y_j be the least element of $H_{\mathbf{x}}$ greater than s_j, and if $j < n-1$, let $s_{j+1} = \max_{i \in \mathbf{x}} \{w_{j+1}(i, z_0, \ldots, z_j) : (\forall k \leq j)[z_k \leq y_k]\}$.

By construction, we have that $h(i, y_0, \ldots, y_{n-1}) = \emptyset^{(n)}(i)$ for all $i \in \mathbf{x}$. Hence, by definition, $f_{\mathbf{x}}(y_0, \ldots, y_{n-1}) = |\mathbf{x} \cap \emptyset^{(n)}|$. But as the y_j were all chosen from $H_{\mathbf{x}}$,

this means that $|\mathbf{x} \cap \emptyset^{(n)}|$ is not a color omitted by $H_{\mathbf{x}}$. This is what was to be shown. $\square$

Unfortunately, the proof of Kummer's theorem is not uniform and hence Corollary 5.17 does not lead to a Weihrauch reduction.

Since each TS_k^n is clearly total and has finite tolerance, we may now apply the Squashing Theorem to obtain the following consequence.

Corollary 5.18. *For all $n \geq 1$ and $j, k \geq 2$, we have $\langle \mathsf{TS}_k^n, \mathsf{TS}_j^n \rangle \not\leq_{\mathrm{W}} \mathsf{TS}_j^n$.*

Proof. By Proposition 5.2, $\mathsf{TS}_j^n \leq_{\mathrm{sW}} \mathsf{TS}_2^n \leq_{\mathrm{sW}} \mathsf{RT}_2^n$. Hence, as described in the proof of Lemma 3.3, every computable instance of TS_k^n has a solution H with $H' \leq_T \emptyset^{(n)}$. As there is a computable instance of SeqTS_k^n all of whose solutions compute $\emptyset^{(n)}$ by Corollary 5.17, it follows that $\mathsf{SeqTS}_k^n \not\leq_{\mathrm{W}} \mathsf{TS}_j^n$. The result follows by applying Theorem 2.5. $\square$

We have shown that for each $k \geq 2$, we can code $\emptyset^{(n)}$ into a computable instance of SeqTS_k^n. However, $\emptyset^{(n)}$ is not able to solve all computable instances. We first prove this in the case when $n = 1$.

Theorem 5.19. *For each $k \geq 2$, there exists a computable instance of SeqTS_k^1 with no $\emptyset'$-computable solution.*

Proof. Using the Limit Lemma, we may fix a computable $g \colon \omega^4 \to 2$ such that for every Δ_2^0 set $D \subseteq \omega^2$, there exists an $e \in \omega$ such that:

- for all $\langle i, a \rangle \in D$, we have $\lim_s g(e, i, a, s) = 1$;
- for all $\langle i, a \rangle \notin D$, we have $\lim_s g(e, i, a, s) = 0$.

Concretely, we may let

$$g(e, i, a, s) = \begin{cases} 1 & \text{if } \Phi_{e,s}^{K_s}(i,a) \downarrow = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We now define our computable instance $\langle f_i : i \in \omega \rangle$ of SeqTS_k^1. We build our sequence so that each f_i is defined independently of the others in such a way that f_e defeats the eth potential Δ_2^0 solution $D = \langle D_i : i \in \omega \rangle$ by ensuring that D_e is not an infinite thin set for f_e.

Construction. For a given i, we define $f_i(s)$ recursively in stages based on s. Fix $i \in \omega$, and suppose that we are at stage s so that we have defined $f_i(t)$ for all $t < s$. Let $A_{i,s}$ be the approximation to those elements in the ith column of the ith possible Δ_2^0 set at stage s, i.e.,

$$A_{i,s} = \{b \in \omega : b < s \text{ and } g(i, i, b, s) = 1\}$$

Let $C_{i,s} = \{f_i(b) : b \in A_{i,s}\}$ be the set of colors used by the elements of this approximation. We have two cases.

Case 1. Suppose that there exists $n < k$ such that $n \notin C_{i,s}$, i.e., suppose that some color is not used on the approximation. We then define $f_i(s)$ to be the least $n < k$ such that $n \notin C_{i,s}$.

Case 2. Suppose that $C_{i,s} = \{0, 1, \ldots, k-1\}$. For each $n < k$, let $b_{n,s} < s$ be least such that $f_i(b_{n,s}) = n$, i.e., $b_{n,s}$ is the first place where color n occurs. Fix $\ell < k$ such that $b_{\ell,s} = \max\{b_{n,s} : n < k\}$, i.e., pick the color whose first occurrence is as late as possible. Define $f_i(s) = \ell$.

Verification. We now verify that there is no Δ_2^0 solution for $\langle f_i : i \in \omega \rangle$. Suppose that $D = \langle D_i : i \in \omega \rangle$ is a Δ_2^0 set. Fix $e \in \omega$ such that:

- for all $\langle i, b \rangle \in D$, we have $\lim_s g(e, i, b, s) = 1$;
- for all $\langle i, b \rangle \notin D$, we have $\lim_s g(e, i, b, s) = 0$.

In particular, we have the following:

- for all $b \in D_e$, we have $\lim_s g(e, e, b, s) = 1$;
- for all $b \notin D_e$, we have $\lim_s g(e, e, b, s) = 0$.

If D_e is finite, then $\langle D_i : i \in \omega \rangle$ is not a solution to $\langle f_i : i \in \omega \rangle$ by definition. Assume then that D_e is infinite. Let $C_e = \{ f_e(b) : b \in D_e \}$ be the set of colors that occur on D_e. We claim that $C_e = \{ 0, 1, \ldots, k-1 \}$. Suppose not. For each $n \in C_e$, let b_n be the least element of D_e such that $f_e(b_n) = n$. Let $m = \max \{ b_n : n \in C_e \}$. Fix $t > m$ such that the approximation to each element of the eth column below m has settled down, i.e., such that:

- for all $b \in D_e$ with $b \leq m$, we have $g(e, e, b, s) = 1$ whenever $s \geq t$;
- for all $b \notin D_e$ with $b \leq m$, we have $g(e, e, b, s) = 0$ whenever $s \geq t$.

