Progress In Electromagnetics Research B, Vol. 18, 165–183, 2009

ELECTROMAGNETIC SOURCE TRANSFORMATIONS AND SCALARIZATION IN STRATIFIED GYROTROPIC MEDIA

P. De Visschere

Ghent University
ELIS, Sint-Pietersnieuwstraat 41, Gent B-9000, Belgium

Abstract—It is known that with restrictions on the type of the constitutive equations, Maxwell's equations in non-uniform media can sometimes be reduced to two 2nd order differential equations for 2 scalar quantities only. These results have previously been obtained in two quite different ways, either by a "scalarization of the sources", where the relevant scalar quantities are essentially vector potential components, and the derivation was limited to isotropic media, or alternatively by using the "scalar Hertz potentials", and this method has been applied to more general media. In this paper, it is shown that both methods are equivalent for gyrotropic media. We show that the scalarization can be obtained by a combination of transformations between electric and magnetic sources and gauge transformations. It is shown that the method based on the vector potential, which previously used a non-traditional definition of the vector potentials, can also be obtained using the traditional definition provided a proper gauge condition is applied, and this method is then extended from isotropic to gyrotropic media. It is shown that the 2 basic scalar Hertz potentials occurring in the second method are invariant under the source scalarization transformations of the first method and therefore are the natural potentials for obtaining scalarization. Finally, it is shown that both methods are also equivalent with a much older third method based on Hertz vectors.

1. INTRODUCTION

Usually problems in electromagnetics are reduced to solving a 2nd order vectorial equation for either electric field or magnetic field [1, 2]. However for some problems, the radiation of a dipole in a stratified

Corresponding author: P. De Visschere (pdv@elis.ugent.be).

medium being a good example, it is still useful to use a representation for the fields in terms of auxiliary functions (vector and scalar potentials or Hertz vector potentials) and then solving equations for these auxiliary functions instead of for the fields directly. It is well-known that for simple uniform electromagnetic media, Maxwell's equations can in this way be reduced to 2 scalar uncoupled 2nd order differential equations, corresponding to the TE/TM modes. For uniform media this has been generalized to more complex (decomposable) media [3]. Also for non-uniform, usually stratified, media such a scalar decomposition has been obtained.

One of the earliest systematic treatments of this "reduction problem" was given by Nisbet [4, 5]. First, a general formulation in terms of 2 Hertz vector potentials was given validly for non-uniform anisotropic media, and then it was noticed that considerable freedom exists for choosing these potentials since they can be subjected to a gauge transformation. Using this freedom the Hertz vectors could be reduced to single component vectors, and conditions were derived under which the resulting differential equations for these components were of 2nd order, at least for isotropic media. The same idea was later extended to anisotropic media [6, 7]. The reduction to two scalar potentials was also extended to gyrotropic media [8, 9] and to even more complicated media [10]. Only rather recently the case of a uniaxial medium was given explicitly [11]. Whereas in these earlier publications [5, 7] a general coordinate system was considered, the later extensions to more complex media usually considered a cartesian coordinate system only, and the non-uniformity was limited to a stratification along e.g., the symmetry axis of the uniaxial medium [11]. Although the 2 scalar potentials are referred to as "scalar Hertz potentials" the link with the Hertz vector potentials is not obvious anymore, and for the stratified uniaxial medium the (initially 4) scalar functions are instead defined by applying a Helmholtz-decomposition to the electric and magnetic field components perpendicular to the symmetry axis [11].

Subsequently Weiglhofer and Georgieva [12, 13] arrived at the same scalar equations, at least for isotropic media, following a completely different method, which rests mainly on the so-called scalarization of sources. It was shown that arbitrary current density distributions can be replaced by equivalent distributions but oriented along a fixed direction. With a proper (unconventional) choice of the vector potentials the latter could then also be scalarized, and the resulting equations are exactly the same as those found using the scalar Hertz potentials. This remarkable correspondence was noticed, but no explanation was given [13].

The main purpose of this paper is to shed some light on this finding, which can not be a coincidence. At the same time, we will extend the source scalarization method explained in [12, 13] to more general (gyrotropic) anisotropic media. We believe that "source scalarization" can be understood best as an application of the well-known equivalence between electric and magnetic charges and currents. This is first presented in § 2. In the following sections the scalarization problem is solved using different potentials. As in most of the referenced papers we consider only cartesian coordinates. A stratified initially uniaxial medium is considered where the symmetry axis is perpendicular to the layers everywhere. The latter condition is necessary to avoid mixing between the longitudinal and transverse field components by applying the constitutive equations. In § 6 the theory is extended to a stratified gyrotropic medium. We will use $\overline{c}$ as a unit vector along the symmetry axis and c as the corresponding coordinate whereas transversal vector components will be labeled by $\perp$; in particular, the transversal nabla operator will be written as $\nabla_\perp$.

2. SOURCE TRANSFORMATIONS

We use Maxwell's equations in the standard form including electric and magnetic charge and current densities, where the latter are labeled by a superscript star

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} - \overline{J}^* \tag{1}$$

$$\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J} \tag{2}$$

$$\nabla \cdot \overline{D} = \rho \tag{3}$$

$$\nabla \cdot \overline{B} = \rho^* \tag{4}$$

The possibly position dependent constitutive properties of the medium are given by $\overline{D} = \epsilon \cdot \overline{E}$ and $\overline{B} = \mu \cdot \overline{H}$. We will write the electric charge and current densities in general as

$$\overline{J} = \frac{\partial \overline{p}}{\partial t} + \nabla \times \overline{m} \tag{5}$$

$$\rho = -\nabla \cdot \overline{p} \tag{6}$$

where $\overline{p}$ and $\overline{m}$ either are given polarization and magnetization densities or must be considered as stream potentials for given ρ and $\overline{J}$ [5]. In either case $\overline{p}$ and $\overline{m}$ can be subjected to a gauge transformation which leaves ρ and $\overline{J}$ invariant [5]

$$\overline{p}' = \overline{p} + \nabla \times \overline{G} \tag{7}$$

$$\overline{m}' = \overline{m} - \frac{\partial \overline{G}}{\partial t} + \nabla g \tag{8}$$

