Hyperplanes of $DW(5,\mathbb{K})$ containing a quad

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Abstract

We prove that every hyperplane of the symplectic dual polar space $DW(5,\mathbb{K})$ that contains a quad is either classical or the extension of a non-classical ovoid of a quad of $DW(5,\mathbb{K})$.

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1 Introduction

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. A set $H \subsetneq \mathcal{P}$ is called a *hyperplane* of $\mathcal{S}$ if every line of $\mathcal{S}$ has either one or all of its points in H. A *(full) projective embedding* of $\mathcal{S}$ is an injective mapping e from $\mathcal{P}$ to the point set of a projective space Σ satisfying (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $\{e(x) \,|\, (x, L) \in \mathrm{I}\}$ is a line of Σ for every line L of $\mathcal{S}$. If $e : \mathcal{S} \to \Sigma$ is a projective embedding of $\mathcal{S}$ and Π is a hyperplane of Σ, then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(\mathcal{P}) \cap \Pi)$ *arises from the embedding* e. A hyperplane of $\mathcal{S}$ is called *classical* if it arises from some projective embedding of $\mathcal{S}$.

Distances $\mathrm{d}(\cdot,\cdot)$ in $\mathcal{S}$ will be measured in its collinearity graph. If x is a point of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points of $\mathcal{S}$ at distance i from x. Similarly, if X is a nonempty set of points and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points at distance i from X, i.e. the set of all points y for which $\min\{\mathrm{d}(y,x) \,|\, x \in X\} = i$.

Consider in the projective space $\mathrm{PG}(5,\mathbb{K})$ a symplectic polarity ζ. The subspaces of $\mathrm{PG}(5,\mathbb{K})$ that are totally isotropic with respect to ζ define a polar space $W(5,\mathbb{K})$. We denote the dual polar space associated to $W(5,\mathbb{K})$ by $DW(5,\mathbb{K})$. The points and lines of $DW(5,\mathbb{K})$ are the totally isotropic planes and lines of $\mathrm{PG}(5,\mathbb{K})$, with incidence being reverse containment. The dual polar space $DW(5,\mathbb{K})$ belongs to the class of *near polygons* introduced by Shult and Yanushka in [25], implying that for every point x and every line L, there exists a unique point on L nearest to x. The maximal distance between two points of $DW(5,\mathbb{K})$ is equal to 3.

If x and y are two points of $DW(5,\mathbb{K})$ at distance 2 from each other, then the smallest convex subspace $\langle x, y \rangle$ of $DW(5,\mathbb{K})$ containing x and y is called a *quad*. A quad Q of

$DW(5,\mathbb{K})$ consists of all totally isotropic planes of $W(5,\mathbb{K})$ containing a given point x_Q of $W(5,\mathbb{K})$. Any two lines L and M of $DW(5,\mathbb{K})$ meeting in a unique point are contained in a unique quad which we denote by $\langle L, M\rangle$. Obviously, we have $\langle L, M\rangle = \langle x, y\rangle$ where x and y are arbitrary points of $L \setminus M$ and $M \setminus L$, respectively. The points and lines of $DW(5,\mathbb{K})$ contained in a given quad Q define a point-line geometry $\widetilde{Q}$ isomorphic to the generalized quadrangle $Q(4,\mathbb{K})$ of the points and lines of a nonsingular quadric of Witt index 2 of $\mathrm{PG}(4,\mathbb{K})$. If Q is a quad of $DW(5,\mathbb{K})$ and x is a point not contained in Q, then Q contains a unique point $\pi_Q(x)$ collinear with x and $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_Q(x), y)$ for every point y of Q. If Q_1 and Q_2 are two distinct quads of $DW(5,\mathbb{K})$, then $Q_1 \cap Q_2$ is either empty or a line of $DW(5,\mathbb{K})$. If $Q_1 \cap Q_2 = \emptyset$, then the map $Q_1 \to Q_2; x \mapsto \pi_{Q_2}(x)$ is an isomorphism between $\widetilde{Q_1}$ and $\widetilde{Q_2}$.

Since $DW(5,\mathbb{K})$ is a near polygon, the set of points of $DW(5,\mathbb{K})$ at non-maximal distance from a given point x is a hyperplane of $DW(5,\mathbb{K})$, the so-called *singular hyperplane with deepest point* x. If Q is a quad of $DW(5,\mathbb{K})$ and G is a hyperplane of $\widetilde{Q} \cong Q(4,\mathbb{K})$, then $Q \cup \{x \in \Gamma_1(Q) \,|\, \pi_Q(x) \in G\}$ is a hyperplane of $DW(5,\mathbb{K})$, the so-called *extension* of G. Every hyperplane of $Q(4,\mathbb{K})$ is either the perp $x^\perp := \{x\} \cup \Gamma_1(x)$ of a point x, a full subgrid or an ovoid, an *ovoid* being a set of points meeting each line in a singleton. The perp of a point x of $Q(4,\mathbb{K})$ is also called the *singular hyperplane of* $Q(4,\mathbb{K})$ *with deepest point* x.

A complete classification of all classical hyperplanes of $DW(5,\mathbb{K})$ is available for all finite fields and all perfect fields of characteristic 2, see Cooperstein & De Bruyn [10], De Bruyn [12, 13] and Pralle [23]. For general (possibly infinite) fields $\mathbb{K}$, it seems not possible to obtain a complete classification of the (not necessarily classical) hyperplanes of $DW(5,\mathbb{K})$ due to the possibility to construct such hyperplanes via transfinite recursion, see Beutelspacher & Cameron [3], Cameron [6] or Cardinali & De Bruyn [7, Section 4]. It might however still be possible to obtain some kind of classification for certain classes of hyperplanes of $DW(5,\mathbb{K})$. The present paper deals with those hyperplanes of $DW(5,\mathbb{K})$ that contain a quad. The following is our main result.

Main Theorem. *Let H be a hyperplane of $DW(5,\mathbb{K})$, $|\mathbb{K}| \neq 2$, containing a quad Q. Then H is one of the following:*

(1) *a singular hyperplane;*
(2) *the extension of a full subgrid of a quad;*
(3) *the extension of an ovoid of $\widetilde{Q}$;*
(4) *a certain hyperplane containing a unique deep point.*

By Pralle [23, Table 1], the dual polar space $DW(5,2) = DW(5,\mathbb{F}_2)$ has eight isomorphism classes of hyperplanes containing deep quads. So, the conclusion of the Main Theorem is false if $|\mathbb{K}| = 2$. The hyperplanes occurring in (1), (2) and (4) of the Main Theorem are classical and unique, up to isomorphism. If Q is a quad of $DW(5,\mathbb{K})$ and O_1 and O_2 are two ovoids of $\widetilde{Q} \cong Q(4,\mathbb{K})$, then the extensions of O_1 and O_2 are isomorphic if and only if the ovoids O_1 and O_2 of $\widetilde{Q}$ are isomorphic. The hyperplanes occurring in (3) of the Main Theorem are therefore not necessarily unique (up to isomorphism), since $Q(4,\mathbb{K})$ can have nonisomorphic ovoids. If $|\mathbb{K}|$ is infinite, then $Q(4,\mathbb{K})$ can even have nonisomorphic classical ovoids, see e.g. Proposition 6.3 of Section 6. For infinite fields $\mathbb{K}$, it is also possible to construct many non-classical ovoids of $Q(4,\mathbb{K})$ by means of transfinite recursion, see Cardinali & De Bruyn [7, Section 4]. The situation for finite fields is completely different. For several prime powers q, it is known that all ovoids of $Q(4,q)=Q(4,\mathbb{F}_q)$ are classical:

Proposition 1.1 • **([2, 20])** *Every ovoid of* $Q(4,4)$ *is classical.*

- **([18, 19])** *Every ovoid of* $Q(4,16)$ *is classical.*
- **([1])** *Every ovoid of* $Q(4,q)$*,* q *prime, is classical.*

For many values of q however, non-classical ovoids of $Q(4,q)$ do exist: (i) $q=p^h$ with p an odd prime and $h\geq 2$ [17]; (ii) $q=2^{2h+1}$ with $h\geq 1$ [27]; (iii) $q=3^{2h+1}$ with $h\geq 1$ [17]; (iv) $q=3^h$ with $h\geq 3$ [26]; (v) $q=3^5$ [22].

The organization of the paper is as follows. The Main Theorem will be proved in Section 4. In Section 2, we define three classes of point-line geometries and provide information about their generation which will be used in the proof of the Main Theorem. In Section 3, we discuss the notions of a pencil of hyperplanes of $Q(4,\mathbb{K})$ and a hyperbolic set of quads of $DW(5,\mathbb{K})$. These two concepts will also play a role in the proof of the Main Theorem. In Section 5, we derive some structural information on those hyperplanes of $DW(5,\mathbb{K})$, $|\mathbb{K}|\neq 2$, that admit deep and ovoidal quads. Finally in Section 6, we discuss the isomorphism problem for classical hyperplanes of $Q(4,\mathbb{K})$.

2 Three classes of point-line geometries

Consider in $\mathrm{PG}(3,\mathbb{K})$ a symplectic polarity ζ. There are two types of lines in $\mathrm{PG}(3,\mathbb{K})$, lines that are totally isotropic (with respect to ζ) and hyperbolic lines. Let L denote a totally isotropic line of $\mathrm{PG}(3,\mathbb{K})$ and let $L_1, L_2=L_1^\zeta$ denote two orthogonal hyperbolic lines. We define the following three point-line geometries $\mathcal{S}_1(\mathbb{K})$, $\mathcal{S}_2(\mathbb{K})$ and $\mathcal{S}_3(\mathbb{K})$.

