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**Title: On the relation between Lebesgue summability and some other summation methods**

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**In: Journal of Mathematical Analysis and Applications, 411 (1), 75-82, 2014.**

**Link to article: <http://dx.doi.org/10.1016/j.jmaa.2013.09.034>**

**To refer to or to cite this work, please use the citation to the published version:**

**J. Vindas (2014). On the relation between Lebesgue summability and some other summation methods. *J. Math. Anal. Appl.* 411(1) 75-82. doi: 10.1016/j.jmaa.2013.09.034**

# ON THE RELATION BETWEEN LEBESGUE SUMMABILITY AND SOME OTHER SUMMATION METHODS

JASSON VINDAS

ABSTRACT. It is shown that if

$$\sum_{n=1}^N n |c_n| = O(N),$$

then Lebesgue summability,  $(C, \beta)$  summability ( $\beta > 0$ ), Abel summability, Riemann summability, and  $(\gamma, \kappa)$  summability ( $\kappa \geq 1$ ) of the series  $\sum_{n=0}^{\infty} c_n$  are all equivalent to one another.

## 1. INTRODUCTION

In this article we establish the equivalence between various methods of summability under a certain hypothesis (condition (1.5) below). Our results extend a recent theorem of Móricz [14, Thm. 1].

We are particularly interested in Lebesgue summability, a summation method that is suggested by the theory of trigonometric series [22]. Consider the formal trigonometric series

$$(1.1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

Formal integration of (1.1) leads to

$$(1.2) \quad L(x) = \frac{a_0 x}{2} + \sum_{n=1}^{\infty} \left( \frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right) .$$

One then says that the series (1.1) is Lebesgue summable at  $x = x_0$  to  $s(x_0)$  if (1.2) is convergent in a neighborhood of  $x_0$  and

$$(1.3) \quad s(x_0) = \lim_{h \rightarrow 0} \frac{\Delta L(x_0; h)}{2h} ,$$

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2010 *Mathematics Subject Classification*. Primary 40E05, 42A24. Secondary 40G05, 40G10.

*Key words and phrases*. Fourier series; Lebesgue summability; Cesàro summability; Abel summability; Riemann summability;  $(\gamma, \kappa)$  summability; Tauberian theorems.

where

$$\begin{aligned} \frac{\Delta L(x_0; h)}{2h} &= \frac{L(x_0 + h) - L(x_0 - h)}{2h} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) \frac{\sin nh}{nh}. \end{aligned}$$

In such a case one writes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx_0 + b_n \sin nx_0) = s(x_0) \quad (\text{L}).$$

Observe that (1.3) tells that the symmetric derivative of the function  $L$  exists and equals  $s(x_0)$  at the point  $x = x_0$ .

The Lebesgue method of summation is somehow complicated, since it is not regular. In fact, if the series (1.1) converges at  $x = x_0$ , then it is not necessarily Lebesgue summable at  $x = x_0$ .

Zygmund investigated conditions under which Lebesgue summability is equivalent to convergence [22, pp. 321-322]. Among other things, he proved the following result. Set  $\rho_n = \sqrt{|a_n|^2 + |b_n|^2}$ .

**Theorem 1** (Zygmund). *If*

$$(1.4) \quad \rho_n = O(1/n),$$

*then, the series  $a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  is convergent at  $x = x_0$  to  $s(x_0)$  if and only if it is Lebesgue summable at  $x = x_0$  to  $s(x_0)$ .*

Móricz has recently studied the role of a certain weaker condition than (1.4) in Lebesgue summability. He has complemented Theorem 1 by showing [14, Thm. 1]:

**Theorem 2** (Móricz). *Suppose that*

$$(1.5) \quad \sum_{n=1}^N n\rho_n = O(N).$$

*If the series  $a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  converges at  $x = x_0$  to  $s(x_0)$ , then it is also Lebesgue summable at  $x = x_0$  to  $s(x_0)$ .*

We have found here that, under condition (1.5), not only may Lebesgue summability be concluded from much weaker assumptions than convergence, but also it becomes equivalent to a number of familiar summability methods. In particular, we shall prove the following theorem, which considerably improves Theorem 2 and may be interpreted as a Tauberian theorem relating various summability procedures. In the next statement  $(\mathfrak{R}, 1)$  and  $(\mathfrak{R}, 2)$  denote the Riemann summability methods [9, Sect. 4.17], while  $(C, \beta)$  stands for Cesàro summability.

