Maximal partial line spreads of non-singular quadrics

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Abstract

For $n \geq 9$, we construct maximal partial line spreads for non-singular quadrics of $PG(n,q)$ for every size between approximately $(cn+d)(q^{n-3}+q^{n-5})\log 2q$ and q^{n-2}, for some small constants c and d. These results are similar to spectrum results on maximal partial line spreads in finite projective spaces by Heden, and by Gács and Szőnyi. These results also extend spectrum results on maximal partial line spreads in the finite generalized quadrangles $W_3(q)$ and $Q(4,q)$ by Pepe, Rößing and Storme.

Keywords: Quadrics; Maximal partial line spreads; Spectrum results

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1 Introduction

A *partial line spread* in $PG(n,q)$ is a set of pairwise disjoint lines. A partial line spread is called *maximal* when it is not contained in a larger partial line spread.
In the literature, there are several articles on spectrum results on maximal partial line spreads in $PG(n,q)$, i.e., for large intervals, it is proven that for every integer k in that interval, there exists a maximal partial line spread of size k in $PG(n,q)$.
Heden performed extensive work on spectrum results for maximal partial line spreads in $PG(3,q)$ [**?**], and Gács and Szőnyi proved spectrum results on maximal partial line spreads in $PG(n,q)$, $n \geq 5$ [**?**].
The techniques of Gács and Szőnyi have now been extended to prove spectrum results on maximal partial line spreads in non-singular quadrics of $PG(n,q)$.
The presented spectrum results on maximal partial line spreads for non-singular quadrics of $PG(n,q)$ extend the spectrum results on maximal partial line spreads in the finite generalized quadrangles $W_3(q)$ and $Q(4,q)$ by Pepe, Rößing and Storme.
These results are as follows.

Theorem 1.1. (Pepe, Rößing and Storme [**?**]) *There exists a maximal partial line spread $\mathcal{S}$ of size k in the generalized quadrangle $W_3(q)$, $q \geq 49$ odd, for every integer k in the interval* $\left[\frac{6q^2+3q-149}{10}, \frac{9q^2-68q+519}{10}\right]$.

Theorem 1.2. (Rößing and Storme [**?**]) *For every integer k in one of the intervals in Table* **??***, there exists a maximal partial line spread $\mathcal{S}$ on the corresponding generalized quadrangles $W_3(q)$, q even, and $Q(4,q)$, q even.*

Note that the upper bounds of the intervals of [**?**] in Table **??** have undergone small corrections. We now summarize some results we need for our construction.

$W_3(q)$, $Q(4,q)$	Interval	
$q=2^{4h}$, $h \geq 2$	$\frac{q^2+194q+10q\lfloor 48\log(q+1)\rfloor-190}{10} \leq k \leq \frac{9q^2-69q+440}{10}$	[?]
$q=2^{4h+1}$, $h \geq 2$	$\frac{q^2+198q+10q\lfloor 48\log(q+1)\rfloor-230}{10} \leq k \leq \frac{9q^2-64q+390}{10}$	[?]
$q=2^{4h+2}$, $h \geq 1$	$\frac{q^2+196q+10q\lfloor 48\log(q+1)\rfloor-210}{10} \leq k \leq \frac{9q^2-66q+410}{10}$	[?]
$q=2^{4h+3}$, $h \geq 1$	$\frac{q^2+192q+10q\lfloor 48\log(q+1)\rfloor-170}{10} \leq k \leq \frac{9q^2-62q+370}{10}$	[?]

Table 1: Spectrum on maximal partial spreads in $Q(4,q)$, q even, and in $W_3(q)$, q even

Result 1.3. [?] *For a bipartite graph with partition $L \cup U$, such that every element in U has degree at least d, and every element in L has maximum degree D, there exists a subset $L' \subseteq L$ such that every element in U is adjacent to at least one element of L', and L' has size*

$$|L'| \leq \min\left(|L|\frac{\log(|U|)}{d}, |L|\frac{1+\log(D)}{d}\right).$$

We always use the form $|L'| \leq |L|\frac{1+\log(D)}{d}$ when applying the previous result. For every case we discuss, when $q > 5$, this is the smallest value.

Result 1.4.

(i) In $PG(3,q)$, $q \geq 7$ odd, there exist maximal partial line spreads for every size between $\frac{q^2+1}{2}+6$ and q^2-q+2 (Heden [?, ?, ?]).

(ii) In $PG(3,q)$, $q > q_0$ even, there exist maximal partial line spreads of every size between $\frac{5q^2+q+16}{8}$ and q^2-q+2 (Govaerts, Heden, Storme [?]; Jungnickel, Storme [?]).

The precise value of q_0 is not yet determined, but most likely q_0 will be 16, 32 or 64.

Result 1.5. *In $PG(3,q)$, $q \geq 3$, there exists a maximal partial line spread with size $\lceil 7\log q\rceil q$* [?].

Remark 1.6. *Whenever we use the previous result, we write $7q\log q$ for convenience.*

In this article we will construct maximal partial line spreads by covering every quadric by two sets of maximal partial line spreads. One set will block the lines of the quadric inside the tangent space $\ell^\perp$ of a line ℓ of the quadric, the other set will block the lines of the quadric outside of $\ell^\perp$. In Section **??** we find a spectrum result on maximal partial line spreads for $PG(n+2,q)\backslash PG(n,q)$. With the lines of this partial spread we will find 3-dimensional spaces through ℓ, blocking the lines outside of $\ell^\perp$. These 3-dimensional spaces intersect the quadric in a hyperbolic quadric $Q^+(3,q)$, and we will consider the lines of one of its reguli.
In Sections **??** and **??** we discuss separately the quadrics that do and do not contain line spreads. The difference lies in the construction of the maximal partial line spreads inside $\ell^\perp$. In both cases we will take several 3-dimensional spaces through ℓ, completely contained in the quadric, and consider both line spreads as maximal partial line spreads of these spaces $PG(3,q)$.
The constructions in this article apply to the parabolic, hyperbolic and elliptic quadrics. However, one may be able to use similar constructions for other polar spaces. From the isomorphism with $Q(2n+2,q)$, q even, we also found spectrum results for symplectic polar spaces $W(2n+1,q)$, q even. For q odd, one can most likely find spectrum results of maximal partial line spreads of symplectic polar spaces in a similar way as for the quadrics. One problem that has to be taken into account is that all points of $PG(2n+1,q)$ are points of the symplectic space $W(2n+1,q)$, but not all lines of $PG(2n+1,q)$ are totally isotropic lines of $W(2n+1,q)$. We are currently

looking at this problem. Using similar techniques for Hermitian varieties is on the contrary less feasible, since a 3-dimensional space intersects a Hermitian variety sometimes in a non-singular Hermitian variety $H(3, q^2)$, which does not contain a line spread, and very little is known on the existence of large maximal partial line spreads in these spaces.

2 Maximal partial line spreads in $PG(n+2, q) \setminus PG(n, q)$

Lemma 2.1. (Beutelspacher [?]) *The space $PG(n, q)$, $n \geq 3$, can be partitioned in one subspace $PG(n-2, q)$ and q^{n-1} lines.*

Lemma 2.2. *Consider a line spread $m_1, \ldots, m_{q^{n-1}}$ of $PG(n, q) \setminus PG(n-2, q)$, with $n \geq 3$. For every integer k such that $(1 + n \log 2q)q^{n-2} \leq k \leq q^{n-1}$, we can choose k lines m_i such that they intersect every line of $PG(n, q) \setminus PG(n-2, q)$, with exception of the not chosen lines m_i. Table **??** shows how large q has to be, such that for a given n the interval is non-empty.*

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
q	11	16	23	25	31	37	41	47	53	59	67	71	79	83

Table 2: Minimal size of q such that the interval of Lemma **??** is non-empty

Proof. Consider a line spread $\mathcal{S}$ in $PG(n, q) \setminus PG(n-2, q)$ (Lemma **??**). We construct a bipartite graph with classes A and B. Class A consists of the q^{n-1} lines of the line spread $\mathcal{S}$. Class B consists of the lines of $PG(n, q) \setminus PG(n-2, q)$ not belonging to the line spread $\mathcal{S}$. Two vertices from different classes are adjacent if their corresponding lines intersect each other. Every vertex of B has degree $d \geq q$. The maximum degree of an element of A is $D = (q+1)\left(\frac{q^n-1}{q-1} - 1\right) \leq 2q^n$ for $q > 3$.
Using Result **??**, we find a subset $A' \subseteq A$ of lines from the line spread $\mathcal{S}$ such that every line of $PG(n, q) \setminus PG(n-2, q)$ that does not belong to $\mathcal{S}$ will intersect a line of A'. We find:

$$|A'| \leq |A| \frac{(1 + \log D)}{q} \leq \frac{q^{n-1}}{q} (1 + \log 2q^n) \leq (1 + n \log 2q)q^{n-2}.$$