Now take any $s \geq t$. Notice that $A_{e,s} \cap \{ 0, 1, \ldots, m \} = D_e \cap \{ 0, 1, \ldots, m \}$, hence $C_e \subseteq C_{e,s}$ and $b_{n,s} = b_n$ for all $n \in C_e$. Furthermore, if $\ell \notin C_e$ and $b_{\ell,s}$ is defined, then we must have $b_{\ell,s} > m$. Now if $C_{e,s} \neq \{ 0, 1, \ldots, k-1 \}$, then we enter Case 1 of the construction and define $f_e(s) \notin C_{e,s}$, so $f_e(s) \notin C_e$. On the other hand, if $C_{e,s} = \{ 0, 1, \ldots, k-1 \}$, then since $b_{n,s} = b_n \leq m$ for all $n \in C_e$ and $b_{n,s} > m$ for all $n \notin C_e$, it follows that the ℓ chosen in Case 2 of the construction must satisfy $\ell \notin C_e$, so $f_e(s) \notin C_e$.

We have therefore shown that $f_e(s) \notin C_e$ for all $s \geq t$. Since D_e is infinite, we may fix $b \in D_e$ with $b \geq t$. We then have have $f_e(b) \notin C_e$, contradicting the definition of C_e. $\qquad\square$

Corollary 5.20. *For each $n \geq 1$ and $k \geq 2$, there exists a computable instance of* SeqTS_k^n *with no $\emptyset^{(n)}$-computable solution.*

Proof. We prove the following stronger claim: For each $n \geq 1$, $k \geq 2$, and $X \in 2^\omega$, there exists an X-computable instance of SeqTS_k^n with no $X^{(n)}$-computable solution. We fix k and prove this result by induction on n. The base case of $n = 1$ is given by the relativized version of Theorem 5.19. Suppose that we know the result for a fixed $n \geq 1$. Let $X \in 2^\omega$ be arbitrary. By induction, we may fix an X'-computable instance $\langle g_i : i \in \omega \rangle$ of SeqTS_k^n with no $X^{(n+1)}$-computable solution. By the relativized Limit Lemma, we may fix an X-computable sequence $\langle f_i : i \in \omega \rangle$ such that $g_i(\mathbf{x}) = \lim_s f_i(\mathbf{x}, s)$ for all i and all $\mathbf{x}$. We may assume that $f_i \colon [\omega]^{n+1} \to k$ for each i, and hence that $\langle f_i : i \in \omega \rangle$ is an X-computable instance of SeqTS_k^{n+1}. Now if $\langle T_i : i \in \omega \rangle$ is a solution to $\langle f_i : i \in \omega \rangle$, then each T_i is an infinite thin set for f_i, so each T_i is an infinite thin set for g_i, and hence $\langle T_i : i \in \omega \rangle$ is a solution to $\langle g_i : i \in \omega \rangle$. Therefore, $\langle f_i : i \in \omega \rangle$ has no $X^{(n+1)}$-computable solution. This completes the induction. $\qquad\square$

After relativization and translation into the language of Weihrauch reducibility, we obtain the following.

Corollary 5.21. *For each $n \geq 1$ and $k \geq 2$, $\mathsf{SeqTS}_k^n \not\leq_{\mathrm{W}} \mathsf{TJ}^n$.*

Here and henceforth, TJ^n denotes the multi-valued function that sends each set X to its Turing jump $X^{(n)}$. From the perspective of reverse mathematics, the

corresponding Π_2^1-principles TJ^n for standard $n \geq 1$, are all equivalent to arithmetic comprehension and hence indistinguishable from each other.

5.4. **Infinitely Many Colors.** Although the principles SeqTS_k^1 for $k \in \omega$ appear to behave similarly with regards to diagonalizing and coding, the situation for SeqTS_ω^1 is very different. We first prove the following result that contrasts with Theorem 5.19.

Proposition 5.22. *Every computable instance of* SeqTS_ω^1 *has a* $\emptyset'$*-computable solution.*

Proof. Let $\langle f_i : i \in \omega \rangle$ be a computable instance of SeqTS_ω^1. Using $\emptyset'$ as an oracle, we compute a sequence of thin sets $\langle A_i : i \in \omega \rangle$. We define each $A_i = \{a_{i,m} : m \in \omega\}$ independently by using $\emptyset'$ to compute an increasing sequence $a_{i,0} < a_{i,1} < \dots$. Given i, start by asking $\emptyset'$ if there exists $b \in \omega$ such that $f_i(b) \neq 0$. Let $a_{i,0}$ be the least such b if one exists, and otherwise let $a_{i,0} = 0$. Suppose that we have defined $a_{i,n}$. Ask $\emptyset'$ if there exists $b > a_{i,n}$ such that $f_i(b) \neq 0$. Let $a_{i,n+1}$ be least such b if one exists, and otherwise let $a_{i,n+1} = a_{i,n} + 1$. Let

$$A_i = \{a_{i,n} : n \in \omega\}$$

and notice that $\langle A_i : i \in \omega \rangle$ is $\emptyset'$-computable.

Let $i \in \omega$. We claim that A_i is thin for f_i. If the set $\{b \in \omega : f_i(b) \neq 0\}$ is infinite, then $A_i \subseteq \{b \in \omega : f_i(b) \neq 0\}$, so A_i is thin for f_i. On the other hand, if $\{b \in \omega : f_i(b) \neq 0\}$ is finite, then $\text{range}(f_i)$ is finite, so A_i is trivially thin for f_i. Therefore, $\langle A_i : i \in \omega \rangle$ is a $\emptyset'$-computable solution to $\langle f_i : i \in \omega \rangle$. $\square$

After relativization and translation into the language of Weihrauch reducibility, we obtain the following.

Corollary 5.23. $\mathsf{SeqTS}_\omega^1 \leq_{\mathrm{sW}} \mathsf{TJ}^1$.

Finally, we have a strong non-coding result for infinitely many colors to contrast with Corollary 5.17. Recall that given two degrees $\mathbf{a}$ and $\mathbf{b}$, the notation $\mathbf{a} \gg \mathbf{b}$ means that every infinite subtree of $2^{<\omega}$ of degree $\mathbf{b}$ has an infinite path of degree at most $\mathbf{a}$. (See [9, pp. 10–11] for some of the basic properties of this relation.)

