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Models for Irreducible Representations of $Spin(m)$

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ABSTRACT In this paper we consider harmonic and monogenic polynomials of simplicial type. It is proved that these polynomials provide explicit realizations of all irreducible representations of $Spin(m)$.

1 Introduction

Let $(e_1, \ldots, e_m)$ be an orthonormal basis of Euclidean space $\mathbb{R}^m$ endowed with the inner product $\langle x, y\rangle = \sum_{i=1}^m x_i y_i,\ x, y \in \mathbb{R}^m$. By $\mathbb{R}_m$ $(\mathbb{C}_m)$ we denote the real (complex) 2^m-dimensional Clifford algebra over $\mathbb{R}^m$ generated by the relations $e_i^2 = -1,\ i = 1, \ldots, m$ and $e_i e_j + e_j e_i = 0,\ i \neq j$. An element of $\mathbb{C}_m$ is of the form $a = \sum_{A\subset M} a_A e_A,\ a_A \in \mathbb{C},\ M = \{1, \ldots, m\}$ and $e_\phi = e_0 = 1$. Reversion on $\mathbb{C}_m$ is the (principal) anti-involution defined by $\tilde{e}_A = (-1)^{\frac{s(s-1)}{2}}, s = \sharp A$ and extended by linearity to $\mathbb{C}_m$. Con-

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jugation on $\mathbb{C}_m$ is the anti-involution on $\mathbb{C}_m$ given by $\bar{a} = \sum_{A\subset M} \bar{a}_A \bar{e}_A$ where $\bar{e}_A = \bar{e}_{\alpha_h} \dots \bar{e}_{\alpha_1}$ and $\bar{e}_j = -e_j$, $j = 1, \dots, m$. Vectors $x \in \mathbb{R}^m$ are identified with Clifford numbers $x = \sum_{j=1}^m x_j e_j$. The following subgroups of the real Clifford algebra $\mathbb{R}_m$ are of interest. The Pin group $Pin(m)$ is the group consisting of products of unit vectors in $\mathbb{R}^m$; the Spin group $Spin(m)$ is the subgroup of $Pin(m)$ consisting of products of an even number of unit vectors in $\mathbb{R}^m$. For an element $s \in Pin(m)$ the map $\chi(s) : \mathbb{R}^m \to \mathbb{R}^m : x \mapsto sx\tilde{s}$ induces an orthogonal transformation of $\mathbb{R}^m$. In this way $Pin(m)$ defines a double covering of the orthogonal group $O(m)$. The restriction of this map to $Spin(m)$, $x :\mapsto sx\bar{s}$ then defines a double covering of the rotation group $SO(m)$. The Dirac operator on $\mathbb{R}^m$ is given by $\partial_x = e_1 \partial_{x_1} + \dots + e_m \partial_{x_m}$. In spherical coordinates $x = \rho\omega,\ \rho = |x| = (x_1^2 + \dots + x_m^2)^{1/2}$ and $\omega \in S^{m-1},\ S^{m-1}$ being the unit sphere in $\mathbb{R}^m$, the Dirac operator admits the polar decomposition $\partial_x = \omega(\partial_\rho + \frac{1}{\rho}\Gamma_\omega)$ where $\Gamma_\omega = -x \wedge \partial_x$ is the spherical Dirac operator on S^{m-1}. In terms of the momentum operators $L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_j}$, $i, j = 1, \dots, m$ on $\mathbb{R}^m$ the Γ-operator is given by $\Gamma = -\sum_{i<j} e_{ij} L_{ij}$ while the Laplace-Beltrami operator $\triangle_S = \sum_{i<j} L_{ij}^2 = \Gamma(m - 2 - \Gamma)$.

The theory of harmonic functions of a matrix variable was presented in detail in [GM]. They consider simplicial harmonics (i.e. harmonic polynomials of a matrix variable invariant under the action of $SL(r)$) which provide models for irreducible representations of $SO(m)$ with integer weight $(k, \dots, k, 0, \dots, 0)$ (r times k). This leads to the idea to look for models of half integer weight irreducible representations of $Spin(m)$ inside spaces of monogenic functions of several vector variables. This theory was already developed to some extent in [Co] (in the case of several quaternionic variables see e.g. [ABLSS], [Pa] and [Pe]). As a matter of fact, to obtain polynomial irreducible representations of $Spin(m)$ we look to spaces of polynomials which are already irreducible with respect to the action of $GL(m)$. These are the so called simplicial polynomials or polynomials of Young type. To obtain models for all integer (half integer) weight representations we then impose harmonicity (monogenicity) conditions. This leads to the notion of simplicial harmonic (monogenic) system. The models for the irreducible representations of $Spin(m)$ arise from the construction of specific highest weight vectors. In the framework of Clifford algebra, weight vectors for the fundamental representations were first constructed in [DS]. Later on weight vectors for arbitrary (half)-integer weights were given in [So3]. Although these weight vectors satisfy the simplicial (monogenic) harmonic system, it took some extra ideas by the first author and basic facts from [FH] to prove that they generate the spaces of simplicial (spinor valued monogenic) scalar valued harmonic polynomials. As a result the simplicial harmonic and monogenic system are (up to isomorphism) the most refined $Spin(m)$-invariant systems of partial differential equations and thus provide the basic building blocks for any $Spin(m)$-invariant system.

