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Quotients of trees for arithmetic subgroups of PGL_2 over a rational function field

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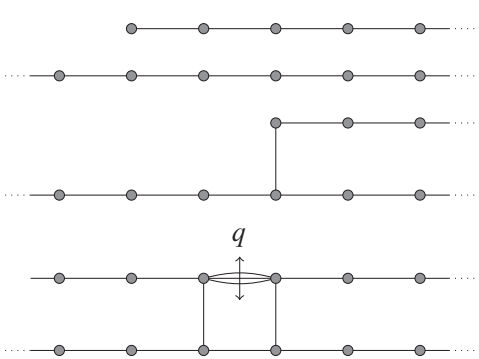
Abstract. In this note we determine the structure of the quotient of the Bruhat–Tits tree of the locally compact group $\mathsf{PGL}_2(F_p)$ with respect to the natural action of its S-arithmetic subgroup $\mathsf{PGL}_2(O_{\{p\}})$, where F is a rational function field over a finite field and p is a place of F.

1 Introduction

Let k be the finite field $\mathbb{F}_q$ of order q and $F := k(t)$ the rational function field over k. Let p be a place of degree d of F corresponding to an irreducible monic polynomial f, inducing the valuation v_p. Let $\mathcal{O}_{\{p\}}$ be the subring of F consisting of the elements of F having poles only at p and let $\mathcal{O}_{\{p,\infty\}}$ be the subring of F consisting of elements having poles only at p and at ∞. Let X be the Bruhat–Tits tree corresponding to the valuation v_p. The vertices of this tree correspond to the homothety classes of rank two $\mathcal{O}_p$-sublattices in F^2.

Serre [8, Chapter II, Section 2.4.2] computed the fundamental domain $\Gamma\backslash X$, where Γ is the arithmetic group $\mathsf{PGL}_2(\mathcal{O}_{\{p\}})$ for degrees $d \in \{1, 2, 3, 4\}$:

$d = 1$:
$d = 2$:
$d = 3$:
$d = 4$:



In this note we compute the fundamental domain $\Gamma\backslash X$ for arbitrary degree d; in Section 2 we state the main result and depict the fundamental domains up

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to degree 7. The approach of our proof is to study the action of the arithmetric group $\mathsf{PGL}_2(\mathcal{O}_{\{p,\infty\}})$ on the product of the Bruhat–Tits tree X of $\mathsf{PGL}_2(\mathbb{F}_q(t)_{v_p})$ and the Bruhat–Tits tree Y of $\mathsf{PGL}_2(\mathbb{F}_q(t)_{v_\infty})$. Strong approximation of PSL_2 allows us to identify the $\mathsf{PGL}_2(\mathcal{O}_{\{p\}})$-orbits on X with the $\mathsf{PGL}_2(\mathcal{O}_{\{\infty\}})$-orbits on Y (cf. Section 3). A subsequent detailed analysis of double coset spaces in Section 4 yields the desired result.

Our approach makes substantial use of the fact that $\mathcal{O}_{\{\infty\}} \cong k[t]$ is Euclidean. Therefore the potential for generalizing our method is limited; we refer to [4] for other situations in which the ring of functions that are regular on a projective curve minus a rational point is at least a principal ideal domain.

Partial results for the rational case under consideration can be found in [5]. The non-rational genus 0 case is studied in [6] and the elliptic curve case in [9].

We point out that, by classical results, the fundamental domain together with information concerning the (finite) stabilizers provides a presentation of the group $\mathsf{PGL}_2(\mathcal{O}_{\{p\}})$ by generators and relations, cf. [3, Section III.$\mathcal{C}$], [1, Chapter 2] and [8, Sections I.4 and II.2].

2 Statement of the main result and examples

In this section we state the main theorem and depict the quotients for d up to 7.

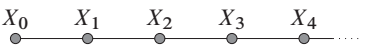
Main Theorem. *Let v_p be a valuation of degree d of the rational function field $\mathbb{F}_q(t)$ and X be the Bruhat–Tits tree of the locally compact group $\mathsf{PGL}_2(\mathbb{F}_q(t)_{v_p})$. Then the orbit space $\mathsf{PGL}_2(\mathcal{O}_{\{p\}})\backslash X$ can be described as follows.*

(1) *If d is odd, then its set of vertices is $\{X_n \mid n \in \mathbb{N}_0\}$ with*
- *one edge between X_n and X_{n+d} ($n \in \mathbb{N}_0$),*
- *one edge between X_n and X_{d-n} ($n \in \mathbb{N}_0$ and $n, d-n \geq 1$),*
- *$q^{2l-1}+q^{2l-2}$ edges between X_n and X_{d-n-2l} ($n, l \in \mathbb{N}_0$, $n, l \geq 1$ and $d-n-2l \geq 1$),*
- *q^{2l-2} edges between X_0 and X_{d-2l} ($l \in \mathbb{N}_0$ and $l, d-2l \geq 1$).*