**Theorem 3.** *Suppose that (1.5) is satisfied. Then, the following statements are equivalent. The trigonometric series  $a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  is:*

- (i) *Lebesgue summable at  $x = x_0$  to  $s(x_0)$ .*
- (ii) *Abel summable at  $x = x_0$  to  $s(x_0)$ .*

- (iii)  $(C, \beta)$  summable, for  $\beta > 0$ , at  $x = x_0$  to  $s(x_0)$ .
- (iv)  $(\mathfrak{R}, 1)$  at  $x = x_0$  to  $s(x_0)$ .
- (v)  $(\mathfrak{R}, 2)$  at  $x = x_0$  to  $s(x_0)$ .

As shown in Section 3, an additional summability method, which naturally generalizes the Riemann methods  $(\mathfrak{R}, 1)$  and  $(\mathfrak{R}, 2)$ , can also be added to the list from Theorem 3 (the so-called  $(\gamma, \kappa)$  summability,  $\kappa \geq 1$ , introduced and studied by Guha in [8]). Furthermore, it should also be noticed that Theorem 3 includes Theorem 1 as a particular instance, as immediately follows from, say, Hardy's elementary Tauberian theorem for  $(C, 1)$  summability [9, 12].

Theorem 3 actually admits a generalization to the class of summability methods discussed in Section 2. The main result of this paper is Theorem 4, stated in Section 3. In Section 3 we will also obtain extensions of Theorem 1.

It should be mentioned that Theorem 3 intersects with the work of Jakimovski. In fact, the equivalence between (ii) and (iii), under the even weaker condition

$$(1.6) \quad \sum_{n=1}^N n (a_n \cos nx_0 + b_n \sin nx_0) = O(N),$$

was already established by him (cf. [1, Thm. 6.2]). On the other hand, the equivalence between the rest of summability methods from Theorem 3 (and that of Guha) appears to be new in the literature. It is also worth pointing out that Jakimovski extensively investigated in [1] Tauberian theorems for Abel-type methods and Borel summability in which the conclusion is Cesàro summability. His Tauberian conditions are in terms of growth comparisons between different higher order Cesàro means, or more generally suitable Hausdorff transforms, of the series. Jakimovski's Tauberian conditions may be regarded as average generalizations of (1.6). Our approach in this article differs from that developed in [1]. In the proof of our main result, Theorem 4, we shall exploit some results by Estrada and the author connecting summability of Fourier series and integrals with local behavior of Schwartz distributions [6, 18, 20] (cf. [21]).

Finally, we mention that Pati [15] and Çanak et al [3] have recently made use of Tauberian conditions involving Cesàro average versions of (1.6) in the study of Tauberian theorems for the so-called  $(A)(C, \alpha)$  summability.

## 2. SUMMABILITY METHODS

We collect here the summability methods that will be studied in Section 3. Let  $\{\lambda_n\}_{n=0}^{\infty}$  be an increasing sequence of non-negative real numbers tending to infinity.

We begin with Riesz summability [9]. Let  $\beta \geq 0$ . We say that the series  $\sum_{n=0}^{\infty} c_n$  is  $(R, \{\lambda_n\}, \beta)$  summable to  $\ell$  if

$$\ell = \lim_{x \rightarrow \infty} \sum_{\lambda_n \leq x} c_n \left(1 - \frac{\lambda_n}{x}\right)^{\beta}.$$

In such a case, we write

$$(2.1) \quad \sum_{n=0}^{\infty} c_n = \ell \quad (\mathbf{R}, \{\lambda_n\}, \beta).$$

In the special case  $\lambda_n = n$ , the summability (2.1) is equivalent to Cesàro  $(C, \beta)$  summability, as follows from the well known equivalence theorem of Marcel Riesz [9, 11].