- The points of $\mathcal{S}_1(\mathbb{K})$ are the points of $\mathrm{PG}(3,\mathbb{K})$ and the lines of $\mathcal{S}_1(\mathbb{K})$ are the hyperbolic lines of $\mathrm{PG}(3,\mathbb{K})$, with incidence being containment.
- The points of $\mathcal{S}_2(\mathbb{K})$ are the points of $\mathrm{PG}(3,\mathbb{K})\setminus L$ and the lines of $\mathcal{S}_2(\mathbb{K})$ are the hyperbolic lines of $\mathrm{PG}(3,\mathbb{K})$ disjoint from L, with incidence being containment.
- The points of $\mathcal{S}_3(\mathbb{K})$ are the points of $\mathrm{PG}(3,\mathbb{K})\setminus(L_1\cup L_2)$ and the lines of $\mathcal{S}_3(\mathbb{K})$ are the hyperbolic lines of $\mathrm{PG}(3,\mathbb{K})$ disjoint from $L_1\cup L_2$, with incidence being containment.

Observe that $\mathcal{S}_2(\mathbb{K})$ and $\mathcal{S}_3(\mathbb{K})$ are full subgeometries of $\mathcal{S}_1(\mathbb{K})$. The point-line geometry $\mathcal{S}_1(\mathbb{K})$ is called the *geometry of the hyperbolic lines* of the symplectic polar space $W(3,\mathbb{K})$ associated with ζ.

If $x_1, x_2, \dots, x_k$ are $k \geq 1$ points of $\mathcal{S}_i(\mathbb{K})$, $i \in \{1,2,3\}$, then $[x_1, x_2, \dots, x_k]_i$ denotes the smallest subspace of $\mathcal{S}_i(\mathbb{K})$ containing $\{x_1, x_2, \dots, x_k\}$. If $[x_1, x_2, \dots, x_k]_i$ coincides with the whole point set of $\mathcal{S}_i(\mathbb{K})$, then we say that $\mathcal{S}_i(\mathbb{K})$ is *generated* by the points $x_1, x_2, \dots, x_k$.

For a proof of the following lemma, see Cooperstein [9, Lemma 2.3] (finite case) and De Bruyn & Pasini [16, Lemma 2.2] (general case).

Lemma 2.1 ([9, 16]) *If $|\mathbb{K}| \geq 3$, then $\mathcal{S}_1(\mathbb{K})$ can be generated by four points.*

Lemma 2.2 *If $|\mathbb{K}| \geq 4$, then $\mathcal{S}_2(\mathbb{K})$ can be generated by four points.*

Proof. Let p be a point of $\mathrm{PG}(3,\mathbb{K})$ not belonging to L and put $\alpha := p^\zeta$. Put $\{p'\} = \alpha \cap L$. Let x_1, x_2 and x_3 be three noncollinear points of α such that x_1x_2 and x_1x_3 meet the line pp' in points distinct from p and p'. Since x_1x_2 and x_1x_3 are hyperbolic lines, we have $x_1x_2 \cup x_1x_3 \subseteq [x_1, x_2, x_3]_2$. Through every point x of α distinct from x_1, p and p', there exists a line distinct from xx_1, xp and xp'. Since this line is a hyperbolic line disjoint from L which contains two distinct points of $x_1x_2 \cup x_1x_3 \subseteq [x_1, x_2, x_3]_2$, it is completely contained in $[x_1, x_2, x_3]_2$. In particular, we have $x \in [x_1, x_2, x_3]_2$. It follows that $\alpha \setminus \{p, p'\} \subseteq [x_1, x_2, x_3]_2$.

Now, let p'' be an arbitrary point of pp' distinct from p and p'. Then $\alpha' := (p'')^\zeta$ is a plane through pp' distinct from α. Let x_4 be an arbitrary point of $\alpha' \setminus pp'$. In the plane α', there are two distinct hyperbolic lines M_1 and M_2 through x_4 not meeting $\{p, p', p''\}$. Each of these lines contains two points of $[x_1, x_2, x_3, x_4]_2$ and hence is contained in $[x_1, x_2, x_3, x_4]_2$. Every point x of α' distinct from x_4, p' and p'' is contained in a line distinct from xx_4, xp' and xp''. Since this line is a hyperbolic line disjoint from L containing two distinct points of $M_1 \cup M_2 \subseteq [x_1, x_2, x_3, x_4]_2$, it is completely contained in $[x_1, x_2, x_3, x_4]_2$. In particular, we have $x \in [x_1, x_2, x_3, x_4]_2$. It follows that $\alpha' \setminus \{p', p''\} \subseteq [x_1, x_2, x_3, x_4]_2$.

Since $\alpha \setminus \{p, p'\} \subseteq [x_1, x_2, x_3]_2 \subseteq [x_1, x_2, x_3, x_4]_2$ and $\alpha' \setminus \{p', p''\} \subseteq [x_1, x_2, x_3, x_4]_2$, we have $(\alpha \cup \alpha') \setminus \{p'\} \subseteq [x_1, x_2, x_3, x_4]_2$. Now, let x be an arbitrary point of $\mathrm{PG}(3,\mathbb{K})$ not contained in $L \cup \alpha \cup \alpha'$. If M is a line through x not contained in $x^\zeta \cup \langle x, L\rangle \cup \langle x, pp'\rangle$, then M is a hyperbolic line disjoint from L containing at least two points of $[x_1, x_2, x_3, x_4]_2$, namely the points in $M \cap \alpha$ and $M \cap \alpha'$. It follows that $x \in [x_1, x_2, x_3, x_4]_2$. So, $[x_1, x_2, x_3, x_4]_2$ coincides with the whole point set of $\mathcal{S}_2(\mathbb{K})$. $\square$

Lemma 2.3 *If $|\mathbb{K}| \geq 5$, then $\mathcal{S}_3(\mathbb{K})$ can be generated by four points.*

Proof. Let x_1 be an arbitrary point of $\mathcal{S}_3(\mathbb{K})$ and let L_{x_1} denote the unique (totally isotropic) line through x_1 meeting L_1 and L_2. Put $\{u_1\} = L_{x_1} \cap L_1$ and $\{u_2\} = L_{x_1} \cap L_2$. Let α be a plane of $\mathrm{PG}(3,\mathbb{K})$ through L_{x_1} distinct from $u_2^\zeta = \langle x_1, L_1\rangle$, $u_1^\zeta = \langle x_1, L_2\rangle$ and x_1^ζ. Then $\alpha = u_3^\zeta$ for some point $u_3 \in L_{x_1} \setminus \{u_1, u_2, x_1\}$. Now, let x_2 and x_3 be two distinct points of $\alpha \setminus L_{x_1}$ such that x_2x_3 meets L_{x_1} in a point different from x_1, u_1, u_2 and u_3. Since x_2x_3 is a line of $\mathcal{S}_3(\mathbb{K})$, every point of x_2x_3 belongs to $[x_1, x_2, x_3]_3$. Now, considering (hyperbolic) lines through x_1 contained in α, we see that every point of $\alpha \setminus L_{x_1}$ is contained in $[x_1, x_2, x_3]_3$. Now, let y denote an arbitrary point of $L_{x_1} \setminus \{u_1, u_2, u_3\}$. By considering a (hyperbolic) line of α through y distinct from L_{x_1}, we see that $y \in [x_1, x_2, x_3]_3$. Hence, $L_{x_1} \setminus \{u_1, u_2, u_3\} \subseteq [x_1, x_2, x_3]_3$.

Now, let α' be a plane of $\mathrm{PG}(3, \mathbb{K})$ through L_{x_1} distinct from $u_2^\zeta = \langle L_{x_1}, L_1 \rangle$, $u_1^\zeta = \langle L_{x_1}, L_2 \rangle$ and $\alpha = u_3^\zeta$. Then $\alpha' = u_4^\zeta$ for some point $u_4 \in L_{x_1} \setminus \{u_1, u_2, u_3\}$. Now, let x_4 denote an arbitrary point of $\alpha' \setminus L_{x_1}$ and let v_1 and v_2 be two distinct points of $L_{x_1} \setminus \{u_1, u_2, u_3, u_4\}$. Such points exist since $|\mathbb{K}| \geq 5$. We will show that $[x_1, x_2, x_3, x_4]_3$ coincides with the whole point set of $\mathcal{S}_3(\mathbb{K})$. Considering the hyperbolic line v_1x_4, we see that $v_1x_4 \subseteq [x_1, x_2, x_3, x_4]_3$. By considering hyperbolic lines through v_2, we now see that $\alpha' \setminus L_{x_1} \subseteq [x_1, x_2, x_3, x_4]_3$. By considering a hyperbolic line through u_3 contained in α', we see that also $u_3 \in [x_1, x_2, x_3, x_4]_3$. Hence, $\alpha \cup \alpha' \setminus \{u_1, u_2\} \subseteq [x_1, x_2, x_3, x_4]_3$. Now, let y be an arbitrary point of $\mathcal{S}_3(\mathbb{K})$ not contained in $\alpha \cup \alpha'$. Let L be a line through y not contained in $\langle y, L_1 \rangle \cup \langle y, L_2 \rangle \cup \langle y, L_{x_1} \rangle \cup y^\zeta$. Then L is a line of $\mathcal{S}_3(\mathbb{K})$ containing two distinct points of $[x_1, x_2, x_3, x_4]_3$, namely the unique points in $L \cap \alpha$ and $L \cap \alpha'$. This implies that $y \in [x_1, x_2, x_3, x_4]_3$. This shows that $\{x_1, x_2, x_3, x_4\}$ generates $\mathcal{S}_3(\mathbb{K})$. □

Lemma 2.4 *If $\{x_1, x_2, x_3, x_4\}$ is a generating set of size four of $\mathcal{S}_i(\mathbb{K})$ ($i \in \{2, 3\}$), then $\{x_1, x_2, x_3, x_4\}$ is also a generating set of points of $\mathcal{S}_1(\mathbb{K})$.*

Proof. Obviously, the point set $[x_1, x_2, x_3, x_4]_i$ of $\mathcal{S}_i(\mathbb{K})$ is contained in $[x_1, x_2, x_3, x_4]_1$. Let x be an arbitrary point of $\mathcal{S}_1(\mathbb{K})$ that is not a point of $\mathcal{S}_i(\mathbb{K})$. Then $x \in L$ if $i = 2$ and $x \in L_1 \cup L_2$ if $i = 3$. Let M denote an arbitrary hyperbolic line through x distinct from L_1 and L_2 if $i = 3$. Since every point of $M \setminus \{x\}$ is a point of $[x_1, x_2, x_3, x_4]_i \subseteq [x_1, x_2, x_3, x_4]_1$, we also have $x \in [x_1, x_2, x_3, x_4]_1$. □

3 Some basic properties

The lines and quads of the dual polar space $DW(5, \mathbb{K})$ through a given point x define a linear space $Res(x) \cong \mathrm{PG}(2, \mathbb{K})$. If H is a hyperplane of $DW(5, \mathbb{K})$ and if $x \in H$, then $\Lambda_H(x)$ denotes the set of lines through x contained in H. We will regard $\Lambda_H(x)$ as a set of points of $Res(x) \cong \mathrm{PG}(2, \mathbb{K})$. If $\Lambda_H(x)$ is the whole set of points of $Res(x)$, then x is called *deep* (*with respect to* H).