The set A' can be enlarged step by step until the whole set A consisting of all the q^{n-1} lines of the line spread $\mathcal{S}$ is reached. $\square$

Construction 2.3. *For $n \geq 3$, consider an $(n+2)$-dimensional projective space $PG(n+2, q)$ with the embedding of two n-dimensional spaces $\Pi_1 \cong PG(n, q)$ and $\Pi_2 \cong PG(n, q)$ that intersect each other in a space $\pi \cong PG(n-2, q)$. Consider a line $\ell \subseteq \Pi_1$ disjoint from π. Consider a line spread $\mathcal{S}$ in $\Pi_2 \setminus \pi$; this line spread $\mathcal{S}$ consists of q^{n-1} lines $m_1, \ldots, m_{q^{n-1}}$.*
Choose k lines m_i of the line spread $\mathcal{S}$, suppose $m_1, \ldots, m_k$, such that their union intersects every line in $\Pi_2 \setminus \pi$, not in $\mathcal{S}$. We consider the q^{n-1} three-dimensional spaces $\langle \ell, m_i \rangle$; these cover all points of $(PG(n+2, q) \setminus \Pi_1) \cup \{\ell\}$ completely.
For $i = 1, \ldots, k$, in the space $\langle \ell, m_i \rangle$, consider a line spread $\mathcal{S}_i$ through ℓ. For $j = k+1, \ldots, q^{n-1}$, in the space $\langle \ell, m_j \rangle$, consider a maximal partial line spread $\mathcal{P}_j$ through ℓ.
The union $\mathcal{S}'$ of the lines of the line spreads $\mathcal{S}_i$, $i = 1, \ldots, k$, and of the maximal partial line spreads $\mathcal{P}_j$, $j = k+1, \ldots, q^{n-1}$, forms a maximal partial line spread in $(PG(n+2, q) \setminus PG(n, q)) \cup \{\ell\}$.

Proof. For two different lines m_i and m_j, we know: $\langle \ell, m_i\rangle \cap \langle \ell, m_j\rangle = \{\ell\}$. So $\mathcal{S}'$ definitely is a partial line spread. Suppose b is a line of $PG(n+2,q)\backslash PG(n,q)$ not belonging to $\mathcal{S}'$; we will show that this line cannot be added to $\mathcal{S}'$.
Suppose b belongs to a three-dimensional space $\langle \ell, m_i\rangle$. When $i \leq k$, we consider a spread in $\langle \ell, m_i\rangle$; every point of b is covered. When $i > k$, we consider a maximal partial spread $\mathcal{P}_i$; there is at least one point of b covered.
Suppose the line b does not belong to such a three-dimensional space. The projection of b from the line ℓ on $\Pi_2\backslash\pi$ gives a line b'; different from every line m_i. Because of the property of the k lines m_i, there is at least one line of $m_1,\ldots,m_k$ that intersects b'; suppose b' intersects m_1. The space $\langle \ell, m_1\rangle$ has one point of b; suppose P. In $\langle \ell, m_1\rangle$, we have considered a line spread $\mathcal{S}_1$; this contains a line that intersects b in P. We have shown that the partial line spread $\mathcal{S}'$ is maximal in $(PG(n+2,q)\backslash PG(n,q)) \cup \{\ell\}$. □

Remark 2.4. *We take a maximal partial line spread in every chosen three-dimensional space through ℓ. Every line in the three-dimensional space has to intersect at least one of the lines of the maximal partial line spread. It could be possible that there exists a line that only intersects with the line ℓ. So it is important to add the line ℓ to the maximal partial line spread. This will not matter in what follows, because we can choose this line ℓ arbitrarily in $PG(n,q)\backslash PG(n-2,q) = \Pi_1\backslash\pi$.*

We define the following parameters:

- k_n: the size of the smallest known subset of lines of a particular line spread of $PG(n,q)\backslash PG(n-2,q)$, such that every line, not in the line spread, intersects a line of the set,
- $[l,u]$: interval of sizes of maximal partial line spreads of $PG(3,q)$,
- s: size of the smallest known maximal partial line spread of $PG(3,q)$.

Theorem 2.5. *Knowing the values of the previously defined variables, we find maximal partial line spreads in $(PG(n+2,q)\backslash PG(n,q)) \cup \{\ell\}$, $n \geq 3$, for every size in the interval:*

$$\left[k_n q^2 + (q^{n-1} - k_n)(s-1) + \left\lceil \frac{q^2 - s + 1}{u - l} \right\rceil (l - s) + 1, q^{n+1} - q^2 + u\right].$$

Proof. We look at the previous construction where k_n three-dimensional spaces contain line spreads through ℓ and every maximal partial line spread $\mathcal{P}_i$ through ℓ in one of the $q^{n-1} - k_n$ other three-dimensional spaces has the smallest known size, namely s. There exists a maximal partial line spread of size

$$k_n q^2 + (q^{n-1} - k_n)(s-1) + 1$$

in $(PG(n+2,q)\backslash PG(n,q)) \cup \{\ell\}$.
We can find an uninterrupted interval of sizes for maximal partial line spreads by using the interval $[l,u]$. Suppose we have a construction where k three-dimensional spaces have a line spread, $q^{n-1} - k - x$ three-dimensional spaces have the smallest known maximal partial line spread $\mathcal{P}_i$, and the other x three-dimensional spaces have a maximal partial line spread of size l, namely the smallest value from the known interval $[l,u]$. We can increase this size step by step by using other partial spreads with their size in the interval $[l,u]$. If we have reached the maximal value u, we have to increase k to $k+1$. We take line spreads and maximal partial line spreads of the following sizes:

number	k	$q^{n-1}-k-x$	x	
size	q^2	$s-1$	$u-1$	1
number	$k+1$	$q^{n-1}-k-x-1$	x	
size	q^2	$s-1$	$l-1$	1

To find an uninterrupted interval, the size corresponding to the second maximal partial line spread has to be smaller than or equal to the size of the first, so we need:

$$\begin{aligned} x(u-1) &\geq q^2 + x(l-1) - (s-1), \\ x &\geq \tfrac{q^2-s+1}{u-l}. \end{aligned}$$

The size of the smallest maximal partial line spread of our interval is therefore:

$$\begin{aligned} & k_n q^2 + (q^{n-1} - k_n - \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-1) + 1 \\ & \quad = k_n q^2 + (q^{n-1} - k_n)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-s) + 1. \end{aligned}$$

Now for our upper bound. A line spread in $PG(n+2,q)\backslash PG(n,q)$ has size q^{n+1}. We find the largest maximal partial line spread of our interval by using $k = q^{n-1}-1$ in our construction. This gives us a maximal partial line spread in $(PG(n+2,q)\backslash PG(n,q)) \cup \{\ell\}$ of size

$$(q^{n-1}-1)q^2 + u - 1 + 1 = q^{n+1} - q^2 + u.$$

□

Corollary 2.6. *Using Result ?? and ?? together with Lemma ??, we know the values:* $k_n = (1 + n \log 2q)q^{n-2}$, $s = 7q \log q$, $[l,u] = \left[(5q^2+q+16)/8, q^2-q+2\right]$ *and find* $x = 3$. *This, together with their conditions* $q \geq 7$, *and if* q *even also* $q > q_0$, *and using the approximation* $7 \log q \leq 7 \log 2q - 4$, *there exist maximal partial line spreads of* $(PG(n+2,q) \setminus PG(n,q)) \cup \{\ell\}$ *of every size in the interval*

$$\left[(-3 + (n+7)\log 2q)q^n,\, q^{n+1} - q + 2\right].$$

Table ?? shows how large q *has to be, such that for a given* n *the interval is non-empty.*

Proof. To prove the lower bound we make some approximations.

$$\begin{aligned} & k_n q^2 + (q^{n-1} - k_n)(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) + 1 \\ & \leq (1 + n\log 2q)q^n + q^{n-1}(7\log 2q - 4)q - q^{n-1} - k_n(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) + 1 \\ & \leq (1 + n\log 2q)q^n + (7\log 2q - 4)q^n \\ & = (-3 + (n+7)\log 2q)q^n \end{aligned}$$

□

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
q	43	47	53	61	67	73	79	89	97	101	107	113	121	127