The extended Abel summation method is defined as follows [9]. We say that the series  $\sum_{n=0}^{\infty} c_n$  is  $(A, \{\lambda_n\})$  summable to  $\ell$  if  $\sum_{n=0}^{\infty} c_n e^{-\lambda_n y}$  converges for  $y > 0$  and

$$\ell = \lim_{y \rightarrow 0^+} \sum_{n=0}^{\infty} c_n e^{-\lambda_n y};$$

we then write

$$(2.2) \quad \sum_{n=0}^{\infty} c_n = \ell \quad (A, \{\lambda_n\}).$$

When  $\lambda_n = n$ , one recovers the usual Abel summability method  $(A)$  in (2.2).

We shall also consider a generalization of Guha's method from [8]. We need to introduce the so-called Young functions [10]. They are given by the Cesàro (integral) means of  $\cos x$ . Let  $\kappa \geq 0$ . We set  $\gamma_0(x) = \cos x$  and, for  $\kappa > 0$ ,

$$(2.3) \quad \gamma_{\kappa}(x) = \frac{\kappa}{x} \int_0^x \left(1 - \frac{t}{x}\right)^{\kappa-1} \cos t \, dt.$$

It is said that  $\sum_{n=0}^{\infty} c_n$  is  $(\gamma, \{\lambda_n\}, \kappa)$  summable to  $\ell$  if the following two conditions hold:

$$\sum_{n=0}^{\infty} c_n \gamma_{\kappa}(\lambda_n h) \quad \text{converges for small } h > 0,$$

and

$$\ell = \lim_{h \rightarrow 0^+} \sum_{n=0}^{\infty} c_n \gamma_{\kappa}(\lambda_n h).$$

We employ the notation

$$(2.4) \quad \sum_{n=0}^{\infty} c_n = \ell \quad (\gamma, \{\lambda_n\}, \kappa)$$

to denote  $(\gamma, \{\lambda_n\}, \kappa)$  summability. If  $\lambda_n = n$ , we write  $(\gamma, \kappa)$  instead of  $(\gamma, \{n\}, \kappa)$ , in accordance with Guha's notation [8]. As explained in [8], the  $(\gamma, \kappa)$  method is intimately connected with certain aspects of the theory of summability of trigonometric series. For instance, if  $\kappa = 1, 2$ , one obtains in (2.3) the functions

$$\gamma_1(x) = \frac{\sin x}{x} \quad \text{and} \quad \gamma_2(x) = \left(\frac{\sin(x/2)}{x/2}\right)^2;$$

so that  $(\gamma, 1) = (\mathfrak{A}, 1)$  and  $(\gamma, 2) = (\mathfrak{A}, 2)$ . We recall that  $(\mathfrak{A}, 1)$  and  $(\mathfrak{A}, 2)$  stand for the Riemann summability methods [9].

Lebesgue summability is of course closely related to the  $(\gamma, 1)$  method, but observe that the convergence of (1.2) is not part of the requirements for  $(\gamma, 1)$  summability. In analogy to the Lebesgue summability method, we say that  $\sum_{n=0}^{\infty} c_n$  is  $(L, \{\lambda_n\})$  summable to  $\ell$  and write  $\sum_{n=0}^{\infty} c_n = \ell (L, \{\lambda_n\})$  if (2.4) holds with  $\kappa = 1$  and additionally

$$(2.5) \quad \sum_{0 < \lambda_n} c_n \frac{e^{i\lambda_n h}}{\lambda_n h} \quad \text{converges for small } |h| > 0.$$

We point out that our convention for this generalization of Lebesgue summability is different from that proposed by Szász in [17, p. 394]. (In fact, Szász' notion coincides with what we call here  $(\gamma, \{\lambda_n\}, 1)$  summability.)