2 Irreducible Representations of $GL(m)$ and Polynomials of Simplicial Variables

Polynomials of k vector variables $x_1, \ldots, x_k$ where $x_l = \sum_{j=1}^m x_{lj} e_j$ can be regarded as polynomials on $\mathbb{R}^{k \times m}$ or on $\mathbb{R}^{km}$ by the identification:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{km} \end{pmatrix} = (x_{lj})$$

The space of these polynomials will be denoted as $\mathcal{P}[x_1, \ldots, x_k]$. Its subspace of polynomials homogeneous of degree l_i in each vector variable x_i will be denoted by $\mathcal{P}_{l_1,\ldots,l_k}[x_1, \ldots, x_k]$. The Fischer inner product on these space is the usual Fischer inner product on polynomials of km scalar variables given by

$$\begin{aligned} \langle P(x_1, \ldots, x_k), Q(x_1, \ldots, x_k) \rangle &= \left[\bar{P}(\frac{\partial}{\partial_{x_1}}, \ldots, \frac{\partial}{\partial_{x_k}}) Q(x_1, \ldots, x_k) \right]_0 (0) \\ &= \left[\bar{P}(\frac{\partial}{\partial x_{lj}}) Q(x_{lj}) \right]_0 (0) \end{aligned}$$

and $\langle R((g^t)^{-1})P, R(g)Q \rangle = \langle P, Q \rangle$. Obviously polynomials of different degree of homogeneity are orthogonal with respect to this inner product. The right regular representation of $GL(m)$ on $\mathcal{P}[x_1, \ldots, x_m]$ or on a subspace of homogeneous polynomials of fixed degree of homogeneity is given by:

$$R(g)P(x_1, \ldots, x_m) = P(Xg) = P(x_1 g, \ldots, x_m g),\ g \in GL(m).$$

Up to equivalence all irreducible representations of $GL(m)$ can be labelled by m-tuples $l = (l_1, \ldots, l_m)$ of integers such that $l_1 \geq \cdots \geq l_m$. An explicit realization of these irreducible representations within the space $\mathcal{P}_{l_1,\ldots,l_m}[x_1, \ldots, x_m]$ can be obtained by imposing row homogeneity conditions on these polynomials. This can be achieved by specifying an extra *left* group action on $\mathcal{P}_{l_1,\ldots,l_m}[x_1, \ldots, x_m]$. This goes as follows (see e.g. [GR]). Let $N_m \subset GL(m)$ be the subgroup of $GL(m)$ consisting of lower triangular matrices such that all elements on the diagonal are one. The subspace of $\mathcal{P}_{l_1,\ldots,l_m}[x_1, \ldots, x_m]$ consisting of polynomials invariant under the left action of N_m, i.e. $P(N_m X) = P(X)$ is denoted by $\mathcal{P}^{N_m}_{l_1,\ldots,l_m}[x_1, \ldots, x_m]$. It can be proved that this space is irreducible for the right regular representation of $GL(m)$ and provides a model for the irreducible representation with weight $l = (l_1, \ldots, l_m)$. We write:

$$\mathcal{P}^{N_m}_{l_1,\ldots,l_m}[x_1, \ldots, x_m] \cong (l_1, \ldots, l_m).$$

This irreducible representation is completely determined by specifying its highest weight vector

$$\begin{aligned} & w_{l_1,\ldots,l_m}(x_1,\ldots,x_m;e_1,\ldots,e_m) \\ = \; & \langle x_1 e_1\rangle^{l_1-l_2}\langle x_1\wedge x_2 e_1\wedge e_2\rangle^{l_2-l_3}\cdots\langle x_1\wedge\cdots\wedge x_m e_1\wedge\cdots\wedge e_m\rangle^{l_m} \end{aligned}$$

where

$$\begin{aligned} \langle x_1\wedge\cdots\wedge x_k e_1\wedge\cdots\wedge e_k\rangle &= -[(x_1\wedge\cdots\wedge x_k)(e_1\wedge\cdots\wedge e_k)]_0 \\ &= \det\begin{pmatrix} \langle x_1e_1\rangle & \cdots & \langle x_1e_k\rangle \\ \vdots & & \vdots \\ \langle x_ke_1\rangle & \cdots & \langle x_ke_k\rangle \end{pmatrix}. \end{aligned}$$

Thus

$$\mathcal{P}^{N_m}_{l_1,\ldots,l_m}[x_1,\ldots,x_m]=\mathrm{span}_{\mathbb{R}}\{R(g)w_{l_1,\ldots,l_m}\}.$$

It follows from Schur's lemma that the weight vector $w_{l_1,\ldots,l_m}$ is the reproducing kernel of $\mathcal{P}^{N_m}_{l_1,\ldots,l_m}[x_1,\ldots,x_m]$, i.e.