Theorem 5.24. *Every computable instance of* SeqTS_ω^1 *has a* low_2 *solution. In fact, if* $\mathbf{d} \gg \mathbf{0}'$, *then every computable instance of* SeqTS_ω^1 *has a solution* A *such that* $\deg(A)' \leq \mathbf{d}$.

Proof. Let $f = \langle f_i : i \in \omega \rangle$ be a computable instance of SeqTS_ω^1, and fix $\mathbf{d} \gg \mathbf{0}'$. We obtain the solution $A = \langle A_i : i \in \omega \rangle$ generically for the following notion of forcing, $\mathbb{P} = (P, \leq)$. An element of P is a pair $\langle \sigma, \tau \rangle$ where $\sigma \in 2^{<\omega}$ and $\tau \in \omega^{<\omega}$, such that:

- for all i and all x, if $\sigma(\langle i, x \rangle) \downarrow = 1$ and $\tau(i) \downarrow$, then $\sigma(\langle i, x \rangle) \neq \tau(i)$;
- for all i, if $\tau(i) \downarrow$, then the set $\{x \in \omega : f_i(x) \neq \tau(i)\}$ is infinite.

We think of σ as being broken into columns $\sigma = \langle \sigma_i : i \in \omega \rangle$ and as being a finite initial segment of the resulting $A = \langle A_i : i \in \omega \rangle$. Thus, $\sigma(\langle i, x \rangle) = \sigma_i(x)$, and σ_i is a initial segment of A_i. The finite sequence τ says which colors are being omitted on a given column, i.e., if $\tau(i) \downarrow$, then the resulting A_i will have the property that $f_i(a) \neq \tau(i)$ for all $a \in A_i$. We define $\langle \sigma^*, \tau^* \rangle \leq \langle \sigma, \tau \rangle$ if both $\sigma \preceq \sigma^*$ and $\tau \preceq \tau^*$.

From a sufficiently generic sequence of conditions

(6) $$\langle \sigma_0, \tau_0 \rangle \geq \langle \sigma_1, \tau_1 \rangle \geq \cdots$$

we can compute the set $A = \bigcup_{i \in \omega} \sigma_i$ and $B = \bigcup_{i \in \omega} \tau_i$, and A will clearly be a solution to f. The appropriate level of genericity corresponds to meeting the following requirements for all $e, i \in \omega$:

$$\begin{aligned} \mathcal{Q}_{e,i} &: \quad |A_i| \geq e; \\ \mathcal{R}_i &: \quad B(i) \text{ is defined.} \end{aligned}$$

We wish to obtain such a sequence with jump of degree at most $\mathbf{d}$, so we also have the requirement:

$$\mathcal{S}_e \quad : A'(e) \text{ is forced.}$$

It thus suffices to show that these requirements are $\mathbf{d}$-effectively dense, i.e., that we can use $\mathbf{d}$ to extend a given condition $\langle \sigma, \tau \rangle$ to meet a given one of the above requirements.

First, suppose we wish to meet $\mathcal{Q}_{i,e}$. If $\tau(i) \uparrow$, we effectively extend σ to σ^* by adding e many 1s in the ith column, and only 0s in the other columns. Then $\langle \sigma^*, \tau \rangle$ is the desired extension. If $\tau(i) \downarrow$, we can computably find distinct $x_0, \dots, x_{e-1} > |\sigma_i|$ with $f_i(x_0) \neq \tau(i), \dots, f_i(x_{e-1}) \neq \tau(i)$. (This is possible because $\langle \sigma, \tau \rangle$ is a condition, so $\{x \in \omega : f_i(x) \neq \tau(i)\}$ is infinite.) We effectively extend σ to σ^* so that $\sigma_i^*(x_0) = \cdots = \sigma_i^*(x_{e-1}) = 1$, and all other new bits of σ^* are 0. We then take $\langle \sigma^*, \tau \rangle$ for the extension. Clearly, in either case, if A extends σ^* then A satisfies $\mathcal{Q}_e$.

Next, suppose we wish to meet $\mathcal{R}_i$. If $\tau(i) \downarrow$, we can just keep $\langle \sigma, \tau \rangle$, so suppose otherwise. Since the set $F = \{x \in \omega : \sigma_i(x) \downarrow = 1\}$ is finite, so is the set $C = \{c \in \omega : (\exists x \in F) f_i(x) = c\}$, and we can find a canonical index for it effectively from i and $\langle \sigma, \tau \rangle$. Fix $c_0, c_1 \notin C$ with $c_0 \neq c_1$. Now at least one of the two sets $\{x \in \omega : f_i(x) \neq c_0\}$ or $\{x \in \omega : f_i(x) \neq c_1\}$ must be infinite. Since these are two effectively given Π_2^0 sets, Lemma 4.2 of [9] implies that we can $\mathbf{d}$-effectively determine a $k \in \{0, 1\}$ such that $\{x \in \omega : f_i(x) \neq c_k\}$ is infinite. If we let τ^* be τ extended so that $\tau^*(i) = c_k$, then $\langle \sigma, \tau^* \rangle$ is a condition by choice of k, and so we can take it to be our extension. Clearly, if B extends τ^* then $B(i)$ is defined.

Finally, suppose we wish to meet $\mathcal{S}_e$. Notice that the set of $\sigma^* \succeq \sigma$ that respect τ, i.e., the set of $\sigma^* \succeq \sigma$ such that $\langle \sigma^*, \tau \rangle$ is a condition, is computable and we can find an index for it as such effectively from e and $\langle \sigma, \tau \rangle$. Since $\mathbf{d} \gg \mathbf{0}'$, we have that $\mathbf{d} \geq \mathbf{0}'$, and hence $\mathbf{d}$ can determine if there exists such a σ^* with $\Phi_e^{\sigma^*}(e) \downarrow$. If so, we extend to the condition $\langle \sigma^*, \tau \rangle$, and otherwise we keep $\langle \sigma, \tau \rangle$. Notice that so long as A extends σ^*, then in the former case we will have $e \in A'$, while in the latter we will have $e \notin A'$.

The argument is now put together in the usual way. We let $\langle \sigma_0, \tau_0 \rangle = \langle \emptyset, \emptyset \rangle$, and then repeatedly use the density of the requirements to $\mathbf{d}$-effectively produce the sequence in (6). $\qquad \square$

After relativization and translation into the language of Weihrauch reducibility, we obtain the following.