where $\overline{G}$, g are arbitrary functions. Since our aim is to transform transversal sources into longitudinal ones, 2 possibilities arise. With $\nabla_\perp g = -\overline{m}_\perp$ a transversal magnetization is turned into a longitudinal one $\frac{\partial g}{\partial c}\overline{c}$. And with a longitudinal $\overline{G} = G_c \overline{c}$ a transversal polarization $\overline{p}_\perp = -\nabla_\perp G_c \times \overline{c}$ is turned into a longitudinal magnetization $-\frac{\partial G_c}{\partial t}\overline{c}$. It is well-known that the polarization $\overline{p}$ and the magnetization $\overline{m}$ can equally well be represented by magnetic charge and current densities, which is most easily seen by rearranging the terms in the Maxwell equations as follows

$$\nabla \times \left(\overline{E} + \epsilon^{-1} \cdot \overline{p}\right) = -\frac{\partial}{\partial t}\left(\overline{B} - \mu \cdot \overline{m}\right) - \frac{\partial \mu \cdot \overline{m}}{\partial t} + \nabla \times \epsilon^{-1} \cdot \overline{p} \tag{9}$$

$$\nabla \times \left(\overline{H} - \overline{m}\right) = \frac{\partial}{\partial t}\left(\overline{D} + \overline{p}\right) \tag{10}$$

$$\nabla \cdot \left(\overline{D} + \overline{p}\right) = 0 \tag{11}$$

$$\nabla \cdot \left(\overline{B} - \mu \cdot \overline{m}\right) = -\nabla \cdot (\mu \cdot \overline{m}) \tag{12}$$

The equivalent magnetic sources are thus given by

$$\overline{J}^* = \frac{\partial \mu \cdot \overline{m}}{\partial t} - \nabla \times \epsilon^{-1} \cdot \overline{p} \tag{13}$$

$$\rho^* = -\nabla \cdot (\mu \cdot \overline{m}) \tag{14}$$

Contrary to the gauge transformations (7) and (8), in this case the source transformation (from electrical charges to magnetic charges) is accompanied by the following field transformations

$$\overline{E}' = \overline{E} + \epsilon^{-1} \cdot \overline{p} \tag{15}$$

$$\overline{B}' = \overline{B} - \mu \cdot \overline{m} \tag{16}$$

In what follows we will label a polarization/magnetization density which is represented by magnetic charges by a superscript star ($\overline{p}^*, \overline{m}^*$). We can thus freely exchange $\overline{p}$ (or $\overline{m}$) for $\overline{p}^*$ (or $\overline{m}^*$) and vice versa as long as $\overline{p} + \overline{p}^*$ (or $\overline{m} + \overline{m}^*$) remains invariant, and we take the field transformations (15) and (16) into account. These “magnetic” stream potentials can also be subjected to a gauge transformation [5]

$$\epsilon^{-1} \cdot \overline{p}'^* = \epsilon^{-1} \cdot \overline{p}^* - \frac{\partial \overline{L}}{\partial t} + \nabla l \tag{17}$$

$$\mu \cdot \overline{m}'^* = \mu \cdot \overline{m}^* - \nabla \times \overline{L} \tag{18}$$

Table 1. The equivalent source contributions due to (external) polarization and magnetization. The starred quantities allow to make a distinction between sources modeled with electric charges/currents and those modeled with magnetic ones.

	polarization	magnetization
$\overline{J}$	$\frac{\partial \overline{p}}{\partial t}$	$\nabla \times \overline{m}$
ρ	$-\nabla \cdot \overline{p}$	
$\overline{J}^*$	$-\nabla \times \epsilon^{-1} \cdot \overline{p}^*$	$\frac{\partial \mu \cdot \overline{m}^*}{\partial t}$
ρ^*		$-\nabla \cdot \mu \cdot \overline{m}^*$

for arbitrary $\overline{L}$, l. With l we can turn a transversal polarization into a longitudinal one, and with $\overline{L} = L_c \overline{c}$ we can turn a transversal magnetization into a longitudinal polarization. For further reference the different representations are tabulated in Table 1.

Using the electric/magnetic charge transformations and gauge transformations if needed we can now scalarize an arbitrary current density[†] along a fixed direction defined by the unit vector $\overline{c}$. The goal of the scalarization process is to replace the current density by equivalent electric and magnetic current densities parallel to $\overline{c}$. For a stratified medium, $\overline{c}$ is perpendicular to the layers, and as mentioned in the "Introduction" this is also the direction of the symmetry axis of the uniaxial medium. The current density can always be written as

$$\overline{J} = J_c \overline{c} + \nabla_\perp v \times \overline{c} + \nabla_\perp u \tag{19}$$

In the context of scalar Hertz potentials the functions u, v are known as auxiliary functions [11], and they can be found by solving the 2-dimensional potential problems

$$\nabla_\perp^2 u = \nabla_\perp \cdot \overline{J}_\perp \tag{20}$$

$$\nabla_\perp^2 v = -\overline{c} \cdot \left(\nabla_\perp \times \overline{J}_\perp\right) \tag{21}$$

Scalarization of the 2nd term in (19) is straightforward, since according to Table 1 it can be attributed to a magnetization $\overline{m} = v\overline{c}$ which can also be represented by a magnetic current density along $\overline{c}$

$$\overline{J}^* = \frac{\partial \mu \cdot \overline{m}^*}{\partial t} = \mu_{//} \frac{\partial v}{\partial t} \overline{c} \tag{22}$$

where $\mu_{//}$ is the permeability along $\overline{c}$. This transformation is accompanied by a field transformation according to (16)

$$\overline{B}' = \overline{B} - \mu_{//} v \overline{c} \tag{23}$$

[†] We will consider an electric current density, but the same method can be applied to a magnetic current density.