$$P^{N_m}_{l_1,\ldots,l_m}(x_1,\ldots,x_m)=D_{l_1,\ldots,l_m}\langle\bar{w}_{l_1,\ldots,l_m}(x_1,\ldots,x_m;\cdot),P^{N_m}_{l_1,\ldots,l_m}(\cdot)\rangle$$

for some non zero constant $D_{l_1,\ldots,l_m}$. Consider now the representation of the upper triangular subgroup (all diagonal elements equal to one) $U_m=N_m^t$ on $\mathcal{P}_{l_1,\ldots,l_m}[x_1,\ldots,x_m]$ given by:

$$\rho(u)P(X)=P(u^tX), u\in U_m$$

and its derived representation given by:

$$(\tilde{\rho}(A)P)(X)=\frac{d}{dt}((\rho(\exp tA)P)(X))\mid_{t=0}$$

where A belongs to the Lie algebra of U_m. This algebra can be identified with the algebra generated by the vector fields $\langle x_i\partial_{x_j}\rangle; j>i$ or equivalently by the algebra generated by $\langle x_i\partial_{x_{i+1}}\rangle$; $i=1,\ldots,m-1$. Summarizing we thus get the following equivalent characterizations of the irreducible representation $l=(l_1,\ldots,l_m)$.

$$\begin{aligned} & \mathcal{P}^{N_m}_{l_1,\ldots,l_m}[x_1,\ldots,x_m] \\ = \; & \{P\in\mathcal{P}_{l_1,\ldots,l_m}[x_1,\ldots,x_m]:\langle x_i\partial_{x_{i+1}}\rangle P=0, i=1,\ldots,m-1\} \\ = \; & \{P_{l_1,\ldots,l_m}(x_1,\ldots,x_m)=P_{l_1,\ldots,l_m}(x_1,x_1\wedge x_2,\ldots,x_1\wedge\cdots\wedge x_m)\} \end{aligned}$$

A pure k-vector of the form $x_1\wedge\cdots\wedge x_k$ is called a simplicial variable and a variable of the form $x_1,x_1\wedge x_2,\ldots,x_1\wedge\cdots\wedge x_m$ is called a flag variable. A polynomial $P(x_1,x_1\wedge x_2,\ldots,x_1\wedge\cdots\wedge x_m)$ depending on a flag variable will be referred to as a simplicial polynomial.

3 Simplicial Harmonic and Monogenic Polynomials

In this section we define some important $SO(m)$ and $Spin(m)$ invariant systems of partial differential equations. Let $P \in \mathcal{P}[x_1, \ldots, x_k]$. Then we call P harmonic if P satisfies the *harmonic system* of equations:

$$\triangle_{x_i} P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k$$

$$\langle \partial_{x_i} \partial_{x_j} \rangle P(x_1, \ldots, x_k) = 0,\ i \neq j$$

where $\triangle_{x_i}$ denotes the Laplacian in the vector variable x_i. The space of these polynomials will be denoted by $\mathcal{H}[x_1, \ldots, x_k]$. This definition of harmonicity corresponds to the notion of harmonic polynomials of matrix variable described by Gilbert and Murray in [GM].

A polynomial P is called monogenic in several vector variables if it satisfies the *monogenic system* (see also [Co], [Pe]):

$$\partial_{x_i} P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k. \tag{1}$$

The space of monogenic polynomials is denoted by $\mathcal{M}[x_1, \ldots, x_k]$. Clearly the monogenic system refines the harmonic system:
$\mathcal{M}[x_1, \ldots, x_k] \subset \mathcal{H}[x_1, \ldots, x_k]$.

The corresponding Fischer decompositions are given by (see also [GM], [Co] and [So1]):

$$\begin{aligned}
&\mathcal{P}[x_1, \ldots, x_k] = \\
&\quad = \mathcal{H}[x_1, \ldots, x_k] \oplus_\perp \left(\sum_{i=1}^{k} |x_i|^2 \mathcal{P}[x_1, \ldots, x_k] + \sum_{i<j} \langle x_i x_j \rangle \mathcal{P}[x_1, \ldots, x_k] \right) \\
&\quad = \mathcal{M}[x_1, \ldots, x_k] \oplus_\perp \left(\sum_{i=1}^{k} x_i \mathcal{P}[x_1, \ldots, x_k] \right).
\end{aligned}$$

It is important to notice that the decompositions between brackets are *not* unique. Only the harmonic or monogenic part of a polynomial are uniquely determined. These systems can be further refined by considering them on homogeneous polynomials of simplicial type, leading to the following definitions.