If H is a hyperplane of $DW(5, \mathbb{K})$ and Q is a quad, then either $Q \subseteq H$ or $Q \cap H$ is a hyperplane of $\widetilde{Q} \cong Q(4, \mathbb{K})$. If $Q \subseteq H$, then Q is called a *deep quad.* If $Q \cap H = x^\perp \cap Q$ for some point $x \in Q$, then Q is called *singular* (*with respect to* H) and x is called the *deep point* of Q. The quad Q is called *ovoidal* (respectively, *subquadrangular*) *with respect to* H if $Q \cap H$ is an ovoid (respectively, a full subgrid) of $\widetilde{Q}$.

A set $\mathcal{G}$ of hyperplanes of $Q(4, \mathbb{K})$ is called a *pencil of hyperplanes* if every point of $Q(4, \mathbb{K})$ is contained in either 1 or all elements of $\mathcal{G}$. If $\mathcal{G}$ is a pencil of hyperplanes of $Q(4, \mathbb{K})$, then $\bigcup_{G \in \mathcal{G}} G$ coincides with the whole point set of $Q(4, \mathbb{K})$ and $G_1 \cap G_2 = G_1 \cap G_3 = G_2 \cap G_3$ for any three distinct hyperplanes G_1, G_2 and G_3 of $\mathcal{G}$.

Lemma 3.1 *Let G_1 and G_2 be two distinct hyperplanes of $Q(4, \mathbb{K})$ containing a line L. Then through every point x not contained in $G_1 \cup G_2$, there exists a unique hyperplane G_x*

such that $G_1 \cap G_2 = G_1 \cap G_x = G_2 \cap G_x$. As a consequence, G_1 and G_2 are contained in a unique pencil $\mathcal{G}$ of hyperplanes of $Q(4, \mathbb{K})$. If G_1 and G_2 are two singular hyperplanes, then every hyperplane of $\mathcal{G}$ is also singular. If at least one of G_1, G_2 is not singular, then precisely one hyperplane of $\mathcal{G}$ is singular.

Proof. Since G_i, $i \in \{1, 2\}$, contains L, it is either a singular hyperplane or a full subgrid. If G is a hyperplane of $Q(4, \mathbb{K})$ satisfying $G_1 \cap G_2 = G_1 \cap G = G_2 \cap G$, then G contains L and hence G is also a singular hyperplane or a full subgrid. We distinguish two cases.

(1) Suppose G_1 and G_2 are singular hyperplanes. Then the deepest points of G_1 and G_2 lie on L and $G_1 \cap G_2 = L$. Any hyperplane G satisfying $G_1 \cap G_2 = G_1 \cap G = G_2 \cap G$ necessarily is a singular hyperplane. So, G_x must be the singular hyperplane whose deepest point is the unique point of L collinear with x.

(2) Suppose at least one of G_1, G_2 is a full subgrid. Then $G_1 \cap G_2$ is the union of two lines L and L'. Put $\{u\} = L \cap L'$. If $x \sim u$, then G_x must be the singular hyperplane of $Q(4, \mathbb{K})$ with deepest point u. If $x \not\sim u$, then G_x must be the unique full subgrid of $Q(4, \mathbb{K})$ containing $L \cup L' \cup \{x\}$. □

As in Section 1, let $W(5, \mathbb{K})$ be the polar space associated with the dual polar space $DW(5, \mathbb{K})$. If L is a hyperbolic line of PG$(5, \mathbb{K})$ (i.e. a line of PG$(5, \mathbb{K})$ that is not a line of $W(5, \mathbb{K})$), then the set $\mathcal{Q}_L$ of the (mutually disjoint) quads of $DW(5, \mathbb{K})$ corresponding to the points of L satisfy the following property: any line M meeting two distinct quads of $\mathcal{Q}_L$ meets every quad of $\mathcal{Q}_L$ in a unique point. Moreover, the quads of $\mathcal{Q}_L$ cover all the points of M. The set $\mathcal{Q}_L$ is called a *hyperbolic set of quads* of $DW(5, \mathbb{K})$. Every two disjoint quads Q_1 and Q_2 of $DW(5, \mathbb{K})$ are contained in a unique hyperbolic set of quads which we denote by $\mathcal{H}(Q_1, Q_2)$.

Let $\mathcal{H}$ be a hyperbolic set of quads of $DW(5, \mathbb{K})$. Let $\mathcal{P}_\mathcal{H}$ denote the set of all quads of $DW(5, \mathbb{K})$ that meet each quad of $\mathcal{H}$ (necessarily in a line). If R_1 and R_2 are two disjoint elements of $\mathcal{P}_\mathcal{H}$, then $\mathcal{H}(R_1, R_2) \subseteq \mathcal{P}_\mathcal{H}$. Put $\mathcal{L}_\mathcal{H} := \{\mathcal{H}(R_1, R_2) \,|\, R_1, R_2 \in \mathcal{P}_\mathcal{H}$ and $R_1 \cap R_2 = \emptyset\}$ and let $\mathcal{S}_\mathcal{H}$ be the point-line geometry with point set $\mathcal{P}_\mathcal{H}$, line-set $\mathcal{L}_\mathcal{H}$ and natural incidence. The following lemma is not so hard to prove, see e.g. Lemmas 3.10, 3.11 and 3.5 of De Bruyn [15] where these claims have been proved in the finite case, but the proofs naturally extend to the infinite case.

Lemma 3.2 (1) *For every hyperbolic set $\mathcal{H}$ of quads of $DW(5, \mathbb{K})$, $\mathcal{S}_\mathcal{H}$ is isomorphic to $\mathcal{S}_1(\mathbb{K})$.*

(2) *If $\mathcal{H}$ is a hyperbolic set of quads of $DW(5, \mathbb{K})$, then $\bigcup_{Q \in \mathcal{P}_\mathcal{H}} Q$ is the whole point set of $DW(5, \mathbb{K})$. Moreover, every point of $DW(5, \mathbb{K})$ not contained in $\bigcup_{Q \in \mathcal{H}} Q$ is contained in a unique element of $\mathcal{P}_\mathcal{H}$.*

(3) *Let $\mathcal{H}$ be a hyperbolic set of quads of $DW(5, \mathbb{K})$ and let H be a hyperplane of $DW(5, \mathbb{K})$ such that $H \cap Q_1$ and $\pi_{Q_1}(H \cap Q_2)$ are distinct hyperplanes of $\widetilde{Q_1}$. Then $\{\pi_{Q_1}(H \cap Q) \,|\, Q \in \mathcal{H}\}$ is a pencil of hyperplanes of $\widetilde{Q_1}$.*

The dual polar space $DW(5, \mathbb{K})$ admits a nice full projective embedding e in PG$(13, \mathbb{K})$ which is called the *Grassmann embedding of $DW(5, \mathbb{K})$*, see e.g. Cooperstein [9, Proposition 5.1]. It is straightforward to verify (see e.g. Cardinali, De Bruyn and Pasini [8, Proposition 4.10]) that if Q is a quad of $DW(5,\mathbb{K})$, then the embedding e_Q of $\widetilde{Q} \cong Q(4,\mathbb{K})$ induced by e is isomorphic to the natural embedding of $\widetilde{Q} \cong Q(4,\mathbb{K})$ in $\mathrm{PG}(4,\mathbb{K})$. So, $\langle e(Q) \rangle$ is 4-dimensional. It is also straightforward to verify (see e.g. the proof of Lemma 4.3 of Blok, Cardinali & De Bruyn [5]) that if Q_1 and Q_2 are two disjoint quads of $DW(5,\mathbb{K})$, then $\langle e(Q_1) \rangle$ and $\langle e(Q_2) \rangle$ are two disjoint subspaces of $\mathrm{PG}(13,\mathbb{K})$.