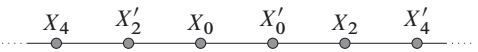
(2) *If d is even, then its set of vertices is $\{X_n, X'_n \mid n \in 2\mathbb{N}_0\}$ with*
- *one edge between X_n and X'_{n+d} and between X'_n and X_{n+d} ($n \in 2\mathbb{N}_0$),*
- *one edge between X_n and X'_{d-n} ($n \in 2\mathbb{N}_0$ and $n, d-n \geq 2$),*
- *$q^{2l-1}+q^{2l-2}$ edges between X_n and X'_{d-n-2l} and between X'_n and X_{d-n-2l} ($n \in 2\mathbb{N}_0$, $l \in \mathbb{N}_0$, $l \geq 1$ and $n, d-n-2l \geq 2$),*
- *q^{2l-2} edges between X_0 and X'_{d-2l} and between X'_0 and X_{d-2l} ($l \in \mathbb{N}_0$, $l \geq 1$ and $d-2l \geq 2$),*
- *$q(q^{d-3}+1)/(q+1)$ edges between X_0 and X'_0 if $d > 2$, and one edge between X_0 and X'_0 if $d = 2$.*

Examples of quotients. We have

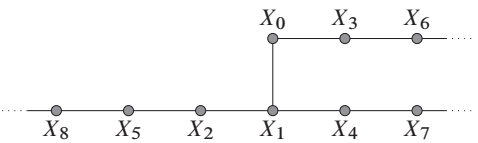
$d = 1$:



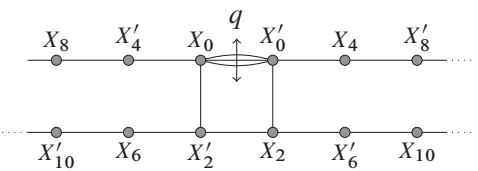
$d = 2$:



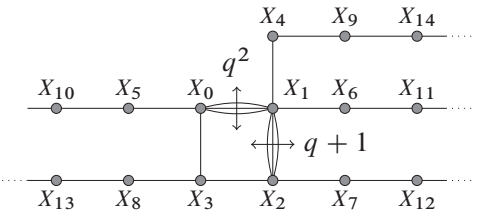
$d = 3$:



$d = 4$:

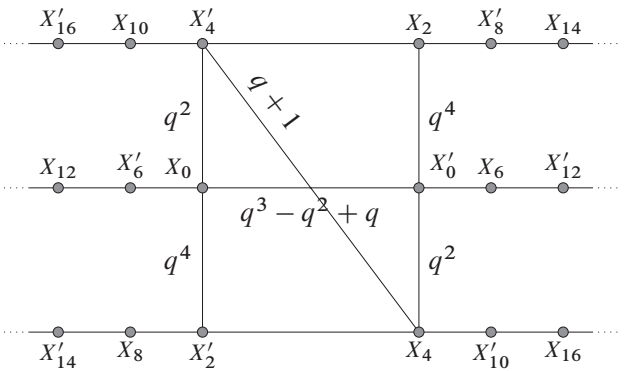


$d = 5$:

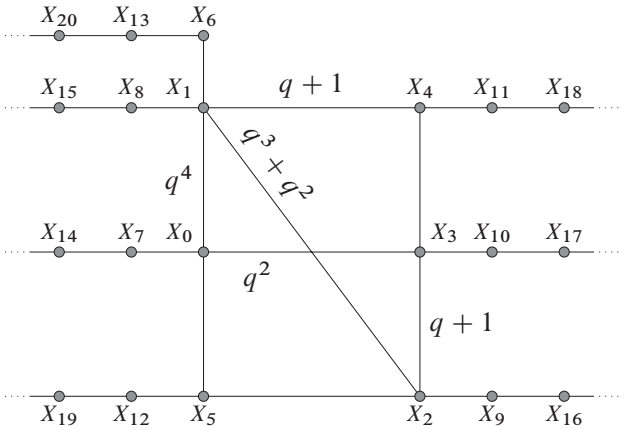


From this point on we depict multi-edges by a label indicating the number of edges in order to avoid cluttering the picture.

$d = 6$:



$d = 7$:



3 Basics and preliminaries

The trees X and Y and their vertices. Let k be the finite field $\mathbb{F}_q$ of order q and let $F := k(t)$ be the rational function field over k. Let p be a place of degree d of F corresponding to an irreducible monic polynomial f, inducing the valuation v_p.

Consider the place ∞ of F, which is a place of degree 1 and corresponds with a valuation $v_\infty(\frac{a}{b}) = \deg b - \deg a$ for $a, b \in k[t]$. The ring $\mathcal{O}_{\{\infty\}}$ of elements with poles only at ∞ then equals the ring of polynomials $k[t]$ in F.