Scalarization of the last contribution in (19) cannot be obtained simply by transforming electric into magnetic sources. However, this transformation can always be combined with a gauge transformation. The last contribution in (19) can then be scalarized by attributing the current density $\nabla_\perp u$ to a transversal polarization density $\overline{p} = \nabla_\perp \int u dt$ which, as we have seen, can be turned into a longitudinal one in the "magnetic" domain by a proper choice of l in (17) (with $\overline{L} = 0$) namely

$$l = -\int \frac{u}{\epsilon_\perp} dt \tag{24}$$

where $\nabla_\perp \epsilon_\perp = 0$ has been used. We then end up with a scalarized current density

$$J'_c = -\epsilon_{//} \frac{\partial}{\partial c} \left(\frac{u}{\epsilon_\perp} \right) \tag{25}$$

Due to the 2 electric/magnetic transformations preceding and following the gauge transformation we must take into account of a transformation of the electric field according to (15)

$$\overline{E}' - \overline{E} = \nabla l = \nabla \left(\int \frac{u}{\epsilon_\perp} dt \right) \tag{26}$$

These results as well as similar ones for the magnetic current density are tabulated in Tables 2 and 3. To conclude this section we make 2 remarks

(i) It can be proved that the scalarizations summarized in Table 2 and Table 3 are unique;

(ii) Under these source transformations the total longitudinal current densities ($J_c + \frac{\partial D_c}{\partial t}$ and $J^*_c + \frac{\partial B_c}{\partial t}$) are invariant, since

$$J_c + \frac{\partial D_c}{\partial t} = \left(\nabla \times \overline{H} \right) \cdot \overline{c} \tag{27}$$

$$J^*_c + \frac{\partial B_c}{\partial t} = -\left(\nabla \times \overline{E} \right) \cdot \overline{c} \tag{28}$$

and $\overline{H}' - \overline{H}$ and $\overline{E}' - \overline{E}$ are either parallel to $\overline{c}$ or equal to a gradient.

3. VECTOR POTENTIALS

We introduce conventional vector and scalar potentials $\overline{A}$, ϕ and also comparable potentials $\overline{A}^*$, ϕ^* for handling the magnetic sources

$$\overline{B} = \nabla \times \overline{A} - \mu \cdot \frac{\partial \overline{A}^*}{\partial t} - \mu \cdot \nabla \phi^* \tag{29}$$

$$\overline{E} = -\frac{\partial \overline{A}}{\partial t} - \nabla \phi - \epsilon^{-1} \cdot \nabla \times \overline{A}^* \tag{30}$$

Table 2. Using proper transformations between electric and magnetic charge representations from Table 1 and possibly gauge transformations of the stream potentials, arbitrary transverse electric current densities can be replaced by equivalent electric or magnetic current densities along the symmetry direction $\overline{c}$. Each transformation is also accompanied by a transformation of the fields, shown in the last 2 rows.

initial current density	$\overline{J} = \nabla_\perp v \times \overline{c}$	$\overline{J} = \nabla_\perp u$
initial stream potential	$\overline{m} = v\overline{c}$	$\overline{p} = \nabla_\perp \int udt$
final stream potential	$\overline{m}^* = v\overline{c}$	$\overline{p} = -\epsilon_{//}\frac{\partial}{\partial c}(\int \frac{u}{\epsilon_\perp} dt)\overline{c}$
final current density	$\overline{J}^* = \mu_{//}\frac{\partial v}{\partial t}\overline{c}$	$\overline{J} = -\epsilon_{//}\frac{\partial}{\partial c}(\frac{u}{\epsilon_\perp})\overline{c}$
$\overline{E}' - \overline{E}$		$\nabla(\int \frac{u}{\epsilon_\perp} dt)$
$\overline{B}' - \overline{B}$	$-\mu_{//}v\overline{c}$	

Table 3. Using proper transformations between electric and magnetic charge representations from Table 1 and possibly gauge transformations of the stream potentials, arbitrary transverse magnetic current densities can be replaced by equivalent electric or magnetic current densities along the symmetry direction $\overline{c}$. Each transformation is also accompanied by a transformation of the fields, shown in the last 2 rows.

initial current density	$\overline{J}^* = \nabla_\perp v^* \times \overline{c}$	$\overline{J}^* = \nabla_\perp u^*$
initial stream potential	$\overline{p}^* = -\epsilon_{//}v^*\overline{c}$	$\overline{m}^* = \nabla_\perp \int \frac{u^*}{\mu_\perp} dt$
final stream potential	$\overline{p} = -\epsilon_{//}v^*\overline{c}$	$\overline{m}^* = -\frac{\partial}{\partial c}(\int \frac{u^*}{\mu_\perp} dt)\overline{c}$
final current density	$\overline{J} = -\epsilon_{//}\frac{\partial v^*}{\partial t}\overline{c}$	$\overline{J}^* = -\mu_{//}\frac{\partial}{\partial c}(\frac{u^*}{\mu_\perp})\overline{c}$
$\overline{E}' - \overline{E}$	$v^*\overline{c}$	
$\overline{B}' - \overline{B}$		$\mu \cdot \nabla(\int \frac{u^*}{\mu_\perp} dt)$

Substitution into Maxwell's curl-equations gives initially

$$\mathcal{L}(\epsilon, \mu)\overline{A} + \epsilon \cdot \nabla \frac{\partial \phi}{\partial t} = \overline{J} \tag{31}$$

$$\mathcal{L}(\mu, \epsilon)\overline{A}^* + \mu \cdot \nabla \frac{\partial \phi^*}{\partial t} = \overline{J}^* \tag{32}$$

where the operator $\mathcal{L}(\epsilon, \mu)$ is defined by

$$\mathcal{L}(\epsilon, \mu) = \epsilon \cdot \frac{\partial^2}{\partial t^2} + \nabla \times \mu^{-1} \cdot \nabla \times \tag{33}$$

The scalar potentials are eliminated using gauge conditions. If these gauge conditions do not mix-up the electric and magnetic quantities then the resulting equations will also remain uncoupled. We first mention the gauge conditions used by Nisbet [5]

$$\nabla \cdot \epsilon \cdot \overline{A} + \alpha \frac{\partial \phi}{\partial t} = 0 \tag{34}$$

$$\nabla \cdot \mu \cdot \overline{A}^* + \alpha^* \frac{\partial \phi^*}{\partial t} = 0 \tag{35}$$

where $\alpha,\ \alpha^*$ are scalars which can still be chosen. As will become clear, these gauge conditions allow scalarization only with additional restrictions on the position dependence of the material parameters. This restriction is eliminated in [12, 13] but only for an isotropic medium by using a different definition for the potentials and using different gauge conditions. At first, this leads to equations for $\overline{A}$ and $\overline{A}^*$ which are coupled but under the restrictions $\nabla_\perp \epsilon = 0$ and $\nabla_\perp \mu = 0$ these equations become uncoupled and scalarizable. Although this method can be extended to uniaxial media we prefer to stick to the conventional decompositions (29) and (30), and we will now derive gauge conditions which allow to scalarize (31) and (32) for a non-uniform uniaxial medium.

