High Precision Evaluation of the Selfpatch Integral for Linear Basis Functions on Flat Triangles

Ignace Bogaert and Daniel De Zutter, *Fellow, IEEE*

***Abstract*—The application of integral equations for the frequency domain analysis of scattering problems requires the accurate evaluation of interaction integrals. Generally speaking, the most critical integral is the selfpatch. However, due to the non-smoothness of the Green function, this integral is also the toughest to calculate numerically. In previous work, the source and test integrals have been determined analytically for the $\frac{1}{R}$ singularity, i.e. the static kernel. In this work we extend this result to the terms of the form $R^n,\ \forall n \in \{0,1,2,3,4\}$ that occur in the Taylor expansion of the Green function. Numerical testing shows that truncating the Taylor series beyond $n = 4$ yields a highly accurate result for $\frac{\lambda}{7}$ and $\frac{\lambda}{10}$ discretizations. These analytical formulas are also very robust when applied to highly irregular triangles.**

***Index Terms*—Selfpatch, Triangular domains, Linear basis functions, Analytical, High accuracy**

I. Introduction

CALCULATING interaction integrals is imperative when solving integral equations using the method of moments. For patches that do not touch at all, the interaction integrals can be computed up to high precision by simply utilizing Gaussian quadrature. For touching patches, the non-smoothness of the Green function makes the calculation of the integrals nontrivial. Although there exists a host of techniques for integrating the Green function over the source triangle [1], [2], the subsequent test integration is usually performed using a simple Gaussian quadrature rule for triangles. The integrand of the test integration is bounded, but still exhibits discontinuities in its first derivative. Therefore, Gaussian quadrature converges relatively slowly with increasing integration order. This is especially true for the selfpatch, i.e. the interaction integral of a triangle with itself. The selfpatch is at the same time the most singular *and* the most dominant integral, so that formulas for its accurate and fast calculation are certainly of value.

In previous work [3], [4], explicit formulas were given for the selfpatch with the static Green function, i.e. the $\frac{1}{R}$ kernel. In this work, we will supply additional formulas for kernels of the form $R^n,\ \forall n \in \{0,1,2,3,4\}$. As will be shown in Section III, this allows the highly accurate and fast evaluation of the selfpatch when the frequency differs from zero.

Let the triangle for which we wish to calculate the selfpatch have vertices $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$. The familiar barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_3 = 1 - \lambda_1 - \lambda_2$ will be used to parameterize this triangle

$$\boldsymbol{r} = \sum_{i=1}^{3} \lambda_i \boldsymbol{r}_i. \tag{1}$$

The following discussion will be limited to linear basis functions, which includes the important cases of RWG and linear-linear basis functions [5], [6]. Therefore, we only need to consider the following integral

$$I_{p,q} = 4A^2 \int_0^1\int_0^{1-\lambda_2}\int_0^1\int_0^{1-\lambda_2'} \lambda_p' \lambda_q G_k(\boldsymbol{r}-\boldsymbol{r}')\, \mathrm{d}\lambda_1' \mathrm{d}\lambda_2' \mathrm{d}\lambda_1 \mathrm{d}\lambda_2, \tag{2}$$

with wavenumber k, Green function

$$G_k(\boldsymbol{r}-\boldsymbol{r}') = \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{3}$$

and A the surface area of the triangle. The position vector $\boldsymbol{r}'$ is defined similarly to (1)

$$\boldsymbol{r}' = \sum_{i=1}^{3} \lambda_i' \boldsymbol{r}_i. \tag{4}$$

Clearly, if $I_{p,q}$ is known, the integrals

$$I_p = 4A^2 \int_0^1\int_0^{1-\lambda_2}\int_0^1\int_0^{1-\lambda_2'} \lambda_q G_k(\boldsymbol{r}-\boldsymbol{r}')\, \mathrm{d}\lambda_1' \mathrm{d}\lambda_2' \mathrm{d}\lambda_1 \mathrm{d}\lambda_2, \tag{5}$$