We represent the vertices of the trees X (the Bruhat–Tits tree of $\mathsf{PGL}_2(F_{v_p})$) and Y (the Bruhat–Tits tree of $\mathsf{PGL}_2(F_{v_\infty})$) by giving two generators spanning a lattice in the homothety class corresponding to the vertex (cf. [8, Chapter II, Section 1]). We will write these two generators as the columns of a (2×2)-matrix with respect to the standard basis of F^2 together with a subscript indicating the place.

As an example and for future use we define the vertices

$$x_0 := \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right)_p \quad \text{and} \quad y_i := \left(\begin{array}{c|c} 1 & t^i \\ 0 & 1 \end{array}\right)_\infty \qquad (i \in \mathbb{N}_0).$$

The first vertex x_0 is a vertex of the tree X, the second series y_i $(i \in \mathbb{N}_0)$ are vertices in Y.

Lemma 3.1. *Let h be an element in $\mathsf{PGL}_2(\mathcal{O}_{\{p,\infty\}})$ represented by a matrix M with entries in $k[t]$, not of all of them divisble by f as polynomials. Then the distance between x_0 and $h(x_0)$ equals $v_p(\det(M))$.*

Proof. Let L be a lattice representing the vertex x_0. Then the definition of distance in [8, Chapter II, Section 1.1] implies that the distance between x_0 and hx_0 equals $b-a$ where b is minimal such that f^bL is contained by ML and a maximal such that f^aL contains ML.

Due to the conditions on the entries of M, we have that L contains ML but fL does not, so $a=0$.

In order to calculate b, first note that $f^bL \subseteq ML$ if and only if $L \supseteq f^bM^{-1}L$. The entries of M^{-1} are, up to minus signs and permutations, the entries of M divided by $\det(M) \in \mathcal{O}^\times_{\{p,\infty\}}$. Hence, if one wants to multiply the matrix M^{-1} with a power f^b of f such that in this product the entries lie in $k[t]$, then the minimal and sufficient such b is $v_p(\det(M))$. □

An analogous statement allows one to compute distances in the tree Y. For instance, the element of $\mathsf{PGL}_2(F)$ represented by the matrix $\left(\begin{smallmatrix} t^{n-m} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ maps y_m to y_n and, accordingly,

$$\mathsf{d}(y_m, y_n) = v_\infty(t^{n-m}) = m-n$$

if $m \geq n$.

The group $\mathsf{PGL}_2(F)$, its subgroups, and their transitivity properties. The group $\mathsf{PGL}_2(F)$ acts faithfully as a group of isometries on both X and Y, where the action is induced by the canonical action from the left of $\mathsf{PGL}_2(F)$ on the 2-dimensional lattices. We represent the elements in $\mathsf{PGL}_2(F)$ by (2×2)-matrices (with respect to the standard basis of F^2).

We will mainly work inside the arithmetic subgroup

$$\Pi := \mathsf{PGL}_2(\mathcal{O}_{\{p,\infty\}})$$

of $\mathsf{PGL}_2(F)$. This group contains the arithmetic groups

$$\Gamma := \mathsf{PGL}_2(\mathcal{O}_{\{p\}}) \quad \text{and} \quad \Xi := \mathsf{PGL}_2(\mathcal{O}_{\{\infty\}})$$

as subgroups.

Recall that a dense subgroup of a topological group acting on a discrete set has the same orbits. Hence, by strong approximation (see [7]), the subgroup $\mathsf{PSL}_2(\mathcal{O}_{\{p,\infty\}}) < \Pi$ acts edge-transitively, if we restrict the action to either X or Y. In particular this group has two orbits (corresponding to the types) on the vertices in either restriction.

A similar fact is true for $\Pi = \mathsf{PGL}_2(\mathcal{O}_{\{p,\infty\}})$: If d is odd, then Π acts non-type preservingly on both X and Y, and hence transitively on the vertices of either tree. If d is even, then Π acts transitively on the vertices of X, but type-preservingly, whence with two orbits on the vertices of Y.

The following are further groups of interest to us:

- $\Pi_{x_0} = \Xi$.
- $\Pi_{y_0} =: \widetilde{\Gamma}$, where $\widetilde{\Gamma}$, in case d is even, acts non-type preservingly on X and contains Γ as an index two subgroup and, in case d is odd, equals Γ.
- $\Xi_{y_0} = \mathsf{PGL}_2(k) =: H_0$.
- $\Xi_{y_i} = H_i$ $(i \in \mathbb{N})$ with

$$H_i := \left\{ \begin{pmatrix} \alpha & b \\ 0 & \delta \end{pmatrix} \;\middle|\; \alpha, \delta \in k^*,\ b \in k[t],\ \deg(b) \leq i \right\}.$$

A fundamental domain for the quotient $\Xi \backslash Y$ is given by the vertices y_i with $i \in \mathbb{N}_0$ and the edges between these, forming a ray (cf. e.g. [8, Chapter II, Section 1.6, Corollary] or the case $d = 1$ in the introduction).

