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ACHIEVING SNAKY

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Abstract

We prove that the polyomino generally known as snaky is a three-dimensional winner, that it loses on an 8×8 board, and that its handicap number is at most one.

1. Introduction and Results

In a polyomino achievement game two players (white and black) alternately mark cells on a k-dimensional rectangular board. The player who marks a set of cells congruent to a given polyomino wins the game. Following the convention of Go, we suppose that black begins the game. Clearly, there is no winning strategy for white (for rigorous proofs see [2], [6]). Thus we consider the weak achievement game in which white wins if she can prevent black from winning. A polyomino is called a winner on a certain board if black wins the weak

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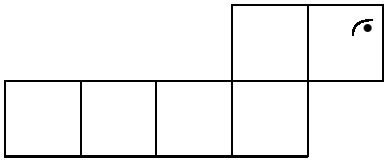


Figure 1: Snaky

achievement game on this board. It is called a k-dimensional winner if black wins the weak achievement game on every k-dimensional board which is sufficiently large in each direction. In two dimensions there are 11 polyominoes which are known to win, whereas for all but 12 polyominoes there exists a paving strategy for white. The remaining polyomino whose status is not decided is commonly called snaky.

Sieben showed that Snaky is a 41-dimensional winner [7]. Harborth and Seemann showed that Snaky is a paving winner; that is, there does not exist a paving strategy for white [6], that Snaky loses on a 6×6-board, and that black loses if her moves are restricted to cells adjacent to cells already marked black [5]. Harary [3],[4] showed that the handicap number of Snaky is at most 2; that is, black wins if she is allowed to put two extra black stones on the board at the beginning of the game.

In this article we will strengthen these results as follows.

Theorem 1 *1. Snaky is a three-dimensional winner.*

2. The handicap number of Snaky is at most 1.

3. Snaky loses on an 8×8 board.

The most general results concerning polyomino achievement games are obtained via the pseudo-probabilistic method developed by Beck [1]. This method constructs strategies as follows. The value of a position is the expected number of polyominoes of the desired type which would be obtained if all cells not yet chosen would be filled at random. A player is said to follow a Beck-strategy if she aims at minimizing respectively maximizing the value of a position at each move. This approach is by far the most successfull method in the theory of achievement games. However, we show that it cannot be used to prove that Snaky is a two-dimensional loser.

Theorem 2 *If white follows a Beck-strategy, then black can achieve Snaky.*

Note that a Beck-strategy is far from unique, since in general there are several cells which induce a maximal change of the value of a position.

2. Snaky Loses on an 8×8 Board

In this section we describe a winning strategy for white on an 8×8 board. This strategy was found with the help of a computer using the following approach. In general, we follow a branch-and-cut algorithm; that is, in each situation we have to find for each possible move of black a suitable answer for white. Clearly, a simple tree-search would collapse after at

most 4 moves. Therefore it has to be supplemented by an efficient cut-algorithm. Instead of playing the game until no snaky is possible anymore, we try to find pavings which provide winning strategies for white. To find such pavings, the algorithm chooses a snaky which is not yet blocked and chooses a pair of cells to block that snaky. This is done recursively for all snakys. For efficiency reasons, the algorithm first takes care of the snakys with the least number of cells left for the blocking pair.

If the algorithm in this way manages to construct a pairing, then most of the time, not all cells are used up, so this pairing yields a winning strategy for white not only in the situation where the pairing was constructed, but also if black has played on one of the unused cells before. We will exploit this phenomenon repeatedly. For example, suppose for a certain first move of black ❶, we have a list of situations consisting of this black stone, two white stones ② and ④, and a paving, such that the following holds true:

1. ② is in the same position on all boards;
2. For each cell C other than the ones of ❶ and ②, there is a situation in our list, such that C is neither taken by ④ nor contained in a pair;
3. Each paving defines a winning strategy for white.

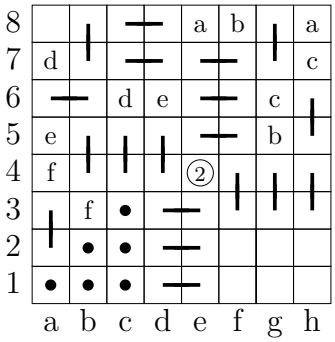
Then white has a winning strategy by first playing ②, and then, after black's move, play according to a situation in the list which does not require the cell taken by black to be free. In this way we can give winning strategies by piling up pavings.

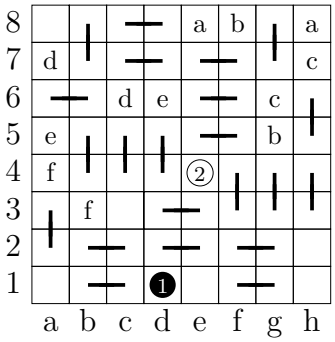
In the sequel, pavings of adjacent cells are denoted by lines linking these cells, whereas pavings of non-adjacent cells are given by the same small letters. Small black stones indicate that white chooses this paving and wins if black took that cell.

Up to symmetry, there are 10 possible first moves for black, 7 of which can be discarded immediately. The other three will be dealt with in seperate subsections.

2.1. The Easy Cases for the First Move

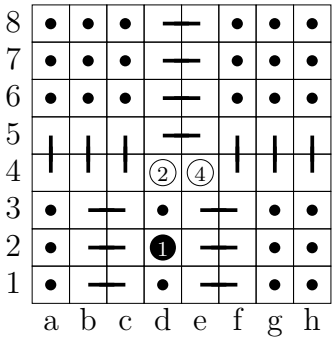
If the first move of black is on one of a1, b1, b2, c1, c2, c3, then the left of the following pavings gives a winning strategy for white. If the first move of black is on d1, the right one gives a winning strategy.

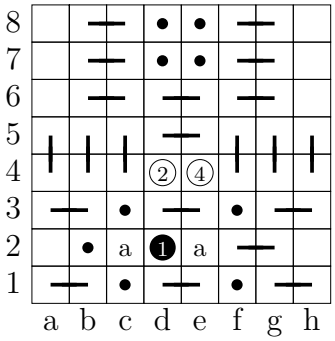


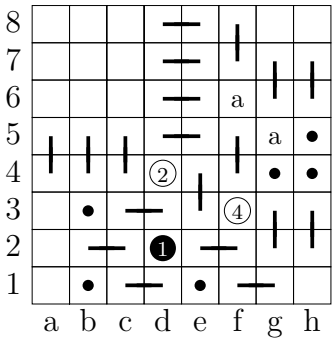


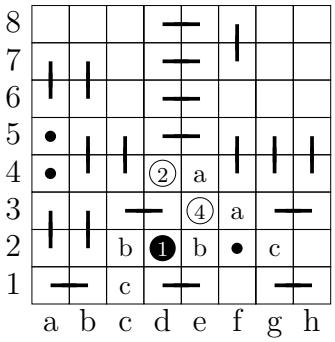
2.2. ❶ in d2

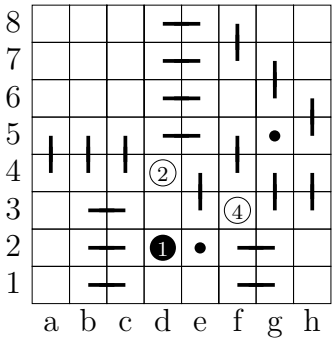
If black plays ❶ in d2, white answers d4, and the following pavings give winnings strategies for all moves ❸ except in e5.