We have

**Proposition 1.** *Suppose that*

$$(2.6) \quad \sum_{\lambda_n \leq x} \lambda_n |c_n| = O(x).$$

*Then, the series  $\sum_{n=0}^{\infty} c_n$  is  $(L, \{\lambda_n\})$  summable if and only if it is  $(\gamma, \{\lambda_n\}, 1)$  summable.*

Thus, under condition (1.5), the trigonometric series (1.1) is Lebesgue summable at  $x = x_0$  to  $s(x_0)$  if and only if it is  $(\mathfrak{R}, 1)$  ( $= (\gamma, 1)$ ) summable at  $x = x_0$  to  $s(x_0)$ . Proposition 1 follows at once from the ensuing simple lemma, which guarantees the absolute and uniform convergence of (2.5) when (2.6) is assumed.

**Lemma 1.** *The condition (2.6) is equivalent to*

$$(2.7) \quad \sum_{x \leq \lambda_n} \frac{|c_n|}{\lambda_n} = O\left(\frac{1}{x}\right).$$

*Proof.* Write  $S(x) = \sum_{\lambda_n \leq x} |c_n|$  for  $x > 0$  and  $S(0) = 0$ . The conditions (2.6) and (2.7) take the form

$$(2.8) \quad T_1(x) := \int_0^x t \, dS(t) = O(x)$$

and

$$(2.9) \quad T_2(x) := \int_x^{\infty} t^{-1} \, dS(t) = O\left(\frac{1}{x}\right),$$

respectively. Assume (2.8). Notice that

$$\int_x^y t^{-1} \, dS(t) = \int_x^y t^{-2} \, dT_1(t) = \frac{T_1(y)}{y^2} - \frac{T_1(x)}{x^2} + 2 \int_x^y \frac{T_1(t)}{t^3} \, dt.$$

Taking  $y \rightarrow \infty$ , we obtain that

$$\int_x^{\infty} t^{-1} \, dS(t) = -\frac{T_1(x)}{x^2} + 2 \int_x^{\infty} \frac{T_1(t)}{t^3} \, dt = O\left(\frac{1}{x}\right).$$

Suppose now that (2.9) holds. Since

$$T_3(x) := \int_{(x,\infty)} t^{-1} dS(t) \leq T_2(x) = O(1/x),$$

we have

$$\int_0^x t dS(t) = - \int_0^x t^2 dT_3(t) = -x^2 T_3(x) + 2 \int_0^x t T_3(t) dt = O(x),$$

as required.  $\square$

### 3. MAIN RESULT

We are now in the position to state our main result:

**Theorem 4.** *If the condition (2.6) holds, then the following six statements are equivalent. The series  $\sum_{n=0}^{\infty} c_n$  is:*

- (a)  $(L, \{\lambda_n\})$  summable to  $\ell$ .
- (b)  $(\gamma, \{\lambda_n\}, \kappa)$  summable to  $\ell$  for some  $\kappa \geq 1$ .
- (c)  $(\gamma, \{\lambda_n\}, \kappa)$  summable to  $\ell$  for all  $\kappa \geq 1$ .
- (d)  $(R, \{\lambda_n\}, \beta)$  summable to  $\ell$  for some  $\beta > 0$ .
- (e)  $(R, \{\lambda_n\}, \beta)$  summable to  $\ell$  for all  $\beta > 0$ .
- (f)  $(A, \{\lambda_n\})$  summable to  $\ell$ .