$$I = 4A^2 \int_0^1\int_0^{1-\lambda_2}\int_0^1\int_0^{1-\lambda_2'} G_k(\boldsymbol{r}-\boldsymbol{r}')\, \mathrm{d}\lambda_1' \mathrm{d}\lambda_2' \mathrm{d}\lambda_1 \mathrm{d}\lambda_2, \tag{6}$$

can be easily computed from integral (2) as follows

$$I_p = \sum_{q=1}^{3} I_{p,q}, \tag{7}$$

$$I = \sum_{p=1}^{3} I_p. \tag{8}$$

We therefore turn to the computation of integral (2). Assuming that the size of the triangle is significantly smaller than the wavelength $\frac{2\pi}{k}$, it makes sense to expand the Green function (3) into a Taylor series in the wavenumber k. This in turn converts the selfpatch integral to a Taylor series

$$I_{p,q} = \frac{A^2}{\pi} \sum_{n=0}^{\infty} (-jk)^n I_{p,q}^{n-1} \tag{9}$$

I. Bogaert and D. De Zutter are with the Department of Information Technology, Ghent University, Belgium. The work of I. Bogaert is supported by a postdoctoral grant from the Research Foundation Flanders (FWO Vlaanderen). E-mail: Ignace.Bogaert@intec.ugent.be and Daniel.Dezutter@intec.UGent.be.

with $I^n_{p,q}$ defined as

$$I^n_{p,q} = \int_0^1\int_0^{1-\lambda_2}\int_0^1\int_0^{1-\lambda'_2} \lambda'_p\lambda_q \frac{|\boldsymbol{r}-\boldsymbol{r}'|^n}{(n+1)!}\mathrm{d}\lambda'_1\mathrm{d}\lambda'_2\mathrm{d}\lambda_1\mathrm{d}\lambda_2. \tag{10}$$

Strictly speaking, series (9) converges absolutely for all complex k. However, if the triangle is electrically large, i.e. if $|k|\,r_{\min} \gg 1$, with $r_{\min}$ the radius of the smallest sphere encircling the triangle, convergence may require a lot of terms and may even be numerically unstable. Therefore this Taylor representation is only practical when the electrical size of the triangle is sufficiently small. Except in the simulation of very good conductors, this is almost always the case in practice.

In the following sections, explicit expressions for $I^n_{1,1}$ and $I^n_{2,3}$, $\forall n \in \{-1, ..., 4\}$ will be given. These expressions depend solely on the three side lengths of the triangle

$$l_1 = |\boldsymbol{r}_3 - \boldsymbol{r}_2|\,, \tag{11a}$$

$$l_2 = |\boldsymbol{r}_1 - \boldsymbol{r}_3|\,, \tag{11b}$$

$$l_3 = |\boldsymbol{r}_2 - \boldsymbol{r}_1|\,, \tag{11c}$$

and cyclically permuting these lengths in the formulas yields the expressions for all $I^n_{p,q}$.

II. Analytical calculation of $I^n_{p,q}$

A. Even n

For even powers of R, the selfpatch is very easily obtained because the integrand is a polynomial of degree n. The following results are obtained:

$$I^0_{1,1} = I^0_{2,3} = \frac{1}{36}. \tag{12}$$

$$I^2_{1,1} = \frac{2l_2^2 + 2l_3^2 + l_1^2}{8640}, \tag{13a}$$

$$I^2_{2,3} = \frac{l_2^2 + l_3^2 + 3l_1^2}{5760}. \tag{13b}$$

$$I^4_{1,1} = \frac{(l_3^2-l_1^2)^2 + (l_1^2-l_2^2)^2 + 4l_2^2l_3^2 + 24(l_3^4+l_2^4) + 11l_1^4}{9072000}, \tag{14a}$$

$$I^4_{2,3} = \frac{4(l_2^2-l_3^2)^2 + 4l_1^2(l_3^2+l_2^2) + 9l_2^4 + 9l_3^4 + 37l_1^4}{4536000}. \tag{14b}$$