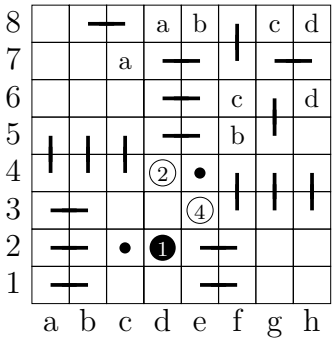


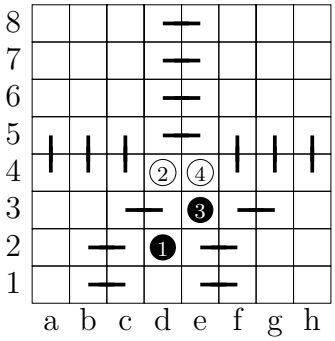


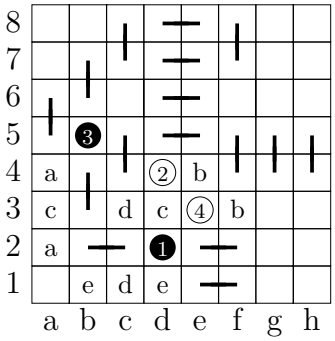


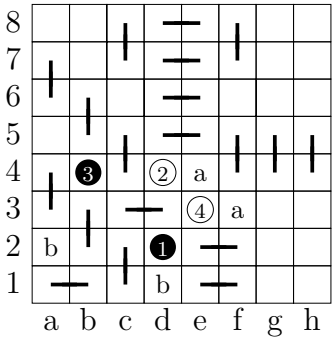


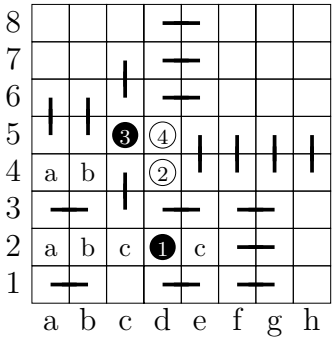


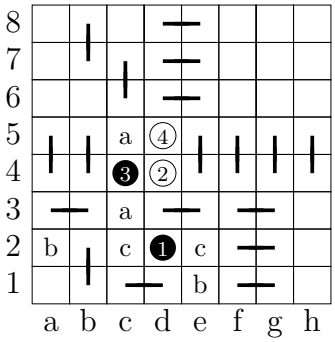


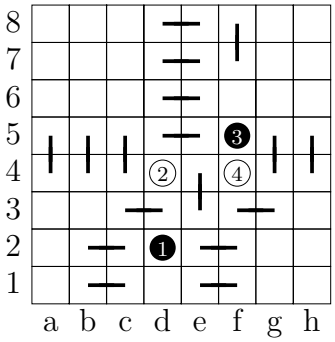


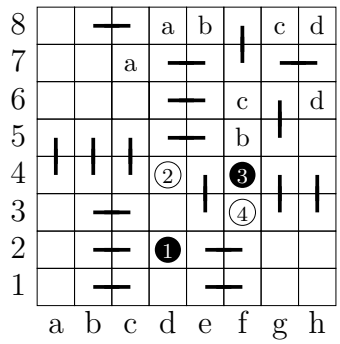


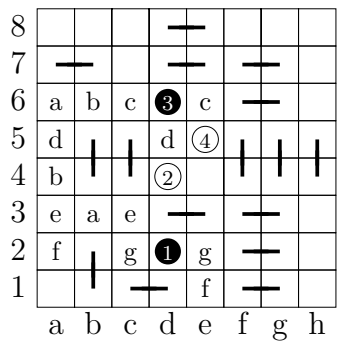


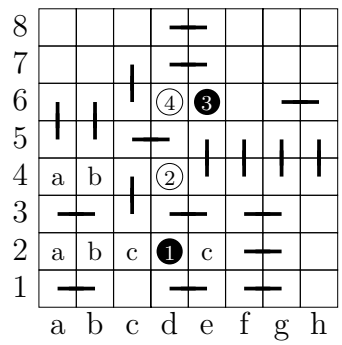


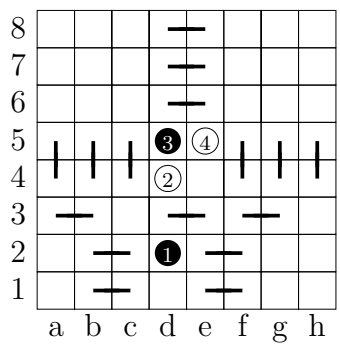






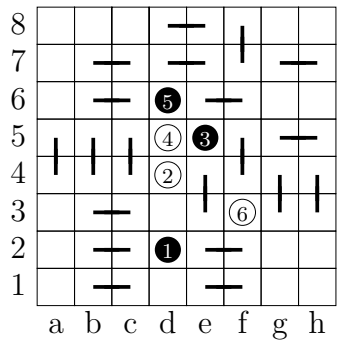


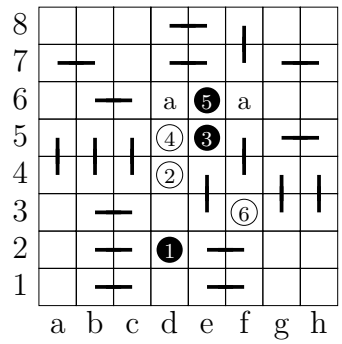




If black plays ❸ in e5, white answers in d5. Note that in all of the above pavings with the exception of the last three, either d5 and e5 are paired or white plays ④ in d5. This means that these pairings can still be used after ❸ in e5 and ④ in d5. (In the pairings where white plays ④ in d5, now white has to play ⑥ in the cell e5 is paired with.)

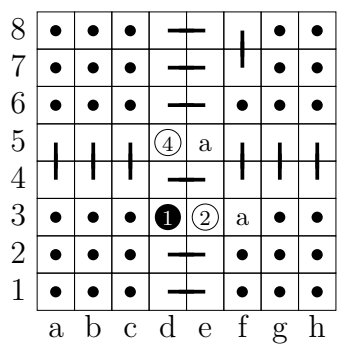
So we already have a winning strategy for all moves ❺ except the ones treated by the last three pairings; that is, d5, d6 and e6. ❺ cannot be on d5 as this cell is occupied by ④, and in the other two cases, white wins using one of the following pairings:

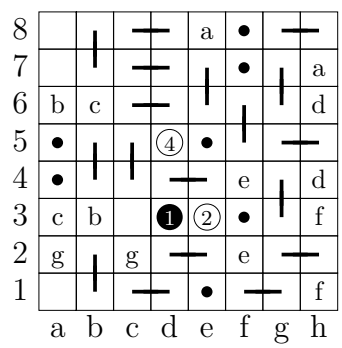


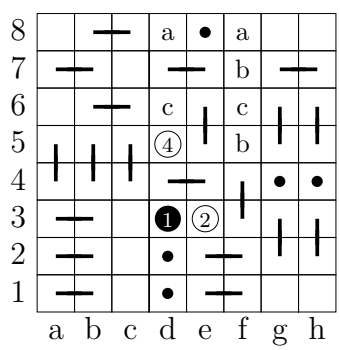


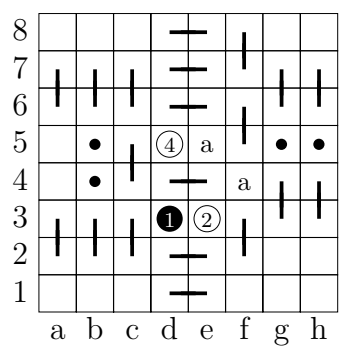
2.3. ❶ in d3

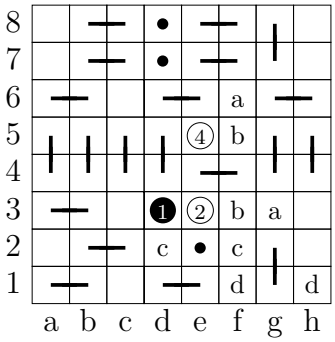
If black starts playing d3, then white answers e3. If black does not play ❸ in d4, then white wins using one of the following pavings:

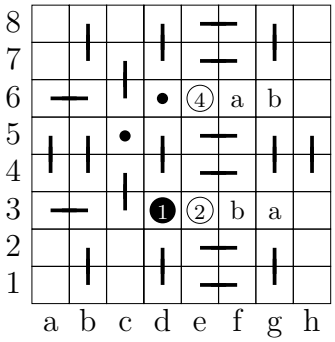


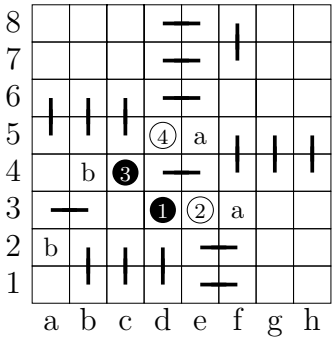


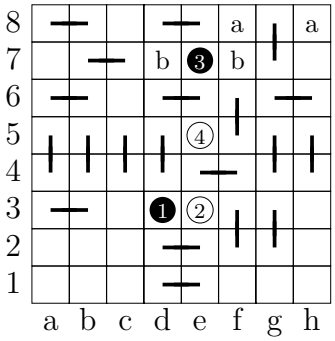





8 7 6 5 4 3 2 1
a b c d e f g h
a
4 b
1 2 b a
c c
d d


8 7 6 5 4 3 2 1
a b c d e f g h
4 a b
1 2 b a


8 7 6 5 4 3 2 1
a b c d e f g h
4 a
b 3
1 2 a
b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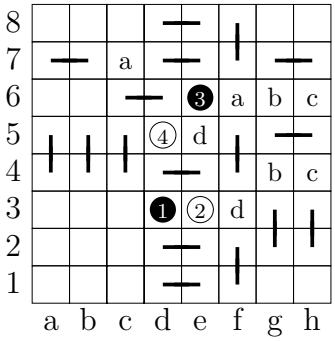
b 3 b
4
1 2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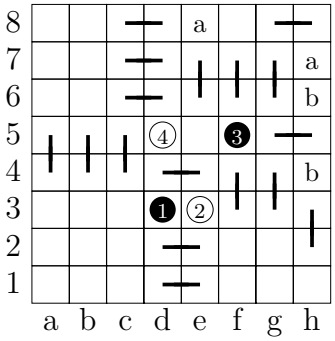
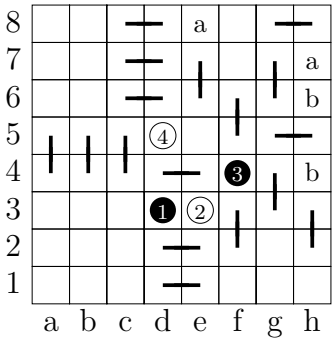
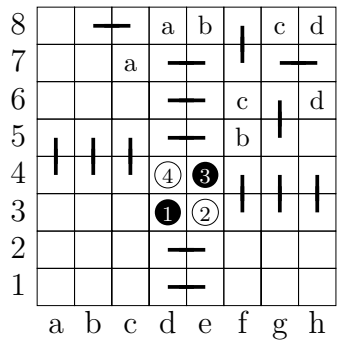
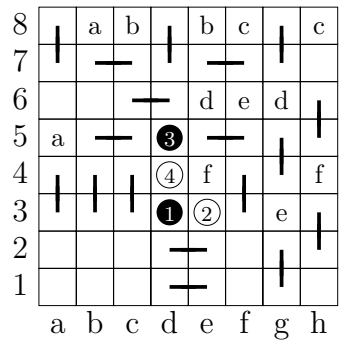
a b c d e f g h
a
3 a b c
4 d
b c
1 2 d

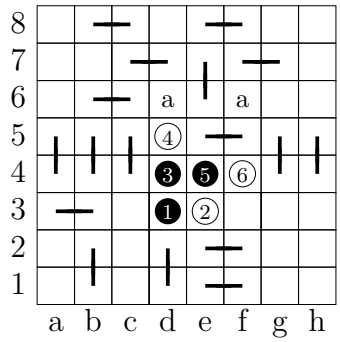

8 7 6 5 4 3 2 1
a b c d e f g h
a
a
b
4 3
b
1 2


8 7 6 5 4 3 2 1
a b c d e f g h
a
a
b
4
3 b
1 2



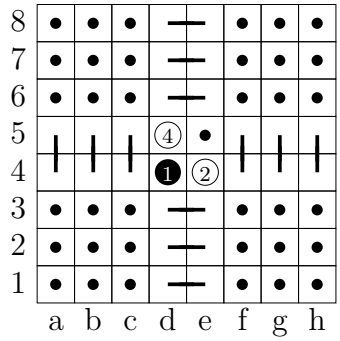


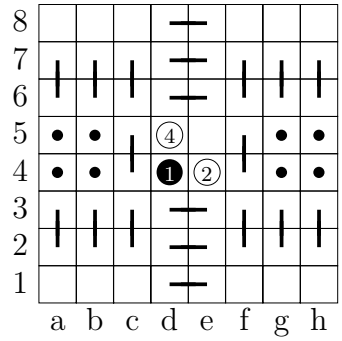
If black plays ❸ in d4, white answers d5. As in the case of ❶ in d2, all of the above pavings with exception of the last two are still usable, as either d4 and d5 are paired or ④ is played in d5. ❺ in d5 is not possible, and the only case which remains to be checked is if black plays ❺ in e4, which is dealt with by the following strategy.

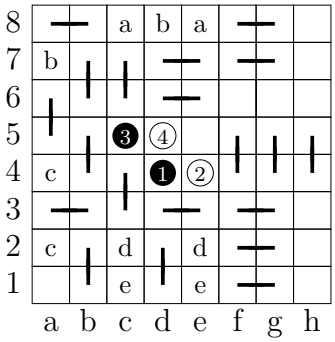


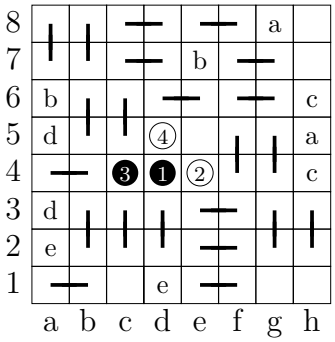
2.4. ❶ in d4

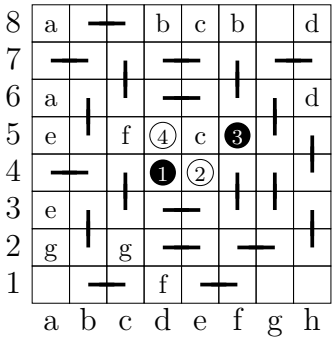
If black starts playing d4, then white answers e4. If black does not play ❸ in d5, then white plays ④ in d5, and the following pavings and their diagonal reflection give winning strategies for white depending on the location of ❸.