Maps sending x_0 to a neighbor and y_m to y_n. Let $h \in \Pi$ such that $h(x_0)$ is adjacent to x_0, represented by a matrix

$$M := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in {\mathcal{O}_{\{p,\infty\}}}^{2\times 2}.$$

By taking the scalar multiple with the appropriate power of f we may assume that α, β, γ and δ all lie in $k[t]$ and are coprime as polynomials.

When this is the case, we have $\mathsf{d}(x_0, h(x_0)) = v_p(\det(M))$ by Lemma 3.1. Hence f is a divisor of $\det(M) \in k[t]$, but f^2 is not. As $\det(M) \in {\mathcal{O}_{\{p,\infty\}}}^{\times}$, this implies that

$$\det(M) = \lambda f \quad \text{with } \lambda \in k^*. \tag{3.1}$$

Next we want to determine the set of elements in $\mathsf{PGL}_2(F)$ which map y_m to y_n with $m, n \in \mathbb{N}_0$. Since the stabilizer of y_0 in $\mathsf{PGL}_2(F)$ is $\mathsf{PGL}_2(\mathcal{O}_\infty)$, this set can be described as $g\mathsf{PGL}_2(\mathcal{O}_\infty)g'$ with arbitrary $g, g' \in \mathsf{PGL}_2(F)$ that satisfy $g'(y_m) = y_0$ and $g(y_0) = y_n$; for instance, the element g can be represented by the matrix $\left(\begin{smallmatrix} t^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the element g' by the matrix $\left(\begin{smallmatrix} t^{-m} & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &:= \begin{pmatrix} at^{n-m} & bt^n \\ ct^{-m} & d \end{pmatrix} \\ &= \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-m} & 0 \\ 0 & 1 \end{pmatrix} \in g\mathsf{PGL}_2(\mathcal{O}_\infty)g'. \end{aligned} \tag{3.2}$$

We conclude from the above discussion that the elements $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathsf{PGL}_2(F)$ which map y_m to y_n are exactly those that satisfy $\alpha, \beta, \gamma, \delta \in F$, $v_\infty(\alpha) \geq m-n$, $v_\infty(\beta) \geq -n$, $v_\infty(\gamma) \geq m$, $v_\infty(\delta) \geq 0$ and $v_\infty(\alpha\delta - \beta\gamma) = m-n$.

In the proof of our main result we will make use of the set $\Upsilon_{n,m}$ of elements of Π that map x_0 to a neighbor and y_m to y_n with $m, n \in \mathbb{N}_0$. We want to describe this set using, as above, matrices

$$M := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathcal{O}_{\{p,\infty\}}{}^{2\times 2}$$

with entries in $k[t]$ that are coprime as polynomials and whose determinant equals a non-zero scalar multiple of f. Since $v_\infty(f) = -d$, this means we have to multiply the matrix from equation (3.2) with the scalar matrix

$$\begin{pmatrix} t^{\frac{d-n+m}{2}} & 0\\ 0 & t^{\frac{d-n+m}{2}}\end{pmatrix}.$$

Using $v_\infty(\frac{a}{b}) = \deg(b) - \deg(a)$ for $a, b \in k[t]$, we arrive at the following description of the desired set:

Proposition 3.2. *The set of elements of Π that map x_0 to a neighbor and y_m to y_n for $m, n \in \mathbb{N}_0$ equals*

$$\Upsilon_{n,m} := \left\{ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \;\middle|\; \alpha, \beta, \gamma, \delta \in k[t],\ \deg(\alpha) \leq \frac{d+n-m}{2},\ \deg(\beta) \leq \frac{d+n+m}{2},\ \deg(\gamma) \leq \frac{d-n-m}{2},\ \deg(\delta) \leq \frac{d-n+m}{2},\ \alpha\delta - \beta\gamma = \lambda f, \lambda \in k^* \right\}. \quad (3.3)$$

Note that, if d and $m+n$ do not have the same parity, then $\Upsilon_{n,m} = \emptyset$. Furthermore note that $\Upsilon_{n,m}$ is stable under multiplication with H_m from the right and with H_n from the left.

Coprime polynomials. At a certain point in the proof we will need to calculate the number of coprime polynomials in $k[t]$ with some degree constraints. For this we use the following result:

Lemma 3.3 ([2, Theorem 3]). *Let $i, j \in \mathbb{N}$, let α be an arbitrary polynomial in $k[t]$ of degree i and let β be an arbitrary polynomial in $k[t]$ of degree j. Then the probability that α and β are coprime is $1 - \frac{1}{q}$.* □

Note that the statement of the preceding lemma is also true, if β is an arbitrary constant polynomial. This has the following immediate corollary.