B. Odd n

For odd n, the integrand R^n is no longer a polynomial. In this case, the dimensionality of the integration can be reduced by first convolving the source and test triangles. Then the remaining two integrations can be performed in a straightforward manner, albeit after lengthy calculations. This technique has been used before, in [3], and has led to analytical expressions for $n = -1$ in equation (10), i.e. the selfpatch with $\frac{1}{R}$ kernel:

$$I^{-1}_{1,1} = \frac{1}{120}\left[6\gamma_1 + (l_1^2 - l_2^2 - l_3^2)\left(\gamma_3 - \frac{\ln_1}{l_1^3}\right)\right] + \frac{l_2-l_1}{60l_3^2} + \frac{l_3-l_1}{60l_2^2} \tag{15a}$$

$$I^{-1}_{2,3} = \frac{1}{80}\left[2\gamma_1 - (l_1^2 - l_2^2 - l_3^2)\left(\gamma_3 - \frac{\ln_1}{l_1^3}\right)\right] - \frac{l_2-l_1}{40l_3^2} - \frac{l_3-l_1}{40l_2^2}. \tag{15b}$$

In these expressions, the logarithms $\ln_i$ are defined as

$$\ln_i = -\ln\left(1 - 2\frac{l_i}{P}\right) \;\; \forall i \in \{1,2,3\} \tag{16}$$

with the triangle perimeter $P = l_1 + l_2 + l_3$. The parameters γ_n

$$\gamma_n = \sum_{i=1}^{3} \frac{\ln_i}{l_i^n}. \tag{17}$$

were also introduced to reduce the notational burden. Note that the γ_n are invariant under permutation of the triangle's vertices, so they remain the same in the formulas for all $I^n_{p,q}$. We will also define the parameter Q as

$$Q = 16A^2, \tag{18}$$

where A is still the surface area of the triangle. One of the reviewers pointed out that, for extremely elongated triangles, the argument $1 - 2\frac{l_i}{P}$ of the logarithm in Eqn. (16) is calculated inaccurately. For these extremely elongated triangles, the following more stable formula for calculating $1 - 2\frac{l_i}{P}$ can be derived

$$\begin{aligned} 1 - 2\frac{l_i}{P} &= \frac{1}{P^2}\left[(l_1+l_2)^2 - l_3^2\right], \\ &= \frac{4l_1l_2l_3}{P^2l_i}\cos^2\left(\frac{\alpha_i}{2}\right), \end{aligned} \tag{19}$$

with the angles α_i associated with vertices $\boldsymbol{r}_i$ calculated by means of the well-known $\mathrm{atan2}$ function

$$\alpha_i = \mathrm{atan2}\left(4A, (l_1^2 + l_2^2 + l_3^2) - 2l_i^2\right). \tag{20}$$

Equation (19) is stable as long as the surface area A of the triangle can be accurately determined from $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$. This is always the case in good meshes, since otherwise no accurate surface integration is possible.

The convolution technique can be applied to integrands of the form R^n without significant change. Indeed, only the two outer integrations are different when n changes, the convolution remains the same. Using this technique, we obtain the following explicit formulas for $n = 1$

$$
\begin{aligned}
I^1_{1,1} = {} & \frac{Q}{40320}\left[10\gamma_3 + 3\left(l_1^2 - l_2^2 - l_3^2\right)\left(\gamma_5 - \frac{\mathrm{ln}_1}{l_1^5}\right)\right] \\
& + \frac{1}{20160}\left(14 l_1 + 17 l_2 + 17 l_3\right) \\
& - \frac{(l_1+l_3)^2(l_3-l_1)^3}{6720 l_2^4} - \frac{(l_1+l_2)^2(l_2-l_1)^3}{6720 l_3^4} \\
& + \frac{(l_3+l_2)(l_2-l_3)^2}{2016 l_1^2} + \frac{(3l_3-l_1)(2l_3+l_1)(l_3-l_1)}{10080 l_2^2} \\
& + \frac{(3l_2-l_1)(2l_2+l_1)(l_2-l_1)}{10080 l_3^2},
\end{aligned}
\tag{21a}
$$