We assume that the sources have already been scalarized so that in (31) and (32) only longitudinal current densities $J_c,\ J_c^*$ occur. Splitting these equations into longitudinal and transversal components the latter equations will only allow the null solution for the transversal components of the vector potentials if their longitudinal components do not occur in these transversal equations. These conditions are easily found by assuming $\overline{A} = A_c \overline{c}$ and equating the transversal component of the LHS of (31) to zero

$$\nabla_\perp \left[\frac{\partial}{\partial c} \left(\frac{A_c}{\mu_\perp} \right) + \epsilon_\perp \frac{\partial \phi}{\partial t} \right] = 0 \tag{36}$$

and a similar "magnetic" equation, which is obtained by replacing unstarred quantities by starred ones and by switching the roles of ϵ and μ. Using the gauge condition (34), this condition can only be met if $\epsilon_\perp,\ \mu_\perp$ do not depend on the longitudinal coordinate c or if $\varepsilon_{//}\mu_\perp$ is independent of c, and then only by choosing $\alpha = \epsilon_{//}\epsilon_\perp\mu_\perp$. For a more general result we must instead choose the following gauge conditions

$$\nabla \cdot \mu_\perp^{-1} \overline{A} + \epsilon_\perp \frac{\partial \phi}{\partial t} = 0 \tag{37}$$

$$\nabla \cdot \epsilon_\perp^{-1} \overline{A}^* + \mu_\perp \frac{\partial \phi^*}{\partial t} = 0 \tag{38}$$

From the longitudinal components of (31), (32) and these gauge conditions, we then find the final scalarized equations

$$\mathcal{L}_s(\epsilon,\mu)\left(\frac{A'_c}{\mu_\perp}\right) = -J'_c \tag{39}$$

$$\mathcal{L}_s(\mu,\epsilon)\left(\frac{A'^*_c}{\epsilon_\perp}\right) = -J'^*_c \tag{40}$$

where scalarized quantities are now explicitly marked by an accent with the (scalar) wave operator defined by

$$\mathcal{L}_s(\epsilon,\mu) = \nabla^2_\perp + \epsilon_{//}\frac{\partial}{\partial c}\epsilon^{-1}_\perp\frac{\partial}{\partial c} - \epsilon_{//}\mu_\perp\frac{\partial^2}{\partial t^2} \tag{41}$$

Taking into account of (29), (30) and the field transformations due to the prior source scalarization, the fields are given by

$$\overline{H} = \left(v - \frac{\partial A'^*_c}{\partial t}\right)\overline{c} + \mu^{-1}_\perp\nabla_\perp A'_c \times \overline{c} - \nabla\left(\mu^{-1}_\perp\int\left[u^* - \frac{\partial}{\partial c}\left(\frac{A'^*_c}{\epsilon_\perp}\right)\right]dt\right) \tag{42}$$

$$\overline{E} = -\left(v^* + \frac{\partial A'_c}{\partial t}\right)\overline{c} - \epsilon^{-1}_\perp\nabla_\perp A'^*_c \times \overline{c} - \nabla\left(\epsilon^{-1}_\perp\int\left[u - \frac{\partial}{\partial c}\left(\frac{A'_c}{\mu_\perp}\right)\right]dt\right) \tag{43}$$

4. HERTZ VECTORS

The main idea behind the use of Hertz vectors is to introduce an additional differentiation in such a way that the new potentials (= Hertz vectors) are governed by equations with the polarization/magnetization (or the stream potentials) as sources instead of the current densities. Since for such Hertz vectors the starred/unstarred stream potentials are equivalent we can eliminate e.g., the "magnetic charge" sources from the start and use instead of (29), (30) the simpler decompositions

$$\overline{B} + \mu\cdot\overline{m}^* = \nabla\times\overline{A} \tag{44}$$

$$\overline{E} - \epsilon^{-1}\cdot\overline{p}^* = -\frac{\partial\overline{A}}{\partial t} - \nabla\phi \tag{45}$$

still complying with (1) and (4). The Hertz vector equations are obtained most easily in the *temporal gauge* [14] ($\phi = 0$), and we then define the Hertz vectors $\overline{\Pi}_e$, $\overline{\Pi}_m$ following [5]

$$\overline{A} = \frac{\partial\overline{\Pi}_e}{\partial t} + \epsilon^{-1}\cdot\nabla\times\overline{\Pi}_m \tag{46}$$

After substituting these equations in (2), (3) and following a standard procedure [15], we find the equations

$$\mathcal{L}(\epsilon,\mu)\overline{\Pi}_e = \overline{p} + \overline{p}^* + \epsilon\cdot\nabla\psi_e \tag{47}$$

$$\mathcal{L}(\mu,\epsilon)\overline{\Pi}_m = \mu\cdot(\overline{m} + \overline{m}^*) + \mu\cdot\nabla\psi_m \tag{48}$$

where the operator $\mathcal{L}(\epsilon, \mu)$ has been defined in (33). The functions ψ_e, ψ_m can be chosen arbitrarily and are in fact redundant due to the gauge transformations (7), (8), (17), and (18). However, an extra term $\nabla\psi_e$ should now be added to the RHS of (45). Substituting (46) in (44) and (45) and replacing the 2nd order time derivative using (47) and (33) we find the following symmetric expressions for the fields

$$\overline{B} + \mu \cdot \overline{m}^* = \nabla \times \frac{\partial \overline{\Pi}_e}{\partial t} + \nabla \times \epsilon^{-1} \cdot \nabla \times \overline{\Pi}_m \tag{49}$$

$$\epsilon \cdot \overline{E} + \overline{p} = \nabla \times \mu^{-1} \cdot \nabla \times \overline{\Pi}_e - \nabla \times \frac{\partial \overline{\Pi}_m}{\partial t} \tag{50}$$

Using gauge transformations (7), (8), (17), and (18), one can try to simplify equations (47) and (48). One possibility is to eliminate the magnetizations ($\overline{m} + \overline{m}^* \rightarrow 0$) so that also $\overline{\Pi}_m \rightarrow 0$. This is the essence of the 1-Hertz-vector method followed by Sein [16]. In this case it is limited to a uniform isotropic medium and therefore using the conventional Lorentz gauge. Another possibility is to scalarize the stream functions so that $\overline{p} + \overline{p}^* \rightarrow p'_c\overline{c}$ and $\overline{m} + \overline{m}^* \rightarrow m'^*_c\overline{c}$ [7]. For these scalarized sources it is now possible to choose the (gauge) functions ψ_e, ψ_m in (47) and (48) in such a way that $\overline{\Pi}_e$, $\overline{\Pi}_m$ are also scalarized and thus have only components along $\overline{c}$. To that end we assume $\overline{\Pi}_{e/m} = \Pi_{(e/m)c}\overline{c}$ and collect the transversal components of e.g., (47)

$$\frac{\partial}{\partial c}\left[\mu_\perp^{-1}\left(\nabla_\perp \Pi_{ec}\right)\right] = \epsilon_\perp \nabla_\perp \psi_e \tag{51}$$