Lemma 3.3 *Suppose $|\mathbb{K}| \geq 3$. Let Q_1 and Q_2 be two disjoint quads of $DW(5,\mathbb{K})$, let $x_1, x_2, \ldots, x_5$ be five points of Q_1 such that $\langle e(x_1), e(x_2), \ldots, e(x_5) \rangle = \langle e(Q_1) \rangle$ and let $y_1, y_2, \ldots, y_5$ be five points of Q_2 such that $\langle e(y_1), e(y_2), \ldots, e(y_5) \rangle = \langle e(Q_2) \rangle$. Put $\mathcal{H} := \mathcal{H}(Q_1, Q_2)$ and let R_1, R_2, R_3, R_4 be four quads of $\mathcal{P}_{\mathcal{H}}$ forming a generating set of size four of $\mathcal{S}_{\mathcal{H}} \cong \mathcal{S}_1(\mathbb{K})$. For every $i \in \{1,2,3,4\}$, let z_i be a point of $R_i \setminus \bigcup_{Q \in \mathcal{H}} Q$. Then the 14 points $e(x_1), e(x_2), \ldots, e(x_5), e(y_1), e(y_2), \ldots, e(y_5), e(z_1), e(z_2), e(z_3), e(z_4)$ form a basis of $\mathrm{PG}(13,\mathbb{K})$.*

Proof. It suffices to prove that the subspace Σ generated by these fourteen points coincides with $\mathrm{PG}(13,\mathbb{K})$. Put $X := e^{-1}(e(\mathcal{P}) \cap \Sigma)$, where $\mathcal{P}$ is the point set of $DW(5,\mathbb{K})$. Then X is a subspace of $DW(5,\mathbb{K})$ containing Q_1, Q_2 and $\{z_1, z_2, z_3, z_4\}$. If S_1 and S_2 are two disjoint quads of $DW(5,\mathbb{K})$ and $S \in \mathcal{H}(S_1, S_2)$, then the points of S are covered by the lines meeting S_1 and S_2. This implies the following:

> $(*)$ If S_1 and S_2 are two disjoint quads contained in X, then also every $S \in \mathcal{H}(S_1, S_2)$ is contained in X.

By Property $(*)$, every $Q \in \mathcal{H} = \mathcal{H}(Q_1, Q_2)$ is contained in X. Now, for every $i \in \{1,2,3,4\}$, let G_i be the set of points of R_i contained in a quad of $\mathcal{H}$. Then G_i is a full subgrid of $\widetilde{R_i}$ which is contained in X. Since also the point $z_i \in R_i \setminus G_i$ belongs to X, we have $R_i \subseteq X$. Since $\{R_1, R_2, R_3, R_4\}$ is a generating set of the point-line geometry $\mathcal{S}_{\mathcal{H}}$, every point of $\mathcal{S}_{\mathcal{H}}$ is contained in X by Property $(*)$. Lemma 3.2(2) now implies that X coincides with the whole point set of $DW(5,\mathbb{K})$. So, $\Sigma = \mathrm{PG}(13,\mathbb{K})$. □

4 Proof of the Main Theorem

In this section, we prove the Main Theorem. We will achieve this goal in three subsections. In Section 4.1, we prove that every non-classical hyperplane of $DW(5,\mathbb{K})$ is the extension of a non-classical ovoid of a quad of $DW(5,\mathbb{K})$. In Section 4.2, we use this result to prove that every hyperplane of $DW(5,\mathbb{K})$, $|\mathbb{K}| \neq 2$, that contains a quad has a deep point. In Section 4.3, we will employ the existence of a deep point to obtain our desired classification. The discussion in Section 4.3 is based on Section 6 of the paper [12].

4.1 The non-classical hyperplanes containing a deep quad

We suppose that H is a hyperplane of $DW(5,\mathbb{K})$, $|\mathbb{K}| \neq 2$, containing a quad Q.

Lemma 4.1 *If every quad disjoint from Q is either deep or ovoidal with respect to H, then H is the extension of an ovoid of $\widetilde{Q}$.*

Proof. (1) We prove that every line L disjoint from Q meets H in precisely one point. Suppose to the contrary that $L \subseteq H$ and let Q' be an arbitrary quad through L disjoint from Q. Then $Q' \subseteq H$. Let x be an arbitrary point of Q'. We prove that $x^\perp \subseteq H$, or equivalently, that every line M through x is contained in H. If M is the unique line through x meeting Q, then $M \subseteq H$ since M contains two points of H, namely the points in $M \cap Q$ and $M \cap Q'$. If M does not meet Q, then M is contained in a quad Q'' disjoint from Q. Since $Q'' \cap Q' \subseteq H$ is at least a line, we have $Q'' \subseteq H$. In particular, $M \subseteq H$.

So, we have that $x^\perp \subseteq H$ for every $x \in Q'$. This would imply that H is the whole point set of $DW(5, \mathbb{K})$, an obvious contradiction.

(2) By (1), no quad is subquadrangular with respect to H. We prove that if a point $x \in Q$ is contained in a line $L \subseteq H$ not contained in Q, then $x^\perp \subseteq H$. Let M denote an arbitrary line through x distinct from L. Then the quad $\langle L, M \rangle$ is either deep or singular with respect to H. Since L and $\langle L, M \rangle \cap Q$ are contained in H, every line of $\langle L, M \rangle$ through x is contained in H. In particular, $M \subseteq H$. Hence, $x^\perp \subseteq H$.

(3) Let X denote the set of all points $x \in Q$ for which $x^\perp \subseteq H$. Then $H = Q \cup \left(\bigcup_{x \in X} x^\perp \right)$ by (2). Since every quad disjoint from Q is ovoidal with respect to H (by (1)), X must be an ovoid of $\widetilde{Q}$. So, H is the extension of an ovoid of $\widetilde{Q}$. □

Lemma 4.2 *If $3 \leq |\mathbb{K}| \leq 7$, then H arises from the Grassmann embedding of $DW(5, \mathbb{K})$.*

Proof. In De Bruyn [15], it was proved that if H is a hyperplane of $DW(5, q)$, $q \neq 2$, such that $Q \cap H$ is a classical ovoid of $\widetilde{Q}$ for every ovoidal quad Q, then H arises from the Grassmann embedding of $DW(5, q)$. The lemma follows from Proposition 1.1 and this fact. □

Lemma 4.3 *If there exists a quad Q' disjoint from Q such that $Q' \cap H$ is a classical hyperplane of $\widetilde{Q'} \cong Q(4, \mathbb{K})$, then H arises from the Grassmann embedding of $DW(5, \mathbb{K})$.*

Proof. By Lemma 4.2, we may assume that $|\mathbb{K}| \geq 8$. Let $e : DW(5, \mathbb{K}) \to \mathrm{PG}(13, \mathbb{K})$ denote the Grassmann embedding of $DW(5, \mathbb{K})$. Put $\mathcal{H} := \mathcal{H}(Q, Q')$. Let X denote the set of all quads $R \in \mathcal{P}_\mathcal{H}$ for which $R \cap Q'$ is a line of Q' not contained in H. Let $\mathcal{Y}$ denote the set of all sets $\mathcal{H}(R, R')$ where R and R' are two disjoint quads of X such that $\mathcal{H}(R, R') \subseteq X$. Let $\mathcal{S}$ be the point-line geometry with point set X and line set $\mathcal{Y}$, with incidence being containment. Then $\mathcal{S} \cong \mathcal{S}_1(\mathbb{K})$ if Q' is an ovoidal quad, $\mathcal{S} \cong \mathcal{S}_2(\mathbb{K})$ if Q' is a singular quad and $\mathcal{S} \cong \mathcal{S}_3(\mathbb{K})$ if Q' is a subquadrangular quad. By Lemmas 2.1, 2.2 and 2.3, the geometry $\mathcal{S}$ has a generating set $\{R_1, R_2, R_3, R_4\}$ of size 4. For every $i \in \{1, 2, 3, 4\}$, let x_i be a point of $R_i \cap H$ not contained in the subgrid $R_i \cap \left(\bigcup_{S \in \mathcal{H}} S \right)$ of $\widetilde{R_i}$. In order to establish the lemma, it suffices to prove the following two claims.

(a) There is a hyperplane H^* arising from e satisfying the following properties: • $Q \subseteq H^*$; • $Q' \cap H^* = Q' \cap H$; • $x_1, x_2, x_3, x_4 \in H^*$.

(b) There is at most one hyperplane H' of $DW(5,\mathbb{K})$ satisfying the following properties: (i) $Q \subseteq H'$; (ii) $Q' \cap H = Q' \cap H'$; (iii) $x_1, x_2, x_3, x_4 \in H'$.

If (a) and (b) are valid, then we can conclude that $H = H^*$ arises from the Grassmann embedding e.

We first prove (a). By Lemmas 2.4 and 3.3, $\mathrm{PG}(13,\mathbb{K}) = \langle e(Q), e(Q'), e(x_1), e(x_2), e(x_3), e(x_4)\rangle$. We also know that $\langle e(Q)\rangle$ and $\langle e(Q')\rangle$ are two disjoint four-dimensional subspaces. The subspace $\langle e(Q' \cap H)\rangle$ is a hyperplane of $\langle e(Q')\rangle$ and hence $\Pi := \langle e(Q), e(Q' \cap H), e(x_1), e(x_2), e(x_3), e(x_4)\rangle$ is a hyperplane of $\mathrm{PG}(13,\mathbb{K})$. If we put $H^* := e^{-1}(e(\mathcal{P}) \cap \Pi)$, where $\mathcal{P}$ is the point set of $DW(5,\mathbb{K})$, then H^* satisfies the required properties.

We next prove (b). We will achieve this goal in a number of steps.