$$
\begin{aligned}
I^1_{2,3} = {} & \frac{Q}{16128}\left[2\gamma_3 - 3\left(l_1^2 - l_2^2 - l_3^2\right)\left(\gamma_5 - \frac{\mathrm{ln}_1}{l_1^5}\right)\right] \\
& + \frac{(l_1+l_3)^2(l_3-l_1)^3}{2688 l_2^4} + \frac{(l_1+l_2)^2(l_2-l_1)^3}{2688 l_3^4} \\
& + \frac{7l_2 + 7l_3 + 10l_1}{8064} + \frac{(l_3+l_2)(l_2-l_3)^2}{4032 l_1^2} \\
& - \frac{l_1(l_3+5l_1)(l_3-l_1)}{4032 l_2^2} - \frac{l_1(l_2+5l_1)(l_2-l_1)}{4032 l_3^2}.
\end{aligned}
\tag{21b}
$$

For $n = 3$ we find

$$
\begin{aligned}
I^3_{1,1} = {} & \frac{Q^2}{15482880}\left[14\gamma_5 + 5(l_1^2 - l_3^2 - l_2^2)\left(\gamma_7 - \frac{\mathrm{ln}_1}{l_1^7}\right)\right] \\
& + \frac{(l_3+l_1)^4(l_3-l_1)^5}{1548288 l_2^6} + \frac{(l_1+l_2)^4(l_2-l_1)^5}{1548288 l_3^6} \\
& - \frac{(l_2+l_3)^3(l_3-l_2)^4}{552960 l_1^4} \\
& - \frac{(14l_1^2 + 5l_1l_3 + 41l_3^2)(l_3+l_1)^2(l_3-l_1)^3}{11612160 l_2^4} \\
& - \frac{(14l_1^2 + 5l_2l_1 + 41l_2^2)(l_1+l_2)^2(l_2-l_1)^3}{11612160 l_3^4} \\
& + \frac{(11l_3^2 + 2l_2l_3 + 11l_2^2)(l_2+l_3)(l_3-l_2)^2}{1658880 l_1^2} \\
& + \frac{(l_2-l_1)(86l_2^4 + 23l_2^3l_1 + 19l_2^2l_1^2 + 5l_2l_1^3 - 13l_1^4)}{11612160 l_3^2} \\
& + \frac{(l_3-l_1)(86l_3^4 + 23l_3^3l_1 + 19l_1^2l_3^2 + 5l_1^3l_3 - 13l_1^4)}{11612160 l_2^2} \\
& + \frac{(l_2+l_3)(268l_3^2 - 169l_2l_3 + 268l_2^2)}{23224320} \\
& + l_1\frac{(56l_1^2 + 8(l_3+l_2)l_1 + 23(l_2^2+l_3^2))}{7741440},
\end{aligned}
\tag{22a}
$$

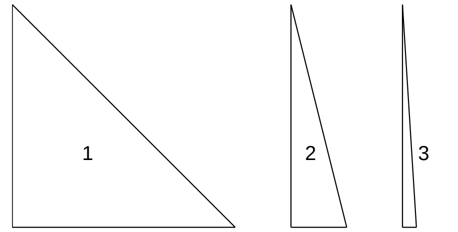

Fig. 1. The four test triangles.