The transversal components of the vector potentials will then vanish if we choose

$$\psi_e = \frac{1}{\epsilon_\perp}\frac{\partial}{\partial c}\frac{\Pi_{ec}}{\mu_\perp} \tag{52}$$

This is a particular form of the condition found by Mohsen [6] for a coordinate system more general than the cartesian system considered here. The longitudinal part of the same equation is given by

$$\epsilon_{//}\frac{\partial^2 \Pi_{ec}}{\partial t^2} - \nabla_\perp^2 \frac{\Pi_{ec}}{\mu_\perp} = p'_c + \epsilon_{//}\frac{\partial \psi_e}{\partial c} \tag{53}$$

and using (52) we obtain

$$\mathcal{L}_s(\epsilon, \mu)\left(\frac{\Pi'_{ec}}{\mu_\perp}\right) = -p'_c \tag{54}$$

where the scalar operator $\mathcal{L}_s$ has been defined in (41). A similar equation can be found starting from (48)

$$\mathcal{L}_s(\mu, \epsilon)\left(\frac{\Pi'_{mc}}{\epsilon_\perp}\right) = -\mu_{//} m'^*_c \tag{55}$$

Unlike the scalarized current densities, which are unique, the scalarized stream potentials and corresponding Hertz vector components are not unique, since they can always be subjected to a gauge transformation. However, we can in particular choose the scalarized stream potentials as follows

$$J'_c = \frac{\partial p'_c}{\partial t} \qquad J'^*_c = \mu_{//}\frac{\partial m'^*_c}{\partial t} \tag{56}$$

where J'_c and J'^*_c are the (unique) scalarized current densities. Comparing (54)and (55) with (39) and (40) we conclude that in that case

$$A'_c = \frac{\partial \Pi'_{ec}}{\partial t} \qquad A'^*_c = \frac{\partial \Pi'_{mc}}{\partial t} \tag{57}$$

This correspondence can also be checked by comparing the field expressions (49) and (50) with (42) and (43) where in (49) and (50) one should also take into account of the field transformations due to the scalarization of the current densities (see Tables 2 and 3).

5. SCALAR HERTZ POTENTIALS

The "scalar Hertz potential" formulation introduced by Weiglhofer [11] starts by decomposing the fields and equations into transversal and longitudinal components and parts. From these equations the longitudinal components E_c, H_c can be eliminated, leaving 4 equations for the 4 unknown transversal components. Up to this point, the method is identical to the 4×4 matrix method used for solving Maxwell's equations in stratified media [17]. However, for dealing with the source terms and unlike the 4×4 matrix method, the transversal field components are then expressed using scalar (Hertz) potential functions

$$\overline{E}_\perp = \nabla_\perp \Phi + \nabla_\perp \times \Theta \overline{c} \tag{58}$$

$$\overline{H}_\perp = \nabla_\perp \Pi + \nabla_\perp \times \Psi \overline{c} \tag{59}$$

In what follows we will give a compact derivation of the scalar Hertz potential equations, following [11]. Splitting Maxwell's curl-equations parallel and perpendicular to $\overline{c}$ we obtain

$$\overline{c} \cdot \left(\nabla_\perp \times \overline{H}_\perp\right) = J_c + \frac{\partial D_c}{\partial t} \tag{60}$$

$$\overline{c} \cdot \left(\nabla_\perp \times \overline{E}_\perp\right) = -\left(J^*_c + \frac{\partial B_c}{\partial t}\right) \tag{61}$$

and

$$\nabla_\perp H_c - \frac{\partial \overline{H}_\perp}{\partial c} = \overline{c} \times \left(\overline{J}_\perp + \frac{\partial \overline{D}_\perp}{\partial t} \right) \tag{62}$$

$$\nabla_\perp E_c - \frac{\partial \overline{E}_\perp}{\partial c} = -\overline{c} \times \left(\overline{J}^*_\perp + \frac{\partial \overline{B}_\perp}{\partial t} \right) \tag{63}$$

Taking the cross-product with $\overline{c}$, the latter 2 equations become

$$\overline{c} \times \nabla_\perp H_c - \frac{\partial \left(\overline{c} \times \overline{H}_\perp \right)}{\partial c} = -\overline{J}_\perp - \frac{\partial \overline{D}_\perp}{\partial t} \tag{64}$$

$$\overline{c} \times \nabla_\perp E_c - \frac{\partial \left(\overline{c} \times \overline{E}_\perp \right)}{\partial c} = \overline{J}^*_\perp + \frac{\partial \overline{B}_\perp}{\partial t} \tag{65}$$

The 4×4 matrix method is based on (63) and (64), without the source terms, where (60) and (61) are used for eliminating E_c and H_c, after inserting the constitutive equations

$$D_c = \epsilon_{//} E_c \qquad \overline{D}_\perp = \epsilon_\perp \overline{E}_\perp \tag{66}$$

$$B_c = \mu_{//} H_c \qquad \overline{B}_\perp = \mu_\perp \overline{H}_\perp \tag{67}$$

Weiglhofer [11] deals with the source terms by operating with $\nabla_\perp \cdot$ on the 4 equations (62)–(65) and by introducing the auxiliary functions already defined in (20) and (21). Using constitutive equations (66), (67) and decompositions (58), (59), all terms then contain the Laplacian $\nabla^2_\perp$ which can be dropped, yielding

$$H_c - \frac{\partial \Pi}{\partial c} - \epsilon_\perp \frac{\partial \Theta}{\partial t} = v \tag{68}$$

$$E_c - \frac{\partial \Phi}{\partial c} + \mu_\perp \frac{\partial \Psi}{\partial t} = -v^* \tag{69}$$

$$\epsilon_\perp \frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial c} = -u \tag{70}$$

$$\mu_\perp \frac{\partial \Pi}{\partial t} + \frac{\partial \Theta}{\partial c} = -u^* \tag{71}$$