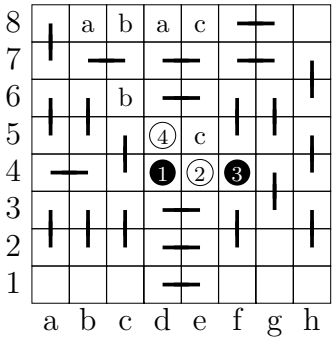




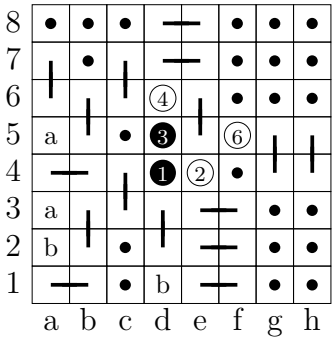


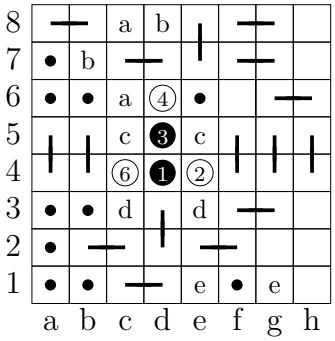


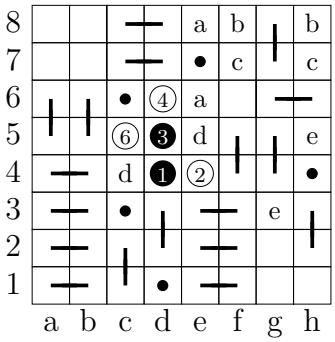


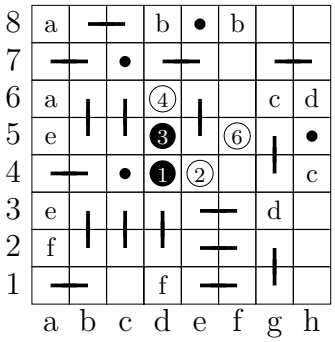


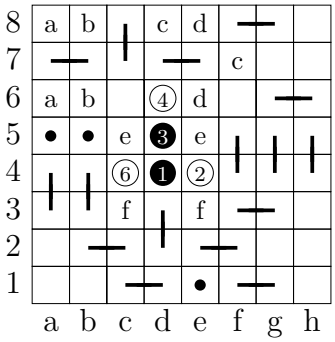
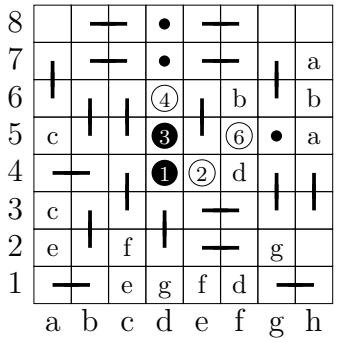
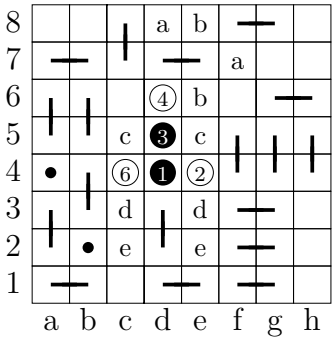
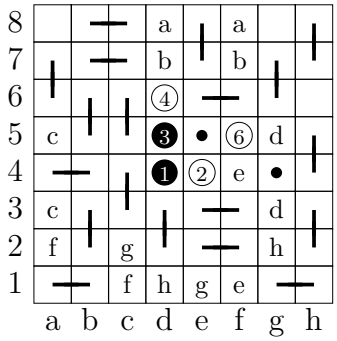
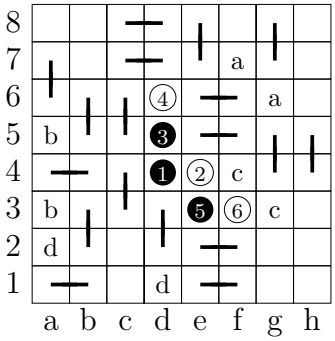
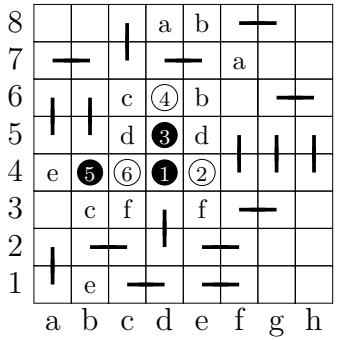
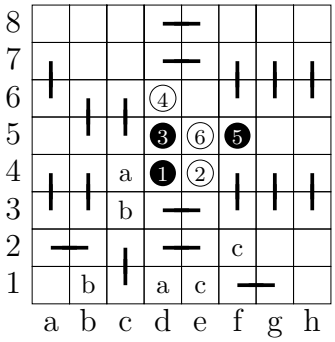
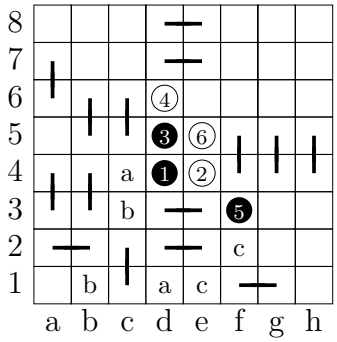
If black plays ❸ in d5, then white answers d6, and the following pavings give winning strategies except for the case that black plays ❺ in d2 or d3.





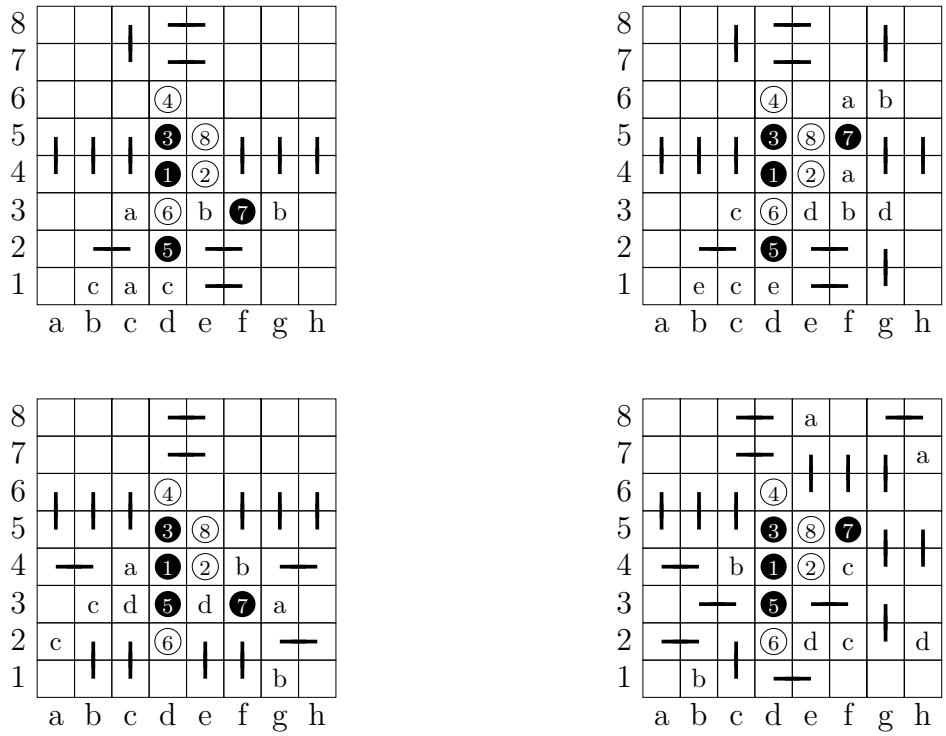




Note that in this list of pavings, d2 and d3 are always paved except in the last two pavings. So if black plays ❺ in d2 or d3, white can play the other one. Again, the above pavings give winning strategies for each location of ❼ except the cases treated in the last two pairings, that is, f3 and f5. Here are winning pavings for the remaining four cases (❺,

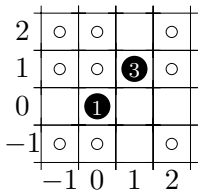
❼) in (d2, f3), (d2, f5), (d3, f3) and (d3, f5):



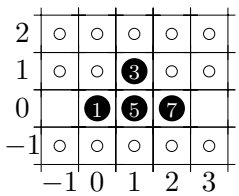
3. Snaky is a 3-dimensional Winner

The fact that Snaky is a 3-dimensional winner follows immediately from the fact that the handicap number of Snaky is at most 1. Since after the first move of black there is an infinite 2-dimensional subfield which contains a black stone and no white stone, restricting the game to this subfield yields a winning strategy for black. However, due to the involved strategy for the game with a handicap of one stone, this approach yields an unnecessarily complicated strategy. Therefore, we give an alternative proof which is fairly simple.

Black plays ❶ at $(0,0,0)$. We may assume that white plays ② at (x,y,z) with $z \neq 0$, and black answers with ❸ at $(1,1,0)$. Next we may suppose that white plays ④ outside the two lines $\mathbb{Z}\times\{0\}\times\{0\}$ and $\{1\}\times\mathbb{Z}\times\{0\}$, for otherwise we reflect the board as $(x,y,z) \mapsto (y,x,z)$. The only two cells contained in both $\mathbb{Z}\times\{0\}\times\{0\} \cup \{1\}\times\mathbb{Z}\times\{0\}$ and its mirror image are the cells $(0,0,0)$ and $(1,1,0)$, which are already taken by black. The situation on the x-y-plane is now the following. Here, and in the sequel, small white stones denote cells which might be taken by white.

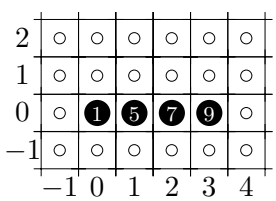


Now black plays ❺ at $(1,0,0)$, and we may assume white does not play in the line $\mathbb{Z}\times\{0\}\times\{0\}$, for otherwise we interchange the roles of the lines $\mathbb{Z}\times\{0\}\times\{0\}$ and $\{1\}\times\mathbb{Z}\times\{0\}$. Now black plays ❼ at $(2,0,0)$, and the board looks as in the following picture.