Before giving a proof of Theorem 4, we would like to discuss two corollaries of it. It is well known that any of the following three assumptions is a Tauberian condition for  $(A, \{\lambda_n\})$  summability, and hence for Riesz  $(R, \{\lambda_n\}, \beta)$  summability,

$$(3.1) \quad c_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

$$(3.2) \quad \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\right)^{p-1} |c_n|^p < \infty \quad (1 < p < \infty),$$

$$(3.3) \quad \sum_{n=1}^N \lambda_n^p (\lambda_n - \lambda_{n-1})^{1-p} |c_n|^p = O(\lambda_N) \quad (1 < p < \infty).$$

Indeed, that convergence follows from  $(A, \{\lambda_n\})$  summability under (3.1) was first shown by Ananda Rau in [2] (see also [9, 19]). The Tauberian theorem related to (3.2) belongs to Hardy and Littlewood, while the one with the Tauberian condition (3.3) to Szász (see [6, Sect. 5] for quick proofs of these two Tauberian theorems).

We can deduce from Theorem 4 the following Tauberian theorem for  $(\gamma, \{\lambda_n\}, \kappa)$  summability.

**Corollary 1.** *Let  $\kappa \geq 1$ . Suppose that*

$$\sum_{n=0}^{\infty} c_n = \ell \quad (\gamma, \{\lambda_n\}, \kappa).$$

Then, any of the Tauberian conditions (3.1), (3.2), or (3.3) implies that  $\sum_{n=0}^{\infty} c_n$  is convergent to  $\ell$ .

*Proof.* Clearly, (3.1) yields (2.6). Furthermore, any of the two conditions (3.2) or (3.3) also implies (2.6), as a straightforward application of the Hölder inequality shows. By Theorem 4, we obtain that the series is  $(A, \{\lambda_n\})$  summable to  $\ell$ . Consequently, the desired convergence conclusion follows from the corresponding Tauberian theorem for  $(A, \{\lambda_n\})$  summability.  $\square$

Combining Corollary 1 and Theorem 4, we obtain the ensuing extension of Zygmund's result (Theorem 1).

**Corollary 2.** *Assume any of the conditions (3.1)–(3.3). Then,  $\sum_{n=0}^{\infty} c_n$  is  $(L, \{\lambda_n\})$  summable to  $\ell$  if and only if it is convergent to  $\ell$ .*

We now set the ground for the proof of Theorem 4. The space  $\mathcal{S}'(\mathbb{R})$  denotes the well known Schwartz space of tempered distributions [4, 16]. We will make use of the notion of distributional point values, introduced by Łojasiewicz in [13]. A distribution  $f \in \mathcal{S}'(\mathbb{R})$  is said to have a distributional point value  $\ell$  of order  $k \in \mathbb{N}$  at the point  $x = x_0$  if there is a locally bounded function  $F$  such that  $F^{(k)} = f$  near  $x = x_0$  and

$$\lim_{x \rightarrow x_0} \frac{k!F(x)}{(x - x_0)^k} = \ell.$$

In such a case one writes  $f(x_0) = \ell$ , distributionally of order  $k$ .

We are more interested in the closely related notion of (distributionally) symmetric point values and its connection with the Fourier inversion formula for tempered distributions [20, Sect. 6] (cf. [16, Chap. 5]). We say that  $f$  has a symmetric point value  $\ell$  of order  $k$  at  $x = x_0$  and write  $f_{\text{sym}}(x_0) = \ell$ , distributionally of order  $k$ , if the distribution

$$(3.4) \quad \chi_{x_0}(h) := \frac{f(x_0 + h) + f(x_0 - h)}{2}$$

satisfies  $\chi_{x_0}(0) = \ell$ , distributionally of order  $k$ . One can show [16, Thm. 5.18] that  $f_{\text{sym}}(x_0) = \ell$ , distributionally, if and only if the pointwise Fourier inversion formula

$$(3.5) \quad \frac{1}{2\pi} \text{p.v.} \langle \hat{f}(u), e^{ix_0 u} \rangle = \ell \quad (\mathbf{C}, \beta)$$

holds for some  $\beta \geq 0$ . The left hand side of (3.5) denotes a principal value distributional evaluation in the Cesàro sense, explained, e.g., in [16, Sect. 5.2.8]. Under additional assumptions on the growth order of  $f$  at  $\pm\infty$ , it is possible to establish a more precise link between the order of summability  $\beta$  and the order of the symmetric point value [20]. We refer to [5, 6, 7, 16, 18, 20] for studies about the interplay between local behavior of distributions and summability of series and integrals.