STEP 1: *If H' is a hyperplane satisfying conditions (i), (ii) and (iii) above, then $H' \cap R_i$, $i \in \{1,2,3,4\}$, is uniquely determined.*
PROOF. Put $L' := R_i \cap Q'$, $L := R_i \cap Q$, let u' be the unique point in $L' \cap H$ and let u be the unique point of L collinear with u'. If x_i is collinear with u, then we necessarily have $H' \cap R_i = u^\perp \cap R_i$. If x_i is not collinear with u, then $H' \cap R_i$ is the unique full subgrid of $\widetilde{R_i}$ containing $uu' \cup L$ and x_i. (QED)

STEP 2: *Suppose H' is a hyperplane satisfying conditions (i), (ii) and (iii) above. Let R and R' be two disjoint quads of X such that $\mathcal{H}(R,R') \subseteq X$. Then for every $R'' \in \mathcal{H}(R,R') \setminus \{R,R'\}$, $H' \cap R''$ is uniquely determined by $H' \cap R$ and $H' \cap R'$.*
PROOF. Put $L := R \cap Q'$, $L' := R' \cap Q'$ and $L'' := R'' \cap Q'$. Put $\{u\} = L \cap H'$, $\{u'\} := L' \cap H'$ and $\{u''\} := L'' \cap H'$. The fact that $\mathcal{H}(R,R') \subseteq X$ implies that the points u, u' and u'' are mutually noncollinear. So, $\pi_{R''}(R \cap H')$ and $\pi_{R''}(R' \cap H')$ are two distinct hyperplanes of $\widetilde{R''}$ containing the line $\pi_Q(L'') = R'' \cap Q$. By Lemma 3.1, there exists a unique hyperplane G of $\widetilde{R''}$ such that $u'' \in G$ and $G \cap \pi_{R''}(R \cap H') = G \cap \pi_{R''}(R' \cap H') = \pi_{R''}(R \cap H') \cap \pi_{R''}(R' \cap H')$. By Lemma 3.2(3), we necessarily have $H' \cap R'' = G$. (QED)

STEP 3: *If H' is a hyperplane satisfying conditions (i), (ii) and (iii) above, then $H' \cap R$ is uniquely determined for every $R \in X$.*
PROOF. This follows from Steps 1 and 2 and the fact that $\{R_1, R_2, R_3, R_4\}$ is a generating set of the geometry $\mathcal{S}$. (QED)

STEP 4: *Suppose H' is a hyperplane satisfying conditions (i), (ii) and (iii) above. If $R \in \mathcal{P}_{\mathcal{H}} \setminus X$, then $R \cap H'$ is uniquely determined.*
PROOF. Consider an element of $\mathcal{H}' \in \mathcal{L}_{\mathcal{H}}$ containing R such that $\mathcal{H}' \setminus \{R\} \subseteq X$. If L is a line meeting each quad of $\mathcal{H}'$, then $L \cap R \cap H'$ is uniquely determined by the intersections $L \cap R' \cap H'$, $R' \in \mathcal{H}' \setminus \{R\}$. If all of $L \cap R' \cap H'$, $R' \in \mathcal{H}' \setminus \{R\}$, are singletons, then $L \cap R \cap H' = L \cap R$. If $L \cap R' \cap H'$ is empty for every $R' \in \mathcal{H}' \setminus \{R\}$, then $L \cap R \cap H' = L \cap R$. If precisely one of $L \cap R' \cap H'$, $R' \in \mathcal{H}' \setminus \{R\}$, is a singleton, then $L \cap R' \cap H' = \emptyset$. (QED)

STEP 5. *There is at most one hyperplane H' satisfying conditions (i), (ii) and (iii) above.*
PROOF. This follows from Steps 3 and 4, and the fact that the quads of $\mathcal{P}_{\mathcal{H}}$ cover all the points of $DW(5,\mathbb{K})$. (QED) □

Lemma 4.4 *The extension H of a classical ovoid O of a quad Q of $DW(5,\mathbb{K})$ arises from the Grassmann embedding of $DW(5,\mathbb{K})$.*

Proof. Take a quad Q' disjoint from Q. Since the map $Q \to Q'; x \mapsto \pi_{Q'}(x)$ defines an isomorphism between $\widetilde{Q}$ and $\widetilde{Q'}$, the set $Q' \cap H = \pi_{Q'}(O)$ necessarily is a classical ovoid of $\widetilde{Q'}$. Lemma 4.3 now implies that H arises from the Grassmann embedding of $DW(5,\mathbb{K})$. $\square$

By Pralle [23] or De Bruyn [12], Lemma 4.4 remains valid if $|\mathbb{K}| = 2$. We proved Lemma 4.4 with the aid of Lemma 4.3. Lemma 4.4 can also be proved with the techniques exposed in Section 3 of De Bruyn [13]. In fact, Lemma 4.4 is precisely Lemma 3.7 of [13] (where the field was supposed to be perfect and of characteristic 2). Lemma 4.4 is also a special case of Theorem 1.2(3) of De Bruyn [14]. However, the machinery necessary to prove Theorem 1.2(3) of [14] is more advanced than the one used in [13] or the present paper.

Since full subgrids and singular hyperplanes of $Q(4,\mathbb{K})$ are classical hyperplanes, Lemmas 4.1, 4.3 and 4.4 imply the following.

Corollary 4.5 *If H is a non-classical hyperplane of $DW(5,\mathbb{K})$ containing a quad, then H is the extension of a non-classical ovoid of a quad.*

Observe that the extension of a non-classical ovoid of a quad of $DW(5,\mathbb{K})$ cannot arise from a projective embedding. Since every hyperplane of $DW(5,2)$ is classical, Corollary 4.5 remains valid if $|\mathbb{K}| = 2$.

4.2 The existence of deep points

In this section, we make use of the following lemma, a proof of which can be found in Pasini [21, Theorem 9.3] and Cardinali & De Bruyn [7, Corollary 1.5].

Lemma 4.6 *Let H be a hyperplane of $DW(5,\mathbb{K})$ arising from the Grassmann embedding and let $x \in H$. Then $\Lambda_H(x)$ is a possibly singular quadric of $Res(x) \cong \mathrm{PG}(2,\mathbb{K})$.*

Observe that in Lemma 4.6, the whole point set of $Res(x)$ should be regarded as a singular quadric.

As before, let Q be a quad of $DW(5,\mathbb{K})$, $|\mathbb{K}| \neq 2$, contained in H. We prove that there exists a point $x \in Q$ for which $x^\perp \subseteq H$. Suppose that this is not the case. Then by Lemmas 4.1, 4.3 and 4.6, for every point $x \in Q$, $\Lambda_H(x)$ is either a line of $Res(x)$ or the union of two distinct lines of $Res(x)$. We distinguish the following two cases and derive a contradiction in each of them.

- There exists a point $x^* \in Q$ such that $\Lambda_H(x^*)$ is a line of $Res(x^*)$.
- For every point $x \in Q$, $\Lambda_H(x)$ is the union of two lines of $Res(x)$.

Case I: There exists a point x^* of Q such that $\Lambda_H(x^*)$ is a line of $Res(x^*)$

Lemma 4.7 (1) *If R is a quad that intersects Q in a line through x^*, then R is singular with deepest point in $(R \cap Q) \setminus \{x^*\}$.*

(2) *If L is a line of Q through x^* and if R_1 and R_2 are two distinct quads intersecting Q in L, then the deepest points of $R_1 \cap H$ and $R_2 \cap H$ are distinct.*

Proof. (1) This follows from the fact that $(x^*)^\perp \cap R \cap H = R \cap Q$.

(2) Suppose $y \in L \setminus \{x^*\}$ is the deepest point of $R_1 \cap H$ and $R_2 \cap H$. Then $\Lambda_H(y)$ contains three lines of $Res(y)$, which is impossible. □

Lemma 4.8 *Every point $y \in ((x^*)^\perp \cap Q) \setminus \{x^*\}$ is the deepest point of a unique singular quad through x^*y. For every point $y \in ((x^*)^\perp \cap Q) \setminus \{x^*\}$, $\Lambda_H(y)$ is the union of two lines of $Res(y)$.*

Proof. Let R_1 and R_2 be two distinct (singular) quads through x^*y and let y_i, $i \in \{1, 2\}$, be the deep point of $R_i \cap H$. Then $y_1 \neq y_2$. Let L_i, $i \in \{1, 2\}$, be a line of Q through y_i distinct from yx^*, and let S_i be a quad through L_i distinct from Q. Since $L_i = S_i \cap Q \subseteq H$, $S_i \cap R_i \subseteq H$ and $S_i \cap R_{3-i} \not\subseteq H$, the quad S_i is subquadrangular with respect to H. Put $\mathcal{H} := \mathcal{H}(S_1, S_2)$ and let S^* be the unique quad of $\mathcal{H}$ containing x^*. By Lemma 3.2(3), $\mathcal{G} := \{\pi_{S^*}(S \cap H) \,|\, S \in \mathcal{H}\}$ is a pencil of hyperplanes of $\widetilde{S^*}$ containing the two distinct subgrids $\pi_{S^*}(S_1 \cap H)$ and $\pi_{S^*}(S_2 \cap H)$. Since the subgrids $\pi_{S^*}(S_1 \cap H)$ and $\pi_{S^*}(S_2 \cap H)$ contain the line $\pi_{S^*}(L_1) = \pi_{S^*}(L_2) = S^* \cap Q$, the pencil $\mathcal{G}$ of hyperplanes of $\widetilde{S^*}$ consists of a unique singular hyperplane by Lemma 3.1. By Lemma 4.7, the unique quad $S \in \mathcal{H}$ for which $\pi_{S^*}(S \cap H)$ is a singular hyperplane of $\widetilde{S^*}$ is precisely S^*. So, the unique quad of $\mathcal{H}$ through y is subquadrangular. So, there exists a line $L_y \subseteq H$ through y not contained in Q. This implies that $\Lambda_H(y)$ is the union of two lines of $Res(y)$. Moreover, the deep point of the singular quad $\langle L_y, x^*y \rangle$ necessarily coincides with y. □

Lemma 4.9 *If R is a quad intersecting Q in a line L not containing x^*, then R is subquadrangular.*

Proof. Let y denote the unique point of L collinear with x^*. Then $\Lambda_H(y)$ is the union of two lines of $Res(y)$. Let Q and S denote the quads through y corresponding to these two lines. Then S contains the line x^*y. There are now two lines of R through y contained in H, namely the lines $L = R \cap Q$ and $R \cap S$. This implies that R is subquadrangular. □

We are now ready to derive a contradiction. Let u be a point of $Q \setminus (x^*)^\perp$. Then by Lemma 4.9, every quad through u distinct from Q is subquadrangular. This is not compatible with the fact that $\Lambda_H(u)$ is either a line or the union of two distinct lines of $Res(u)$.