Corollary 5.25. $\mathsf{TJ}^1 \bullet \mathsf{SeqTS}_\omega^1 \leq_{\mathrm{sW}} \mathsf{WKL} \bullet \mathsf{TJ}^1$.

Using the uniform low basis theorem of Brattka, de Brecht, and Pauly [4], we also obtain the following.

Corollary 5.26. $\mathsf{SeqTS}_\omega^1 \leq_{\mathrm{sW}} \mathfrak{L}_2$.

Where the low_2 principle $\mathfrak{L}_2$ is the composition of the inverse function of TJ^2 with the iterated limit operator lim^2 in a manner similar to the definition of $\mathfrak{L}$ in [4].

By looking for splittings instead of forcing the jump, one can also prove a cone-avoidance theorem saying that if $\langle C_j : j \in \omega \rangle$ is a sequence of non-computable sets, then every computable instance of SeqTS_ω^1 has a solution A such that $C_j \not\leq_T A$ for all j. As in [15, Theorem 3.4], it is also possible to combine these arguments to produce low_2 cone-avoiding solutions to computable instances in the case that the sequence $\langle C_j : j \in \omega \rangle$ is $\emptyset'$-computable. We omit the details, which are standard.

5.5. **Weihrauch Reductions for Thin Sets.** We are not able to prove an analog of Theorem 3.1 for TS_k^n using the Squashing Theorem because we lack an analog of Proposition 2.1 for thin sets. However, we can give a direct proof that $\mathsf{TS}_j^1 \not\leq_\mathrm{W} \mathsf{TS}_k^1$ when $j < k$.

Theorem 5.27. *For all $j, k \geq 2$ with $j < k$, we have $\mathsf{TS}_j^1 \not\leq_\mathrm{W} \mathsf{TS}_k^1$.*

Proof. Suppose instead that $\mathsf{TS}_j^1 \leq_\mathrm{W} \mathsf{TS}_k^1$, and fix reductions Φ and Ψ witnessing this fact. We shall build a computable coloring $f : \omega \to j$ and an infinite thin set $T \subseteq \omega$ for $\Phi(f) : \omega \to k$ such that $\Psi(T)$ cannot be an infinite thin set for f. For convenience, we assume that for any finite $F \subseteq \omega$, if $\Psi(F)(x) \downarrow$ for some $x \in \omega$ then the use of the computation is bounded by $\max F$. As the construction will be uniformly computable, we may use the recursion theorem as in the proof of Proposition 4.7 so as to not have to consider the instance f in the oracle for Ψ.

The idea of the proof is to start by defining f arbitrarily, monitoring the coloring $\Phi(f)$ as it forms alongside, and waiting to find a finite homogeneous set F_0 for $\Phi(f)$ that is large enough so that $\Psi(F_0)$ contains some number x_0. Once this happens, we change how we define f so that all future numbers have a different color from x_0. In this way, we force any sufficiently large set extending $\Psi(F_0)$ to contain numbers of at least two different colors. We then repeat the process, looking for a finite homogeneous set $F_1 > F_0$ for $\Phi(f)$ large enough so that $\Psi(F_0 \cup F_1)$ contains some number x_1 colored differently from x_0. We then change how we define f again so that all future numbers have a different color from x_0 and from x_1. In this way, we force any sufficiently large set extending $\Psi(F_0 \cup F_1)$ to contain numbers of at least three different colors.

Continuing in this way, we build $F_0 < \cdots < F_{j-2}$ such that any sufficiently large set extending $\Psi(\bigcup_{i<j-1} F_i)$ contains numbers of all j many colors. Thus, to define the desired set T, we have only to produce an infinite set extending $\bigcup_{i<j-1} F_i$ that is thin for $\Phi(f)$. But since each F_i was chosen to be homogeneous for $\Phi(f)$, this coloring assumes at most $j-1$ many colors on $\bigcup_{i<j-1} F_i$. So, if we let $H > F_{j-2}$ be any infinite homogeneous set for $\Phi(f)$, then $\Phi(f)$ assumes at most j many colors on $\bigcup_{i<j-1} F_i \cup H$, which we take to be T. Then T is thin for $\Phi(f)$ since $j < k$.

We proceed to the formal details.

Construction. We proceed by stages. At stage s, we define an initial segment f_s of f on $[0, s]$. During the construction, we also define $j-1$ many sets $F_0, \ldots, F_{j-2}$ that will be used in the definition of T.

At stage $s = 0$ we set $f_0 = \emptyset$, and declare all colors $c < j$ *valid*.

At stage $s > 0$, let $l \in \omega$ be such that we have already defined F_i for each $i < l$. Call s an *action stage* if $l < j-1$, and if there exists a finite set $F \leq s$ and number $x \leq s$ such that

- $\bigcup_{i<l} F_i < F$;
- F is homogeneous for $\Phi(f_{s-1})$;
- $f_{s-1}(x)$ is some currently valid color;
- $\Psi(\bigcup_{i<l} F_i)(x) \uparrow$ and $\Psi(\bigcup_{i<l} F_i \cup F)(x) \downarrow= 1$.

In this case, let F_l and x_l be the least such F and x, respectively, and declare the color $f_{s-1}(x)$ to no longer be valid. By induction, this leaves at least one valid color.

Regardless of whether s is an action stage or not, we extend f_{s-1} to f_s by choosing the least color $c < j$ that is still valid, and letting $f_s(y) = c$ for all $y \leq s$ on which f_{s-1} has not yet been defined.

Verification. It is clear that $f = \bigcup_s f_s$ is a computable j-coloring. We begin with an observation. Note that there can be no more than $j - 1$ many action stages, since the number of F_i defined at the start of such a stage must be fewer than $j-1$, and a new F_i is then defined. So, let $l \leq j - 1$ be the total number of action stages; we claim that $l = j - 1$. Since the number of valid colors is reduced by one at every action stage, this implies that there is precisely one color that is permanently valid.