Table 3: Minimal size of q such that the interval of Corollary ?? is non-empty

3 First construction: $Q(4n,q)$, q even, $Q^-(4n+1,q)$, $Q^+(4n+3,q)$

3.1 Line spreads

Lemma 3.1. *Consider a non-singular quadric $Q_n(q)$ in $PG(n,q)$. If $Q_n(q)$ has a line spread, then also the non-singular quadric $Q_{n+4}(q)$ in $PG(n+4,q)$, of the same type as $Q_n(q)$, that is, elliptic, parabolic or hyperbolic, has a line spread.*

Proof. Consider a line ℓ of the quadric $Q_{n+4}(q)$ and its tangent space $\ell^\perp \cong PG(n+2,q)$. We have $\ell^\perp \cap Q_{n+4}(q) = \ell Q_n(q)$, where $Q_n(q)$ is of the same type as $Q_{n+4}(q)$.
The quadric $Q_n(q)$ is non-singular in a space $\pi \cong PG(n,q)$ which we can embed in a space $\Pi \cong PG(n+2,q)$ disjoint from ℓ. Using Lemma **??**, we find a line spread $\mathcal{S} = \{m_1, \ldots, m_{q^{n+1}}\}$ of $\Pi\backslash\pi$. Every line m_i spans a three-dimensional space together with ℓ. Such a three-dimensional space intersects the quadric $Q_{n+4}(q)$ in a hyperbolic quadric $Q^+(3,q)$. In every such quadric $Q^+(3,q)$ we take the regulus through ℓ; this gives us $q^{n+1}q = q^{n+2}$ lines of $Q_{n+4}(q)$ different from ℓ. Every point P of $Q_{n+4}(q)\backslash\ell^\perp$ spans a plane together with ℓ. This plane intersects $\Pi\backslash\pi$ in one point; this point belongs to exactly one line of the line spread $\mathcal{S}$. The q^{n+2} lines form a partition of the points of $Q_{n+4}(q)\backslash\ell^\perp$.
Now look at $Q_n(q) \subseteq \pi$. We supposed this quadric has a line spread $\{t_1, \ldots, t_s\}$. Every three-dimensional space $\langle \ell, t_i\rangle$ is completely contained in $\ell Q_n(q)$. Every point of $\ell^\perp \cap Q_{n+4}(q)$, not on ℓ, belongs to exactly one of these three-dimensional spaces. In every space $\langle \ell, t_i\rangle$, we consider a line spread through ℓ. The union of all these lines forms a partition of the points of $\ell^\perp \cap Q_{n+4}(q)$.
The union of the two kinds of lines forms a line spread for $Q_{n+4}(q)$. □

Corollary 3.2. *Since the quadrics $Q(4,q)$, q even, $Q^+(3,q)$ and $Q^-(5,q)$ all contain a line spread, see [**?**], we know that for $n \geq 1$, every parabolic quadric $Q(4n,q)$, q even, every hyperbolic quadric $Q^+(4n-1,q)$, and every elliptic quadric $Q^-(4n+1,q)$ contains a line spread.*

Remark 3.3. *The parabolic quadric $Q(4,q)$, q odd, has no spreads. Very little is known about partial line spreads of $Q(4,q)$, q odd, and no large maximal partial line spreads have yet been found. This is why our results are only valid for quadrics $Q(4n+4,q)$ when q is even, and it seems difficult to find a similar construction for q odd.*

3.2 The cone $\ell Q_n(q)$

Lemma 3.4. *Consider a non-singular quadric $Q_n(q)$ in $PG(n,q)$; suppose it has a line spread $t_1, \ldots, t_r$. We can choose c lines t_i such that every line of $Q_n(q)$, not in the line spread, intersects at least one of these c lines. The integer c can be chosen for every value in the following interval:*

$$c \in \left[q^{n-3}(1+(n-2)\log 2q), \frac{|Q_n(q)|}{q+1} \right].$$

Proof. Consider a line spread $\mathcal{S}$ in $Q_n(q)$. We construct a bipartite graph with two classes A and B. Class A consists of the lines of the line spread $\mathcal{S}$. Consider B to contain the lines of $Q_n(q)$ not in the line spread. Two vertices of different classes are adjacent if their corresponding lines intersect each other. Every vertex of B has degree $q+1$.
The quadric $Q_n(q)$ contains $\frac{|Q_n(q)||Q_{n-2}(q)|}{q+1}$ lines. We can check for every type of quadric that the maximal degree of an element of A is $D \leq 2q^{n-2}$ for $q > 3$. The set A consists of $\frac{|Q_n(q)|}{q+1}$ lines. Using Result **??**, we find a subset $A' \subseteq A$ of lines of the line spread $\mathcal{S}$ such that every

line of $Q_n(q)$ not in $\mathcal{S}$ intersects at least one of the lines of A'. We can check for every type of quadric that $\frac{|Q_n(q)|}{q+1} \leq q^{n-3}(q+1)$. We find:

$$\left|A'\right| \leq q^{n-3}(1+(n-2)\log 2q).$$

Step by step we can add extra lines of the line spread until we have all the lines of the line spread $\mathcal{S}$. $\square$

Corollary 3.5. *The previous lemma leads to the following intervals for specific quadrics:*

- $Q(4m,q), q$ *even* $: c \in \left[q^{4m-3}(1+(4m-2)\log 2q), \frac{q^{4m}-1}{q^2-1}\right]$,
- $Q^-(4m+1,q) : c \in \left[q^{4m-2}(1+(4m-1)\log 2q), \frac{(q^{2m}-1)(q^{2m+1}+1)}{q^2-1}\right]$,
- $Q^+(4m+3,q) : c \in \left[q^{4m}(1+(4m+1)\log 2q), \frac{(q^{2m+1}+1)(q^{2m+2}-1)}{q^2-1}\right]$.

Table **??** *shows how large q has to be such that for a given m the constructed interval is non-empty.*

$q\backslash m$	1	2	3	4	5	6	7	8	9	10	11	12	13
$Q(4m,q)$, q even	16	64	64	128	128	256	256	256	256	256	512	512	512
$Q^-(4m+1,q)$	23	41	67	97	121	149	173	199	229	257	289	331	349
$Q^+(4m+3,q)$	31	53	79	103	131	157	191	223	243	277	307	337	367

Table 4: Minimal size of q such that the interval of Corollary **??** is non-empty

Construction 3.6. *Suppose $Q_n(q)$ contains a line spread.*
Consider a non-singular quadric $Q_{n+4}(q)$ and take an arbitrary line ℓ of $Q_{n+4}(q)$. Consider its tangent space $\ell^\perp \cong PG(n+2,q)$. We find: $\ell^\perp \cap Q_{n+4}(q) = \ell Q_n(q)$; this quadric $Q_n(q)$ lies in a space $\pi \cong PG(n,q)$.
Consider a line spread $\mathcal{S} = \{t_1,\dots,t_r\}$ on $Q_n(q)$.
Look at the three-dimensional spaces $\langle \ell, t_1\rangle,\dots,\langle \ell, t_r\rangle$; these are completely contained in the quadric $Q_{n+4}(q)$. Every point of $\ell^\perp \cap Q_{n+4}(q)$, not on ℓ, belongs to exactly one of these three-dimensional spaces.
Suppose there exist c lines t_i such that every line of the quadric $Q_n(q)$, not in $\mathcal{S}$, intersects at least one of these c lines. For $i=1,\dots,c$, take a line spread $\mathcal{S}_i$ in the space $\langle \ell, t_i\rangle$ through ℓ. For $j=c+1,\dots,r$, take a maximal partial line spread $\mathcal{P}_j$ in $\langle \ell, t_j\rangle$ through ℓ.
The union $\mathcal{S}'$ of the lines of the line spreads $\mathcal{S}_i$ together with the lines of the maximal partial line spreads $\mathcal{P}_j$ forms a maximal partial line spread in $\ell Q_n(q)$.

Proof. This proof is similar to the proof of Construction **??**. $\square$

If there does not exist a line spread in $Q_n(q)$, then the previous construction does not work. This case will be discussed later in Section **??**.