Corollary 3.4. *Let $i \in \mathbb{N}$, $j \in \mathbb{N}_0$, let α be an arbitrary polynomial in $k[t]$ of degree i and let β be an arbitrary polynomial in $k[t]$ of degree* at most *j. Then the probability that α and β are coprime is $1 - \frac{1}{q}$.* □

4 The orbit space $\widetilde{\Gamma}\backslash X$

Vertices of $\widetilde{\Gamma}\backslash X$. By the transitivity properties discussed in Section 3 we may identify the coset space Π/Ξ with the set of vertices of X. Moreover, if we consider the natural left action of Π on Y as a right action via inversion, then we may identify the coset space $\widetilde{\Gamma}\backslash\Pi$ with the set of vertices of Y, in case d is odd, and the set Y_{even} of vertices of Y with the same type as y_0, in case d is even. Hence

$$\begin{aligned}\widetilde{\Gamma}\backslash X &\cong \widetilde{\Gamma}\backslash(\Pi/\Xi)\\ &= (\widetilde{\Gamma}\backslash\Pi)/\Xi\\ &\cong \begin{cases} Y_{\text{even}}/\Xi, & \text{if } d \text{ is even,}\\ Y/\Xi, & \text{if } d \text{ is odd.}\end{cases}\end{aligned}$$

The sets $\{y_0, y_2, y_4, \dots\}$ resp. $\{y_0, y_1, y_2, \dots\}$ from Section 3 form a system of representatives for the Ξ-orbits on Y_{even} resp. Y. Hence the above correspondence provides a labeling of each $\widetilde{\Gamma}$-orbit on X as X_i if and only if it corresponds to the Ξ-orbit containing y_i. If d is even, of course, only even indices i occur.

Edges of $\widetilde{\Gamma}\backslash X$. Next we describe the number of edges in the quotient between the orbits X_n and X_m (where $n, m \in \mathbb{N}$, $m \geq n$ and both even if d is even).

Let x be a vertex in the orbit X_n. It corresponds to a double coset $\widetilde{\Gamma} g \Xi$ for some $g \in \Pi$ and, by definition, there exists an element $g \in \Pi$ such that $g(x_0) = x$ and $g^{-1}(y_0) = y_n \iff g(y_n) = y_0$. Similarly, for each vertex x' in the orbit X_m there exists an element $g' \in \Pi$ with $g'(x_0) = x'$ and $g'(y_m) = y_0$.

Assume $x' \in X_m$ is adjacent to x. Then $z := g^{-1}x'$ is a neighbor of x_0 and, moreover, $g^{-1}g'(x_0, y_m) = (z, y_n)$, whence $h := g^{-1}g' \in \Pi$ is an element of $\Upsilon_{n,m}$. Two elements $h_1 = g^{-1}g_1'$ and $h_2 = g^{-1}g_2'$ of $\Upsilon_{n,m}$ determine the same neighbor of x if and only if they are in the same left coset in $\Upsilon_{n,m}/H_m$, as H_m is the stabilizer of the pair (x_0, y_m).

Next we have to account for the orbits of the stabilizer of $\widetilde{\Gamma}_x = \Pi_{(x,y_0)}$ on the neighbors of x in the orbit X_m. In fact, we will study the orbits on the neighbors of x_0 in $g^{-1}\widetilde{\Gamma}_x g = g^{-1}\Pi_{(x,y_0)}g = \Pi_{(x_0,y_n)} = H_n$ instead: Because of the

natural left action of H_n on X two neighbors z and z' of x_0 in X_m are in the same H_n-orbit if and only if their corresponding left cosets hH_m and $h'H_m$ are contained in the same double coset in $H_n \backslash \Upsilon_{n,m} / H_m$.

We conclude the following:

Proposition 4.1. *The number of edges between the orbits X_n and X_m in the quotient $\widetilde{\Gamma}\backslash X$ equals $|H_n \backslash \Upsilon_{n,m} / H_m|$.*

An alternative approach to Proposition 4.1 can be found in [8, Exercise 2, p. 116]. From this point on the main difficulty lies in calculating $|H_n \backslash \Upsilon_{n,m} / H_m|$.

We will distinguish between three cases in order to determine this number. Note that we assume that $m \geq n$, and we can additionally assume that d and $m+n$ have the same parity as otherwise $\Upsilon_{n,m}$ is empty (see Section 3).

