PARTITIONS WHICH ARE p- AND q-CORE

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Abstract

Let p and q be distinct primes, n an integer with $n > p^2q^2$. Then there is no partition of n which is at the same time p- and q-core. Hence there is no irreducible representation of S_n which is of p- and q-defect zero at the same time.

Let n be an integer. Then there is a natural bijection between the set of partitions of n and the irreducible representations of the symmetric group on n letters S_n. A representation of a finite group G with character χ is called of p-defect zero, if $|G|_p|\chi(1)$. In the case of the symmetric group this is known to be equivalent to the statement that the corresponding partition has no hook-number divisible by p, in this case the partition is called a p-core partition. Granville and Ono [2] proved that for any $t \geq 7$ and any n there is a t-core partition of n, thus for every $p \geq 7$ there is an irreducible representation of S_n with p-defect zero, an easier proof was given by Kiming [4].

In a recent paper Navarro and Willems [5] asked for relations between the p- and the q-blocks of representations. In this note we will show that the property of having defect zero exclude each other, if n is large enough compared to p and q. More precisely we will prove the following theorem.

Theorem 1. Let p and q be primes, n an integer with $n > p^2q^2$. Then there is no irreducible representation of S_n with p- and q-defect zero.

By the correspondence between irreducible representations of the S_n and partitions of n this will follow from the following statement.

Theorem 2. Let s and t be relatively prime integers, n an integer with $n > s^2t^2$. Then there is no partition of n which is at the same time s- and t-core.

Especially, the number of partitions which are simultaneously s- and t-core is finite. J. Kohles Anderson [3] proved a more precise version of this statement: The number of partitions with this property is in fact equal to $\frac{1}{s+t}\binom{s+t}{t}$. However, the proof we give here seems to be simpler then the one given by her.

I would like to thank the referee for making me aware of [3].

The proof will use the description of t-core partitions introduced by Garvan, Kim and Stanton [1].

For the sequel we choose an arbitrary partition $n = \lambda_1 + \ldots + \lambda_k$ of n and assume that it is t-core and s-core at the same time. We thus have to show that $n < s^2t^2$.

Consider the diagram of the partition, i.e. the set of cells whose first row consists of λ_1 cells $(1,1),(1,2),\ldots,(1,\lambda_1)$, the second of λ_2 cells and so on. Label a cell (i,j) with $j-i \pmod{st}$, cells in column 0 are labeled in the same way. A cell at the end of a row is called exposed. Now divide the diagram into regions S_k, such that a cell belongs to S_k if and only if $s(k-1) \leq j-i < sk$, in the same way T_k denotes the cells with $t(k-1) \leq j-i < tk$. Now by [1], paragraph 2, we know that if the partition is s-core, and there is an exposed cell labeled with i in the region S_k, then there is an exposed cell labeled with $\tilde{i} \equiv i \pmod{s}$ in every region S_l with $l \leq k$. Especially, there is some sequence $k_\nu, 0 \leq \nu \leq l, k_0 = 1$, such that $\lambda_{k_\nu} \equiv \lambda_1 - (k_\nu - 1) \pmod{s}$, $(k_{\nu+1} - k_\nu) < \lambda_{k_\nu} - \lambda_{k_{\nu+1}} < 2s - (k_{\nu+1} - k_\nu)$ and $\lambda_{k_l} < s$, i.e. $\lambda_{k_\nu} = \lambda_1 - \nu s + k_\nu$. Assume that $l < t$. Since $\lambda_{k_\nu} \leq \lambda_{k_{\nu+1}}$, we have $k_{\nu+1} \leq k_\nu + s$, thus the partition under consideration consists of at most $ls < st$ summands, each being st at most, thus we have $n \leq s^2t^2$.

Now if $l > t$, then the labels of the exposed cells in the rows k_ν run through a complete remainder system $\pmod{t}$, since s and t are coprime, the remainders of $\lambda_{k_\nu} - k_\nu = \lambda_1 - \nu s$, $0 \leq \nu < t$ are therefore all different. However, by [1] we know that if the partition is t-core, and there is an exposed cell in region T_k with the label i, then there is no exposed cell with a label $\bar{i} \equiv t-i-1 \pmod{t}$ in any region T_l with $l \geq 1-k$. If λ_1 is in region T_k, then $\lambda_{k_{t-1}}$ is in region T_l with $l \geq k-s$, thus $k-s < 1-k$, i.e. $k \leq s/2$. By the definition of T_k we have $\lambda_1 < t(s/2+1) \leq st$.

Since the property of being a t-core partition is unchanged under conjugation, by the same reasoning we get that there are less than st summands, thus we obtain $n < s^2t^2$ again.

Thus in any case the assumption that our partition is at the same time s-core and t-core leads to the estimate $n < s^2t^2$ which proves our theorem.

References

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