A polynomial $P \in \mathcal{P}_{l_1, \ldots, l_k}[x_1, \ldots, x_k]$ satisfies the *simplicial harmonic system* if P is harmonic and of simplicial type, i.e:

$$\triangle_{x_i} P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k$$

$$\langle \partial_{x_i} \partial_{x_j} \rangle P(x_1, \ldots, x_k) = 0,\ i \neq j$$

$$\langle x_i \partial_{x_{i+1}} \rangle P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k-1.$$

The Lie algebra generated by the operators $\triangle_1, \langle x_i, \partial_{x_{i+1}} \rangle$, $i = 1, \ldots, k-1$ is the algebra consisting of the operators determining the harmonic system, together with the operators coming from the action of the upper triangular group U_k. Hence the simplicial harmonic system is equivalent to

$$x_1 \mapsto P_{l_1,\ldots,l_k}(x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \cdots \wedge x_k) \tag{2}$$

is harmonic in x_1. This space will be denoted by $\mathcal{H}^{N_k}_{l_1,\ldots,l_k}[x_1, \ldots, x_k]$. Simplicial harmonic polynomials of the form $P(x_1 \wedge \cdots \wedge x_k)$ are exactly the harmonics studied by Gilbert and Murray in connection with equal weight representations of $SO(m)$.

A polynomial $P \in \mathcal{P}_{l_1,\ldots,l_k}[x_1, \ldots, x_k]$ satisfies the *simplicial monogenic system* if P is monogenic and of simplicial type, i.e:

$$\begin{array}{l} \partial_{x_i} P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k \\ \langle x_i \partial_{x_{i+1}} \rangle P(x_1, \ldots, x_k) = 0,\ i = 1, \ldots, k-1, \end{array}$$

or equivalently, by taking commutators of $\partial_{x_1}, \langle x_i, \partial_{x_{i+1}} \rangle$

$$x_1 \mapsto P_{l_1,\ldots,l_k}(x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \cdots \wedge x_k) \tag{3}$$

is monogenic x_1. This space will be denoted by $\mathcal{M}^{N_k}_{l_1,\ldots,l_k}[x_1, \ldots, x_k]$.

4 Irreducible Representations of $Spin(m)$

Let us recall some facts related to the algebraic construction of irreducible representations of $Spin(m)$ (see also [GM] and [FH]). Up to equivalence the unitary irreducible $Spin(m)$-modules can be labelled by considering the action of the maximal torus of $Spin(m)$:

$$T = \{s = \exp(\frac{1}{2} e_{12} t_1) \cdots \exp(\frac{1}{2} e_{2M-1,2M} t_M), t_j \in \mathbb{R}, M = [\frac{m}{2}]\}.$$

Let $R(s) : Spin(m) \to V$ be an irreducible representation of $Spin(m)$. If we restrict this representation to the maximal abelian subgroup T of $Spin(m)$, the space V splits into weight subspaces generated by eigenvectors v satisfying

$$R(\exp(\frac{1}{2} e_{12} t_1) \cdots \exp(\frac{1}{2} e_{2M-1,2M} t_M))v = \exp i(l_1 t_1 + \cdots + l_M t_M)v.$$

The eigenvalues are determined by M-tuples $l = (l_1, \ldots, l_M)$ consisting entirely of either integer or half integer numbers. They are the so called weights of the representation. These weights can be ordered lexicographically: $l = (l_1, \ldots, l_M) > l' = (l'_1, \ldots, l'_M)$ if the first non zero difference $l_i - l'_i$ is positive. In this way V may be identified with an ordered set of M-tuples which is the same for equivalent representations. In this set of M-tuples a unique weight can be singled out by considering the action of the

Weyl group on the ordered weights. The Weyl group acts as a permutation group on the numbers determining the weights together with an arbitrary or even number of changes of signs when m is odd or even. Factoring out this action one can see that V contains a unique *highest* weight with respect to the ordering defined above. These are called highest weights and are of the form:

$$\begin{array}{lcl} l = (l_1, \ldots, l_M) & : & l_1 \geq l_2 \geq \ldots \geq l_M \quad \text{if} \quad m = 2M+1 \\ l = (l_1, \ldots, l_M) & : & l_1 \geq l_2 \geq \ldots \geq |l_M| \quad \text{if} \quad m = 2M \end{array}$$

where all $l_i \in \mathbb{Z}$ or all $l_i \in \frac{1}{2}\mathbb{Z}$. By a theorem of Cartan the weight subspace corresponding to the highest weight is one dimensional; it is generated by the highest weight vector (defined up to a multiple). Moreover each M-tuple of the form above is actually the highest weight of exactly one irreducible representation of $Spin(m)$. This gives the correspondence between highest weights or highest weight vectors and unitary irreducible $Spin(m)$-modules. Of particular importance are the so called fundamental (the notion of fundamental we use is not the standard one) weights. These are the highest weights of the form:

$$(1, 0, \ldots, 0), \ldots, (1, \ldots, 1), (\frac{1}{2}, \ldots, \frac{1}{2})$$

if $m = 2M+1$, and

$$(1, 0, \ldots, 0), \ldots, (1, \ldots, 1), (\frac{1}{2}, \ldots, \frac{1}{2}), (1, \ldots, 1, -1), (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$$

if $m = 2M$.