Before proving the claim, we define T. Since each F_i for $i < l$ is homogeneous for $\Phi(f)$, it follows that $\Phi(f)$ assumes at most l many colors on $\bigcup_{i<l} F_i$. Thus if $H > F_{l-1}$ is any infinite homogeneous set for $\Phi(f)$, then $\Phi(f)$ assumes at most $l + 1 \leq j < k$ many colors on $\bigcup_{i<l} F_i \cup H$. It follows that $T = \bigcup_{i<l} F_i \cup H$ is thin for $\Phi(f)$.

Now to see the claim, let t be 0, or any action stage before the $(j-1)$st. Seeking a contradiction, suppose there is no action stage greater than t. In particular, all the F_i for $i < l$ are defined at or before stage t. If $c < j$ is the least color still valid at the end of stage t, then all sufficiently large numbers are colored c by f. Thus, since T is an infinite thin set for $\Phi(f)$, it follows that $\Psi(T)$, being an infinite thin set for f, contains some number x colored c by f on which $\Psi(\bigcup_{i<l} F_i)$ diverges. But now if F is a sufficiently long initial segment of H so that $\Psi(\bigcup_{i<l} F_i \cup F)(x) \downarrow= 1$, then any stage $s > t$ with $s \geq F$ and $s \geq x$ will be an action stage. The proof is complete. $\square$

Since the circulation of a pre-print of the present article, Hirschfeldt and Jockusch have extended the above argument to all $n > 1$. Their proof will appear in [18].

Theorem 5.28 (Hirschfeldt and Jockusch [18]). *For all $n > 1$ and all $j, k \geq 2$ with $j < k$, we have $\mathsf{TS}_j^n \not\leq_{\mathrm{W}} \mathsf{TS}_k^n$.*

6. The Rainbow Ramsey's Theorem and measure

For our final results, we turn to the rainbow Ramsey's theorem and further connections with randomness.

Definition 6.1. Fix $n, k \geq 1$.

(1) A coloring $f: [\omega]^n \to \omega$ is *k-bounded* if for each $c \in \omega$, there are at most k many $\mathbf{x} \in [\omega]^n$ such that $f(\mathbf{x}) = c$.
(2) A set $S \subseteq \omega$ is a *rainbow for f* if f is injective on $[S]^n$.

Statement 6.2 (Rainbow Ramsey's Theorem). Given $n, k \geq 1$, let RRT_k^n denote the statement every k-bounded $f: [\omega]^n \to \omega$ has an infinite rainbow. Let $\mathsf{RRT}_{<\infty}^n$ denote $(\forall k \geq 1) \mathsf{RRT}_k^n$.

Just as for RT^1_k and TS^1_k, every computable instance of RRT^1_k has a computable solution. However, in contrast to the situations for SeqRT^1_k and SeqTS^1_k, every computable instance of SeqRRT^1_k also has a computable solution. In fact, we have the following stronger fact.

Proposition 6.3. *Every computable instance of $\mathsf{SeqRRT}^1_{<\infty}$ has a computable solution.*

Proof. Let $\langle f_i : i \in \omega \rangle$ be a computable instance of $\mathsf{SeqRRT}^1_{<\infty}$. We then have that for each i and each c, the set $\{x \in \omega : f_i(x) = c\}$ is finite. From this it follows that for each i and each *finite* set $C \subseteq \omega$, the set $\{x \in \omega : f_i(x) \in C\}$ is finite. We can now define a computable sequence $\langle A_i : i \in \omega \rangle$ by choosing the elements of each A_i recursively so that the color of a new element is distinct from all previous elements already chosen to be A_i. $\square$

Theorem 6.4 (Csima and Mileti [12], Theorem 3.10). *For all $k \geq 1$, if $X \subseteq \omega$ is 2-random then every computable k-bounded coloring $f \colon [\omega]^2 \to \omega$ has an infinite X-computable rainbow.*

The proof of this theorem proceeds by constructing a $\emptyset'$-computable subtree T of $2^{<\omega}$ of positive measure, each infinite path through which computes an infinite rainbow for f. This proof is very nearly uniform. The tree T can be obtained uniformly $\emptyset'$-computably from an index for f, and the reduction from the infinite paths through T to the infinite rainbows for f is uniform as well. The only non-uniformity stems from the way 2-random sets pick out infinite paths through T. We begin by showing that this non-uniformity is essential.

For each $i \in \omega$ and each bounded coloring $f \colon [\omega]^2 \to \omega$, let

$$\mathcal{S}_{f,i} = \{S \subseteq \omega : \Phi_i(S) \text{ is an infinite rainbow for } f\}.$$

Let μ denote the uniform measure on Cantor space.

Proposition 6.5. *There is no computable function h such that for all $i \in \omega$ and all 2-random $R \subseteq \omega$, if Φ_i is a 2-bounded coloring $[\omega]^2 \to \omega$ then $\Phi_{h(i)}(R)$ is an infinite rainbow for f.*

Proof. First, fix any $i \in \omega$. Let $w(i)$ be the least $\sigma \in 2^{<\omega}$, if one exists, such that

$$(7) \qquad\qquad \Phi_i(\sigma)(x) \downarrow = \Phi_i(\sigma)(y) \downarrow = 1$$

for some $x < y$. Then, define a coloring $f_i \colon [\omega]^2 \to \omega$ by stages, as follows. At stage s, we define f_i on $[0, s) \times \{s\}$. If $w(i)$ has not yet converged, let $f_i(z, s) = \langle z, s \rangle$ for all $z < s$. Otherwise, choose the least $x < y < s$ satisfying (7) above for $\sigma = w(i)$, and define

$$f_i(z, s) = \begin{cases} \langle x, s \rangle & \text{if } z = x \text{ or } z = y, \\ \langle z, s \rangle & \text{else}, \end{cases}$$

for all $z < s$.