Case II: For every point $x \in Q$, $\Lambda_H(x)$ is the union of two lines of $Res(x)$

Lemma 4.10 *For every point $x \in Q$, there exists a unique line $L_x \subseteq Q$ through x such that: (i) if R is a quad intersecting Q in L_x, then R is deep or singular; (ii) if R is a quad through x intersecting Q in a line $L \neq L_x$, then R is subquadrangular.*

Proof. The set $\Lambda_H(x)$ is the union of two distinct lines of $Res(x)$ through a given point u of $Res(x)$. The line L_x is precisely the line through x corresponding to the point u of $Res(x)$. □

The following is an immediate consequence of Lemma 4.10.

Corollary 4.11 *The lines L_x, $x \in Q$, form a spread $\mathcal{A}$ of $\widetilde{Q}$, i.e. a set of lines of $\widetilde{Q}$ partitioning the point set.*

Now, let L_1 and L_2 be two distinct lines of $\mathcal{A}$. Then L_1 and L_2 are contained in a unique full subgrid G of $\widetilde{Q}$. Let S_i, $i \in \{1,2\}$, be a quad of $DW(5,\mathbb{K})$ intersecting Q in L_i. Then S_i is deep or singular with respect to H. Put $\mathcal{H} := \mathcal{H}(S_1,S_2)$. If S_1 and S_2 are deep, then every $S \in \mathcal{H}$ is also deep. If for a certain $i \in \{1,2\}$, S_i is deep and S_{3-i} is singular, then every $S \in \mathcal{H} \setminus \{S_1,S_2\}$ is also singular and $\pi_{S_{3-i}}(S \cap H) = S_{3-i} \cap H$. If S_1 and S_2 are singular and $\pi_{S_1}(H \cap S_2) = H \cap S_1$, then there exists a unique $S^* \in \mathcal{H}$ that is deep with respect to H. Moreover, $\pi_{S_1}(S \cap H) = S_1 \cap H$ for every $S \in \mathcal{H} \setminus \{S^*\}$. Finally, if S_1 and S_2 are singular and $\pi_{S_1}(S_2 \cap H) \neq S_1 \cap H$, then every quad of $\mathcal{H}$ is singular by Lemmas 3.1 and 3.2(3). So, we see that also every line of G disjoint from L_1 and L_2 belongs to $\mathcal{A}$. This is however impossible: the spread $\mathcal{A}$ has lines not contained in G an each such line contains a unique point of G.

4.3 The classical hyperplanes of $DW(5,\mathbb{K})$ containing a deep point

In this section, we suppose that H is a classical hyperplane of $DW(5,\mathbb{K})$, $|\mathbb{K}| \neq 2$, containing a quad Q. By Section 4.2, we know that there exists a point $x \in Q$ for which $x^\perp \subseteq H$. Observe that every quad through x is either deep or singular with respect to H. If $\mathcal{D}(x) \neq \emptyset$ denotes the set of quads through x contained in H, then by Lemma 6.1 of De Bruyn [12], we know that the following holds.

Lemma 4.12 *Suppose $y \in H \cap \Gamma_3(x)$, then there exists an isomorphism from $Res(y)$ to the dual $Res^D(x)$ of $Res(x)$ mapping $\Lambda_H(y)$ to $\mathcal{D}(x)$.*

In view of Lemmas 4.6 and 4.12, we can then consider the following cases:

(1) $\Gamma_3(x) \cap H = \emptyset$;

(2) $\Gamma_3(x) \cap H \neq \emptyset$ and $\Lambda_H(y)$ is a point of $Res(y)$ for every $y \in H \cap \Gamma_3(x)$;

(3) $\Gamma_3(x) \cap H \neq \emptyset$ and $\Lambda_H(y)$ is a line of $Res(y)$ for every $y \in H \cap \Gamma_3(x)$;

(4) $\Gamma_3(x) \cap H \neq \emptyset$ and $\Lambda_H(y)$ is the union of two distinct lines of $Res(y)$ for every $y \in H \cap \Gamma_3(x)$;

(5) $\Gamma_3(x) \cap H \neq \emptyset$ and $\Lambda_H(y)$ is a nonsingular nonempty conic of $Res(y)$ for every $y \in H \cap \Gamma_3(x)$;

(6) $\Gamma_3(x) \cap H \neq \emptyset$ and $\Lambda_H(y)$ is the whole point set of $Res(y)$ for every $y \in H \cap \Gamma_3(x)$.

If case (1) occurs, then H is the singular hyperplane with deepest point x. This is a consequence of the fact that every hyperplane of $DW(5, \mathbb{K})$ is a maximal proper subspace (see Blok & Brouwer [4, Theorem 7.3] or Shult [24, Lemma 6.1]). Indeed, the fact that $\Gamma_3(x) \cap H = \emptyset$ implies that H is contained in the singular hyperplane of $DW(5, \mathbb{K})$ with deepest point x.

If case (2) occurs, then by Proposition 6.5 of De Bruyn [12], H is the extension of a (necessarily classical) ovoid of a quad Q of $DW(5, \mathbb{K})$. This quad Q is obtained as follows. Let $y \in \Gamma_3(x) \cap H$, let L_y denote the unique line through y contained in H and let z be the unique point of L_y at distance 2 from x. Then $Q = \langle x, z \rangle$.

If case (3) occurs, then by Proposition 6.4 of De Bruyn [12], H is a singular hyperplane whose deepest point is contained in $\Gamma_1(x)$.

If case (4) occurs, then by Proposition 6.6 of De Bruyn [12], H is the extension of a full subgrid G of a quad Q of $DW(5, \mathbb{K})$. The quad Q and its subgrid G can be constructed as follows. Let $y \in \Gamma_3(x) \cap H$. By Lemma 4.12, there are two distinct lines L_1 and L_2 through x such that $\mathcal{D}(x)$ consists of the quads through x containing L_1 or L_2 or both. Then $Q = \langle L_1, L_2 \rangle$ and G is the unique full subgrid of $\widetilde{Q}$ containing $L_1 \cup L_2 \cup \{\pi_Q(y)\}$.

By Lemma 6.8 and Proposition 6.10 of De Bruyn [12] (see also [13, p. 580]), $DW(5, \mathbb{K})$ has, up to isomorphism, a unique classical hyperplane corresponding to case (5) above. The proof relies on properties of the Grassmann embedding of $DW(5, \mathbb{K})$.

Observe that there exists no hyperplane as in case (6) above. Indeed, if case (6) occurs, then by Lemma 4.12, $\mathcal{D}(x)$ consists of all quads through x. This would imply that the singular hyperplane H_x with deepest point x is contained in H. Since H_x is a maximal proper subspace, we would have $H = H_x$, in contradiction with $\Gamma_3(x) \cap H \neq \emptyset$.

We wish to observe that Propositions 6.4, 6.5, 6.6, 6.10 and Lemmas 6.1, 6.8 in [12] are stated for the finite dual polar space $DW(5, q)$, but that their proofs remain valid if $\mathbb{F}_q$ is replaced by any field $\mathbb{K}$.

Suppose now that case (5) occurs. Then $\Lambda_H(x)$ is the whole point set of $Res(x)$ and $\Lambda_H(y)$ is a nonsingular nonempty conic of $Res(y)$ for every $y \in \Gamma_3(x) \cap H$. Recall that $\mathcal{D}(x)$ is a nonsingular nonempty conic of the dual projective plane $Res^D(x)$ of $Res(x)$.

Let $y \in \Gamma_1(x)$. If the line yx is a line of $Res^D(x)$ exterior to $\mathcal{D}(x)$, then $\Lambda_H(y)$ is a singleton of $Res(y)$. If the line yx is a line of $Res^D(x)$ tangent to $\mathcal{D}(x)$, then $\Lambda_H(y)$ is a line of $Res(y)$. If the line yx is a line of $Res^D(x)$ that is secant to $\mathcal{D}(x)$, then $\Lambda_H(y)$ is the union of two distinct lines of $Res(y)$.

Let $y \in \Gamma_2(x) \cap H$. Then the quad $\langle x, y \rangle$ belongs to $\mathcal{D}(x)$. Let L be a line through x contained in $\langle x, y \rangle$ that is a line of $Res^D(x)$ secant to $\mathcal{D}(x)$. Let z denote the unique point of L collinear with y. Then $\Lambda_H(z)$ is the union of two distinct lines of $Res(z)$ through the point of $Res(z)$ corresponding to L. This implies that every quad through yz distinct from Q is subquadrangular. This latter fact implies that $\Lambda_H(y)$ is the union of two distinct lines of $Res(y)$.

The above implies that x is the unique point of H that is deep with respect to H. In fact, among all the hyperplanes of $DW(5, \mathbb{K})$, $|\mathbb{K}| \neq 2$, containing a quad, the ones with a unique deep point are precisely those hyperplanes isomorphic to the hyperplane H considered here.

5 On the structure of hyperplanes of $DW(5, \mathbb{K})$ admitting deep and ovoidal quads

At the end of Section 4, we derived some structural information on the hyperplanes of $DW(5, \mathbb{K})$, $|\mathbb{K}| \neq 2$, that contain a quad and a unique deep point. In the present section, we derive some additional structural information if the hyperplane is also known to admit ovoidal quads.

The setting is as follows. We suppose that H is a hyperplane of $DW(5, \mathbb{K})$, $|\mathbb{K}| \neq 2$, admitting a deep quad Q_1 and an ovoidal quad Q_2. Then $Q_1 \cap Q_2 = \emptyset$. Put $\mathcal{H} := \mathcal{H}(Q_1, Q_2)$. Observe that for every $Q \in \mathcal{H} \setminus \{Q_1\}$, the quad Q is ovoidal with respect to H and $\pi_{Q_1}(Q \cap H) = \pi_{Q_1}(Q_2 \cap H)$.