Remark that we also consider $(1, \ldots, 1)$, (m odd) and $(1, \ldots, 1, 0)$, $(1, \ldots, \pm 1)$, (m even) to be fundamental. Strictly speaking they are not fundamental in the standard sense because they can be realized inside tensor products of the other (standard) fundamental weights. All other irreducible representations of $Spin(m)$ can be built from these fundamental representations by a procedure called Cartan product. Let (V, R) and (V', R') be irreducible representations with weights $l = (l_1, \ldots, l_M)$ and $l' = (l'_1, \ldots, l'_M)$ and corresponding weight vectors w_l and $w_{l'}$. Then the tensor product $(V \otimes V', R \otimes R')$ is also a representation of $Spin(m)$ and usually splits in a lot of irreducible subpieces. However one piece is canonically defined. By considering the action of the Weyl group on the weight decomposition of this tensor product one arrives at the highest weight occurring in $V \otimes V'$. This weight is given by $(l_1 + l'_1, \ldots, l_M + l'_M)$ and has weight vector $w_l \otimes w_{l'}$. By a theorem of Cartan it occurs exactly *once* in the decomposition of $V \otimes V'$. The projection of $V \otimes V'$ on the highest weight subspace is the Cartan product $V[\times]V$ of two irreducible representations. Now any highest weight can be written uniquely as a linear combination of our fundamental highest weights where the coefficient of the fundamental half integer weight representation is either zero or one (in case m is even we take the convention that the weights $(1,\dots,1,-1)$, $(\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2})$ only occur if the last number in a general highest weight is negative). Therefore an arbitrary highest weight can be obtained in a canonical way inside a tensor product of (symmetric) tensor powers of the fundamental representations by means of the Cartan projection. If for example $m = 2M+1$, the irreducible representation $s_1(1,0,\dots,0)+s_2(1,1,0,\dots,o)+\dots+s_M(1,\dots,1)$ can be realized inside

$$E_{s_1,\dots,s_M} = \ \mathrm{Sym}^{s_1}(1,0,\dots,0)\otimes\ \mathrm{Sym}^{s_2}(1,1,0,\dots,0)\otimes\cdots\otimes \\ \mathrm{Sym}^{s_M}(1,\dots,1)$$

while the representation $s_1(1,0,\dots,0)+s_2(1,1,0,\dots,0)+\dots+s_M(1,\dots,1)$ $+(\frac{1}{2},\dots,\frac{1}{2})$ can be realized inside

$$E'_{s_1,\dots,s_M} = E_{s_1,\dots,s_M}\otimes(\frac{1}{2},\dots,\frac{1}{2}),$$

or in the submodule

$$(l_1,\dots,l_M)\otimes(\frac{1}{2},\dots,\frac{1}{2}).$$

Let E be a representation space of $Spin(m)$ corresponding to a representation R. The Lie algebra of $Spin(m)$ can be identified with the space $\mathbb{R}_{m,2}$ of bivectors in $\mathbb{R}_m$. Its infinitesimal representation is given by

$$dR(w)f = \lim_{\epsilon\to 0}\frac{1}{\epsilon}(R(\exp(\epsilon w)-1)f.$$

The Casimir operator of the representation R is then defined by

$$C(R) = \frac{1}{4}\sum_{i<j} dR(e_{ij})^2.$$

The Casimir operator $C(R)$ acts by scalar multiplication on the R-irreducible pieces occurring in E. Its spectrum depends only on the highest weights characterizing the irreducible pieces and not on the specific way how the irreducible pieces are realized inside E. But there is no $1-1$ correspondence between eigenspaces of the Casimir operator and highest weights because different highest weights can produce the same eigenvalue for the action of the Casimir operator. This has to do with the fact that also higher order operators (which commute with all $Spin(m)$-invariant operators) are needed to determine the highest weights in a unique way. However, on the canonical representation space $E = E_{s_1,\dots,s_M}$ the Casimir operator $C(R)$ behaves much better. We know that

$$E_{s_1,\dots,s_M} = (s_1+\dots+s_M, s_2+\dots+s_M,\dots,s_M)\oplus \text{ lower highest weights }.$$

Now $C(R)$ acts by scalar multiplication on each irreducible submodule occurring in $E_{s_1,\ldots,s_M}$. In particular $C(R)$ acts by multiplication with some constant $C_{s_1,\ldots,s_M}$ on the leading weight space. It can be proved (see [FH]) that the action of $C(R)$ on the remaining highest weights is scalar multiplication with constants which are *all different* from $C_{s_1,\ldots,s_M}$. This means that inside $E_{s_1,\ldots,s_M}$ the irreducible representation $(s_1+\cdots+s_M,\ldots,s_M)$ is completely determined by the action of the Casimir operator. The same is also true for the realization of $(s_1+\cdots+s_M+\frac{1}{2},\ldots,s_M+\frac{1}{2})$ inside $E'_{s_1,\ldots,s_M}$. This result will prove to be very helpfull in the sequel.