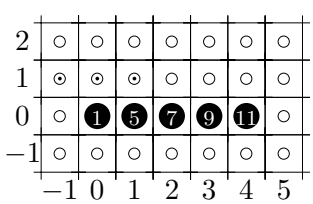


This figure is symmetric with respect to the axis $\{1\}\times\mathbb{Z}\times\{0\}$, thus we may suppose that white does not play $(0,0,3)$, and that black can take $(0,0,3)$ as ❾ at $(3,0,0)$. We do not make any assumptions on ⑩ right now.

What we have obtained so far is a string of 4 black stones in a row within 5 moves. The only use of ❸ was to create a threat which allows us to obtain this string. Therefore, to increase the symmetry of our situation, we remove this stone from the board. Obviously, if black can enforce victory without this stone, she can do so with it.



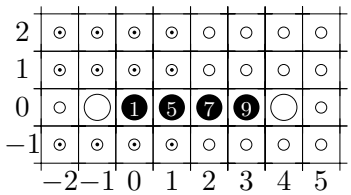
Now, we distinguish two cases: Either there is a white stone on both of $(-1,0,0)$ and $(4,0,0)$, or (at least) one of those fields is empty. We start with the latter case. Without loss of generality, let the empty cell be $(4,0,0)$. Then black takes this cell as ⓫ and white plays ⑫ anywhere.



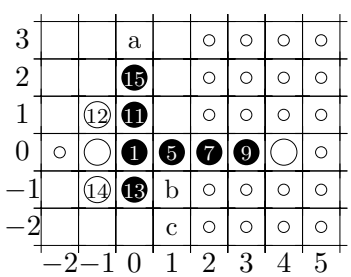
Now we consider the set of fields $S := \{(-1,1,0),(0,1,0),(1,1,0)\}$ which is dotted in the picture above. There are seven other symmetric copies of that. As up to now, there are only six white stones on the board, and all of these copies are disjoint, at least one of those

sets is free of white stones, and without loss of generality we may suppose that there are no white stones in S. Then black plays ⓭ at $(0,1,0)$, which creates two threats on $(-1,1,0)$ and $(1,1,0)$, only one of which white can cancel. So black wins with ⓯ on one of these two cells.

We now consider the second case, that is, among her first five moves, white took both $(-1,0,0)$ and $(4,0,0)$, and the board looks as in the following picture.

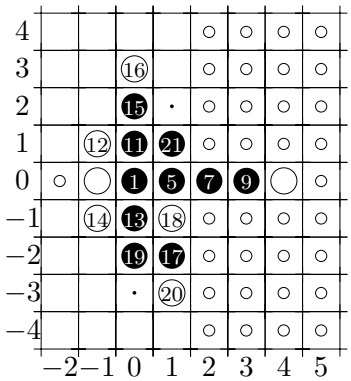


This time, we focus on the set $S' := \{\leq 1\} \times \{\neq 0\} \times \{0\}$, which is dotted in the diagram above. There are three other symmetric copies of this, and all these copies are disjoint. As there are only five white stones on the board and two of them are at $(-1,0,0)$ resp. $(4,0,0)$, at least one of these copies is empty, and without loss of generality we may suppose that there are no white stones in S'. Now black plays ⓫ at $(0,1,0)$, which forces white to play ⑫ at $(-1,1,0)$ or lose immediatelly. Similarly, black plays ⓭ at $(0,-1,0)$ which forces white to play ⑭ at $(-1,-1,0)$. Now black plays ⓯ at $(0,2,0)$, and the board looks as follows.

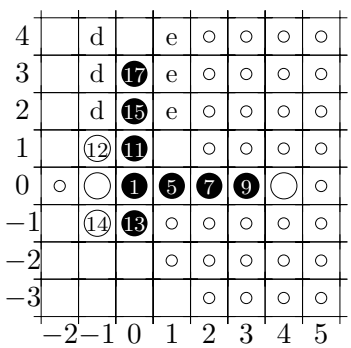


Assume that white plays ⑯ *not* on one of the three marked fields a, b, c. Then black plays ⓱ at b, which creates two imminent threats on a and c, only one of which white can counter, and black wins.

If white plays ⑯ at a, then black plays ⓱ at $(1,-2,0)$, which forces white to play ⑱ at $(1,-1,0)$ as shown in the following picture. Now black plays ⓳ at $(0,-2,0)$, white has to answer with ⑳ at $(1,-3,0)$, black plays ㉑ at $(1,1,0)$, and has two threats on the marked fields $(1,2,0)$ and $(0,-3,0)$ and wins.

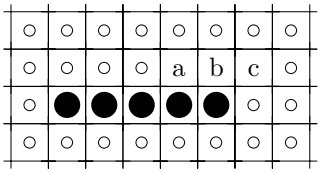


It remains to check the cases where ⑯ is played at $(1, -1, 0)$ or $(1, -2, 0)$. That is, black plays ⓱ at $(0, 3, 0)$. Now ⑱ can be placed in at most one of the two marked sets $\{(-1, 2, 0), (-1, 3, 0), (-1, 4, 0)\}$, and $\{(1, 2, 0), (1, 3, 0), (1, 4, 0)\}$, that is, black obtains 5 cells in a row with free space in at least one direction, and wins as described above.



4. Snaky with Handicap 1 in Two Dimensions

In this section we show that the handicap number of Snaky is at most 1. To keep the presentation as simple as possible, we begin by describing certain situations leading to victory for black. We will use the following notation: A small white stone like the ones in the following diagram means that we are not interested in that field; there may be anything there. Cells labeled with small letters may be occupied or not depending on the context.

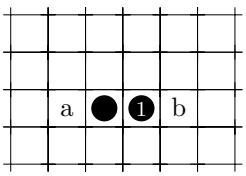


We will often encounter the situation above. In fact, we have already seen it in the previous section. If it is black's turn, she plays b; white can block at most one of a and c and black wins on the other one. This situation is so common that in the future, we won't say anything about it at all.

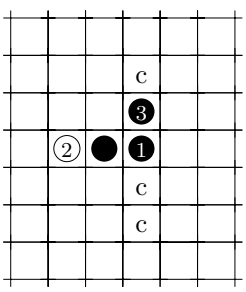
The other positions leading to black victory we need are more complex and will be dealt with in Subsections 4.2–4.6.

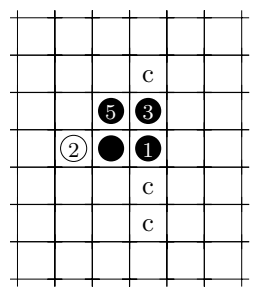
4.1. The Main Strategy

The main strategy is the following. Black starts playing a neighbour of the handicap stone:



If white does not answer ② on a or b, black is able to get four in a row on that line. This can be turned to a winning strategy for black (see Section 4.4), which we will call R (for "row"). Otherwise, without loss of generality, white plays ② in a. Then, black answers ❸ as follows.





④ in one of the a.

If in the situation of the left board, white does not play one of the fields marked c, then black again wins using strategy R. Otherwise, black plays ❺, and has achieved a block of 4 black cells with relatively few white stones around it, and we will finish our proof by constructing a winning strategy starting from this block.

In the next two subsections we give two strategies T and T^+, which will prove useful throughout this section. In the following 3 subsections we describe strategy R and thereby establish our claim that black either wins immediately or obtains a block of 4 cells as in the right picture above. Subsection 4.7 continues from this block and describes the necessary sub-cases, which will be dealt with in Subsections 4.8–4.12.

4.2. Strategy T

(This strategy is named T because in the end, the black stones form a kind of T.) Suppose the situation is as in the picture on the left where at most one white stone is placed in the dotted area. We claim that black can win the game. By symmetry we may assume that the additional white stone lies in the area dotted in the right diagram.