Case 1: $m + n > d$. This assumption implies, using the description from (3.3), that γ is zero, whence $\alpha\delta = \det(M) = \lambda f$ for some $\lambda \in k^*$ by equation (3.1). As f is irreducible, this is only possible if one of α or δ is of degree d and equals f times a constant while the other is a constant. Since $m \geq n$, the description from equation (3.3) implies that, in fact, α has degree 0 and δ has degree d and equals f times a constant. We conclude that $m - n = d$ or otherwise $\Upsilon_{n,m} = \emptyset$. In particular, if $\Upsilon_{n,m} \neq \emptyset$, then $\deg(\beta) \leq \frac{d+n+m}{2} = d + n$.

Altogether

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mu_1 & \beta \\ 0 & \mu_2 f \end{pmatrix} = \begin{pmatrix} \mu_1 & qf + r \\ 0 & \mu_2 f \end{pmatrix}$$

with $\mu_1, \mu_2 \in k^*$ and $\beta = qf + r$ via Euclidean division with $\deg(r) < d \leq m$ and $\deg(q) = \deg(\beta) - \deg(f) \leq n$. We conclude

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \mu_1 & qf + r \\ 0 & \mu_2 f \end{pmatrix} \\ &= \begin{pmatrix} 1 & q \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} \mu_1 & r \\ 0 & 1 \end{pmatrix} \in H_n \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix} H_m. \end{aligned}$$

Since – as long as $\Upsilon_{n,m} \neq \emptyset \iff m - n = d$ – indeed $\left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right) \in \Upsilon_{n,m}$, the double coset space $H_n \backslash \Upsilon_{n,m} / H_m$ consists of a single double coset.

Case 2: $m + n = d$. Note that $\left(\begin{smallmatrix} 0 & -f \\ 1 & 0 \end{smallmatrix}\right) \in \Upsilon_{n,m}$, i.e., this set is non-empty. Again using the description from equation (3.3), it follows that γ is in k. If $\gamma = 0$, as in Case 1 we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mu_1 & \beta \\ 0 & \mu_2 f \end{pmatrix},$$

which implies $n = 0$ via the condition $d = \deg(\lambda f) \leq \frac{d-n+m}{2}$ from (3.3). Hence $\deg(\beta) \leq d = m$. We compute

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mu_1 & \beta \\ 0 & \mu_2 f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -f \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 & \beta \\ 0 & \mu_2 \end{pmatrix} \in H_n \begin{pmatrix} 0 & -f \\ 1 & 0 \end{pmatrix} H_m.$$

If $\gamma \in k^*$, the fact $\det(M) = \lambda f$ for some $\lambda \in k^*$ (cf. equation (3.1)) allows us to normalize to $\gamma = 1$, so that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\delta - \lambda f \\ 1 & \delta \end{pmatrix}$$

with $\deg(\alpha) \leq n$, $\deg(\delta) \leq m$ and $\deg(\beta) = \deg(\alpha\delta - \lambda f) \leq d$. We compute

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\delta - \lambda f \\ 1 & \delta \end{pmatrix} = \begin{pmatrix} \lambda & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -f \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \in H_n \begin{pmatrix} 0 & -f \\ 1 & 0 \end{pmatrix} H_m.$$

We conclude that $H_n \backslash \Upsilon_{n,m} / H_m$ consists of a single double coset.

Case 3: $m + n < d$. Define $l := (d - m - n)/2$.

We start by calculating the size of the set $\Upsilon_{n,m}$ using the description of equation (3.3). The polynomial β is contained in an $(l + n + m + 1)$-dimensional subspace V of $k[t]$, the polynomial δ in an $(l + m + 1)$-dimensional subspace W. The canonical projection $k[t] \to k[t]/(f)$ maps the subspaces V, W isomorphically on subspaces $\overline{V}$, $\overline{W}$ of $k[t]/(f)$, because

$$\deg(\beta) \leq l + n + m = \frac{d + m + n}{2} < d = \deg(f)$$

and

$$\deg(\delta) \leq l + m = \frac{d + m - n}{2} < d = \deg(f).$$

Multiplications with $\overline{\alpha} := \alpha + (f)$ and with $\overline{\gamma} := \gamma + (f)$ in $k[t]/(f)$ induce bijective k-linear maps $k[t]/(f) \to k[t]/(f)$, as α and f resp. γ and f are coprime (for degree reasons, as f is irreducible).

We have

$$\begin{aligned}\dim \overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma} &\geq \dim_k(\overline{W}) + \dim_k(\overline{V}) - \dim_k(k[t]/(f)) \\ &= 2l + 2m + n + 2 - d = m + 2.\end{aligned}$$

It describes the choices of polynomials $\alpha, \beta, \gamma, \delta \in k[t]$ subject to the degree restraints in equation (3.3) that satisfy $\alpha\delta - \beta\gamma \equiv 0 \mod f$. Among those, precisely the choices with $\alpha\delta - \beta\gamma = \lambda f$ with $\lambda \in k^*$ lead to elements of $\Upsilon_{n,m}$. We observe that $\alpha\delta - \beta\gamma$ has degree at most d, i.e., it suffices to exclude the polynomials α, β, γ, δ satisfying $\alpha\delta - \beta\gamma = 0$.