We will now show how this abstract considerations can be made concrete in the language of Clifford algebra (see also [DS] and [So3]). Let $s \in Spin(m)$; consider the following two unitary representations of $Spin(m)$:

$$\begin{array}{rcl} H(s)P(x_1,\ldots,x_k) & = & sP(\bar{s}x_1s,\ldots,\bar{s}x_ks)\bar{s} \\ L(s)P(x_1,\ldots,x_k) & = & sP(\bar{s}x_1s,\ldots,\bar{s}x_ks). \end{array}$$

To define representations of $Pin(m)$ one just replaces $\bar{s}$ by $\tilde{s}$. The H-representation may act on the space of harmonic polynomials and actually defines a representation of $SO(m)$ while the L-representation will act on monogenic polynomials. Both representations can be restricted to the corresponding subspaces of homogeneous simplicial harmonic or monogenic polynomials:

$$H(s)P_{l_1,\ldots,l_k}(x_1,\ldots,x_1\wedge\ldots\wedge x_k) = sP_{l_1,\ldots,l_k}(\bar{s}x_1s,\ldots,\bar{s}x_1\wedge\ldots\wedge x_ks)\bar{s}$$

and

$$L(s)P_{l_1,\ldots,l_k}(x_1,\ldots,x_1\wedge\ldots\wedge x_k) = sP_{l_1,\ldots,l_k}(\bar{s}x_1s,\ldots,\bar{s}x_1\wedge\ldots\wedge x_ks).$$

In case of *scalar* valued simplicial harmonic polynomials the H-representation is the usual representation of $SO(m)$. The Casimir operators corresponding to this representations were already considered in [So2]. Let $L_{x_l,ij} = x_{li}\partial_{x_{lj}} - x_{lj}\partial_{x_{li}}$ be the ij-momentum operator in the variable x_l. Let $\triangle_{S,x_l} = \sum_{ij} L^2_{x_l,ij}$ be the Laplace-Beltrami operator in the variable x_l and $\triangle_{S,x_rx_s} = \sum_{i<j} L_{x_r,ij}L_{x_s,ij}$ be the "mixed" Laplace-Beltrami operator. The Casimir operators of both representations are then given by

$$\begin{array}{rcl} \frac{1}{4}C(H) & = & \sum_{i<j}(L_{x_1,ij}+\cdots+L_{x_k,ij})^2 \\ & = & \sum_{l=1}^{k}\triangle_{S,x_l} + 2\sum_{1\le i<j\le k}\triangle_{S,x_ix_j} \\ & = & \sum_{l=1}^{k}\triangle_{S,x_l} + 2\sum_{1\le i<j\le k}\langle x_i,x_j\rangle\langle\partial_{x_i},\partial_{x_j}\rangle - \\ & & -\langle x_j,\partial_{x_i}\rangle\langle x_i,\partial_{x_j}\rangle + \langle x_j,\partial_{x_j}\rangle \end{array}$$

while
$$\begin{aligned}\frac{1}{4}C(L) &= \sum_{i<j}(L_{x_1,ij}+\cdots+L_{x_k,ij}+\frac{1}{2}e_{ij})^2\\ &= \frac{1}{4}C(H)+\sum_{i=1}^{k}\Gamma_{x_i}-\frac{m(m-1)}{8}.\end{aligned}$$

From this it easily follows that the spaces of harmonic and monogenic simplicial polynomials are eigenspaces of the $C(H)$ and $C(L)$ Casimir operators respectively. The action of $\frac{1}{4}C(H)$ on $\mathcal{H}^{N_k}_{l_1,\ldots,l_k}[x_1,\ldots,x_k]$ produces the eigenvalue
$$-\sum_{j=1}^{k} l_j(l_j+m-2j)\,,$$
while the action of $\frac{1}{4}C(L)$ on $\mathcal{M}^{N_k}_{l_1,\ldots,l_k}[x_1,\ldots,x_k]$ gives the eigenvalue
$$-\sum_{j=1}^{k} l_j(l_j+m-2j+1)-\frac{m(m-1)}{8}\,.$$

Models for fundamental representations can be realized inside the complex Clifford algebra $\mathbb{C}_m$. To see this, consider the actions of $Spin(m)$ on $\mathbb{C}_m$ given by
$$l(s)a = sa \quad \text{or} \quad h(s)a = sa\bar{s}.$$
This leads to the fundamental representations l of $Spin(m)$ on spinor spaces S and h of $Spin(m)$ on k-vector spaces $\mathbb{C}_{m,k}, k \leq M$. In the odd dimensional case we consider the basic isotropic vectors
$$T_j = \frac{1}{2}(e_{2j-1}-ie_{2j}), \bar{T}_j = -\frac{1}{2}(e_{2j-1}+ie_{2j})$$
and the idempotents $I_j = T_j\bar{T}_j$; then the product $I = I_1\ldots I_M$ is primitive idempotent; the ideal $\mathbb{C}^+_m I$ is minimal and gives a model for the spinor space (see e.g. [DSS]). The action of the maximal torus gives
$$\begin{aligned} l(s)I &= \exp\frac{1}{2}(t_1e_{12}+\cdots+t_Me_{2M-1,2M})I\\ &= \exp\frac{i}{2}(t_1+\cdots+t_M)I;\end{aligned}$$
and the weight is given by $(\frac{1}{2},\ldots,\frac{1}{2})$. Next for the representation h one uses the null-k-vectors
$$T_1\wedge\cdots\wedge T_k,\ k=1,\ldots,M$$
and the representation is given by
$$h(s)T_1\wedge\cdots\wedge T_k = sT_1\wedge\cdots\wedge T_k\bar{s} = \exp i(t_1+\cdots+t_k)T_1\wedge\cdots\wedge T_k$$

