GREEN'S FUNCTION OF POTENTIAL PROBLEMS IN LENS SHAPED GEOMETRIES

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The Kelvin inversion method will be outlined to determine the potential distribution due to a point charge (or the Green's function) in geometries bounded by flat and spherical surfaces.

1. **INTRODUCTION**

For two dimensional potential problems, the conformal transformation is a very powerful technique to obtain analytical or semi analytical solutions.

For the boundary element method, it is very useful if the Green's function is known analytically. Especially when the boundary included sharp corners or spines, a numerical solution with the classical fundamental solution will give rise to larger errors. If one could use a Green's function which fulfills the boundary condition around a corner or a spine, the latter need not to be discretised. As a consequence a higher accuracy will be obtained. The most straightforward way to find such a Green's function is certainly the conformal mapping technique.

A lot of papers have been devoted to singularities in the neighborhood of corners or other parts where the boundary is not that smooth. Mostly the boundary potential or flux is then written as the sum of an analytical function representing the singular behavior and a non singular remainder which is evaluated numerically [1, 2, 3, 4]. This kind of singular behavior is often found by a simple conformal mapping of the corner or spine into a half plane. Recently efforts have been undertaken to attack this problem for three dimensional problems [5].

For three dimensional potential problems there exists also a method called inversion which transforms geometry into another one so that in both cases the Laplace equations still holds. This method, invented by Lord Kelvin, is not so well known and is only described in a few textbooks [6]. In many textbooks the method is not even mentioned. Also a limited number of papers are devoted to this topic [7,

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8, 9]. The reason is that the inversion has limited applicability. Only when spherical geometries are involved, the method has proved its usefulness because a sphere can be transformed into a plane when the inversion centre is located on it.

In this paper we will use the inversion method in order to determine a Green's function which satisfies the Dirichlet boundary conditions on a lens shaped structure *i.e.* bounded by two spherical surfaces (or a flat plane and a spherical surface). Needless to mention that the Green's function for an electrostatic problem is nothing else than the potential distribution due to a single point charge.

2. **BASIC THEORY**

Consider a point P with spherical coordinates (r,φ,θ) as shown in Fig. 1.

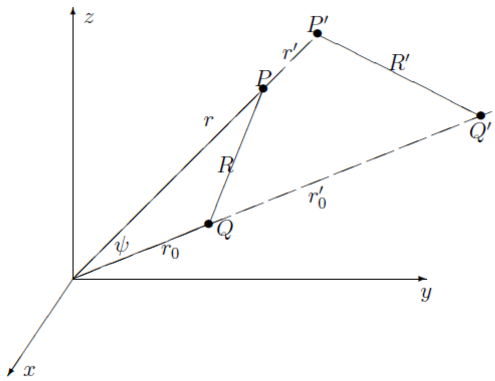

Fig. 1 – Kelvin inversion.

To any point P corresponds another point P' with coordinates (r',φ,θ), where r' is given by the inversion formula with a^2 the power of the inversion:

$$r' = \frac{a^2}{r} \ . \tag{1}$$

Remark that the angular coordinates φ and θ remain unaltered, a being a fixed distance. It has been proved that the inversion of a sphere can give rise to another sphere or a flat plane if the originating sphere goes through the origin of the coordinate system. If a function $\Phi(r,\varphi,\theta)$ exist, one can define a second function $\Phi'(r',\varphi,\theta)$ in the inverted domain by the following formula:

$$\Phi'(r',\varphi,\theta) = \frac{a}{r'}\Phi(r,\varphi,\theta) = \frac{a}{r'}\Phi(\frac{a^2}{r'},\varphi,\theta) \ . \tag{2}$$

It has been proved that the function Φ and Φ' satisfy the relation [6]:

$$\nabla^2_{r'}\Phi'(r',\varphi,\theta) = \frac{a^5}{r'^5}\nabla^2_r\Phi(r,\varphi,\theta)\,. \tag{3}$$

Hence, (3) means that if Φ satisfies the Laplace equation, Φ' will be a harmonic function too.

If Φ satisfies the Dirichlet boundary condition $\Phi = 0$ on a surface S the potential Φ' will satisfy $\Phi' = 0$ on the inverted surface S' due to (2). If one has solved the Laplace equation with $\Phi = 0$ on S, the solution Φ' is known immediately by using (2).