$$
\begin{aligned}
I^3_{2,3} = {} & \frac{Q^2}{4423680}\left[2\gamma_5 - 5(l_1^2 - l_2^2 - l_3^2)\left(\gamma_7 - \frac{\mathrm{ln}_1}{l_1^7}\right)\right] \\
& - \frac{(l_1+l_2)^4(l_2-l_1)^5}{442368 l_3^6} - \frac{(l_1+l_3)^4(l_3-l_1)^5}{442368 l_2^6} \\
& - \frac{(l_2+l_3)^3(l_3-l_2)^4}{1105920 l_1^4} \\
& + \frac{(17l_2^2 + 5l_2l_1 + 38l_1^2)(l_1+l_2)^2(l_2-l_1)^3}{3317760 l_3^4} \\
& + \frac{(17l_3^2 + 5l_3l_1 + 38l_1^2)(l_1+l_3)^2(l_3-l_1)^3}{3317760 l_2^4} \\
& - \frac{(l_2-l_1)(l_2^2 + 3l_1^2)(25l_1^2 + 7l_2l_1 - 2l_2^2)}{3317760 l_3^2} \\
& - \frac{(l_3-l_1)(l_3^2 + 3l_1^2)(25l_1^2 + 7l_3l_1 - 2l_3^2)}{3317760 l_2^2} \\
& + \frac{(l_2+l_3)(11l_2^2 + 2l_3l_2 + 11l_3^2)(l_3-l_2)^2}{3317760 l_1^2} \\
& + l_1\frac{(l_3^2+l_2^2)}{245760} + l_1^2\frac{(l_2+l_3)}{92160} + l_1^3\frac{23}{829440} \\
& + \frac{(l_2+l_3)(28l_2^2 - 29l_3l_2 + 28l_3^2)}{2211840}.
\end{aligned}
\tag{22b}
$$

Although these formulas are more lengthy than those in (15), they do not require the evaluation of additional logarithms or square roots. Apparently, only three logarithms and three square roots are required for the explicit evaluation of at least the six first terms in series (9), which is very efficient.

III. Numerical results

The previously given formulas were implemented in Matlab and compared to a brute-force adaptive integration routine. Four different, increasingly sharp, triangles were used:

$$\boldsymbol{r}_1 = \begin{bmatrix}0 & 0 & 0\end{bmatrix}, \tag{23}$$

$$\boldsymbol{r}_2 = \begin{bmatrix}4^{-m+1} & 0 & 0\end{bmatrix}, \tag{24}$$

$$\boldsymbol{r}_3 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}. \tag{25}$$

with $m \in \{1, 2, 3, 4\}$. These triangles are shown in Figure 1 and are designed to investigate the robustness of the analytical formulas when the triangle becomes highly irregular.

The convergence of the Taylor series (9) is now investigated. For this, we truncate the series after the R^n term and compute the relative error with respect to the adaptive routine. In Figures 2 and 3, the relative error for integral (6) is plotted as a function of the truncation bound n. For Figures 2 and 3, the wavenumbers 1.333m^{-1} and 0.8889m^{-1} are respectively used. This corresponds to a $\frac{\lambda}{7}$ and $\frac{\lambda}{10}$ discretization for the largest

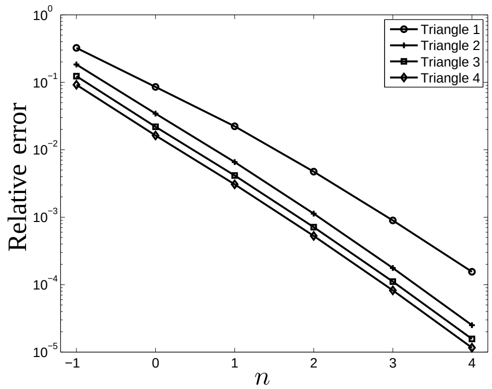

Fig. 2. The relative error on the selfpatch with $k = 1.333\text{m}^{-1}$ when all terms up to R^n are calculated analytically.