so that the weights are given by $(1,0,\ldots,0)$, $(1,1,0,\ldots,0),\ldots,(1,\ldots,1)$. Alternatively, these fundamental representations can also be realized by the highest weight vectors $\langle x_1,T_1\rangle, \langle x_1\wedge x_2, T_1\wedge T_2\rangle,\ldots,\langle x_1\wedge\cdots\wedge x_m, T_1\wedge\cdots\wedge T_M\rangle$. These weight vectors then simply generate the spaces of 1- up to M-linear alternating forms. To produce highest weight vectors for irreducible representations where the numbers determining the weight are all equal we now take symmetric tensor powers of these fundamental highest weight vectors $T_1,\ldots,T_1\wedge\cdots\wedge T_m$ or equivalently of $\langle x_1,T_1\rangle,\ldots,\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle$. This may be done in a concrete way using polynomial functions of simplicial variables

$$\langle x_1\wedge\cdots\wedge x_k, T_1\wedge\cdots\wedge T_k\rangle^{s_k}$$

on which the spin group acts like

$$H(s)F(x_1\wedge\cdots\wedge x_k)=F(\bar{s}x_1\wedge\cdots\wedge x_k s)$$

and the weight is found from the action of the maximal torus on this highest weight vector

$$H(s)\langle x_1\wedge\cdots\wedge x_k, T_1\wedge\cdots\wedge T_k\rangle^{s_k}=\exp(si(t_1+\cdots+t_k))$$

$$\langle x_1\wedge\cdots\wedge x_k, T_1\wedge\cdots\wedge T_k\rangle^{s},$$

i.e. the weight is given by $(s,s,\ldots,s,0,\ldots,0)$ (where s appears k times). It is not hard to see that this highest weight vector is simplicially harmonic or equivalently, harmonic of a matrix variable. The space of simplicially harmonic functions $\mathcal{H}^{N_k}_{s,\ldots,s,0,\ldots,0}[x_1,\ldots,x_k]$ is then the irreducible space to which this highest weight vector belongs. Models for all irreducible representations with integer weight are obtained by taking further tensor products of these highest weight vectors, i.e. by considering the simplicial functions

$$F(x_1,\cdots,x_1\wedge\cdots\wedge x_M)=\langle x_1,T_1\rangle^{s_1}\cdots\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M},$$

whereby the representation of $Spin(m)$ on simplicial scalar functions is given by

$$H(s)F(x_1,\cdots,x_1\wedge\cdots\wedge x_M)=F(\bar{s}x_1 s,\ldots,\bar{s}x_1\wedge\cdots\wedge x_M s).$$

For the action of the maximal torus on the highest weight vectors we find

$$\begin{aligned}&H(s)\langle x_1,T_1\rangle^{s_1}\cdots\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M}\\ =\;&\exp i((s_1+\cdots+s_M)t_1+\cdots+s_M t_M)\langle x_1,T_1\rangle^{s_1}\cdots\\ &\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M}\end{aligned}$$

i.e. the weight is given by $(s_1+\cdots+s_M, s_2+\cdots+s_M, \ldots, s_M)$. Models for all irreducible representations with half integer weight are now easily obtained by multiplying this highest weight vector further with the primitive idempotent I , i.e. to consider the spinor valued function

$$F(x_1, \cdots, x_1\wedge\cdots\wedge x_M) = \langle x_1, T_1\rangle^{s_1}\cdots\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M} I$$

whereby the representation of $Spin(m)$ on spinor valued simplicial functions is given by

$$L(s)F(x_1, \cdots, x_1\wedge\cdots\wedge x_M) = sF(\bar{s}x_1 s, \ldots, \bar{s}x_1\wedge\cdots\wedge x_M s).$$

For the action of the maximal torus on the highest weight vector we obtain

$$\begin{aligned} & L(s)\langle x_1, T_1\rangle^{s_1}\cdots\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M} I \\ = \quad & \exp i((s_1+\cdots+s_M+\frac{1}{2})t_1+\cdots+(s_M+\frac{1}{2})t_M) \\ & \langle x_1, T_1\rangle^{s_1}\cdots\langle x_1\wedge\cdots\wedge x_M, T_1\wedge\cdots\wedge T_M\rangle^{s_M} I \end{aligned}$$

i.e. the weight is $(s_1+\cdots+s_M+\frac{1}{2}, \ldots, s_M+\frac{1}{2})$. If we make *this* choice for the highest weight vectors, there are the following observations that can be made in the odd dimensional case. Using the H-representation each weight vector