In this paper some potential problems will be solved where the surface S is a sphere or part of a sphere. The potential Φ is created by a point charge giving rise to a potential distribution which can be considered as the Green's function of the Laplace equation. After inversion, the sphere transforms into a flat plane. In the inverted domain, the potential distribution Φ' due to a point charge is easily found by introducing image charges.

Due to the coefficient a^5/r'^5 in (3), the Poisson equation is not invariant for inversion.

Hence, we have first of all to check how a Coulomb potential of a single point charge q is transformed by inversion. Image a point charge q in the point Q with coordinates (r_0,φ_0,θ_0) as shown in Fig. 1. The potential in the point $P(r,\varphi,\theta)$ is then:

$$\Phi = \frac{q}{4\pi\varepsilon_0 R} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\Psi}}\,, \tag{4}$$

where ε_0 is the dielectric constant of vacuum and Ψ is the angle between the $\vec{r}_0$ and $\vec{r}$ vectors. After inversion the point P is transformed into P' and Q into Q'. The potential Φ' is then found by using (2):

$$\begin{aligned}\Phi'(r',\varphi,\theta) &= \frac{a}{r'}\Phi\left(\frac{a^2}{r'},\varphi,\theta\right) = \frac{a}{r'}\cdot\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{\frac{a^4}{r'^2}+\frac{a^4}{r_0'^2}-2\frac{a^2}{r'}\frac{a^2}{r_0'}\cos\Psi}} = \\ &= \frac{q}{4\pi\varepsilon_0}\frac{r_0'}{a}\frac{1}{\sqrt{r'^2+r_0'^2-2r'r_0'\cos\Psi}}\,,\end{aligned} \tag{5}$$

where $r' = a^2/r$ and $r_0' = a^2/r_0$ have been used. Remark that the angle Ψ does not vary by the inversion. (5) is still a Coulomb potential in the inverted domain provided the charge q is replaced by $q' = q\cdot r_0'/a$.

3. A FEW EXAMPLES

A first example is shown in Fig. 2a. Only the a cross sectional view across the (y, z) plane is displayed

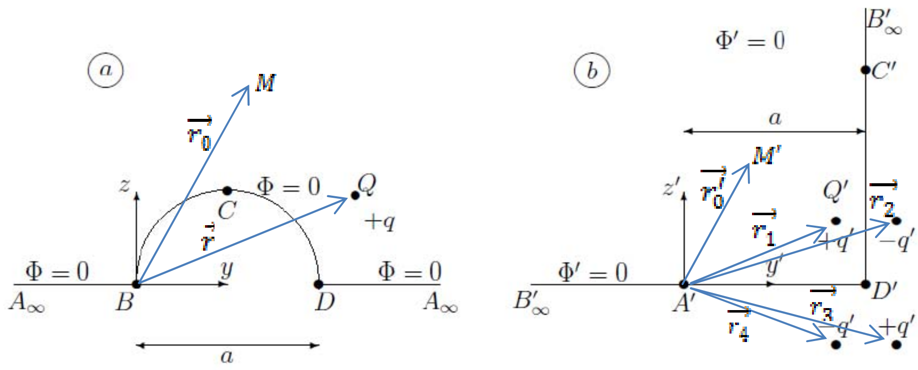

Fig. 2 – Geometry bounded by flat plane and half sphere.

The geometry we are interested in is the upper part above the plane $z = 0$ and above the half sphere. The sphere has diameter a and its centre point is located in $(0, a/2, 0)$ in the (x, y) plane. Both the plane $z = 0$ and the half sphere are at zero potential. In order to solve the problem of Fig. 2a, a Kelvin inversion will be used taking B as the centre and a^2 as the power of the inversion. The geometry $A_\infty BCDA_\infty$ is then transformed into two planes $B'_\infty A'D'$ and $D'C'B'_\infty$, perpendicular to each other as shown in Fig. 2b. The index ∞ is used for points at infinity. Obviously $A' \equiv B$ and $D' \equiv D$. If $Q \equiv Q(\vec{r})$ then $Q' \equiv Q'(\vec{r}_1)$ when $\vec{r}_1 = \vec{r}a^2/r^2$. The charge $q = 1$ (we search for the Green formula) is then also transformed into a charge $q' = qa/r = a/r$ in the point Q'.

For the geometry of Fig. 2b, the potential is easily calculated by introducing two negative $-q'$ images charges (in the points defined by the $\vec{r}_2$ and $\vec{r}_4$ vectors) and one positive $+q'$ image charge (in the point defined by $\vec{r}_3$). The $\vec{r}_2$, $\vec{r}_3$ and $\vec{r}_4$ vectors are calculated with respect to a and $\vec{r}_1$. Let be the arbitrary $M \equiv M(\vec{r}_0)$ point.