○	○	○	○	○	○	○
○	○	○	·	○	○	○
○	○	○	·	·	·	○
○	○	○	·	·	·	○
○	○	○	·		·	○
○	○	○	·			○
○	●	●	●	●	○	○
○	○	○	·			○
○	○	○	·		·	○
○	○	○	·	·	·	○
○	○	○	·	·	·	○
○	○	○	·	○	○	○
○	○	○	○	○	○	○

○	○	○	○	○	○	○
○	○	○	·	○	○	○
○	○	○		·		○
○	○	○				○
○	○	○				○
○	○	○	·			○
○	●	●	●	●	○	○
○	○	○				○
○	○	○	·		·	○
○	○	○	·	·	·	○
○	○	○	·	·	·	○
○	○	○	·	○	○	○
○	○	○	○	○	○	○

The strategy of black begins as depicted in the following diagrams. If white does not play ⑥ in a in the left diagram, then black plays ❼ in a, and wins in one of the cells marked b, c or d. After ⑥, the strategy goes on as depicted in the right diagram.

○	○	○	○	○	○	○
○	○	○	·	○	○	○
○	○	○	b	·	c	○
○	○	○	b	a	c	○
○	○	○	b	❺	c	○
○	○	○	·	❶	②	○
○	●	●	●	●	○	○
○	○	○	d	❸	④	○
○	○	○	·		·	○
○	○	○	·	·	·	○
○	○	○	·	·	·	○
○	○	○	·	○	○	○
○	○	○	○	○	○	○

○	○	○	⑧	⑥		○
○	○	○	❼	❺		○
○	○	○	a	❶	②	○
○	●	●	●	●	○	○
○	○	○	b	❸	④	○
○	○	○	c	❾	·	○
○	○	○	d	·	·	○
○	○	○	·	·	·	○
○	○	○	·	○	○	○
○	○	○	○	○	○	○

If the additional white stone is not on one of the fields a, c or d in the right diagram (note that it cannot be on b due to our initial assumption), the black move ❾ forces white to play ⑩ in a, and black wins in the area b, c, d. If the additional stone is on one of a, c, or d, black still plays ❾ at the same place, and white has to play either a or one of b, c, d. In any case, after ⑩ there has to be a white stone both on a and in the area b, c, d, and the situation is as in the left diagram. The three following diagrams show the winning strategy in these three cases.

○	○	○	⑧	⑥		○
○	○	○	❼	❺		○
○	○	○	◯	❶	②	○
○	●	●	●	●	○	○
○	○	○	b	❸	④	○
○	○	○	c	❾		○
○	○	○	d			○
○	○	○				○
○	○	○		○	○	○
○	○	○	○	○	○	○

One white stone on one of b, c or d.

○	○	○	⑧	⑥		○
○	○	○	❼	❺		○
○	○	○	◯	❶	②	○
○	●	●	●	●	○	○
○	○	○	ⓑ	❸	④	○
○	○	○	e	❾	f	○
○	○	○	e	⓫	f	○
○	○	○	e		f	○
○	○	○		○	○	○
○	○	○	○	○	○	○

After ⑫, black wins in either e or f.

○	○	○	⑧	⑥		○
○	○	○	❼	❺		○
○	○	○	◯	❶	②	○
○	●	●	●	●	○	○
○	○	○	⑭	❸	④	○
○	○	○	ⓒ	❾	e	○
○	○	○	f	⓫	e	○
○	○	○	f	⓭	e	○
○	○	○	f	○	○	○
○	○	○	○	○	○	○

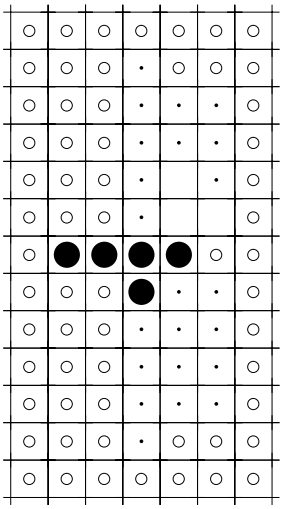
⑫ in e. Black wins in f.

○	○	○	⑧	⑥		○
○	○	○	❼	❺		○
○	○	○	◯	❶	②	○
○	●	●	●	●	○	○
○	○	○	⓭	❸	④	○
○	○	○	⓯	❾	e	○
○	○	○	ⓓ	⓫	e	○
○	○	○		⑭	e	○
○	○	○		○	○	○
○	○	○	○	○	○	○

⑫ in e.

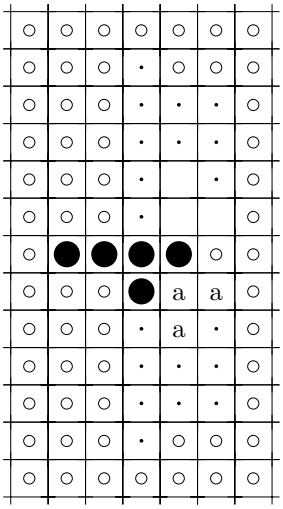
4.3. Strategy T^+

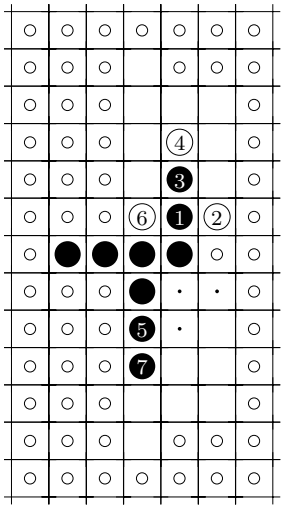
In this subsection we describe a variant of strategy T which requires black to have an additional stone, but needs less space not occupied by white. We will show that in the following situation black can win.



At most one white stone in the dotted area.

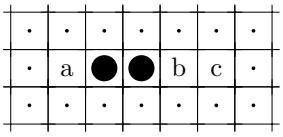
If the extra stone is not in the area a of the following left board, then we can use strategy T. Otherwise the strategy is the one on the right board:





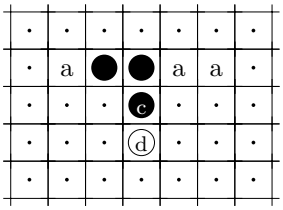
4.4. Strategy R

This strategy is used in two situations (see Section 4.1): (i) If white does not react on a or b after the first move of black. Without loss of generality we may also assume that white did not play in c.



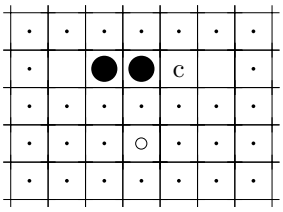
One white stone in the dotted area.

(ii) If, after the second move of black, white did not react in one of the fields marked a. (Note that the board has been turned 90° to the left, this will allow us to identify these two cases.)



One white stone in the dotted area.

To treat both cases together, we ignore ❍ and do not assume anything about the field with ⓓ:

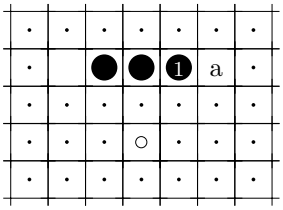


One extra white stone in the dotted area.
We will call this stone ⓪.

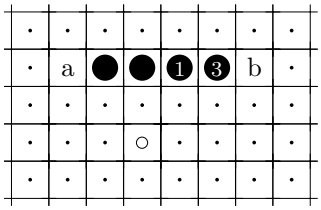
Black plays on c (we will call that move ❶) and white answers somewhere. Now, the rough idea is the following. If up to now white did not play on the row with the three black stones, then black will finally win by extending this row (see Section 4.6). Otherwise, white does not have enough stones to prevent black from winning outside of that row. In the following section, we consider the second case.

4.5. Strategy R_1

By symmetry, we can at least assume that ② is not played on a of the left diagram. So black can play ❸ there, and ④ is played anywhere.



⓪ in the dotted area, ② anywhere, at least one of these in the line with the black row.



⓪, ② and ④ anywhere, with at least one of them on the line with the black row.

The remainder of the strategy depends on whether white did play on both a and b or not.

First consider the case where white did not play on both of these cells. In this case, the remainder of the game will only take place in the top three rows of the diagram. So we assume that white did not play on b. Then black plays ❺ on b and white plays ⑥ anywhere.

○	a	a	a	○	b	b	b	○
○	○	●	●	❶	❸	❺	○	○
○	c	c	c	○	d	d	d	○

After ⑥, at most four white stones in the rows shown, at least one of them in the middle row.

Now at most three of the areas a, b, c, and d contain a white stone, so black wins.