We now proceed to show our main result.



*Proof of Theorem 4.* The equivalence between (d), (e), and (f) has been established by Estrada and the author in [6, Cor. 4.16] under the still weaker assumption

$$\sum_{\lambda_n \leq x} \lambda_n c_n = O(x).$$

(The case  $\lambda_n = n$  of this result is due to Jakimovski [1, Thm. 6.2].) Taking Proposition 1 into account, it therefore suffices to show the implications (b) $\Rightarrow$ (d) and (e) $\Rightarrow$ (c). We first need to show the following claim:

**Claim 1.** Let  $\kappa \geq 1$ . Under the assumption (2.6),

$$(3.6) \quad \sum_{n=0}^{\infty} c_n = \ell \quad (\gamma, \{\lambda_n\}, \kappa) \implies \sum_{n=0}^{\infty} c_n = \ell \quad (\gamma, \{\lambda_n\}, \tau) \quad \text{for } \tau \geq \kappa.$$

*Proof of Claim 1.* Set  $G_\tau(h) = \sum_{n=0}^{\infty} c_n \gamma_\tau(\lambda_n h)$ . Lemma 1 ensures that all these series are absolutely convergent for  $h > 0$  if  $\tau \geq 1$ . Let  $\tau > \kappa$ . Since

$$\Gamma(\kappa + 1)x^\tau \gamma_\tau(x) = \frac{\Gamma(\tau + 1)}{\Gamma(\tau - \kappa)} \int_0^x (x - t)^{\tau - \kappa - 1} t^\kappa \gamma_\kappa(t) dt,$$

we have

$$\begin{aligned} G_\tau(h) &= \frac{\Gamma(\tau + 1)}{\Gamma(\kappa + 1)\Gamma(\tau - \kappa)h^\tau} \sum_{n=0}^{\infty} c_n \int_0^h (h - t)^{\tau - \kappa - 1} t^\kappa \gamma_\kappa(\lambda_n t) dt \\ &= \frac{\Gamma(\tau + 1)}{\Gamma(\kappa + 1)\Gamma(\tau - \kappa)h^\tau} \int_0^h (h - t)^{\tau - \kappa - 1} t^\kappa G_\kappa(t) dt \\ &= \frac{\Gamma(\tau + 1)}{\Gamma(\kappa + 1)\Gamma(\tau - \kappa)} \int_0^1 (1 - t)^{\tau - \kappa - 1} t^\kappa G_\kappa(ht) dt \\ &= \ell + o(1), \quad h \rightarrow 0^+, \end{aligned}$$

where we have used Lemma 1 and the bound  $\gamma_\kappa(x) = O(1/x)$  to exchange integration and summation in the second equality.  $\square$

We aboard the proof of (b)  $\Rightarrow$  (d). Define the tempered distribution

$$f(x) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x}.$$

By (3.6), we can assume that the series is  $(\gamma, \lambda_n, k)$  summable to  $\ell$  for an integer  $k \geq 1$ , namely,

$$(3.7) \quad F(h) := \frac{h^k}{k!} \sum_{n=0}^{\infty} c_n \gamma_k(\lambda_n h) = \ell \frac{h^k}{k!} + o(|h|^k), \quad h \rightarrow 0.$$

It is clear that  $F^{(k)} = \chi_0$ , where  $\chi_0$  is the distribution given by (3.4). Thus, (3.7) leads to the conclusion  $f_{\text{sym}}(0) = \ell$ , distributionally of order  $k$ . Therefore, applying

[16, Thm 5.18], we obtain

$$\begin{aligned}\ell &= \frac{1}{2\pi} \text{p.v.} \left\langle \hat{f}(u), 1 \right\rangle \quad (\text{C}, \beta) \\ &= \lim_{x \rightarrow \infty} \sum_{\lambda_n \leq x} c_n \quad (\text{C}, \beta) \\ &= \sum_{n=0}^{\infty} c_n \quad (\text{R}, \{\lambda_n\}, \beta),\end{aligned}$$

for some  $\beta > 0$ . (It actually follows from the stronger result [20, Thm. 6.7] that this holds for every  $\beta > k$ .) Hence, the summability (d) has been established.