3.3 The quadric $Q_{n+4}(q)$

Construction 3.7. *Take an arbitrary line ℓ of $Q_{n+4}(q)$. Consider the tangent space $\ell^\perp \cong PG(n+2,q)$, we have: $\ell^\perp \cap Q_{n+4}(q) = \ell Q_n(q)$. This quadric $Q_n(q)$ lies in a space $\pi \cong PG(n,q)$ which can be embedded in a space $\Pi \cong PG(n+2,q)$ disjoint from ℓ.*

Starting from a line spread $\mathcal{S}_1 = \{t_1, \ldots, t_r\}$ of $Q_n(q)$ and using Construction ??, we find a maximal partial line spread $\mathcal{S}'_1$ in $\ell^\perp \cap Q_{n+4}(q)$ that contains the line ℓ.
In the three-dimensional space $\langle \ell, t_1 \rangle$, we have considered a line spread. Take a maximal partial line spread $\mathcal{S}_2 = \{m_1, \ldots, m_s, t_1\}$ of $(\Pi \backslash \pi) \cup \{t_1\}$, for example by Construction ??. Every three-dimensional space $\langle \ell, m_1 \rangle, \ldots, \langle \ell, m_s \rangle$ intersects $Q_{n+4}(q)$ in a hyperbolic quadric $Q^+(3, q)$ through ℓ. In every such three-dimensional space we take the corresponding regulus through ℓ. All these lines, without ℓ, form an appropriate partial line spread $\mathcal{S}'_2$ for $Q_{n+4}(q) \backslash \ell^\perp$ of size qs. The union $\mathcal{S}'$ of lines of $\mathcal{S}'_1$ and $\mathcal{S}'_2$ forms a maximal partial line spread for the quadric $Q_{n+4}(q)$.

Proof. A line intersecting ℓ cannot be added. A line in $\ell^\perp$ cannot be added since $\mathcal{S}'_1$ is a maximal partial line spread. Consider two different lines m_i and m_j, we know $\langle \ell, m_i \rangle \cap \langle \ell, m_j \rangle = \{\ell\}$. So we have a partial line spread $\mathcal{S}'_2$. Consider a line b in $Q_{n+4}(q) \backslash \ell^\perp$ disjoint from ℓ; we will show that this line cannot be added to $\mathcal{S}'$.
When b belongs to a three-dimensional space $\langle \ell, m_i \rangle$, then it belongs to a regulus contained in $\mathcal{S}'_2$. Suppose b does not belong to such a three-dimensional space. Projection of b from ℓ on $\Pi \backslash \pi$ gives a line b', different from every line m_i. Since we have considered a maximal partial line spread in $(\Pi \backslash \pi) \cup \{t_1\}$, there is at least one line in $\{m_1, \ldots, m_s, t_1\}$ that intersects b'.
Suppose b' intersects the line t_1 in a point. The space $\langle \ell, t_1 \rangle$ contains a point of b, suppose P. In $\langle \ell, t_1 \rangle$, we have considered a line spread through ℓ; this spread contains a line that intersects b in P. So the line b cannot be added to $\mathcal{S}'$. Suppose b' intersects a line m_i, suppose m_1. The space $\langle \ell, m_1 \rangle$ contains one point of b, suppose P. In $\langle \ell, m_1 \rangle$, we have taken the regulus of $Q_{n+4}(q)$ through ℓ; this spread contains a line which intersects b in P. The line b cannot be added to $\mathcal{S}'_2$; we have shown that the partial line spread $\mathcal{S}'$ is maximal. $\square$

Suppose $Q_n(q)$ is a non-singular quadric in $PG(n, q)$ containing a line spread $\mathcal{L}$. Define c_n to be the smallest number of lines of $\mathcal{L}$ such that every line of the quadric, not in the line spread, intersects at least one of the c_n lines. For an upper bound on c_n we refer to Lemma ??.

Theorem 3.8. *For $n \geq 3$, when the non-singular quadric $Q_n(q)$ in $PG(n, q)$ has a line spread, we can find maximal partial line spreads $\mathcal{S}'$ in $\ell Q_n(q)$ for every size in the following interval:*

$$\left[c_n q^2 + \left(\frac{|Q_n(q)|}{q+1} - c_n \right)(s-1) + \left\lceil \frac{q^2 - s + 1}{u - l} \right\rceil (l - s) + 1, \frac{|Q_n(q)|}{q+1} q^2 - q^2 + u \right].$$

Proof. We use Construction ?? with $c = c_n$ and take for every $\mathcal{P}_j$ the smallest known maximal partial line spreads; these have size s. This gives us a maximal partial line spread of size:

$$c_n q^2 + \left(\frac{|Q_n(q)|}{q+1} - c_n \right)(s-1) + 1.$$

To find an uninterrupted interval we proceed in the same way as in Theorem ??. In at least x three-dimensional spaces, we will consider maximal partial line spreads with their size in the interval $[l, u]$. Similarly as in the earlier proof we find that $x \geq \frac{q^2-s+1}{u-l}$. By making the correct choices we find an uninterrupted interval with lower bound:

$$c_n q^2 + \left(\frac{|Q_n(q)|}{q+1} - c_n - \left\lceil \frac{q^2 - s + 1}{u - l} \right\rceil \right)(s-1) + \left\lceil \frac{q^2 - s + 1}{u - l} \right\rceil (l-1) + 1.$$

Now for our upper bound. A line spread of $Q_n(q)$ has size $\frac{|Q_n(q)|}{q+1}$. We find the largest maximal partial line spread of our interval by using $c = \frac{|Q_n(q)|}{q+1} - 1$ in our Construction ??. This gives us a maximal partial line spread of $\ell Q_n(q)$ of size

$$\left(\frac{|Q_n(q)|}{q+1} - 1 \right) q^2 + u - 1 + 1 = \frac{|Q_n(q)|}{q+1} q^2 - q^2 + u.$$

$\square$

Corollary 3.9. *Using Result* **??** *and* **??** *together with Corollary* **??***, we know the values for our variables, with restrictions $n \geq 3$, $q \geq 7$ and for even q also $q > q_0$, so for particular quadrics there exists a maximal partial line spread for every size in the interval:*

$$\ell Q(4n-4,q), q \text{ even } : \left[(-3+(4n+1)\log 2q)q^{4n-5}, \frac{q^{4n-2}-1}{q^2-1} - q + 1\right],$$

$$\ell Q^+(4n-5,q) : \left[(-3+4n\log 2q)q^{4n-6}, \frac{q^{4n-3}+q^{2n}-q^{2n-1}-1}{q^2-1} - q + 1\right],$$

$$\ell Q^-(4n-7,q) : \left[(-3+(4n-2)\log 2q)q^{4n-8}, \frac{q^{4n-5}-q^{2n-1}+q^{2n-2}-1}{q^2-1} - q + 1\right].$$

Theorem 3.10. *In $Q_{n+4}(q)$, $n \geq 3$, there exist maximal partial line spreads $\mathcal{S}'$ for every size in the interval:*

$$\left[(qk_n + c_n)q^2 + (q^n + \tfrac{|Q_n(q)|}{q+1} - qk_n - c_n)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (q+1)(l-s) + 1, \right.$$
$$\left. q^{n+2} + \tfrac{|Q_n(q)|}{q+1} q^2 - q^2 + u \right].$$

Proof. We are looking at maximal partial line spreads of Construction **??**.
Using Theorem **??**, we find maximal partial line spreads $\mathcal{S}_1'$ in $\ell Q_n(q)$ of every size in the interval

$$\left[c_n q^2 + \left(\frac{|Q_n(q)|}{q+1} - c_n\right)(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) + 1, \frac{|Q_n(q)|}{q+1} q^2 - q^2 + u \right].$$

Using Theorem **??**, we can find a maximal partial line spread $\mathcal{S}_2$ in $(PG(n+2,q) \backslash PG(n,q)) \cup \{t_1\}$; its size lies in the following interval:

$$\left[k_n q^2 + (q^{n-1} - k_n)(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) + 1, q^{n+1} - q^2 + u \right].$$

Since the three-dimensional space $\langle \ell, t_1 \rangle$ is completely covered by a line spread, every line in $Q_{n+4}(q) \backslash \ell^\perp$ that after projection from ℓ is projected onto a line that intersects t_1, is also covered by the maximal partial line spread $\mathcal{S}_1'$, see Construction **??**. So from now on we will talk about these appropriate partial line spreads in $PG(n+2,q) \backslash PG(n,q)$ that intersect all the lines not in the partial spread and disjoint from t_1, and we still use the name $\mathcal{S}_2$.
The constructed maximal partial line spread has size $|\mathcal{S}'| = |\mathcal{S}_1'| + q|\mathcal{S}_2|$. When the size of $\mathcal{S}_2$ increases by one, the size of $\mathcal{S}'$ increases by q.
We have uninterrupted intervals for the sizes of $\mathcal{S}_1'$ and $\mathcal{S}_2$. So an uninterrupted interval for the sizes of $\mathcal{S}'$ can be found if the length of the interval of $\mathcal{S}_1'$ is larger than q; this leads to the condition:

$$\frac{|Q_n(q)|}{q+1} q^2 - q^2 + u - c_n q^2 - \left(\frac{|Q_n(q)|}{q+1} - c_n\right)(s-1) - \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) - 1 \geq q. \quad (1)$$