Clearly, f_i is 2-bounded for each i. Moreover, if there exists an $S \subseteq \omega$ such that $\Phi_i(S)$ is an infinite rainbow for f_i, then $w(i)$ is defined. Say $w(i) = \sigma$. Then for the least $x < y$ satisfying (7), we have $f_i(x, s) = f_i(y, s)$ for all sufficiently large s, so x and y cannot belong to any infinite rainbow for f_i. In particular, if $S \succeq \sigma$ then $\Phi_i(S)$ is not such a rainbow for f_i. It follows that

$$(8) \qquad\qquad \mu(\mathcal{S}_{f_i,i}) \leq 1 - 2^{-|\sigma|} < 1.$$

Now note that f_i is uniformly computable in i. So let g be a computable function such that $f_i = \Phi_{g(i)}$ for all i. Seeking a contradiction, suppose a function h as in the statement exists. By the recursion theorem, we may fix an $i \in \omega$ such that $\Phi_{h(g(i))}(S) = \Phi_i(S)$ for all $S \subseteq \omega$. In particular, $\mathcal{S}_{f,h(g(i))} = \mathcal{S}_{f,i}$, so by assumption, $\mathcal{S}_{f,i}$ contains all 2-random subsets of ω. But since the set of all 2-random subsets of ω has measure 1, this contradicts (8). $\qquad\square$

We wish to know whether Theorem 6.4 carries over to ω applications, i.e., whether every computable instance of SeqRRT^2_k also has a solution computable in each 2-random. By the preceding proposition, the most direct way of obtaining this fails, as the theorem cannot be proved uniformly. Nevertheless, we are able to give an affirmative answer to the question.

Theorem 6.6. *If $k \geq 1$ and $X \subseteq \omega$ is 2-random, then every computable instance of SeqRRT^2_k has an X-computable solution.*

Proof. This is a small adaptation of the proof of Theorem 3.10 in [12], and we refer to results in that article. Let $f = \langle f_i : i \in \omega \rangle$ be a computable instance of SeqRRT^2_k, so $f_i \colon [\omega]^2 \to \omega$ is k-bounded for all i. The proof of Proposition 3.3 is uniform, so we may assume that each f_i is normal. When defining φ_f and T_f in Definition 3.7 and Definition 3.8, instead interleave the process of working on the various f_i across the levels of the tree, i.e., at level $\langle i, n \rangle$, work on the function f_i. Proposition 3.9 still applies so that any 2-random X will compute a path through this combined tree, and any such path computes a solution to $\langle f_i : i \in \omega \rangle$. $\qquad\square$

To translate the above result into the language of Weihrauch reducibility or the language of reverse mathematics, we first need to isolate the assertion of the existence of 2-randoms as a Π^1_2 principle. To do so in a formal setting takes some care, since this is intrinsically a statement about paths through non-computable trees. A detailed account of this and associated difficulties in the specific context of second-order arithmetic is presented in Avigad, Dean, and Rute [2, Section 3], where they also introduce the principle 2-WWKL as one possible formalization (not to be confused with the principle q-WWKL discussed in Section 4). Here we shall content ourselves with the informal definition below (which agrees with theirs in ω-models) and refer the reader to their paper for technical details.

Statement 6.7 (2-WWKL). *For every set X, there is a 2-random set relative to X.*

Thus, after relativization and translation into the language of Weihrauch reducibility, we obtain from Theorem 6.6 the following:

Corollary 6.8. *For each $k \geq 1$, $\mathsf{SeqRRT}^2_k \leq_{\mathrm{W}}$ 2-WWKL and $\mathsf{RCA}_0 \vdash$ 2-WWKL $\to$ SeqRRT^2_k.*

The following consequence complements a result of Conidis and Slaman [11, Theorem 2.1 and Corollary 4.2] that RRT^2_k does not imply $\mathsf{B}\Sigma^0_2$ over RCA_0.

Corollary 6.9. *For each $k \geq 1$, $\mathsf{RT}^1_2 \not\leq_{\mathrm{W}} \mathsf{RRT}^2_k$.*

Proof. Suppose instead that $\mathsf{RT}^1_2 \leq_{\mathrm{W}} \mathsf{RRT}^2_k$. By Proposition 2.11, this would imply that $\mathsf{SeqRT}^1_2 \leq_{\mathrm{W}} \mathsf{SeqRRT}^2_k$. By Lemma 3.2, there is a computable instance of SeqRT^1_2 such that every solution computes $\emptyset'$. By Theorem 6.6, every 2-random

$X \subseteq \omega$ computes a solution to every computable instance of SeqRRT^2_k. This is a contradiction because there is 2-random that does not compute $\emptyset'$ (in fact, no 2-random computes $\emptyset'$). $\qquad\square$

For our final result, we exhibit a degree-theoretic difference between SeqRRT^2_k and $\mathsf{SeqRRT}^2_{<\infty}$. This contrasts with the situation between SeqRT^2_k and $\mathsf{SeqRT}^2_{<\infty}$, i.e., the sequential forms of Ramsey's Theorem for k many colors and finitely many colors. Specifically, it is not difficult to see that if X is a set with $\deg(X) \gg \mathbf{0}''$ then every computable instance of either of these principles has an X-computable solution. That this bound is sharp follows by recent work of Wang [31, Section 3.1].

Lemma 6.10. *For each rational number $q > 0$ and each $i \in \omega$, there exists a bounded coloring $f\colon [\omega]^2 \to \omega$ such that $\mu(\mathcal{S}_{f,i}) < q$. Moreover, an index for f as a computable function can be found uniformly computably from q and i.*

Proof. The idea is to elaborate on the proof of Proposition 6.5. For all $i, n \in \omega$, we inductively define $w(i, n)$ to be the least canonical index of a finite subset F of $2^{<\omega}$ such that

(1) $[\![F]\!] \cap \bigcup_{m<n}[\![D_{w(i,m)}]\!] = \emptyset$;
(2) $\mu([\![F]\!]) \geq q$;
(3) for each $\sigma \in F$, there exist $x < y$ such that $\Phi_i(\sigma)(x) \downarrow= \Phi_i(\sigma)(y) \downarrow= 1$, and x and y are not used by $D_{w(i,m)}$ for any $m < n$, as defined below.

For each σ in F, choose the least x and y satisfying condition 3, and say these are *used* by σ and by F.

The coloring f is now defined by stages. At stage s, we define f on $[0, s) \times \{s\}$. Choose the least n such that $w(i, n)$ has not yet converged. For each $m < n$, and each $\sigma \in D_{w(i,m)}$, choose the $x < y$ used by σ, and define $f(x, s) = f(y, s) = \langle x_0, s \rangle$ for the least x_0 used by $D_{w(i,m)}$. (We may assume that if $w(i, m)$ has converged by stage s then all numbers used by $D_{w(i,m)}$ are smaller than s.) For $z < s$ not used by any $D_{w(i,m)}$, let $f(z, s) = \langle z, s \rangle$.