In other words, there exists a k-linear map

$$\Psi : \overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma} \to k, \quad \overline{\alpha}\overline{\delta} = \overline{\beta}\overline{\gamma} \mapsto \frac{\alpha\delta - \beta\gamma}{f}, \tag{4.1}$$

which is well-defined as $\deg(\alpha), \deg(\beta), \deg(\gamma), \deg(\delta) < d$. We are looking for choices of polynomials outside $\ker(\Psi)$.

If $\alpha\delta - \beta\gamma = \lambda f$ for some $\lambda \in k^*$, then α and γ are coprime (again for degree reasons, as f is irreducible). Moreover,

$$\deg(\alpha) = l + n = \frac{d - m + n}{2} \quad \text{or} \quad \deg(\gamma) = l = \frac{d - m - n}{2},$$

since $\deg(\alpha\delta - \beta\gamma) = d$. By Corollary 3.4 there exist $(q-1)^2(q^{2l+n} + q^{2l+n-1})$ choices of pairs (α, γ) satisfying these two conditions.

Let (α, γ) be such a pair. If nevertheless $\alpha\delta - \beta\gamma = 0 \iff \alpha\delta = \beta\gamma$, then $\alpha \mid \beta$ and $\gamma \mid \delta$, as α and γ are coprime. Reduction of the equality $\alpha\delta = \beta\gamma$ by $\alpha\gamma$ yields

$$\frac{\delta}{\gamma} = \frac{\beta}{\alpha} =: \epsilon \in k[t],$$

i.e., $\beta = \alpha\epsilon$ and $\delta = \gamma\epsilon$. Since

$$\deg(\alpha) = l + n = \frac{d - m + n}{2} \quad \text{or} \quad \deg(\gamma) = l = \frac{d - m - n}{2},$$

we have

$$\deg(\epsilon) = \deg(\beta) - \deg(\alpha) = \deg(\delta) - \deg(\gamma) \leq m.$$

Conversely, any $\epsilon \in k[t]$ with $\deg(\epsilon) \leq m$ provides suitable $\beta := \alpha\epsilon$ and $\delta := \gamma\epsilon$ satisfying $\alpha\delta - \beta\gamma = 0$.

The collection of all these choices of ϵ provides an $(m+1)$-dimensional subspace U consisting of $\overline{\alpha}\overline{\delta} = \overline{\alpha\gamma\epsilon} = \overline{\beta}\overline{\gamma} \in \overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma}$. Using the linear map Ψ introduced in (4.1), we have $U = \ker(\Psi)$. Since

$$\begin{aligned}1 = (m+2) - (m+1) &\leq \dim(\overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma}/U) \\ &= \dim(\overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma}/\ker(\Psi)) \leq \dim(k) = 1,\end{aligned}$$

we have

$$\dim(\overline{\alpha}\overline{W} \cap \overline{V}\overline{\gamma}) = m + 2.$$

Therefore for each of the $(q-1)^2(q^{2l+n} + q^{2l+n-1})$ viable choices of pairs (α, γ) we obtain $q^{m+2} - q^{m+1}$ viable choices of pairs (β, δ). We conclude that

$$\begin{aligned} |\Upsilon_{n,m}| &= (q-1)(q^{2l+n} + q^{2l+n-1})(q^{m+2} - q^{m+1}) \\ &= q^{n+m}(q^{2l+1} + q^{2l})(q-1)^2. \end{aligned}$$

Note that we divided by $q-1$ to take into account the fact that we work in $\mathsf{PGL}_2(F)$ and not in $\mathsf{GL}_2(F)$, so that for fixed α, β, γ, δ the matrices $\left(\begin{smallmatrix} \lambda\alpha & \lambda\beta \\ \lambda\gamma & \lambda\delta \end{smallmatrix}\right)$, $\lambda \in k^*$, all describe the same element of $\Upsilon_{n,m}$.

We will calculate $|H_n \backslash \Upsilon_{n,m} / H_m|$ by distinguishing the following three subcases.

Subcase 3 (a): $m, n > 0$. In order to approach this subcase we take a look at the general form of a double coset in $H_n \backslash \Upsilon_{n,m} / H_m$ by considering the following product:

$$\begin{pmatrix} \kappa & \zeta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \kappa\alpha + \zeta\gamma & * \\ \gamma & \lambda\delta + \eta\gamma \end{pmatrix}.$$

Here $\kappa, \lambda \in k^*$ and $\zeta, \eta \in k[t]$ with $\deg \zeta \leq n$ and $\deg \eta \leq m$. As we are working in PGL_2, we are able to choose some entries equal to 1.