If the size of the interval for $\mathcal{S}_1'$ is larger than $(q^2-u)q$, then we can make the jump from $\mathcal{S}_2$ of size $q^{n+1} - q^2 + u$ to a line spread of size q^{n+1}. This leads to the condition:

$$\frac{|Q_n(q)|}{q+1} q^2 - q^2 + u - c_n q^2 - \left(\frac{|Q_n(q)|}{q+1} - c_n\right)(s-1) - \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) - 1 \geq (q^2-u)q. \quad (2)$$

For given n, the inequalities in (??) and (??) are valid for the respective quadrics $Q_n(q)$ if q is large enough. For smaller n the lower bounds on q are presented in Table ??.
We only have to look for the maximal and minimal size of $\mathcal{S}'$.
When we take for $\mathcal{S}'_1$ the largest possible maximal partial line spread, that is not a line spread, and for $\mathcal{S}_2$ a line spread, then we find size:

$$\frac{|Q_n(q)|}{q+1}q^2 - q^2 + u + \left(q^{n+1}\right) q.$$

This is the largest possible maximal partial line spread for $Q_{n+4}(q)$ we can find by our construction.
The smallest by us constructed appropriate partial line spread of $PG(n+2,q)\backslash PG(n,q)$ has size $k_n q^2 + (q^{n-1} - k_n)(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s)$. Every line of such a partial line spread gives us q lines for the sought maximal partial line spread $\mathcal{S}'$. We add the lines of the smallest maximal partial line spread in $\ell Q_n(q)$; this has size $c_n q^2 + \left(\frac{|Q_n(q)|}{q+1} - c_n\right)(s-1) + \left\lceil \frac{q^2-s+1}{u-l} \right\rceil (l-s) + 1$. The smallest size is thus

$$\begin{aligned}
&\left(k_n q^2 + \left(q^{n-1} - k_n\right)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-s)\right) q \\
&\quad + c_n q^2 + \left(\tfrac{|Q_n(q)|}{q+1} - c_n\right)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-s) + 1 \\
&\qquad = (qk_n + c_n)q^2 + (q^n + \tfrac{|Q_n(q)|}{q+1} - qk_n - c_n)(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (q+1)(l-s) + 1.
\end{aligned}$$

□

We now summarize the spectrum result for this construction method. Due to the isomorphism between $W(2n-1,q)$, q even, and $Q(2n,q)$, q even, the spectrum result for $Q(4n,q)$, q even, is also valid for $W(4n-1,q)$, q even.

Corollary 3.11. *Using Result ?? and ??, Lemma ?? and Corollary ??, we know the values of our variables together with the constraints $n \geq 3$, $q \geq 7$ and for q even also $q > q_0$, so for particular quadrics there exist maximal partial line spreads $\mathcal{S}'$ for every size in the following interval:*
$W(4n-1,q), q$ even $\cong Q(4n,q), q$ even :

$$\left[(-3 + (4n+3)\log 2q)q^{4n-3} + (-3 + (4n+1)\log 2q)q^{4n-5},\ \frac{q^{4n}-1}{q^2-1} - q + 1\right],$$

$Q^+(4n-1,q)$:

$$\left[(-3 + (4n+2)\log 2q)\, q^{4n-4} + (-3 + 4n\log 2q)q^{4n-6},\ \frac{(q^{2n-1}+1)(q^{2n}-1)}{q^2-1} - q + 1\right],$$

$Q^-(4n-3,q)$:

$$\left[(-3 + 4n\log 2q)q^{4n-6} + (-3 + (4n-2)\log 2q)q^{4n-8},\ \frac{(q^{2n-1}+1)(q^{2n-2}-1)}{q^2-1} - q + 1\right].$$

Table ?? shows how large q has to be such that for a given n, the interval determined is non-empty, and such that the conditions we have put on the length of the interval for $|\mathcal{S}'_1|$ in Theorem ?? are fulfilled.

$q\backslash n$	3	4	5	6	7	8	9	10	11	12	13	14
$Q(4n,q), q$ even	128	128	128	256	256	256	256	512	512	512	512	512
$Q^+(4n-1,q)$	67	97	125	149	173	211	233	263	293	331	353	383
$Q^-(4n-3,q)$	53	79	107	131	163	191	223	251	277	307	337	367

Table 5: Minimal size of q such that the interval corresponding to Corollary **??** is non-empty

4 Second construction: $Q^+(4n+1,q)$, $Q(4n+2,q)$, $Q^-(4n+3,q)$

The quadrics $Q^+(4n+1,q)$ and $Q^-(4n+3,q)$ cannot contain a line spread since $q+1$ does not divide $|Q^+(4n+1,q)| = (q^{2n}+1)(q^{2n+1}-1)/(q-1)$, nor $|Q^-(4n+3,q)| = (q^{2n+1}-1)(q^{2n+2}+1)/(q-1)$. So the previous construction does not work here. Even though $q+1$ divides $|Q(4n+2,q)| = (q^{4n+2}-1)/(q-1)$, there are not many results known about the existence of line spreads in the non-singular quadric $Q(4n+2,q)$. From [?], we know a line spread in $Q(6,2)$ and $Q(6,3)$ exists, but we do not know the situation for other q.
We will construct large maximal partial line spreads, and from this we will find a spectrum of maximal partial line spreads similarly as in the first construction. Note that the following method does not work for $Q(4,q)$, $Q(8,q)$, nor for any parabolic quadric $Q(4n,q)$, $n \geq 1$, q odd.

4.1 Large maximal partial line spreads

Lemma 4.1. *Consider the non-singular quadric $Q_{4n+i}(q)$ in $PG(4n+i,q)$, with $i \in \{1,2,3\}$, corresponding to a quadric in the set $\{Q^+(4n+1,q), Q(4n+2,q), Q^-(4n+3,q)\}$. Define $\delta(1)=0$ and $\delta(2)=\delta(3)=1$. For these quadrics, there exists a maximal partial line spread $\mathcal{M}$ for which $\mathcal{M} = \mathcal{S} \cup \mathcal{P}$. Here the set $\mathcal{P}$ is a maximal partial line spread of the tangent space $\pi^\perp$, with π a $(2n-1)$-dimensional space contained in the quadric. Every point of the quadric outside of $\pi^\perp$ is covered by the line spread $\mathcal{S}$. The smallest and largest maximal partial line spread of this form have respectively size*

$$(q^{2n+i}+1)\left(\tfrac{q^{2n}-1}{q^2-1}\right) = \left\{q^{2n+i}\tfrac{q^{2n}-1}{q^2-1}\right\} + \left\{\tfrac{q^{2n}-1}{q^2-1}\right\}$$

$$\text{and } (q^{2n+i}+1)\left(\tfrac{q^{2n}-1}{q^2-1}\right) + q^{2n-2+\delta(i)} = \left\{q^{2n+i}\tfrac{q^{2n}-1}{q^2-1}\right\} + \left\{\tfrac{q^{2n}-1}{q^2-1} + q^{2n-2+\delta(i)}\right\}.$$

The value inside the first pair of braces shows the size of $\mathcal{S}$, and the value inside the second pair of braces shows the size of $\mathcal{P}$.

The construction of these two maximal partial line spreads on these quadrics $Q_{4n+i}(q)$ is as follows:

- We fix a line ℓ and a particular $(2n-1)$-dimensional space π of $Q_{4n+i}(q)$, passing through ℓ.
- Outside of $\ell^\perp$, we consider the same line spread as in the first construction.
- In $\ell^\perp \backslash \pi^\perp$, we consider a line spread of $Q_{4n+i}(q)$, defined in the following way. Consider the quotient geometry $\ell^\perp/\ell$ of ℓ in $\ell^\perp$ and consider in this quotient geometry of ℓ in $\ell^\perp$ the quotient geometry π' of π. There exists a line spread of $(\ell^\perp/\ell) \setminus \pi'$. By taking line spreads through ℓ in the 3-spaces defined by the lines of this line spread and ℓ, a line spread of $Q_{4n+i}(q)$ in $\ell^\perp \setminus \pi^\perp$ is obtained. The union of the two preceding line spreads, not including the line ℓ, is the line spread $\mathcal{S}$.

- The tangent space $\pi^\perp$ of π intersects $Q_{4n+i}(q)$ into the union of $q^{i-1}+1$ generators, of dimension $2n$. For the smallest maximal partial line spread, $\mathcal{P}$ is a line spread of π. For the largest maximal partial line spread, $\mathcal{P}$ is the union of line spreads of $q^{\delta(i)}+1$ hyperplanes of generators through π. These hyperplanes each cover a $(2n-2)$-dimensional space of π and are chosen such that they overlap in a common $(2n-3)$-dimensional space of π. Their corresponding line spreads coincide inside this common $(2n-3)$-dimensional space.