The inverse of the point M is $M' \equiv M'(\vec{r}'_0)$ where $\vec{r}'_0 = \vec{r}_0 a^2/r_0^2$.

The potential Φ established in the $M(\vec{r}_0)$ point by the $q = 1$ charge placed in the $Q(\vec{r})$ we are looking for, can be found by using (2). This potential is given by Green's formula in the Dirichlet problem for Laplace equation:

$$G(\vec{r},\vec{r}_0)=\frac{a}{r_0}\sum_{k=1}^{4}\frac{a}{r}\frac{(-1)^{k+1}}{4\pi\varepsilon_0\left|\vec{r}_0'-\vec{r}_k\right|}. \tag{6}$$

Another possibility is to transform the image charges back to the original domain (Fig. 2a) taking (5) into account but this is more difficult.

If $M(\vec{r}_0)$ is in the boundary surface then $G(\vec{r},\vec{r}_0)$ = 0. If we know the potential $\Phi=\Phi_f(\vec{r})$ of the boundary surface Σ (with condition $\Phi_f(\vec{r})\,\big|_{r>R_0}=0$, where $R_0 > a$ is fixed value) the potential in the arbitrary $M(\vec{r}_0)$ point is:

$$\Phi(\vec{r}_0)=-\varepsilon_0\int_\Sigma \Phi_f(\vec{r})\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n_\Sigma}\mathrm{d}A. \tag{7}$$

For the computation the following observation is useful:

If $G(\vec{r}_1,\vec{r}_2)=\dfrac{1}{\left|\vec{r}_1-\vec{r}_2\right|}$ and $\vec{n}_1=\dfrac{\vec{r}_1}{r_1}$ then $\dfrac{\partial G(\vec{r}_1,\vec{r}_2)}{\partial n_1}=\dfrac{(\vec{r}_2-\vec{r}_1)\vec{r}_1}{r_1\left|\vec{r}_1-\vec{r}_2\right|^3}$.

The second example (Fig. 3a) is quite similar except that the centre of the sphere is now above the $z = 0$ plane.

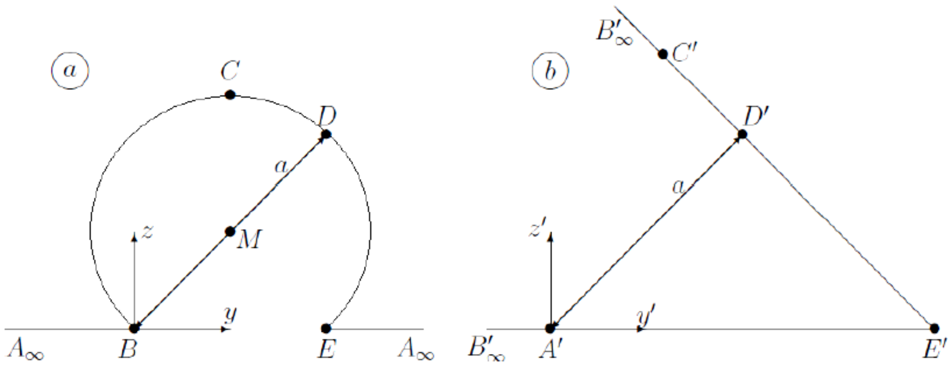

Fig. 3 – Geometry bounded by flat plane and sphere.

The z-coordinate of M was chosen in such a way, that the geometry after inversion is transformed into the domain bounded by two planes $B'_\infty A'E'$ and $E'D'C'B'_\infty$ intersecting each other at an angle of 45°.

In this configuration the image charges can be easily found to determine the potential distribution Φ'.

A third example is shown in Fig. 4a. The domain has the shape of a lens, bounded by two spherical surfaces.