On the other hand the longitudinal equations (60) and (61) become

$$-\nabla^2_\perp \Psi = J_c + \epsilon_{//} \frac{\partial E_c}{\partial t} \tag{72}$$

$$\nabla^2_\perp \Theta = J^*_c + \mu_{//} \frac{\partial H_c}{\partial t} \tag{73}$$

Finally $\partial E_c / \partial t$ and $\partial H_c / \partial t$ can be calculated from (68)–(71) as a function of Ψ, Θ only, and when substituted in (72) and (73) one obtains two uncoupled 2nd order equations in Ψ, Θ. We remind the reader that the RHSs of (60), (61) and thus also of (72), (73) are invariant under the source scalarization transformations shown in Tables 2 and 3, meaning that $\nabla_\perp \times \overline{E}_\perp$ and $\nabla_\perp \times \overline{H}_\perp$, and also Ψ, Θ are invariant under these transformations. Therefore, the scalar potentials Ψ, Θ can only depend on the unique scalarized current densities J'_c and J'^*_c. Whereas using the vector potential method in § 3 or the Hertz vector method in § 4, scalarization could only be obtained with some effort by applying appropriate source and gauge transformations; the scalar Hertz potentials Ψ and Θ are the natural potentials for obtaining scalarization since they are invariant under the required transformations. The final equations can then also be obtained immediately by making the RHSs of (68)–(71) zero and replacing J_c, J_c^* in (72) and (73) by the scalarized versions J'_c, J'^*_c leading to

$$\mathcal{L}_s(\epsilon, \mu)\,(\Psi) \,=\, -J'_c \tag{74}$$

$$\mathcal{L}_s(\mu, \epsilon)\,(-\Theta) \,=\, -J'^*_c \tag{75}$$

However note that E_c, H_c and Φ, Π are not invariant. From (70) and (71), we find

$$\Phi \,=\, \epsilon_\perp^{-1} \int \left(-u + \frac{\partial \Psi}{\partial c} \right) dt \tag{76}$$

$$\Pi \,=\, -\mu_\perp^{-1} \int \left(u^* + \frac{\partial \Theta}{\partial c} \right) dt \tag{77}$$

and subsequently from (68) and (69)

$$E_c \,=\, -v^* + \frac{\partial \Phi}{\partial c} - \mu_\perp \frac{\partial \Psi}{\partial t} \tag{78}$$

$$H_c \,=\, v + \frac{\partial \Pi}{\partial c} + \epsilon_\perp \frac{\partial \Theta}{\partial t} \tag{79}$$

The total fields can then be written as

$$\overline{E} \,=\, -\left(v^* + \mu_\perp \frac{\partial \Psi}{\partial t} \right) \overline{c} + \nabla_\perp \Theta \times \overline{c} + \nabla \Phi \tag{80}$$

$$\overline{H} \,=\, \left(v + \epsilon_\perp \frac{\partial \Theta}{\partial t} \right) \overline{c} + \nabla_\perp \Psi \times \overline{c} + \nabla \Pi \tag{81}$$

These expressions confirm the field transformations in Tables 2 and 3, and they correspond term for term with the expressions in (42) and (43) with the correspondence

$$\Psi = \frac{A'_c}{\mu_\perp} = \frac{\partial}{\partial t} \frac{\Pi'_{ec}}{\mu_\perp} \qquad -\Theta = \frac{A'^*_c}{\epsilon_\perp} = \frac{\partial}{\partial t} \frac{\Pi'_{mc}}{\epsilon_\perp} \tag{82}$$

$$\Phi = -\phi - \int \frac{u}{\epsilon_\perp} dt \qquad \Pi = -\phi^* - \int \frac{u^*}{\mu_\perp} dt \tag{83}$$

which was already (partially) apparent from (74), (75) and (39), (40).

6. GYROTROPIC MEDIA

For a gyrotropic medium the transversal dielectric tensor is given by [8]

$$\epsilon_\perp = \begin{bmatrix} \epsilon_\perp & j\epsilon'_\perp \\ -j\epsilon'_\perp & \epsilon_\perp \end{bmatrix} = \epsilon_\perp I + j\epsilon'_\perp \overline{c} \times I \tag{84}$$

with a similar expression for the permittivity tensor.

Since the scalarization of the solenoidal parts of $\overline{J}_\perp$ and $\overline{J}'_\perp$ does not involve these transversal constitutive tensors no changes are needed here. However, for the irrotational parts the reasoning leading to (24), (25) and (26) must be extended with additional terms. Introducing the gauge function G_c in the "electric" domain (7) and (8) and as before l in the "magnetic" domain (17) we obtain the transformed stream potentials

$$\overline{p} = \nabla_\perp \int u dt + \epsilon \cdot \nabla l + \nabla_\perp G_c \times \overline{c} \tag{85}$$

$$\overline{m}^* = -\frac{\partial G_c}{\partial t} \overline{c} \tag{86}$$

If $\epsilon'_\perp \neq 0$ then the 2nd term on the RHS of (85) contains an extra term which can be compensated by the last term if

$$G_c = j\epsilon'_\perp l \tag{87}$$

In this way the extra transversal term in the polarization is transformed into a longitudinal magnetization, which can be represented by a longitudinal magnetic current density

$$J'^*_c = -\mu_{//} \frac{\partial^2 G_c}{\partial t^2} \tag{88}$$

Since (24), (25) and (26) remain valid it suffices thus to add a magnetic current density

$$J'^*_c = j \frac{\epsilon'_\perp}{\epsilon_\perp} \mu_{//} \frac{\partial u}{\partial t} \tag{89}$$

and due to the electric/magnetic switch ($\overline{m} \to \overline{m}^*$) in (86) we must also add a matching field transformation

$$\overline{B}' - \overline{B} = \mu_{//} \frac{\partial G_c}{\partial t} \overline{c} = -j \frac{\epsilon'_\perp}{\epsilon_\perp} \mu_{//} u \overline{c} \tag{90}$$

Again $J_c + \frac{\partial B_c}{\partial t}$ remains invariant for this additional transformation, and therefore the formulation using the *scalar Hertz potentials* should still automatically lead to scalarized equations as shown in [9].