Proof. We will prove this by induction on n. Suppose $n=1$. We consider the quadric $Q_{4+i}(q) \in \{Q^+(5,q), Q(6,q), Q^-(7,q)\}$, so $i \in \{1,2,3\}$ for the respective quadrics.
Consider a line ℓ of the quadric and its tangent space $\ell^\perp$, we know $\ell^\perp \cap Q_{4+i}(q) = \ell Q_i(q)$, which consists of $q^{i-1}+1$ planes intersecting each other in the line ℓ.
Using the construction seen in the proof of Lemma **??**, we find a line spread $\mathcal{S}$ for the quadric outside of $\ell^\perp$ consisting of q^{i+2} lines. These lines arise from reguli of quadrics $Q^+(3,q)$ through ℓ contained in the original quadric.
For the points of the quadric in $\ell^\perp$, we can find a maximal partial line spread $\mathcal{P}$ consisting of either the line ℓ or respectively 2, $q+1$ or $q+1$ lines belonging to different planes, and intersecting the line ℓ in different points.
We find a suitable maximal partial line spread of the quadric $Q_{4+i}(q)$ of size $q^{i+2}+1$ or $q^{i+2}+q^{\delta(i)}+1$.
Suppose the lemma is true for $n-1$. Consider the quadric $Q_{4n+i}(q)$, take a line ℓ and the tangent space $\ell^\perp$, we have $\ell^\perp \cap Q_{4n+i}(q) = \ell Q_{4(n-1)+i}(q)$. Again, we can partition the quadric outside of $\ell^\perp$ by q^{4n-2+i} pairwise disjoint lines arising from reguli through ℓ contained in the quadric.
Using the induction hypotheses, consider an appropriate maximal partial line spread $\mathcal{M}' = \mathcal{S}' \cup \mathcal{P}'$ for the quadric $Q_{4(n-1)+i}(q)$. Suppose $\pi'^\perp$ is the tangent space to a $(2n-3)$-dimensional space π' inside this quadric $Q_{4(n-1)+i}(q)$ for which $\mathcal{P}'$ is a maximal partial line spread. We have $\pi'^\perp \cap Q_{4(n-1)+i}(q) = \pi' Q_i(q)$. This space consists of $q^{i-1}+1$ generators of the quadric $Q_{4(n-1)+i}(q)$. Every point of the quadric $Q_{4(n-1)+i}(q)$ outside of $\pi'^\perp$ is covered by the line spread $\mathcal{S}'$.
Suppose $\pi = \langle \ell, \pi' \rangle$, we have $\pi^\perp = \langle \ell, \pi'^\perp \rangle$, where $\pi^\perp$ is the tangent space of π with respect to the quadric $Q_{4n+i}(q)$ and $\pi'^\perp$ is the tangent space of π' with respect to the quadric $Q_{4(n-1)+i}(q)$. Every line of $\mathcal{M}'$ spans a three-dimensional space together with ℓ; these spaces intersect each other only in ℓ. In every such space, we take a line spread through ℓ. The points of the quadric, not on ℓ, inside the 3-spaces generated by ℓ and a line of $\mathcal{S}'$ will be partitioned by $q^2|\mathcal{S}'|$ lines. These lines, together with the lines found outside of $\ell^\perp$, form a line spread $\mathcal{S}$ of $Q_{4n+i}(q) \backslash \pi^\perp$. We have $|\mathcal{S}'| = q^{2n-2+i}\left(\frac{q^{2n-2}-1}{q^2-1}\right)$.
From this, it follows:

$$|\mathcal{S}| = q^{4n-2+i} + q^2|\mathcal{S}'| = \frac{q^{4n+i} - q^{4n-2+i} + q^{4n-2+i} - q^{2n+i}}{q^2-1} = q^{2n+i}\frac{q^{2n}-1}{q^2-1}.$$

The space $\pi^\perp$ consists of $q^{i-1}+1$ $(2n)$-dimensional spaces of the quadric $Q_{4n+i}(q)$ that intersect each other in the $(2n-1)$-dimensional space π.
Inside the space $\pi^\perp$, we consider a maximal partial line spread $\mathcal{P}$ consisting of either a line spread of π or the union of line spreads of $q^{\delta(i)}+1$ hyperplanes of generators. These hyperplanes each cover a $(2n-2)$-dimensional space of π and are chosen such that they overlap in a common $(2n-3)$-dimensional space of π. Their corresponding line spreads coincide inside this common $(2n-3)$-dimensional space; this is possible because of Lemma **??**. We note that for the parabolic and elliptic quadric all the points of π are covered. It is clear that $\mathcal{P}$ has the appropriate size,

namely:

$$|\mathcal{P}| = \frac{q^{2n}-1}{q^2-1} \text{ or } |\mathcal{P}| = (q^{\delta(i)}+1)q^{2n-2} + \frac{q^{2n-2}-1}{q^2-1} = \frac{q^{2n}-1}{q^2-1} + q^{2n-2+\delta(i)}.$$

Since $\mathcal{P}$ is a maximal partial line spread in $\pi^\perp \cap Q_{4n+i}(q)$, it is clear that the union $\mathcal{M} = \mathcal{S} \cup \mathcal{P}$ is a maximal partial line spread of $Q_{4n+i}(q)$, that furthermore meets the conditions of the lemma. □

4.2 The cone $\ell Q_n(q)$

Lemma 4.2. *Consider a non-singular quadric $Q_{4n+i}(q)$ of $PG(4n+i,q)$. Suppose we have a maximal partial line spread of $Q_{4n+i}(q)$ of the form $\mathcal{M} = \mathcal{S} \cup \mathcal{P}$, such as considered in Lemma* **??**. *Suppose $Q_{\mathcal{S}}$ is the part of the quadric covered by $\mathcal{S}$, so outside of $\pi^\perp$, where π is the particular $(2n-1)$-dimensional space contained in $Q_{4n+i}(q)$. We can choose d lines from $\mathcal{S}$ such that every line of the quadric $Q_{4n+i}(q)$ outside of $\pi^\perp$, not in $\mathcal{S}$, intersects at least one of these d lines. We can choose such $d = d_{4n+i}$ lines for every integer d in the interval:*

$$d_{4n+i} \in \left[(2+(8n-4+2i)\log 2q)q^{4n-3+i},\ q^{2n+i}\frac{q^{2n}-1}{q^2-1}\right].$$

Table **??** *shows how large q has to be such that for a quadric with given n the determined interval for d is non-empty.*

$q\backslash n$	1	2	3	4	5	6	7	8	9	10	11
$Q^+(4n+1,q)$	27	73	127	179	239	307	359	421	487	557	617
$Q(4n+2,q)$	37	89	137	193	251	311	373	439	503	569	631
$Q^-(4n+3,q)$	49	97	151	211	269	331	389	457	521	587	647

Table 6: Minimal size of q for every quadric corresponding to Lemma **??**

Proof. The proof is analogous to that of Lemma **??**. Consider the line spread $\mathcal{S}$. We construct a bipartite graph with vertex classes A and B. Class A consists of all the lines of $\mathcal{S}$ and class B consists of all the lines of $Q_{\mathcal{S}}$ not belonging to the line spread $\mathcal{S}$. Every line of $Q_{\mathcal{S}}$ not belonging to $\mathcal{S}$ intersects at least q lines of $\mathcal{S}$, so every vertex of B has degree at least q.
We know $|A| = q^{2n+i}\frac{q^{2n}-1}{q^2-1} \leq 2q^{4n-2+i}$ and for every type of quadric, $D \leq (q+1)(|Q_{4n+i-2}(q)|-1) \leq 2q^{4n-2+i}$ for $q > 3$.
Using Result **??**, we find a subset $A' \subseteq A$ such that every line of $Q_{\mathcal{S}}$ not belonging to $\mathcal{S}$ intersects a line of A'. We find:

$$|A'| \leq |A|\frac{(1+\log D)}{q} \leq 2q^{4n-3+i}(1+(4n-2+i)\log 2q).$$

The set A' can be enlarged step by step until we find the whole set A, consisting of all the lines of the line spread $\mathcal{S}$. □

Construction 4.3. *Suppose the non-singular quadric $Q_n(q)$ in $PG(n,q)$ has a maximal partial line spread $\mathcal{M} = \mathcal{S} \cup \mathcal{P}$, as constructed in Lemma* **??**.
Consider a non-singular quadric $Q_{n+4}(q)$, of the same type as $Q_n(q)$, and take an arbitrary line ℓ of $Q_{n+4}(q)$. Consider its tangent space $\ell^\perp \cong PG(n+2,q)$, we know: $\ell^\perp \cap Q_{n+4}(q) = \ell Q_n(q)$. The quadric $Q_n(q)$ lies in a space $\pi \cong PG(n,q)$.