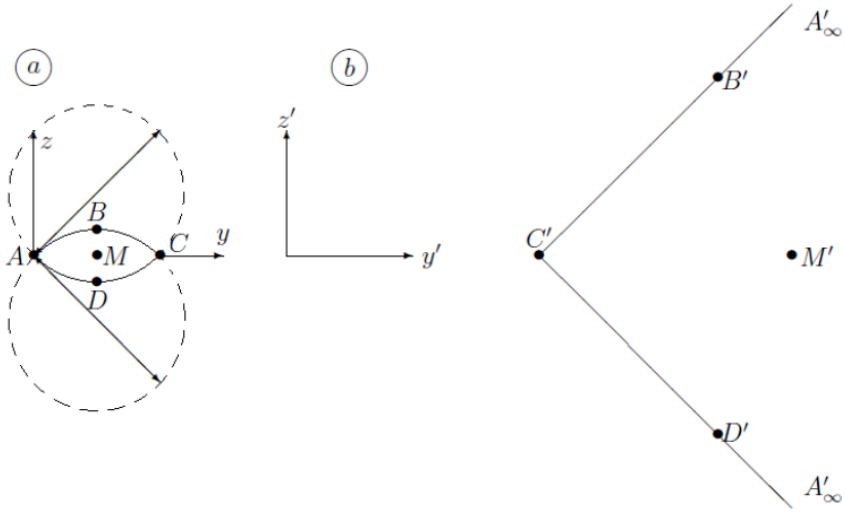

Fig. 4 – Lens shaped geometry bounded by two spheres.

After inversion the interior domain in transformed into a quarter space as shown in Fig. 4b. For the latter configuration, image sources can be easily found as already explained in the first example. If one considers the lens bounded by the upper sphere and the plane $z = 0$ or the structure ABCM the inversion method is still applicable. After transformation one obtains the geometry $A_\infty'B'C'M'$ or two planes with an intersecting angle of 45°. The method of image charges is still applicable.

4. **GREEN'S FUNCTION FOR ARBITRARY INTERSECTING ANGLE**

The examples in the foregoing section were all carefully chosen in such a way that the two planes after inversion have an intersecting angle of 180°/n where n = 2, 3, 4, ... [8]. It is then quite easy to use the method of image charges to determine the potential distribution. Generally, this is not always possible. First of all one can consider the example of Fig. 2a again, but this time one is interested in the complementary domain, *i.e.* the half space $z < 0$ and the hollow half sphere. This geometry is mapped into two the space bounded by the two half planes but this time one has to consider the other side or in other words the intersecting angle is not 270° and the method of image charges is not possible.

There is however a solution available for a point charge between two planes at an arbitrary angle β (Fig. 5).

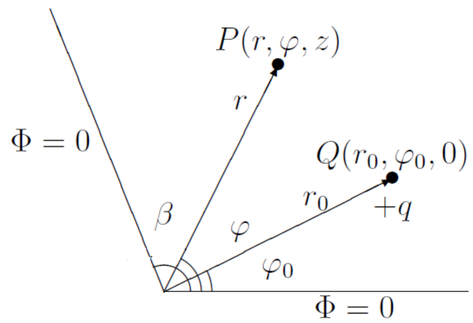

Fig. 5 – Geometry with arbitrary intersection angle β .

Using polar coordinates for $P(r, \varphi, z)$ and $Q(r_0, \varphi_0, 0)$ the potential Φ is given by [10]:

$$\Phi(r,\varphi,z) = \frac{q}{4\pi\varepsilon_0\sqrt{rr_0}} \int_\eta^\infty [\frac{\sinh\frac{\pi\zeta}{\beta}}{\cosh\frac{\pi\zeta}{\beta} - \cosh\frac{\pi(\varphi-\varphi_0)}{\beta}} - \\ - \frac{\sinh\frac{\pi\zeta}{\beta}}{\cosh\frac{\pi\zeta}{\beta} - \cosh\frac{\pi(\varphi+\varphi_0)}{\beta}}]\frac{d\zeta}{\sqrt{\cosh\zeta - \cosh\eta}}, \tag{8}$$

where:

$$\cosh\eta = \frac{r^2 + r_0^2 + z^2}{2rr_0}. \tag{9}$$

The evaluation of the Green's function requires a numerical integration of (8). However the integrand of (8) does not involve oscillating functions of the variable ζ , which is an advantage from numerical point of view.

It is important to mention that the observations in this paper can be extended to other time variation regimes for the sizes [11, 12].

The results are also useful for the hybrid finite element-boundary element method technique for unbounded domains, namely in the establishment of the boundary element method equation [13, 14].

We also mention its usefulness in building functions that model the singularities for surface edges. These functions can be added to the ones of the form finite element method [15].

5. **CONCLUSIONS**

It has been shown that the potential distribution caused by a point charge can be found by the Kelvin inversion for geometries where the boundary involves spheres and flat planes. This potential distribution is also known as the Green's function which can be used in numerical methods such as the boundary element method and in analytical computation of potential in the Dirichlet problem for the Laplace equation.

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