By the Main Theorem, we know that H is either the extension of an ovoid of a quad or a classical hyperplane containing a unique deep point x^*. In the latter case, the set $\mathcal{D}(x^*)$ of deep quads through x^* defines a nonsingular nonempty conic in the dual plane $Res^D(x^*)$ of $Res(x^*)$. The existence of ovoidal quads implies that $Res^D(x^*)$ has lines which are exterior to $\mathcal{D}(x^*)$, or equivalently that $\mathbb{K}$ has quadratic extensions.

Lemma 5.1 *If $R \in \mathcal{P}_{\mathcal{H}}$, then R is subquadrangular or singular with respect to H.*

Proof. Since $(R \cap Q_2) \cap H$ is a singleton, R cannot be deep and since $(R \cap Q_1) \cap H$ is the line $R \cap Q_1$, R cannot be ovoidal. □

Lemma 5.2 (1) *Let R_1 and R_2 be two disjoint quads of $\mathcal{P}_{\mathcal{H}}$ that are singular with respect to H. Then every quad of $\mathcal{H}(R_1, R_2)$ is singular with respect to H with deep point belonging to Q_1.*

(2) *Let R_1 and R_2 be two disjoint quads of $\mathcal{P}_{\mathcal{H}}$ such that at least one of R_1, R_2 is subquadrangular with respect to H. Then there exists a unique quad in $\mathcal{H}(R_1, R_2)$ that is singular with respect to H (with deep point belonging to Q_1).*

Proof. This is a consequence of Lemmas 3.1 and 3.2(3). □

Let $X_{\mathcal{H}}$ denote the set of quads of $\mathcal{P}_{\mathcal{H}}$ that are singular with respect to H. By Lemmas 5.1 and 5.2, we immediately have

Corollary 5.3 *The set $X_{\mathcal{H}}$ is either $\mathcal{P}_{\mathcal{H}}$ or a hyperplane of $\mathcal{S}_{\mathcal{H}}$.*

Proposition 5.4 *If $X_{\mathcal{H}} = \mathcal{P}_{\mathcal{H}}$, then H is the extension of the ovoid $\pi_{Q_1}(Q_2 \cap H)$ of $\widetilde{Q_1}$.*

Proof. Every quad $R \in \mathcal{P}_{\mathcal{H}}$ is singular with respect to H with deep point $\pi_{Q_1}(R \cap Q_2 \cap H)$.

Now, by Lemma 3.2(2), every point of $DW(5, \mathbb{K})$ is contained in an element of $\mathcal{P}_{\mathcal{H}}$. Hence, the points of $H \setminus Q_1$ are precisely the points x not contained in Q_1 for which $\pi_{Q_1}(x) \in \pi_{Q_1}(Q_2 \cap H)$. So, H is the extension of the ovoid $\pi_{Q_1}(Q_2 \cap H)$ of $\widetilde{Q_1}$. □

Lemma 5.5 *Suppose $X_{\mathcal{H}}$ is a hyperplane of $\mathcal{S}_{\mathcal{H}}$. Then the following holds:*

(1) *There exists a unique line L^* of Q_1 such that the singular quads of $\mathcal{P}_{\mathcal{H}}$ are precisely those quads of $\mathcal{P}_{\mathcal{H}}$ meeting L^*.*

(2) *If x^* is the unique point of L^* contained in $\pi_{Q_1}(Q_2 \cap H)$, then x^* is deep with respect to H.*

Proof. (1) If $|\mathbb{K}| \geq 3$, then for every hyperplane G of the point-line geometry $\mathcal{S}_1(\mathbb{K})$, then there exists a unique point u of $PG(3, \mathbb{K})$ such that G consists of all points of $W(3, \mathbb{K})$ collinear with or equal to u, see e.g. Proposition 1.4 of De Bruyn [11]. Since $\mathcal{S}_{\mathcal{H}} \cong \mathcal{S}_1(\mathbb{K})$ and $X_{\mathcal{H}}$ is a hyperplane of $\mathcal{S}_{\mathcal{H}}$, this implies that there exists a unique line L^* of Q_1 such that the singular quads of $\mathcal{P}_{\mathcal{H}}$ are precisely those quads of $\mathcal{P}_{\mathcal{H}}$ meeting L^*.

(2) Let L denote the unique line through x^* meeting Q_2. Then $L \subseteq H$. Let M denote a line through x^* distinct from L. The quad $\langle L, M \rangle$ of $\mathcal{P}_{\mathcal{H}}$ is singular with respect to H and its deep point equals x^* (since $L \subseteq H$ and $\langle L, M \rangle \cap Q_1 \subseteq H$). Hence, $M \subseteq H$. □

By using Lemmas 5.1 and 5.5, it is now not so hard to prove that H satisfies the following properties without relying on Lemma 4.6 or any other property of the Grassmann embedding of $DW(5, \mathbb{K})$.

(i) If $x \in Q_1 \cap \Gamma_2(x^*)$, then $\Lambda_H(x)$ is the union of two lines of $Res(x)$.

(ii) Every line through x^* is contained in at most two deep quads.

(iii) Every deep quad Q through x^* contains a unique line through x^* that is contained in a unique deep quad (namely, Q itself). Every other line of Q through x^* is contained in precisely two deep quads.

We omit the proofs since by Section 4 we already know that these properties are valid. Observe that (ii) and (iii) imply that $\mathcal{D}(x^*)$ is a so-called oval of $Res^D(x^*)$. By Section 4, we know that this oval is in fact a nonsingular conic.

6 The classical ovoids of $Q(4, \mathbb{K})$

All hyperplanes mentioned in the statement of the main theorem are unique, up to isomorphism, except possibly for the extensions of the ovoids of the quads. If Q is a quad of $DW(5, \mathbb{K})$ and O_1 and O_2 are two ovoids of $\widetilde{Q}$, then the extensions of O_1 and O_2 are isomorphic if and only if the ovoids O_1 and O_2 of $\widetilde{Q} \cong Q(4, \mathbb{K})$ are isomorphic. So, it suffices to discuss the isomorphism problem for ovoids of $Q(4, \mathbb{K})$. In the discussion below, we restrict ourselves to classical ovoids of $Q(4, \mathbb{K})$.

Suppose Q is the quadric of $PG(4, \mathbb{K})$ defining the generalized quadrangle $Q(4, \mathbb{K})$. We choose a reference system in $PG(4, \mathbb{K})$ with respect to which Q has equation $X_0^2 + X_1X_2 + X_3X_4 = 0$. Let $\overline{\mathbb{K}}$ be a fixed algebraic closure of $\mathbb{K}$. For every subfield $\mathbb{F}$ of $\overline{\mathbb{K}}$ containing $\mathbb{K}$ we consider a projective space $PG(4, \mathbb{F})$ having the same reference system as $PG(4, \mathbb{K})$. So, we regard $PG(4, \mathbb{K})$ as a subgeometry of $PG(4, \mathbb{F})$ which itself will be regarded as a subgeometry of $PG(4, \overline{\mathbb{K}})$. We denote by Λ the set of all non-squares in $\mathbb{K}$ and by Λ' the set of all $\lambda \in \mathbb{K}$ for which the polynomial $X^2 + X + \lambda \in \mathbb{K}[X]$ is irreducible.

If G_1 and G_2 are two distinct classical hyperplanes of $Q(4, \mathbb{K})$, then there exists a unique hyperplane Π_i, $i \in \{1, 2\}$, of $\mathrm{PG}(4, \mathbb{K})$ such that $G_i = \Pi_i \cap Q$, and we denote by $\mathcal{V}$ the set of all hyperplanes of $\mathrm{PG}(4, \mathbb{K})$ through $\Pi_1 \cap \Pi_2$. We define $[G_1, G_2] := \{\Pi \cap Q \,|\, \Pi \in \mathcal{V}\}$ and $(G_1, G_2) := [G_1, G_2] \setminus \{G_1, G_2\}$.

Lemma 6.1 *Let G_1 and G_2 be two singular hyperplanes of $Q(4, \mathbb{K})$ whose deepest points are noncollinear.*

(1) *If $\mathbb{K}$ is a perfect field of characteristic 2, then every hyperplane of (G_1, G_2) is singular.*

(2) *If $\mathbb{K}$ is a nonperfect field of characteristic 2, then every hyperplane of (G_1, G_2) is either singular or ovoidal, with both possibilities occurring.*

(3) *If $\mathbb{K}$ is a field of characteristic distinct from 2 in which each element is a square, then every hyperplane of (G_1, G_2) is subquadrangular.*

(4) *If $\mathbb{K}$ is a field of characteristic distinct from 2 in which not each element is a square, then every hyperplane of (G_1, G_2) is subquadrangular or ovoidal, with both possibilities occurring.*

Proof. Since the automorphism group of $Q(4, \mathbb{K})$ acts transitively on the ordered pairs of noncollinear points, we may assume that the deepest points p_1 and p_2 of G_1 and G_2 are equal to $(0, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$. It is then easily verified that (G_1, G_2) consists of the hyperplanes G'_λ, $\lambda \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$, where G'_λ is the hyperplane of $Q(4, \mathbb{K})$ described by the equations $X_0^2 - \lambda X_3^2 + X_1X_2 = 0$, $X_4 = -\lambda X_3$. If $\lambda \in \Lambda$, then G'_λ is an ovoid of $Q(4, \mathbb{K})$. If $\lambda \notin \Lambda$, then G'_λ is a singular hyperplane or a full subgrid of $Q(4, \mathbb{K})$ depending on whether $char(\mathbb{K})$ is equal to 2 or not. The claims of the lemma now easily follow. $\square$

Lemma 6.2 *Let O be a classical ovoid of $Q(4, \mathbb{K})$. Then at least one of the following two cases occurs:*

(I) $O \in (x^\perp, G)$ for some full subgrid G of $Q(4, \mathbb{K})$ and some point x of $Q(4, \mathbb{K})$ not contained in G;

(II) $\mathbb{K}$ is a nonperfect field of characteristic 2 and $O \in (x_1^\perp, x_2^\perp)$ for two noncollinear points x_1 and x_2 of $Q(4, \mathbb{K})$.