Considering the formulation using *Hertz vectors*, extra (transversal) terms in $j\epsilon'_\perp$ will also occur in (51). However, the functions $\psi_{e/m}$ in (47) and (48) are only part of the gauge transformation, and more in general we can also add terms in $\overline{G}$ respectively in $\overline{L}$ as in (7), (8) and (17), (18) to the RHS of (47), (48). With in particular $\overline{G} = G_c\overline{c}$ (and $\overline{L} = L_c\overline{c}$) we then obtain instead of (51) condition

$$\begin{aligned} &\frac{\partial}{\partial c}\left(\frac{\mu_\perp}{|\mu_\perp|}\nabla_\perp\Pi_{ec}\right) - \overline{c}\times\frac{\partial}{\partial c}\left(\frac{j\mu'_\perp}{|\mu_\perp|}\nabla_\perp\Pi_{ec}\right) \\ &= \epsilon_\perp\nabla_\perp\psi_e + j\epsilon'_\perp\overline{c}\times\nabla_\perp\psi_e + \nabla G_c\times\overline{c} \end{aligned} \tag{91}$$

where $|\mu_\perp| = \det\,\mu_\perp = \mu_\perp^2 - \mu'^2_\perp$. This condition is fulfilled by choosing

$$\psi_e = \epsilon_\perp^{-1}\frac{\partial}{\partial c}\left(\frac{\mu_\perp}{|\mu_\perp|}\Pi_{ec}\right) \tag{92}$$

$$G_c = j\left[\frac{\partial}{\partial c}\frac{\mu'_\perp}{\mu_\perp} + \frac{\epsilon'_\perp}{\epsilon_\perp}\frac{\partial}{\partial c}\right]\frac{\mu_\perp}{|\mu_\perp|}\Pi_{ec} \tag{93}$$

The longitudinal Equation (53) is replaced by

$$\epsilon_{//}\frac{\partial^2\Pi_{ec}}{\partial t^2} - \nabla_\perp^2\left(\frac{\mu_\perp}{|\mu_\perp|}\Pi_{ec}\right) = p'_c + \epsilon_{//}\left(\frac{\partial\psi_e}{\partial c} - \frac{\partial L_c}{\partial t}\right) \tag{94}$$

where $-L_c$ is given by a similar expression as in (93). We notice that except for the replacement of $\mu_\perp^{-1}$ by $\mu_\perp/|\mu_\perp|$ the main change is the occurrence of a cross-coupling term due to the additional gauge functions G_c, L_c which indeed mix between electric and magnetic stream functions. If we replace (41) by the more general expression

$$\mathcal{L}_s(\epsilon,\mu) = \nabla_\perp^2 + \epsilon_{//}\frac{\partial}{\partial c}\epsilon_\perp^{-1}\frac{\partial}{\partial c} - \epsilon_{//}\frac{|\mu_\perp|}{\mu_\perp}\frac{\partial^2}{\partial t^2} \tag{95}$$

then the scalarized equations for a gyrotropic medium (omitting the accents) are given by

$$\mathcal{L}_s(\epsilon,\mu)\Sigma_e + j\epsilon_{//}\left[\frac{\partial}{\partial c}\frac{\epsilon'_\perp}{\epsilon_\perp} + \frac{\mu'_\perp}{\mu_\perp}\frac{\partial}{\partial c}\right]\frac{\partial\Sigma_m}{\partial t} = -p_c \tag{96}$$

$$\mathcal{L}_s(\mu,\epsilon)\Sigma_m - j\mu_{//}\left[\frac{\partial}{\partial c}\frac{\mu'_\perp}{\mu_\perp} + \frac{\epsilon'_\perp}{\epsilon_\perp}\frac{\partial}{\partial c}\right]\frac{\partial\Sigma_e}{\partial t} = -\mu_{//}m_c^* \tag{97}$$

where $\Sigma_e = \frac{\mu_\perp}{|\mu_\perp|}\Pi_{ec}$ and $\Sigma_m = \frac{\epsilon_\perp}{|\epsilon_\perp|}\Pi_{mc}$ and as for the uniaxial case these Hertz vector components are equivalent to the scalar Hertz potentials with $\Psi = \partial\Sigma_e/\partial t$ and $-\Theta = \partial\Sigma_m/\partial t$.

Finally, we consider the formulation using ordinary but standard *vector potentials*. Comparing (31), (32) with (47), (48) the close resemblance between both formulations is apparent where in particular $-\partial\phi^{(*)}/\partial t \Leftrightarrow \psi_{e/m}$ and therefore scalarization of (31), (32) might also be possible. However, whereas for a uniaxial medium the functions $\phi^{(*)}$ or $\psi_{e/m}$ are sufficient for obtaining scalar equations, for a gyrotropic medium the additional freedom offered by the gauge function G_c in (91) (and L_c) is needed, and these functions do not occur in (31) and (32). With some hindsight we realize that transformations of the "electric" and "magnetic" stream potentials according to

$$\epsilon^{-1}\cdot\overline{p}' = \epsilon^{-1}\cdot\overline{p} - \overline{\mathcal{E}} \tag{98}$$

$$\epsilon^{-1}\cdot\overline{p}'^{*} = \epsilon^{-1}\cdot\overline{p}^{*} + \overline{\mathcal{E}} \tag{99}$$

$$\overline{m}' = \overline{m} + \overline{\mathcal{H}} \tag{100}$$

$$\overline{m}'^{*} = \overline{m}^{*} - \overline{\mathcal{H}} \tag{101}$$

correspond with the following transformations of the current densities

$$\overline{J}' = \overline{J} + \nabla\times\overline{\mathcal{H}} - \epsilon\cdot\frac{\partial\overline{\mathcal{E}}}{\partial t} \tag{102}$$

$$\overline{J}'^{*} = \overline{J}^{*} - \mu\cdot\frac{\partial\overline{\mathcal{H}}}{\partial t} - \nabla\times\overline{\mathcal{E}} \tag{103}$$

These transformations are thus legitimate, provided the fields are transformed according to

$$\overline{E}' = \overline{E} + \overline{\mathcal{E}} \qquad \overline{H}' = \overline{H} + \overline{\mathcal{H}} \tag{104}$$

With additional transformations (102) and (103) of the current densities the formulations based on the standard vector potentials on one hand and on the Hertz vectors on the other hand become fully equivalent. The former can thus also be scalarized with a proper choice of the derivatives of the potentials $\partial\phi^{(*)}/\partial t$ and of the longitudinal fields $\mathcal{E}\overline{c}$ and $\mathcal{H}\overline{c}$. As a result, the scalar equations (96) and (97) also hold for the vector potentials with the proper substitutions $A_c^{(*)} \Leftrightarrow \Pi_{(e/m)c}$, $p_c \Leftrightarrow J_c$ and $\mu_{//}m_c^* \Leftrightarrow J_c^*$ (omitting the accents). Also for this transformation $\nabla_\perp\times\overline{E}_\perp$ and $\nabla_\perp\times\overline{H}_\perp$ are invariant, explaining why it is also automatically included using the scalar Hertz potentials.