Now consider the case where white did play on both a and b within her first three moves. Then, the situation is as in the following diagram.

·	·	·	·	·	·	·	·
·	ⓐ	●	●	❶	❸	ⓑ	·
·	·	·	·	·	·	·	·
·	·	·	○	·	·	·	·
·	·	·	·	·	·	·	·

One more white stone anywhere.

In this situation, black can always win using the strategy T – either on the left-hand side or on the right-hand side of the board – the opposite side of the one with the extra white stone.

4.6. Strategy R_2

The situation is the one in the following diagram. We claim that black can win by playing only in these three rows. Hence we may assume that white plays only in these three rows as well, and we can ignore the rest of the board completely.

·	·	·	·	·	·	·	·	·
			●	●	❶			
·	·	·	·	·	·	·	·	·

At most two white stones (which we call ⓪ and ②) in the dotted area.

The strategy for black is to extend the black row until she can force a win. However, she has to be a bit careful in which direction to extend the row; the right direction has to be chosen depending on the locations of ⓪ and ②.

White cannot prevent black from getting five in a row. Moreover, this happens at a stage

where white has at most four stones in the three rows shown, and to prevent black from winning, there must be one of them in each of the areas a, b, c and d of the following left diagram. In particular, if white plays in the middle row at any time, then black wins, so we can assume that she does not.

○	a	a	a	○	b	b	b	○
○	○	●	●	●	●	●	○	○
○	c	c	c	○	d	d	d	○

·	·	e	·	·	·	e	·	·
			●	●	❶			
·	·	e	·	·	·	e	·	·

If any of ⓪ or ② is *not* on a field marked e in the right board, then black is able to choose directions for extending his row in such a way that this stone is not part of any of the areas a, b, c and d, so black wins.

So now assume that both ⓪ and ② are on a field marked e. Black extends his row two times to the right and white has to play in the remaining two of the areas a, b, c and d. Then, we have the situation depicted in the left diagram, and the right diagram shows how black wins.

	a	a	a		b	b	b	
		●	●	❶	❸	❺		
	c	c	c		d	d	d	

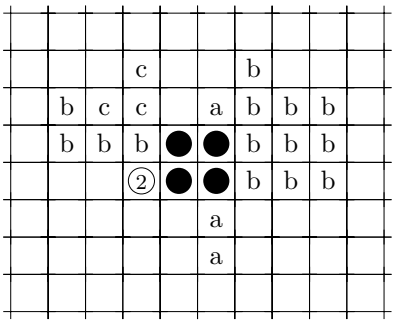
One white stone in each of the areas a, b, c and d; two of them at the left most end of the area.

	○	e	e	e	○	f	f	f
		●	●	❶	❸	❺	❼	
	○	g	g	g	○	h	h	h

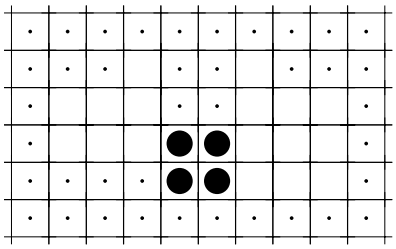
After ❼, at most two of the areas e, f, g, h contain a white stone. ⑧ in one of them, black wins in the other one.

4.7. Contiuation of the Main Strategy

Now, we continue with the main strategy. Remember that we have a black 2×2-block on the board, together with three white stones around it, as depicted in the left diagram below. In most cases, the remainder of the strategy for black is the one described in Section 4.9 and which we call B_1 (B stands for "block"), the prerequisites of which are depicted in the diagram on the right. However, depending on the second and third move of white, we have to turn and/or reflect the board in an appropriate way; we will therefore first describe when to use which strategy



④ in one of the a. ⑥ anywhere.



Preriquisites for strategy B_1: All three white stones in the dotted area.

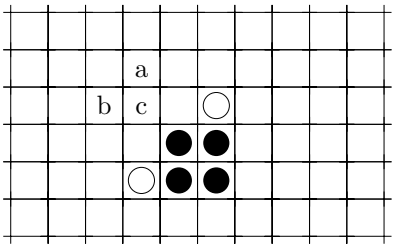
Comparing these two diagrams, we can describe when to use which strategy.

If ⑥ is outside of regions b and c, then black can use strategy B_1.

If ⑥ is somewhere in region b, then again stragegy B_1 can be used, this time after turning the board 90° to the right and maybe flipping horizontally (depending on wether ④ was above or below the black stones).

If ⑥ is in region c and ④ is below the black stones, then black plays stragegy B_2 (see Section 4.11).

The remaining possibilities are:



One extra white stone in a, b or c.

If the extra white stone is in a or b, then black plays strategy B_1', possibly flipped diagonally (confer Section 4.10). If the extra white stone is in c, then black plays strategy B_3 (confer Section 4.12).

In fact, black wins in almost any situation consisting of a black 2×2-block together with three white stones. We will however only describe the strategy in the cases that are relevant to us.

4.8. Strategy B_0

Most of the strategies B_i will start in the same way. In this section, we describe this common beginning. Assume the situation on the left board, with three white stones somewhere in the dotted area or in e or f. Assume additionally that either field e or area f is empty (or

both). These are the prerequisites for this strategy (which we will call B_0).

·	·	·	·	·	·	f	f	f	·
·				●	●				·
·	·	·	e	●	●				·
·	·	·	·	·	·	·	·	·	·

·	·	·	·	·	·	f	f	f	·
·		a	❼	●	●				·
·	·	·	e	●	●	b			·
·	·	·	·	·	·	·	·	·	·

Black starts playing ❼. White has to answer ⑧ in a or b; otherwise black wins in one of the following ways:

○	○	○	○	○	○	○	○	○	○
○		❾	❼	●	●	○	○	○	○
○	○	○	○	●	●		○	○	○
○	○	○	○	○	○	○	○	○	○

○	○	○	○	○	○	○	○	○	○
○	○		❼	●	●	○	○	○	○
○	○	○	○	●	●	❾		○	○
○	○	○	○	○	○	○	○	○	○

If ⑧ in b, then ❾ and ⑩ as in the following left board. Let's call this "outcome 1". If ⑧ in a, then ❾ as in the following right board.

·	·	·	·	·	·	·	·	·	·
·	⑩	❾	❼	●	●				·
·	·	·	·	●	●	⑧			·
·	·	·	·	·	·	·	·	·	·

·	·	·	·	·	·	f	f	f	·
·		⑧	❼	●	●	❾	d	g	·
·	·	·	e	●	●	h	h	h	·
·	·	·	·	·	·	·	·	·	·

Now, remember that in the right board, either e or all three f are empty by assumption. If white did not answer ⑩ in d, then ⓫ in d and black has at least two of the four winning threads e, f, g and h. So we can assume ⑩ in d. We will call the resulting situation "outcome 2".

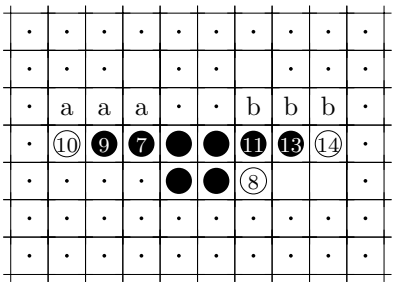
4.9. Strategy B_1

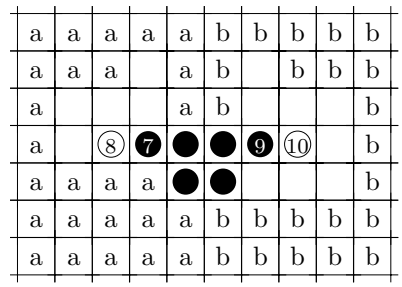
Suppose the situation is as in the following diagram.

·	·	·	·	·	·	·	·	·	·
·	·	·		·	·		·	·	·
·				·	·				·
·				●	●				·
·	·	·	·	●	●				·
·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·

Three white stones in the dotted area.

Black starts playing strategy B_0. The left diagram shows what to do in outcome 1. The right one shows outcome 2. In this case, either area a or area b contains at most one white stone. Black plays strategy T^+ on the corresponding side of the board.