We now prove (e) $\Rightarrow$ (c). We will actually show that if  $\sum_{n=0}^{\infty} c_n$  is  $(\text{R}, \{\lambda_n\}, 1)$  summable to  $\ell$ , then the series is  $(\gamma, \{\lambda_n\}, 1)$  summable. By (3.6), (c) will automatically follow. We may assume that  $\ell = 0$ . Set  $S(x) = \sum_{\lambda_n \leq x} c_n$  for  $x > 0$  and  $S(0) = 0$ . Our assumption is then

$$S_1(x) = \int_0^x S(t) dt = o(x), \quad x \rightarrow \infty.$$

Employing (2.6), we obtain

$$|S(x)| = \left| \frac{S_1(x)}{x} + \frac{1}{x} \int_0^x t dS(t) \right| = O(1).$$

Let  $\mu$  and  $y$  be two positive numbers to be chosen later. We keep  $h < \mu/y$ . Write

$$\sum_{n=0}^{\infty} c_n \gamma_1(\lambda_n h) = \left( \sum_{\lambda_n \leq \mu/h} + \sum_{\mu/h < \lambda_n} \right) c_n \gamma_1(\lambda_n h) =: I_1(h, \mu) + I_2(h, \mu).$$

By using Lemma 1, we can estimate  $I_2(h, \mu)$  as

$$|I_2(h, \mu)| \leq \frac{1}{h} \sum_{\mu/h < \lambda_n} \frac{|c_n|}{\lambda_n} < \frac{C_1}{\mu},$$

where  $C_1$  does not depend on  $h$ . Integrating by parts twice, we get

$$\begin{aligned}I_1(h, \mu) &= (\gamma_1(\mu)S(\mu/h) - h\gamma_1'(\mu)S_1(\mu/h)) + h^2 \left( \int_0^y + \int_y^{\mu/h} \right) S_1(t)\gamma_1''(ht) dt \\ &=: I_{1,1}(h, \mu) + h^2 I_{1,2}(h, \mu, y) + h^2 I_{1,3}(h, \mu, y).\end{aligned}$$

We can find constants  $C_2, C_3, C_4, C_5 > 0$ , independent of  $h, \mu$ , and  $y$ , such that

$$|I_{1,1}(h, \mu)| < \frac{C_2}{\mu} + C_3 \frac{|S_1(\mu/h)|}{\mu/h},$$

$$|I_{1,2}(h, \mu, y)| < C_4 h^2 y^2,$$

and

$$|I_{1,3}(h, \mu, y)| < C_5 h^2 \int_y^{\mu/h} |S_1(t)| dt \leq C_5 h \mu \max_{t \in [y, \mu/h]} |S_1(t)|.$$

Given  $\varepsilon > 0$ , we fix  $\mu$  larger than  $4(C_1 + C_2)/\varepsilon$ . Next, we can choose  $y$  such that  $|S_1(x)| \leq \varepsilon x / (4 \max\{C_3, \mu^2 C_5\})$  for all  $x \geq y$ . Finally, if we choose  $h_0 < \min\{\mu/y, \sqrt{\varepsilon/(4C_4 y^2)}\}$ , we obtain

$$\left| \sum_{n=0}^{\infty} c_n \gamma_1(\lambda_n h) \right| < \varepsilon \quad \text{for } 0 < h < h_0.$$

This completes the proof of Theorem 4.  $\square$

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