7. CONCLUSIONS

We have compared three "potential" methods for solving Maxwell's equations with arbitrary sources in a lineair stratified gyrotropic medium where the longitudinal symmetry axis (sometimes referred to as the distinguished axis) is perpendicular to the strata. In particular, we studied the reduction of Maxwell's equations to two scalar equations (scalarization). A prerequisite for scalarization to be possible and which we have accepted without proof, is that the constitutive tensors should not introduce mixing between longitudinal and transversal field components. A second condition which was often required in the derivations is that the transversal constitutive tensors should not depend on the transverse coordinates. It is perhaps interesting to note that there are no restrictions on the position dependence of the longitudinal properties $\epsilon_{//}$ and $\mu_{//}$.

Introducing the conventional *vector/scalar potentials* or the *Hertz vectors* two uncoupled vector equations are obtained. We have shown that with the limitations already mentioned these equations can always be scalarized. First the sources must be scalarized: Using the equivalence between electric and magnetic sources and gauge transformations for the polarizations (stream potentials), the current densities can always be replaced by unique current densities along the distinguished axis. The sources for the Hertz vector equations (the stream potentials) can be scalarized with the gauge transformations only, but it is easier to use (56) and the scalarized current densities. At this stage the problem cannot yet be reduced to two scalar differential equations because there is still cross-coupling between the longitudinal and transversal components of the vector potentials or Hertz vectors. However, using gauge transformations (see (36), (51) and (91)) these cross-coupling terms can always be eliminated, and two scalar equations are obtained. When using the vector potentials, for a gyrotropic medium, an additional transformation between electric/magnetic sources must be performed (see (98)–(104)), and for such a gyrotropic medium the final scalar equations are also coupled. The two methods are found to be fully equivalent, and the "vector potential" quantities are merely the time derivatives of the corresponding "Hertz vector" quantities.

We noticed that the appropriate electric/magnetic source transformations leave the longitudinal "total" current densities $\nabla_\perp \times \overline{E}_\perp$ and $\nabla_\perp \times \overline{H}_\perp$ invariant, and this obviously holds also for the gauge transformations. In a third method, 4 *scalar Hertz potentials* are defined based on Helmholtz decompositions (58) and (59) in the transversal plane, and it immediately follows that 2 of those scalar potentials, Ψ and Θ, are also invariant under all transformations needed to obtain scalarization. These are the natural potentials for obtaining scalarization, and this invariance explains why, when using these scalar Hertz potentials, the scalarized current densities emerge effortlessly. These scalar Hertz potentials are also equivalent with the longitudinal components of the scalarized traditional Hertz vectors used in the 2nd method, and eventually we conclude that the three methods are fully equivalent. It remains to be investigated whether this conclusion still holds for more complex media for which scalarization has been obtained using the scalar Hertz potentials [10].

REFERENCES

1. Joannopoulos, J. D., S. G. Johnson, J. N. Winn, and R. D. Meade, *Photonic Crystals*, 3nd edition, Princeton University Press, 2008.
2. Monk, P., *Finite Element Methods for Maxwell's Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, 2003.
3. Olyslager, F. and I. Lindell, "Field decomposition and factorization of the Helmholtz determinant operator for bianisotropic media," *IEEE Transactions on Antennas and Propagation*, Vol. 49, No. 4, 660–665, Apr. 2001.
4. Nisbet, A., "Hertzian electromagnetic potentials and associated gauge transformations," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1934–1990)*, Vol. 231, No. 1185, 250–263, Aug. 1955.
5. Nisbet, A., "Electromagnetic potentials in a heterogeneous nonconducting medium," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences (1934–1990)*, Vol. 240, No. 1222, 375–381, Jun. 1957.
6. Mohsen, A., "Electromagnetic field representation in inhomogeneous anisotropic media," *Applied Physics A: Materials Science & Processing*, Vol. 2, No. 3, 123–128, Sep. 1973.
7. Mohsen, A., "Electromagnetic-field representation in source region," *Applied Physics*, Vol. 10, No. 1, 53–55, Jan. 1976.
8. Przeźiecki, S. and R. A. Hurd, "A note on scalar Hertz potentials for gyrotropic media," *Applied Physics*, Vol. 20, No. 4, 313–317, Dec. 1979.
9. Weiglhofer, W. S. and W. Papousek, "Scalar Hertz potentials for anisotropic media," *Aeu-Arch Elektron. Ub*, Vol. 39, No. 6, 343–346, Jan. 1985.
10. Weiglhofer, W. and S. Hansen, "Faraday chiral media revisited. I. Fields and sources," *IEEE Transactions on Antennas and Propagation*, Vol. 47, No. 5, 807–814, May 1999.
11. Weiglhofer, W. S., "Scalar Hertz potentials for nonhomogeneous uniaxial dielectric-magnetic mediums," *Int. J. Appl. Electrom.*, Vol. 11, No. 3, 131–140, Jan. 2000.

12. Georgieva, N. and W. S. Weiglhofer, "Electromagnetic vector potentials and the scalarization of sources in a nonhomogeneous medium," *Physical Review E*, Vol. 66, No. 4, Oct. 8, 2002.
13. Weiglhofer, W. S. and N. Georgieva, "Vector potentials and scalarization for nonhomogeneous isotropic mediums," *Electromagnetics*, Vol. 23, No. 5, 387–398, Jul. 2003.
14. Jackson, J. D., "From Lorenz to Coulomb and other explicit gauge transformations," *Am. J. Phys.*, Vol. 70, No. 9, 917–928, 2002.
15. Jackson, J. D., *Classical Electrodynamics*, 3rd edition, 280–282, John Wiley & Sons, 1999.
16. Sein, J. J., "Solutions to time-harmonic Maxwell equations with a hertz vector," *Am. J. Phys.*, Vol. 57, No. 9, 834–839, Jan. 1989.
17. Berreman, D. W., "Optics in stratified and anisotropic media — 4×4 — matrix formulation," *J. Opt. Soc. Am.*, Vol. 62, No. 4, 502–510, Jan. 1972.