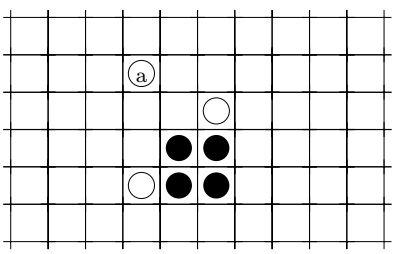




⑫ in one of the a. After ⑭, black wins in b.

4.10. Strategy B_1'

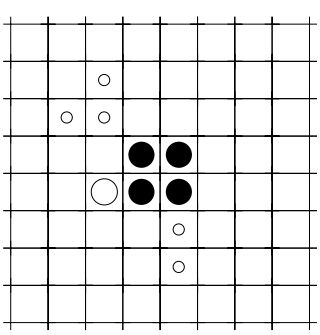
Strategy B_1 also works in the following situation.



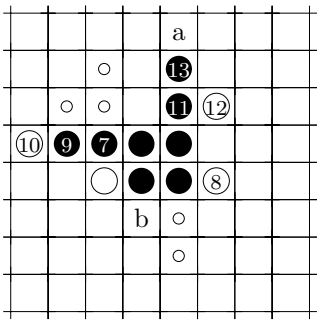
The only reason strategy B_1 needed the field a to be empty was to be able to play strategy T^+ in the left half of the board in the case where B_0 reached outcome 2. However, in this situation, the majority of the white stones are on the left, so T^+ is played on the right-hand side of the board anyway.

4.11. Strategy B_2

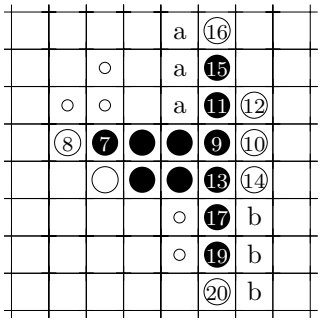
Suppose the situation is as in the following diagram.



Black starts playing strategy B_0. The left diagram shows what to do in outcome 1, the right one what to do in outcome 2.



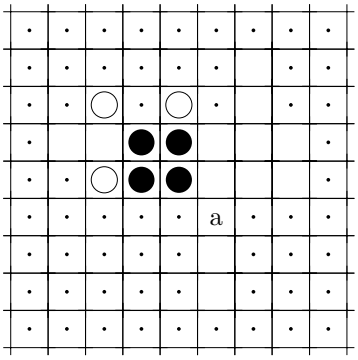
After 14, black wins in a or b.

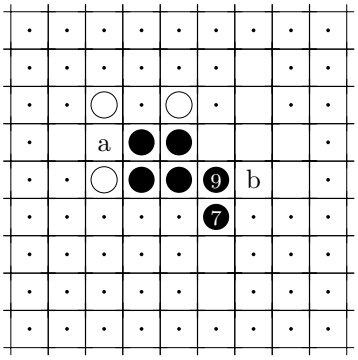


18 in a. Black wins in b.

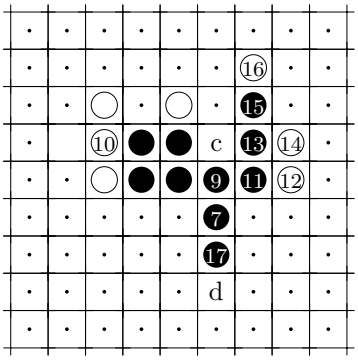
4.12. Strategy B_3

Suppose the situation is the one on the left board. Then black plays 7 on a, and by symmetry we may assume that white answers 8 in the dotted area.

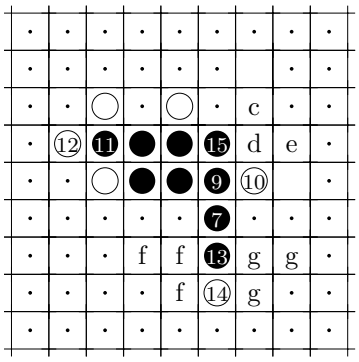




After 9, white has to play in one of the cells marked a and b in the right board, for otherwise black wins by adapting strategy B_0. If white plays a, then black wins as described in the left board:



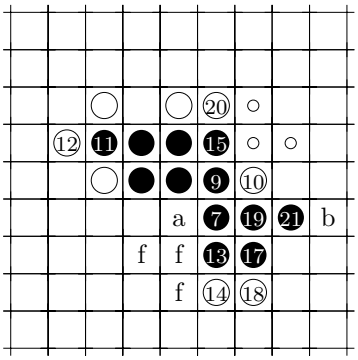
After 18, black wins in either c or d.



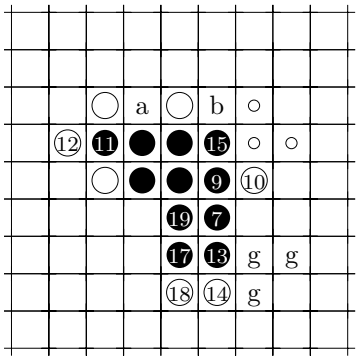
So now assume 10 in b. Black now plays as on the right board above. After 15, white has to answer with 16 in c, d or e; otherwise 17 in d and either c or e wins.

Now, if 8 was played outside of the regions f and g, then black plays strategy T on the bottom of the column 15 9 7 13 and wins. The following two diagrams give winning strategies

for the other cases, thus concluding the proof that black can always win the game.



⑧ somewhere in f. After ㉒, black wins in either a or b.



⑧ somewhere in g. After ⑳, black wins in either a or b.

5. There is No Beck-strategy for White

The most successful approach to polyomino achievement games is the pseudo-probabilistic method developed by Beck [1]. In each situation, define the value of a cell not yet occupied to be the expected number of black snakys passing through this cell if all cells not yet taken by either black or white are filled with black and white chips at random. For example, the value of a cell far away from all occupied cells is $\frac{3}{4}$, since there are 48 possibilities for snaky to pass through this cell (6 cells in snaky $\times$ 4 directions $\times$ 2 orientations), and each is totally black with probability $\frac{1}{64}$. The value of a cell adjacent to a black cell and far away from all other occupied cell is $\frac{17}{16}$, since there are 28 snakys passing through the adjacent cell, but not through the black cell, and 20 snakys passing both through the cell under consideration and the black cell, thus the total value is $\frac{28}{64} + \frac{20}{32} = \frac{17}{16}$.

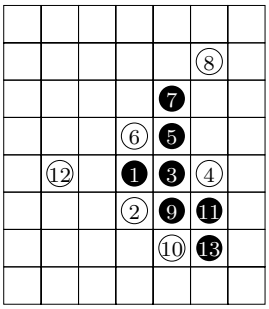
Now the strategy of white is to play in such a way that the value of the whole board, which is defined as the sum of the values of all empty cells, is minimized. By an easy computation (see [1]), one checks that the difference of the board values before and after a move of white is equal to the value of the cell white played on; therefore, an equivalent definition of the strategy for white is to play on a cell with maximal value. For games in high dimensions this method not only yields quite general results, but is often the only feasible approach available.

In general the cell with the highest value is not necessarily unique. We say that white plays a Beck strategy, if she always plays on one of the cells with highest value.

One might be tempted to believe that this strategy could be used to prove that white wins, thus deciding whether snaky is a winner or not. Unfortunately, this is not the case; the following example shows that white does not win when playing a Beck strategy. Note that in this case, apart from the initial symmetry for the first move of white, Beck's strategy

strictly determines the white moves.

The table on the right shows the values of the moves of white: The value a_i in the column denoted by i is the number of snakys passing through the white stone containing already i black stones (and no white ones). The number on the right is the value of the move multiplied by 64, that is, $\sum_i a_i 2^i$.



	0	1	2	3	4	
2	28	20				68
4	27	10	7			75
6	12	16	8			76
8	35	5	2	2		69
10	23	6	2	3		67
12	34	2	2	1	1	70

Of course, especially 12 is surprising. However, one easily computes that the Beck value of the cell of 13, for example, would have been only $\frac{66}{64}$.

References

[1] J. Beck, Remarks on positional games. I. *Acta Math. Acad. Sci. Hungar.* **40** (1982), 65–71.

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