Consider the line spread $\mathcal{S} = \{t_1, \ldots, t_s\}$ and the maximal partial line spread $\mathcal{P} = \{m_1, \ldots, m_r\}$ on $Q_n(q)$. Suppose $Q_{\mathcal{S}}$ to be the part of the quadric $Q_n(q)$ covered by the lines of $\mathcal{S}$.
Suppose we can choose d lines $t_1, \ldots, t_d$ of $\mathcal{S}$, such that every line of $Q_{\mathcal{S}}$, not in $\mathcal{S}$, intersects at least one of these d lines.
*Consider the three-dimensional spaces $\langle \ell, t_1 \rangle, \ldots, \langle \ell, t_s \rangle, \langle \ell, m_1 \rangle, \ldots, \langle \ell, m_r \rangle$; these are completely contained in $Q_{n+4}(q)$. Unlike Construction **??**, not every point of $\ell^{\perp} \cap Q_{n+4}(q)$ belongs to one of these three-dimensional spaces; this because $\mathcal{M}$ is not a line spread of $Q_n(q)$.*
For $i = 1, \ldots, d$, take a line spread $\mathcal{S}_i$ in the space $\langle \ell, t_i \rangle$ containing ℓ. For $j = d+1, \ldots, s$, take a maximal partial line spread $\mathcal{P}_j$ in the space $\langle \ell, t_j \rangle$ through ℓ. For $h = 1, \ldots, r$, take a line spread $\mathcal{R}_h$ in the space $\langle \ell, m_h \rangle$ through ℓ.
The union $\mathcal{S}'$ of the line spreads $\mathcal{S}_i$ and $\mathcal{R}_h$, together with the lines of the maximal partial line spreads $\mathcal{P}_j$, forms a maximal partial line spread in $\ell Q_n(q)$.

Proof. The intersection of two considered three-dimensional spaces through ℓ is the line ℓ. So the set $\mathcal{S}'$ is a partial line spread. Suppose b is a line of $\ell Q_n(q)$ not belonging to $\mathcal{S}'$; we will show that this line cannot be added to $\mathcal{S}'$. We can assume that b is skew to ℓ.
Suppose b belongs to a three-dimensional space $\langle \ell, t_i \rangle$, then at least one point of b is covered.
Suppose b belongs to a three-dimensional space $\langle \ell, m_h \rangle$, then every point of b is covered.
Suppose b does not belong to one of these three-dimensional spaces. The projection of b from the line ℓ on π gives a line b', different from every line t_i and m_h.
Suppose b' belongs to $Q_{\mathcal{S}}$. Because of the property of the d lines, there is at least one line from $\{t_1, \ldots, t_d\}$ that intersects b', suppose t_1. The space $\langle \ell, t_1 \rangle$ contains one point of b, suppose P. In $\langle \ell, t_1 \rangle$, we have considered a line spread $\mathcal{S}_1$ which contains a line that intersects b in P.
Suppose b' does not belong to $Q_{\mathcal{S}}$, then at least one line m_i intersects the line b'; suppose m_1. We have considered a line spread $\mathcal{R}_1$ in $\langle \ell, m_1 \rangle$; this spread contains a line intersecting b in a point.
We have shown that the partial line spread $\mathcal{S}'$ is maximal. □

If we use this construction for $\ell Q_n(q)$ in Construction **??**, then we obtain again a maximal partial line spread of $Q_{n+4}(q)$. The proof proceeds in exactly the same way as in that of Construction **??**.

4.3 The quadric $Q_{4n+i}(q)$

Suppose $Q_n(q)$ is a non-singular quadric in $PG(n, q)$ containing a maximal partial line spread $\mathcal{M} = \mathcal{S} \cup \mathcal{P}$, as constructed in Lemma **??**. Define d_n to be the smallest number of lines of $\mathcal{S}$ such that every line of $Q_{\mathcal{S}}$, not in $\mathcal{S}$, intersects at least one of the d_n lines.
Recall the following notations:

- k_n: the size of the smallest known subset of lines of a particular line spread of $PG(n, q) \backslash PG(n-2, q)$, such that every line, not in the line spread, intersects a line of the set,
- $[l, u]$: interval of sizes of maximal partial line spreads of $PG(3, q)$,
- s: size of the smallest known maximal partial line spread of $PG(3, q)$,
- d_n: the smallest number of lines of $\mathcal{S}$ such that every line of $Q_{\mathcal{S}}$, not in $\mathcal{S}$, intersects at least one of the d_n lines.

Theorem 4.4. *Consider the non-singular quadric $Q_{4n-4+i}(q)$ in $PG(4n-4+i, q)$, $n \geq 2$, with $i \in \{1, 2, 3\}$, corresponding to a quadric in the set $\{Q^+(4n-3, q), Q(4n-2, q), Q^-(4n-1, q)\}$. In $\ell Q_{4n-4+i}(q)$, there exist maximal partial line spreads of every size in the interval*

$$\left[d_{4n-4+i}(q^2-s+1)+\left(\frac{q^{2n-2}-1}{q^2-1}\right)\left(q^{2n-2+i}(s-1)+q^2\right)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)+1\right.$$
$$\left., \frac{q^{2n-2}-1}{q^2-1}\left(q^{2n+i}+q^2\right)+q^{2n-2+\delta(i)}-q^2+u\right].$$

Proof. In Lemma **??** we constructed two maximal partial line spreads in $Q_{4n-4+i}(q)$. From each of these we can construct an uninterrupted interval of maximal partial line spreads in $\ell Q_{4n-4+i}(q)$. One can calculate that these intervals overlap if they are non-empty. So we will use the smallest form to calculate the lower bound and the largest form to calculate the upper bound of our interval.

Consider in $Q_{4n-4+i}(q)$ a maximal partial line spread of size $(q^{2n-2+i}+1)\left(\frac{q^{2n-2}-1}{q^2-1}\right)$ and use Construction **??** with value d_{4n-4+i}. Choose for every $\mathcal{P}_i$ the smallest known maximal partial line spread of $PG(3,q)$; this has size s. This gives a maximal partial line spread of size:

$$d_{4n-4+i}q^2+\left(q^{2n-2+i}\frac{q^{2n-2}-1}{q^2-1}-d_{4n-4+i}\right)(s-1)+\left(\frac{q^{2n-2}-1}{q^2-1}\right)q^2+1.$$

To find an uninterrupted interval we proceed in the same way as in Theorem **??**. In at least x three-dimensional spaces, we will consider maximal partial line spreads with their sizes in the interval $[l,u]$. Similarly as in the earlier proof, we find that $x\geq\frac{q^2-s+1}{u-l}$. We find the lower bound:

$$d_{4n-4+i}q^2+\left(q^{2n-2+i}\frac{q^{2n-2}-1}{q^2-1}-d_{4n-4+i}\right)(s-1)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)+\left(\frac{q^{2n-2}-1}{q^2-1}\right)q^2+1.$$

Step by step we can enlarge the maximal partial line spreads $\mathcal{P}_i$ or exchange them for a line spread $\mathcal{S}_i$. It is clear that we find an uninterrupted interval.