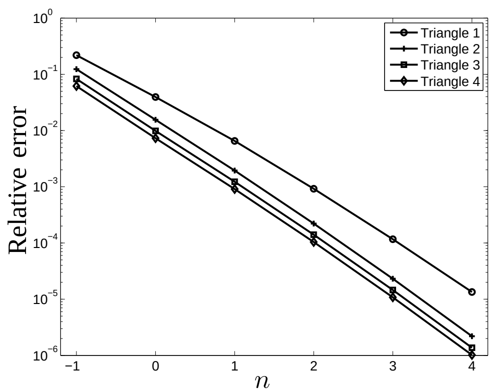

Fig. 3. The relative error on the selfpatch with $k = 0.8889\text{m}^{-1}$ when all terms up to R^n are calculated analytically.

test triangle (Triangle 1 in Figure 1). This explains why series (9) converges faster in Figure 3. When the Taylor series is truncated after the R^4 term, utilizing the analytical formulas from this paper, an accuracy around $2\ 10^{-4}$ and $2\ 10^{-5}$ is obtained for a $\frac{\lambda}{7}$ and $\frac{\lambda}{10}$ discretization respectively. For the thinner triangles, the convergence is even faster since these have a smaller electrical size. The thinner triangles also show that the formulas from this paper are as effective for elongated triangles as for more regular triangles.

For testing the computational efficiency, the formulas from this paper were implemented in the C programming language. Our numerical experiments show that, on an AMD Opteron 244 processor (1.8 GHz), the evaluation of 5 million selfpatches (all nine components of $I_{p,q}$) required only 19 seconds. It is useful to compare this to the time needed for evaluating a complex exponential (= the Green function with canceled singularity), which is one of the dominant costs in state of the art singularity cancellation schemes [1]. In 19 seconds, the same AMD processor was able to calculate around 65 million exponentials. This means that calculating the entire selfpatch using the formulas from this paper is roughly equivalent to evaluating only 13 exponentials. Clearly this is very efficient, especially when considering that both the source and test integration have been taken into account exactly.

IV. Conclusion

Exact, closed form expressions for the first six terms in the Taylor expansion of the selfpatch integral have been presented. The evaluation of these terms requires only three logarithms and three square roots plus elementary operations. For our test cases, it was shown that these first six terms are sufficient to obtain a $2\ 10^{-4}$ and $2\ 10^{-5}$ relative accuracy for $\frac{\lambda}{7}$ and $\frac{\lambda}{10}$ discretizations respectively. Therefore, no additional integration of a nonsingular part is necessary. Hence this way of evaluating the selfpatch is extremely efficient. The analytical formulas also have the additional advantage of being very robust for highly irregular triangles.

References

[1] M.A. Khayat and D.R. Wilton, "Numerical evaluation of singular and near-singular potential Integrals," *IEEE Transactions on Antennas and Propagation*, vol. 53, no. 10, pp. 3180–3190, October 2005.

[2] R.D. Graglia and G. Lombardi, "Machine Precision Evaluation of Singular and Nearly Singular Potential Integrals by Use of Gauss Quadrature Formulas for Rational Functions," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 4, pp. 981–998, April 2008.

[3] T.F. Eibert and V. Hansen, "On the calculation of potential integrals for linear source distributions on triangular domains," *IEEE Transactions on Antennas and Propagation*, vol. 43, no. 12, pp. 1499–1502, Dec 1995.

[4] P. Arcioni and M. Bressan and L. Perregrini, "On the evaluation of the double surface integrals arising in the application of the boundary integral method to 3-D problems," *IEEE Transactions on Microwave Theory and Techniques*, vol. 45, no. 3, pp. 436–439, Mar 1997.

[5] S.D. Rao and D.R. Wilton and A.W. Glisson, "Electromagnetic scattering by surfaces of arbitrary shape," *IEEE Transactions on Antennas and Propagation*, vol. 30, no. 3, pp. 409–418, May 1982.

[6] L.C. Trintinalia and H. Ling, "First Order Triangular Patch Basis Functions for Electromagnetic Scattering Analysis," *Journal of Electromagnetic Waves and Applications*, vol. 15, no. 11, pp. 1521–537, May 2001.