Proof. Let x_1 be an arbitrary point of $Q(4, \mathbb{K})$ not contained in O, let $y \in O \cap x_1^\perp$ and let L be an arbitrary line through y distinct from yx_1. Then y is contained in every hyperplane of $(O, x_1^\perp)$ and L is contained in a unique hyperplane G' of $(O, x_1^\perp)$. The hyperplane G' of $Q(4, \mathbb{K})$ is either a singular hyperplane or a full subgrid. Observe that x_1 is not contained in any of the hyperplanes of $(O, x_1^\perp)$. If $(O, x_1^\perp)$ contains a full subgrid G, then $O \in (x_1^\perp, G)$ with $x_1 \notin G$ and case (I) of the lemma occurs. Suppose therefore that no hyperplane of $(O, x_1^\perp)$ is a full subgrid. Then G' is a singular hyperplane of $Q(4, \mathbb{K})$. Since $x_1 \notin G'$, the deepest point x_2 of G' is not collinear with x_1. Since $O \in (x_1^\perp, x_2^\perp)$ and none of the elements of $(x_1^\perp, x_2^\perp)$ is a full subgrid, $\mathbb{K}$ must then be a nonperfect field of characteristic 2 by Lemma 6.1. $\square$

Let O be a classical ovoid of $Q(4, \mathbb{K})$. There are two possibilities by Lemma 6.2.

(I) Suppose there exists a full subgrid G of $Q(4,\mathbb{K})$ and a point $x \notin G$ such that $O \in (G, x^\perp)$. Since the automorphism group of $Q(4,\mathbb{K})$ acts transitively on the pairs (G', x') were G' is a full subgrid of $Q(4,\mathbb{K})$ and x' is a point of $Q(4,\mathbb{K})$ not contained in G' (see De Bruyn [13, Lemma 2.3]), we may without loss of generality suppose that $x = (1,0,0,1,-1)$ and that G is described by the equations $X_0 = 0$, $X_1X_2 + X_3X_4 = 0$. It is then easy to verify that there exists a $\lambda \in \mathbb{K}$ such that O is described by the equations $X_4 = X_3 + \lambda X_0$, $X_0^2 + X_3^2 + \lambda X_0X_3 + X_1X_2 = 0$.

Suppose $\lambda \neq 0$ and $char(\mathbb{K}) \neq 2$. Then the map $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0 + \frac{\lambda}{2}X_3, X_1, X_2, X_3, X_4 - \lambda X_0 - \frac{\lambda^2}{4}X_3)$ determines an automorphism of $Q(4,\mathbb{K})$ mapping O to the ovoid O_μ, $\mu = \frac{\lambda^2}{4} - 1$, of $Q(4,\mathbb{K})$ with equations $X_4 = -\mu X_3$, $X_0^2 - \mu X_3^2 + X_1X_2 = 0$. Since O_μ is an ovoid, $\mu \in \Lambda$. Now, two irreducible polynomials $X^2 - \mu_1$ and $X^2 - \mu_2$ of $\mathbb{K}[X]$ determine the same quadratic extension of $\mathbb{K}$ in $\overline{\mathbb{K}}$ if and only if there exists an $a \in \mathbb{K}^*$ such that $\mu_2 = a^2\mu_1$. If this is the case, then the map $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0, X_1, X_2, aX_3, a^{-1}X_4)$ induces an isomorphism of $Q(4,\mathbb{K})$ mapping O_{μ_2} to O_{μ_1}.

Suppose $char(\mathbb{K}) = 2$. Then $\lambda \neq 0$. The map $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0, X_1, X_2, \lambda X_3, \lambda^{-1}X_4)$ defines an automorphism of $Q(4,\mathbb{K})$ mapping O to the ovoid O'_μ, $\mu = \frac{1}{\lambda^2}$, of $Q(4,\mathbb{K})$ with equations $X_4 = \mu X_3 + X_0$, $X_0^2 + X_0X_3 + \mu X_3^2 + X_1X_2 = 0$. Since O'_μ is an ovoid, $\mu \in \Lambda'$. If $\mu_1, \mu_2 \in \Lambda'$, then the irreducible polynomials $X^2 + X + \mu_1$ and $X^2 + X + \mu_2$ of $\mathbb{K}[X]$ define the same quadratic extension of $\mathbb{K}$ in $\overline{\mathbb{K}}$ if and only if there exists an $a \in \mathbb{K}$ such that $\mu_2 = \mu_1 + a^2 + a$. If this is the case, then the map $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0 + aX_3, X_1, X_2, X_3, X_4 + a^2X_3)$ induces an automorphism of $Q(4,\mathbb{K})$ mapping O'_{μ_2} to O'_{μ_1}.

(II) Suppose $\mathbb{K}$ is a nonperfect field of characteristic 2 and there exist two noncollinear points x_1 and x_2 of $Q(4,\mathbb{K})$ such that $O \in (x_1^\perp, x_2^\perp)$. Since the automorphism group of $Q(4,\mathbb{K})$ acts transitively on the ordered pairs of noncollinear points of $Q(4,\mathbb{K})$, we may without loss of generality assume that $x_1 = (0,0,0,1,0)$ and $x_2 = (0,0,0,0,1)$. Then there exists a $\lambda \in \mathbb{K}^*$ such that O is equal to the ovoid O_λ of $Q(4,\mathbb{K})$ described by the equations $X_4 = \lambda X_3$, $X_0^2 + \lambda X_3^2 + X_1X_2 = 0$. Since O_λ is an ovoid, $\lambda \in \Lambda$. Now, two irreducible polynomials $X^2 + \lambda_1$ and $X^2 + \lambda_2$ of $\mathbb{K}[X]$ determine the same quadratic extension of $\mathbb{K}$ in $\overline{\mathbb{K}}$ if and only if there exist $a, b \in \mathbb{K}$ with $a \neq 0$ such that $\lambda_2 = a^2\lambda_1 + b^2$. If this is the case, then the map $(X_0, X_1, X_2, X_3, X_4) \mapsto (X_0 + bX_3, X_1, X_2, aX_3, a^{-1}(X_4 + b^2X_3))$ induces an automorphism of $Q(4,\mathbb{K})$ mapping O_{λ_2} to O_{λ_1}.

Suppose π_1 and π_2 are two hyperplanes of $\mathrm{PG}(4,\mathbb{K})$ intersecting Q in classical ovoids of $Q(4,\mathbb{K})$ and there exists a projectivity μ of $\mathrm{PG}(4,\mathbb{K})$ stabilizing $Q(4,\mathbb{K})$ mapping $Q_1 := Q \cap \pi_1$ to $Q_2 := Q \cap \pi_2$. Let $\mathbb{F}_i$, $i \in \{1,2\}$, be the unique quadratic extension of $\mathbb{K}$ contained in $\overline{\mathbb{K}}$ such that the quadratic equation defining Q_i defines a nonsingular hyperbolic quadric or a cone Q'_i of the 3-space π'_i of $\mathrm{PG}(4,\mathbb{F}_i)$ determined by π_i. Let π''_2 denote the 3-space of $\mathrm{PG}(4,\mathbb{F}_1)$ determined by π_2. The projectivity μ' of $\mathrm{PG}(3,\mathbb{F}_1)$ that naturally extends μ maps Q'_1 to a nonsingular hyperbolic quadric or cone Q''_2 of π''_2, implying that $\mathbb{F}_2 = \mathbb{F}_1$.

Not all collineations of $Q(4,\mathbb{K})$ are necessarily induced by projectivities of the ambient projective space $\mathrm{PG}(4,\mathbb{K})$. Taking also those collineations into account that are related to field-automorphisms, the above discussion easily yields the following.

Proposition 6.3 (1) *Let $\mathbb{K}$ be a field of characteristic distinct from 2. Then every classical ovoid of $Q(4,\mathbb{K})$ is isomorphic to some ovoid O_λ where $\lambda \in \Lambda$. If $\lambda_1, \lambda_2 \in \Lambda$, then the classical ovoids O_{λ_1} and O_{λ_2} of $Q(4,\mathbb{K})$ are isomorphic if and only if there exists an $a \in \mathbb{K}^*$ and an automorphism σ of $\mathbb{K}$ such that $\lambda_2 = a^2 \cdot \lambda_1^\sigma$.*

(2) *Let $\mathbb{K}$ be a field of characteristic 2. Then every classical ovoid of $Q(4,\mathbb{K})$ is isomorphic to some ovoid O_λ where $\lambda \in \Lambda$, or some ovoid O'_λ where $\lambda \in \Lambda'$. If $\lambda_1, \lambda_2 \in \Lambda$, then the classical ovoids O_{λ_1} and O_{λ_2} of $Q(4,\mathbb{K})$ are isomorphic if and only if there exist $a, b \in \mathbb{K}$ with $a \neq 0$ and an automorphism σ of $\mathbb{K}$ such that $\lambda_2 = a^2 \cdot \lambda_1^\sigma + b^2$. If $\lambda_1, \lambda_2 \in \Lambda'$, then the classical ovoids O'_{λ_1} and O'_{λ_2} of $Q(4,\mathbb{K})$ are isomorphic if and only if there exists an $a \in \mathbb{K}$ and an automorphism σ of $\mathbb{K}$ such that $\lambda_2 = \lambda_1^\sigma + a^2 + a$.*

Final Remark. We can now see that either case I or case II in Lemma 6.2 occurs. In case I, the field extension of degree 2 associated with O is separable, while in case II the field extension is not separable.

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