Now for our upper bound. This we find by taking $d=q^{2n-2+i}\frac{q^{2n-2}-1}{q^2-1}-1$ and the part $\mathcal{P}$ of its largest form of size $\frac{q^{2n-2}-1}{q^2-1}+q^{2n-4+\delta(i)}$. Then the largest value in the uninterrupted interval is:

$$\left(q^{2n-2+i}\frac{q^{2n-2}-1}{q^2-1}-1\right)q^2+u-1+\left(\frac{q^{2n-2}-1}{q^2-1}+q^{2n-4+\delta(i)}\right)q^2+1$$
$$=\frac{q^{2n-2}-1}{q^2-1}\left(q^{2n+i}+q^2\right)+q^{2n-2+\delta(i)}-q^2+u.$$

□

Corollary 4.5. *Using Result* **??** *and* **??** *together with Corollary* **??** *and the estimate* $7\log q\leq 7\log 2q-4$*, we know the values for our variables, with restrictions* $n\geq 2$*,* $q\geq 7$ *and for even* q *also* $q>q_0$*. For the following particular quadrics, there exist maximal partial line spreads for every size in the interval:*

$$\ell Q^+(4n-3,q):\left[(-2+(8n-3)\log 2q)\,q^{4n-4},\ \frac{q^{4n-1}+q^{2n}-q^{2n+1}-1}{q^2-1}+q^{2n-2}-q+1\right],$$

$$\ell Q(4n-2,q):\left[(-2+(8n-1)\log 2q)\,q^{4n-3},\ \frac{q^{4n}-q^{2n+2}+q^{2n}-1}{q^2-1}+q^{2n-1}-q+1\right],$$

$$\ell Q^-(4n-1,q):\left[(-2+(8n+1)\log 2q)\,q^{4n-2},\ \frac{q^{4n+1}-q^{2n+3}+q^{2n}-1}{q^2-1}+q^{2n-1}-q+1\right].$$

Theorem 4.6. *Consider the non-singular quadric* $Q_{4n+i}(q)$ *in* $PG(4n+i,q)$*,* $n\geq 2$*, with* $i\in\{1,2,3\}$*, corresponding to a quadric in the set* $\{Q^+(4n+1,q),Q(4n+2,q),Q^-(4n+3,q)\}$*. There exist maximal partial line spreads* $\mathcal{S}'$ *in* $Q_{4n+i}(q)$ *of every size in the interval:*

$$\left[(qk_{4n-4+i}+d_{4n-4+i})(q^2-s+1)+\tfrac{q^{2n}-1}{q^2-1}q^{2n-2+i}(s-1)+\left\lceil\tfrac{q^2-s+1}{u-l}\right\rceil(q+1)(l-s)+\tfrac{q^{2n}-1}{q^2-1}\right.$$
$$\left., \tfrac{(q^{2n+i}+1)(q^{2n}-1)}{q^2-1}+q^{2n-2+\delta(i)}+u-q^2-1\right].$$

Proof. We look at the maximal partial line spread from Construction **??**, but using Construction **??** for the cone $\ell Q_{4n-4+i}(q)$.
We use Theorem **??** to find a maximal partial line spread $\mathcal{S}_1'$ in $\ell Q_{4n-4+i}(q)$ for every size in the interval

$$\left[d_{4n-4+i}(q^2-s+1)+\left(\tfrac{q^{2n-2}-1}{q^2-1}\right)\left(q^{2n-2+i}(s-1)+q^2\right)+\left\lceil\tfrac{q^2-s+1}{u-l}\right\rceil(l-s)+1\right.$$
$$\left., \tfrac{q^{2n-2}-1}{q^2-1}\left(q^{2n+i}+q^2\right)+q^{2n-2+\delta(i)}-q^2+u\right].$$

In $(PG(4n-2+i,q)\backslash PG(4n-4+i,q))\cup\{t_1\}$, using Theorem **??** we find a maximal partial line spread $\mathcal{S}_2$ of every size in the interval:

$$\left[k_{4n-4+i}q^2+(q^{4n-5+i}-k_{4n-4+i})(s-1)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)+1, q^{4n-3+i}-q^2+u\right].$$

Since the three-dimensional space $\langle\ell,t_1\rangle$ is completely covered by a line spread, every line of $Q_{4n+i}(q)\backslash\ell^\perp$ that after projection from ℓ is projected on a line that intersects t_1, is also covered by the maximal partial line spread $\mathcal{S}_1'$, see Construction **??**. We will concentrate on the particular partial line spread in $PG(4n-2+i,q)\backslash PG(4n-4+i,q)$ that covers all the lines disjoint from t_1, and we still use the notation $\mathcal{S}_2$.
The constructed maximal partial line spread has size $|\mathcal{S}'|=|\mathcal{S}_1'|+q|\mathcal{S}_2|$. If the length of the interval of $\mathcal{S}_1'$ is larger than $(q^2-u)q$, we find an uninterrupted interval and then we can make the jump from the maximal partial line spread $\mathcal{S}_2$ of size $q^{4n-3+i}-q^2+u$ to a line spread of size q^{4n-3+i}. This gives the condition:

$$\begin{aligned}&\frac{q^{2n-2}-1}{q^2-1}\left(q^{2n+i}+q^2\right)+q^{2n-2+\delta(i)}-q^2+u\\&-\left(d_{4n-4+i}(q^2-s+1)+\left(\frac{q^{2n-2}-1}{q^2-1}\right)\left(q^{2n-2+i}(s-1)+q^2\right)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)+1\right)\geq(q^2-u)q.\end{aligned}\tag{3}$$

For given n, the inequality in (**??**) is valid for the respective quadrics $Q_{4n+i}(q)$ if q is large enough. For small n, the lower bounds on q are presented in Table **??**.
If we take $\mathcal{S}_1'$ to be the largest possible maximal partial line spread and for $\mathcal{S}_2$ a line spread, then we find size:

$$\frac{q^{2n-2}-1}{q^2-1}\left(q^{2n+i}+q^2\right)+q^{2n-2+\delta(i)}-q^2+u+q^{4n-3+i}q=\frac{(q^{2n+i}+1)(q^{2n}-1)}{q^2-1}+q^{2n-2+\delta(i)}+u-q^2-1.$$

This is the largest maximal partial line spread for $Q_{4n+i}(q)$ found by our construction.
The smallest appropriate partial line spread in $PG(4n-2+i,q)\backslash PG(4n-4+i,q)$ has size $k_{4n-4+i}q^2+(q^{4n-5+i}-k_{4n-4+i})(s-1)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)$. Every such line gives us q lines of the sought maximal partial line spread $\mathcal{S}'$. We add the lines of the smallest maximal partial line spread in $\ell Q_{4n-4+i}(q)$; this has size $d_{4n-4+i}(q^2-s+1)+\left(\frac{q^{2n-2}-1}{q^2-1}\right)\left(q^{2n-2+i}(s-1)+q^2\right)+\left\lceil\frac{q^2-s+1}{u-l}\right\rceil(l-s)+1$. The lower bound of our uninterrupted interval is

$$(k_{4n-4+i}q^2 + (q^{4n-5+i} - k_{4n-4+i})(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-s))q$$
$$+d_{4n-4+i}(q^2 - s + 1) + \left(\tfrac{q^{2n-2}-1}{q^2-1}\right)\left(q^{2n-2+i}(s-1) + q^2\right) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (l-s) + 1$$
$$= (qk_{4n-4+i} + d_{4n-4+i})(q^2 - s + 1) + \left(q^{2n-2} + \tfrac{q^{2n-2}-1}{q^2-1}\right) q^{2n-2+i}(s-1) + \left\lceil \tfrac{q^2-s+1}{u-l} \right\rceil (q+1)(l-s) + \tfrac{q^{2n}-1}{q^2-1}.$$
$\square$

We now summarize the spectrum result for Construction 2. The second interval is valid for $Q(4n+2,q)$, q even and odd, but for $W(4n+1,q)$, only for q even.

Corollary 4.7. *Using Result* **??** *and* **??**, *Lemma* **??** *and Corollary* **??** *we know the values of our variables together with the constraints $n \geq 2$, $q \geq 7$, and for q even also $q > q_0$, so for particular quadrics there exist maximal partial line spreads of every size in the following interval:*
$Q^+(4n+1,q)$:

$$\left[(-3 + (4n+4)\log 2q)q^{4n-2} + (-2 + (8n-3)\log 2q)\, q^{4n-4}, \frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1} + q^{2n-2} - q + 1\right].$$

$W(4n+1,q)$, q even, and $Q(4n+2,q)$:

$$\left[(-3 + (4n+5)\log 2q)q^{4n-1} + (-2 + (8n-1)\log 2q)\, q^{4n-3}, \frac{(q^{2n+2}+1)(q^{2n}-1)}{q^2-1} + q^{2n-1} - q + 1\right].$$

$Q^-(4n+3,q)$:

$$\left[(-3 + (4n+6)\log 2q)q^{4n} + (-2 + (8n+1)\log 2q)\, q^{4n-2}, \frac{(q^{2n+3}+1)(q^{2n}-1)}{q^2-1} + q^{2n-1} - q + 1\right].$$

Table **??** *shows how large q has to be such that for a given n the determined interval is non-empty and such that the condition we have put on the length of the interval of $|\mathcal{S}_1'|$ is fulfilled.*

$q\backslash n$	2	3	4	5	6	7	8	9	10	11	12
$Q^+(4n+1,q)$	53	79	107	131	163	191	223	251	277	307	337
$Q(4n+2,q)$	61	89	113	139	167	197	223	257	283	313	347
$Q^-(4n+3,q)$	67	97	127	149	173	211	233	263	293	331	353

Table 7: Minimal size of q for quadric corresponding to Corollary **??**

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