If the degree of γ is at least 1, the fact that α and γ are coprime allows us to compute κ (and subsequently ζ) via the Chinese Remainder Theorem by considering $\kappa\alpha + \zeta\gamma$ modulo γ. If $\gamma \in k$, then necessarily $\deg(\alpha) = l + n > 0$. As $l > 0$ and $\deg \zeta \leq n$, one can derive κ and subsequently ζ from the leading coefficient of $\kappa\alpha + \zeta\gamma$. Analogously, one can compute λ and η. (Or one inverts the two, now known, matrices on the left hand side of the equation in order to obtain the third.)

This implies that the size of a double coset in $H_n \backslash \Upsilon_{n,m} / H_m$ is $|H_n||H_m|$ and, hence,

$$\begin{aligned} |H_n \backslash \Upsilon_{n,m} / H_m| &= \frac{|\Upsilon_{n,m}|}{|H_n||H_m|} \\ &= \frac{q^{n+m}(q^{2l+1} + q^{2l})(q-1)^2}{(q-1)q^{n+1} \cdot (q-1)q^{m+1}} = q^{2l-1} + q^{2l-2}. \end{aligned}$$

Subcase 3 (b): $m > 0$ and $n = 0$. We adopt a similar strategy as in the previous subcase, trying to determine the factors of a product of matrices. Here we have to

consider the product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & * \\ c\alpha + d\gamma & * \end{pmatrix}$$

with $a, b, c, d \in k$ such that $ad - bc \neq 0$, $\lambda \in k^*$, and $\eta \in k[t]$ with $\deg \eta \leq m$.

As before we can compute c and d from $c\alpha + d\gamma$ and a and b from $a\alpha + b\gamma$, as α and γ are coprime. The values of λ and η are then obtained again by inverting the two, now known, matrices on the left hand side of the equation in order to obtain the third. We again conclude that the size of a double coset in $H_n \backslash \Upsilon_{n,m} / H_m$ is $|H_n||H_m|$ and so

$$|H_n \backslash \Upsilon_{n,m} / H_m| = \frac{|\Upsilon_{n,m}|}{|H_n||H_m|} = \frac{q^m(q^{2l+1} + q^{2l})(q-1)^2}{(q-1)q(q+1) \cdot (q-1)q^{m+1}} = q^{2l-2}.$$

Subcase 3 (c): $n = m = 0$. This final subcase will be handled differently from the previous ones. Note that we necessarily have that d is even.

We first count the total number of orbits of edges containing the vertex x_0 under its vertex stabilizer $\widetilde{\Gamma}_{x_0} = \Xi_{y_0} = \mathsf{PGL}_2(k)$. This is equivalent to the study of the $\mathsf{PGL}_2(k)$-orbits of points on the projective line $\mathbb{P}^1(\mathbb{F}_{q^d})$. One counts

- one orbit of length $q+1$ (corresponding to the embedding $\mathbb{P}^1(\mathbb{F}_q) \subset \mathbb{P}^1(\mathbb{F}_{q^d})$),
- one orbit of length $q^2 - q$ (corresponding to $\mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q) \subset \mathbb{P}^1(\mathbb{F}_{q^d})$),
- $q + q^3 + \cdots + q^{d-3}$ other orbits of length $q(q-1)(q+1)$ if $d \geq 4$.

Indeed, $\mathsf{PGL}_2(\mathbb{F}_q)$ acts on $\mathbb{P}^1(\mathbb{F}_{q^d})$ via Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. A fixed point $z = \frac{az+b}{cz+d}$ corresponds to a solution of a quadratic equation, whence an element $z \in \mathbb{F}_{q^d} \setminus \mathbb{F}_{q^2}$ has trivial stabilizer and therefore necessarily lies in an orbit of length $q(q-1)(q+1) = |\mathsf{PGL}_2(\mathbb{F}_q)|$.

This leads to a total of two orbits if $d = 2$, and $2 + q + q^3 + \cdots + q^{d-3}$ orbits if $d \geq 4$. Each of these orbits corresponds with an edge in the quotient $\widetilde{\Gamma} \backslash X$ containing X_0.

As in Case 2 and Subcase 3 (b) above we already have accounted for a total of $1 + 1 + q^2 + q^4 + \cdots + q^{d-4}$ edges if $d \geq 4$, and one edge if $d = 2$, from X_0 to other vertices, the number of edges from X_0 to itself is the difference of both numbers, which is $q(q^{d-3} + 1)/(q+1)$.

Conclusion. The main result now follows: If d is odd, then Γ equals $\widetilde{\Gamma}$, so our claims are immediate from the previously determined quotient $\widetilde{\Gamma} \backslash X$. If d is even, then $\Gamma \backslash X$ is the bipartite double cover of $\widetilde{\Gamma} \backslash X$, as $\widetilde{\Gamma}$ does not preserves types in X whereas Γ does, forcing us to introduce the vertices X'_n.

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