Clearly, f is computable. We claim that it is bounded. To this end, observe that $w(i, n)$ is defined for only finitely many n, since otherwise

$$D_{w(e,0)}, \ldots, D_{w(e,\lceil 1/q\rceil)}$$

would determine $\lceil 1/q \rceil + 1$ many disjoint subsets 2^ω, each of measure at least q. So let n be least such that $w(i, n)$ is undefined, and for each $m < n$, let k_m be the number of elements used by $D_{w(i,m)}$. The only colors used more than once by f are of the form $\langle x_0, s \rangle$, where x_0 is the least number used by some $D_{w(i,m)}$, and in this case, $f(x, t) = \langle x_0, s \rangle$ only if $s = t$ and x is used by $D_{w(i,m)}$. Thus, f uses each such color $\langle x_0, s \rangle$ at most k_m many times, implying that f is k-bounded for $k = \sup_{m<n} k_m$.

Now with n as above, notice that if an $S \subseteq \omega$ extends some $\sigma \in D_{w(i,m)}$ for $m < n$, then $\Phi_i(S)(x) \downarrow= \Phi_i(S)(y) \downarrow= 1$ for the $x < y$ used by σ. By construction, $f(x, s) = f(y, s)$ for all sufficiently large s, so $\Phi_i(S)$ cannot be an infinite rainbow for f. Thus, any S such that $\Phi_i(S)$ is such a rainbow must lie outside of $\bigcup_{m<n}[\![D_{w(i,m)}]\!]$. But this means that the measure of all such S is less than q, because otherwise we could find a finite set F satisfying conditions 1, 2, and 3 in the definition of w, and $w(i, n)$ would be defined. $\qquad\square$

Proposition 6.11. *There exists a computable instance of $\mathsf{SeqRRT}^2_{<\infty}$ such that not every 2-random $X \subseteq \omega$ computes a solution.*

Proof. Let g be a computable function such that

$$\Phi_{g(e,j)}(S)(x) = \Phi_e(S)(\langle x, \langle e, j \rangle \rangle)$$

for all $e, j \in \omega$ and all $S \subseteq \omega$. In other words, $\Phi_{g(e,j)}(S)$ is the restriction of $\Phi_e(S)$ to the $\langle e, j \rangle$th column. For all $e, j \in \omega$, apply Lemma 6.10 to get a computable bounded coloring $f_{\langle e,j \rangle} \colon [\omega]^2 \to \omega$ such that

$$\mu(\mathcal{S}_{f_{\langle e,j \rangle}, g(e,j)}) < 2^{-j}.$$

Then $\langle f_i : i \in \omega \rangle$ is a computable sequence of colorings, and further, for all $e \in \omega$ and $S \subseteq \omega$, if $\Phi_e(S)$ is a sequence of infinite rainbows for the f_i, then $\Phi_{g(e,j)}(S)$ is an infinite rainbow for $f_{\langle e,j \rangle}$. Thus for each e, it must be that

$$\mu(\{S \subseteq \omega : \Phi_e(S) \text{ is a sequence of infinite rainbows for the } f_i\}) = 0,$$

for if this measure were at least 2^{-j} then so would $\mu(\mathcal{S}_{f_{\langle e,j \rangle}, g(e,j)})$, which cannot be. Since the measure of the 2-randoms is 1, it follows that there is a 2-random $X \subseteq \omega$ that computes no sequence of infinite rainbows for the f_i. $\square$

After relativization and translation into the language of Weihrauch reducibility, we obtain the following.

Corollary 6.12. $\mathsf{SeqRRT}^2_{<\infty} \not\leq_{\mathrm{W}} 2\text{-}\mathsf{WWKL}$.

7. Questions

We close by listing a few questions left open by our work. Chief among these is whether the analogue of Theorem 3.1 holds for (general) Weihrauch reducibility.

Question 7.1. If $n, j, k \geq 2$ and $j < k$, is it the case that $\mathsf{RT}^n_k \not\leq_{\mathrm{W}} \mathsf{RT}^n_j$?

Though not our focus here, our results naturally lead to questions about non-uniform reductions as well. In particular, we can ask the following about a non-uniform version of Theorem 3.1, which is closely related to Question 5.5.3 of [27].

Question 7.2. If $n, j, k \geq 2$ and $j < k$, does every $f : [\omega]^n \to k$ compute a $g : [\omega]^n \to j$, such that every infinite homogeneous set for g computes an infinite homogeneous set for f?

We also have the following question about thin sets and rainbows.

Question 7.3. Are there analogues of Proposition 2.1 for TS^n_k and RRT^n_k?

Appendix A. Equivalence of definitions

In this section, we provide a proof of the equivalence of Definition 1.5 with the definition of (strong) Weihrauch reducibility employed in computable analysis, in the limited context of where both reductions make sense. We do not include here the technicalities particular to that field, and instead focus on the following primary definition that is used to extend the notion of Weihrauch reducibility to more specific settings.

Our discussion below will be limited to functions from Cantor space to Cantor space, but nothing would be lost by considering instead functions on Baire space.

Definition A.1. Let $\mathcal{F}$ and $\mathcal{G}$ be sets of partial functions $2^\omega \to 2^\omega$.

(1) $\mathcal{F}$ is *Weihrauch reducible* to $\mathcal{G}$, written $\mathcal{F} \leq_W \mathcal{G}$, if there exist Turing functionals Φ and Ψ such that

$$(\forall G \in \mathcal{G})(\exists F \in \mathcal{F}) \ F = \Psi \circ \langle \mathrm{id}, G \circ \Phi \rangle.$$

(2) $\mathcal{F}$ is *strongly Weihrauch reducible* to $\mathcal{G}$, written $\mathcal{F} \leq_{sW} \mathcal{G}$, if there exist Turing functionals Φ and Ψ such that

$$(\forall G \in \mathcal{G})(\exists F \in \mathcal{F}) \ F = \Psi \circ G \circ \Phi.$$

The order of quantifiers here may at first appear to be reversed from that used in our definition of $\leq_W$ and $\leq_{sW}$. In order to explain this, we shall use the following notation.

(1) Given a Π_2^1 principle P of second-order arithmetic, let $\mathcal{F}_\mathsf{P}$ be the set of all partial $F : 2^\omega \to 2^\omega$ whose domain includes the set of instances of P, and $F(A)$ for each instance A of P is a solution to the that instance.
(2) Given an arithmetically-definable set $\mathcal{F}$ of partial functions $2^\omega \to 2^\omega$, let $\mathsf{P}_\mathcal{F}$ be the Π_2^1 principle of second-order arithmetic whose instances are the members of the intersection of the domains of the functions in $\mathcal{F}$, and the solutions to any such instance A are the sets $F(A)$ for $F \in \mathcal{F}$.

We begin with the following general Galois connection.

Proposition A.2. *Let P be a Π_2^1 principle of second-order arithmetic, and let $\mathcal{F}$ be an arithmetically-definable set of partial functions $2^\omega \to 2^\omega$ with common domain. Then:*

(1) $\mathsf{P} \leq_W \mathsf{P}_\mathcal{F}$ *if and only if* $\mathcal{F}_\mathsf{P} \leq_W \mathcal{F}$;
(2) $\mathsf{P} \leq_{sW} \mathsf{P}_\mathcal{F}$ *if and only if* $\mathcal{F}_\mathsf{P} \leq_{sW} \mathcal{F}$.

Proof. We prove (1), the proof of (2) being analogous. First, suppose $\mathsf{P} \leq_W \mathsf{P}_\mathcal{F}$ via Φ and Ψ. We claim that Φ and Ψ also witness that $\mathcal{F}_\mathsf{P} \leq_W \mathcal{F}$. Indeed, fix any $G \in \mathcal{F}$, and define $F = \Psi \circ \langle \mathrm{id}, G \circ \Phi \rangle$. We have only to verify that $F \in \mathcal{F}_\mathsf{P}$. If A is an instance of P, then $\Phi(A)$ is an instance of $\mathsf{P}_\mathcal{F}$, meaning an element of the intersection of the domains of the members of $\mathcal{P}$, and so in particular, a member of the domain of G. Thus, $G(\Phi(A))$ is defined, and by definition of $\mathsf{P}_\mathcal{F}$, this is a solution to the instance $\Phi(A)$, meaning $\Psi(A, G(\Phi(A)))$ is a solution to A. Thus, A is in the domain of F, and $F(A) = \Psi(A, G(\Phi(A)))$ is a solution to A, as needed.

In the other direction, suppose $\mathcal{F}_\mathsf{P} \leq_W \mathcal{F}$ via Φ and Ψ. Let A be any instance of P, and so a member of the intersection of the domains of the functions in $\mathcal{F}_\mathsf{P}$. By the definition of $\leq_W$ above, $\Phi(A)$ is an element of the domain of every function in $\mathcal{F}$, and so an instance of $\mathsf{P}_\mathcal{F}$. Let S be any solution to this instance, so that $S = G(\Phi(A))$ for some $G \in \mathcal{F}$, and then let $F \in \mathcal{F}_\mathsf{P}$ be as given for G by the definition of $\leq_W$. We have that $F(A)$ is a solution to A, and $F(A) = \Psi(A, G(\Phi(A))) = \Psi(A, S)$. This completes the proof. $\square$

The proposition allows us to translate results employing Definition 1.5 into results employing Definition A.1.

Corollary A.3. *Let P and Q be Π_2^1 principles of second-order arithmetic. Then:*

(1) $\mathsf{P} \leq_W \mathsf{Q}$ *if and only if* $\mathcal{F}_\mathsf{P} \leq_W \mathcal{F}_\mathsf{Q}$;
(2) $\mathsf{P} \leq_{sW} \mathsf{Q}$ *if and only if* $\mathcal{F}_\mathsf{P} \leq_{sW} \mathcal{F}_\mathsf{Q}$.

Proof. By the proposition, we have that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{P}_{\mathcal{F}_\mathsf{Q}}$ if and only if $\mathcal{F}_\mathsf{P} \leq_{\mathrm{W}} \mathcal{F}_\mathsf{Q}$. But it is easily checked that $\mathsf{P}_{\mathcal{F}_\mathsf{Q}} = \mathsf{Q}$. This gives us (1), and the proof of (2) is analogous. $\square$

Translations in the reverse direction require an additional assumption.

Corollary A.4. *Let $\mathcal{F}$ and $\mathcal{G}$ each be an arithmetically-definable set of partial functions $2^\omega \to 2^\omega$ with common domain. If $\mathcal{F}_{\mathsf{P}_\mathcal{F}} = \mathcal{F}$ then:*

(1) $\mathcal{F} \leq_{\mathrm{W}} \mathcal{G}$ *if and only if* $\mathsf{P}_\mathcal{F} \leq_{\mathrm{W}} \mathsf{P}_\mathcal{G}$*;*
(2) $\mathcal{F} \leq_{\mathrm{sW}} \mathcal{G}$ *if and only if* $\mathsf{P}_\mathcal{F} \leq_{\mathrm{sW}} \mathsf{P}_\mathcal{G}$*.*

Proof. Again, we only prove (1). By the proposition, we have that $\mathcal{F} = \mathcal{F}_{\mathsf{P}_\mathcal{F}} \leq_{\mathrm{W}} \mathcal{G}$ if and only if $\mathsf{P}_\mathcal{F} \leq_{\mathrm{W}} \mathsf{P}_\mathcal{G}$, as desired. $\square$

The additional assumption above is natural, as it encompasses most results from computable analysis. Indeed, the primary objects of study in that context are not reductions between arbitrary sets of functions, but rather, reductions between sets of realizers of multi-functions. For completeness, we include definitions of these concepts (see [5, Definition 2.2] for a more technical version better suited for work with represented spaces).

Definition A.5. Let $f : 2^\omega \to 2^\omega$ be a partial multi-valued function. A partial single-valued function $F : 2^\omega \to 2^\omega$ is a *realizer* for f, written $F \vdash f$, if $F(A) \in f(A)$ for each A in the domain of f.

Notice now that if $f : 2^\omega \to 2^\omega$ is a partial multi-valued function, then the set $\mathcal{F} = \{F : F \vdash f\}$ satisfies $\mathcal{F}_{\mathsf{P}_\mathcal{F}} = \mathcal{F}$.

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