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1,\ldots,x_m;T_1,\ldots,T_M) \\ = \quad & \langle x_1T_1\rangle^{l_1-l_2}\langle x_1\wedge x_2T_1\wedge T_2\rangle^{l_2-l_3}\ldots\langle x_1\wedge\ldots\wedge x_MT_1\wedge\ldots\wedge T_M\rangle^{l_M} \end{aligned}$$

and the corresponding irreducible representation belong to one space $\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ of simplicial harmonic polynomials. This however is not enough to conclude that this space of simplicial harmonic polynomials itself is irreducible. To establish this we need to go back to our construction of the irreducible representation $(l_1,\ldots,l_M)$. By Cartan projection this irreducible representation is canonically realized inside a tensor product of symmetric powers of the fundamental representations. This tensor product contains in particular the space of simplicial harmonic polynomials $\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$. Let $C_{l_1,\ldots,l_M}$ be the eigenvalue of the Casimir operator $C(H)$ acting on the irreducible representation $(l_1,\ldots,l_M)$. As pointed out before, *inside* this tensor product $\mathrm{Ker}(C(H)-C_{l_1,\ldots,l_M}) \cong (l_1,\ldots,l_M)$. Because the highest weight vectors belong to exactly one space of harmonic polynomials of simplicial type and these polynomials are already eigenspaces of $C(H)$ we thus obtain

$$\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cong (l_1,\ldots,l_M)$$

where we consider *scalar* valued polynomials. If we now consider the L-representation, then

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1,\ldots,x_m;T_1,\ldots,T_M)I_1\ldots I_M \\ = \quad & \langle x_1T_1\rangle^{l_1-l_2}\langle x_1\wedge x_2T_1\wedge T_2\rangle^{l_2-l_3}\ldots \\ & \langle x_1\wedge\ldots\wedge x_MT_1\wedge\ldots\wedge T_M\rangle^{l_M} I_1\ldots I_M \,, \end{aligned}$$

together with the irreducible representation it generates under the action of L, belongs to exactly one space $\mathcal{M}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ of simplicial monogenic polynomials. Because this space is an eigenspace of the Casimir operator $C(L)$, it is now sufficient to consider spinor valued simplicial monogenic polynomials and to apply the same line of thinking as for the H-representation. We thus obtain

$$\mathcal{M}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cong (l_1+\frac{1}{2},\ldots,l_M+\frac{1}{2})$$

where the polynomials under consideration take values in the *spinors*. In the even dimensional case ($m=2M$) the construction of highest weight vectors is similar except for the fact that there are now two inequivalent spinor spaces which lead to inequivalent basic representations of $Spin(m)$ namely the spinor spaces $\mathbb{C}^+_m I_+$ and $\mathbb{C}^+_m I_-$ whereby the primitive idempotents I_+ and I_- are given by

$$I_+ = I_1\ldots I_{M-1}I_M,\; I_- = I_1\ldots I_{M-1}I'_M,$$

and

$$I'_M = \bar{T}_M T_M = \frac{1}{2}(1+ie_{m-1}e_m).$$

This has to do with the fact that the pseudoscalar $E = e_1\ldots e_m$ is actually $Spin(m)$-invariant and has square $(-1)^M$. Hence there are two invariant projectors

$$P_+ = \frac{1}{2}(1+(-i)^M E) \text{ and } P_- = \frac{1}{2}(1-(-i)^M E)$$

onto the eigenspaces of E and, as we also have that $I_+ = P_+ I_+$ and $I_- = P_- I_-$, the spinor spaces $\mathbb{C}^+_m I_+$ and $\mathbb{C}^+_m I_-$ are inequivalent under the action of the representation l of $Spin(m)$. The weights are obtained from the action of the maximal torus and given by $(\frac{1}{2},\cdots,\frac{1}{2})$ resp. $(\frac{1}{2},\cdots,-\frac{1}{2})$. Remark that we also have that $\mathbb{C}^-_m I_+ \cong \mathbb{C}^+_m e_m I_+ \cong \mathbb{C}^+_m I_- e_m$ as equivalent spin representations. In the same way, the space of M-vectors in $\mathbb{C}_m$ splits into two inequivalent representations. The M-null frame $T_1\wedge\cdots\wedge T_M$ satisfies

$$P_+(T_1\wedge\cdots\wedge T_M) = T_1\wedge\cdots\wedge T_M \text{ and } P_-(T_1\wedge\cdots\wedge T_M) = 0$$

while the M-null frame $T_1\wedge\cdots\wedge\bar{T}_M$ satisfies

$$P_-(T_1\wedge\cdots\wedge\bar{T}_M) = T_1\wedge\cdots\wedge\bar{T}_M \text{ and } P_+(T_1\wedge\cdots\wedge\bar{T}_M) = 0.$$

These null frames provide weight vectors for representations of weight $(1,\ldots,1,1)$ and $(1,\ldots,1,-1)$ respectively. In terms of M-linear alternating forms $F(x_1\wedge\cdots\wedge x_M)$, these representations are given by forms satisfying respectively the scalar system of equations

$$P_-(\partial_{x_1}\wedge\cdots\wedge\partial_{x_M})F(x_1\wedge\cdots\wedge x_M) = 0$$

and

$$P_+(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}) F(x_1 \wedge \cdots \wedge x_M) = 0,$$

generating together the space of M-linear alternating forms. For the construction of models for irreducible representations of $Spin(m)$ with half integer weight we now use two types of highest weight vectors

$$\begin{aligned} F_+ &= \langle x_1, T_1\rangle^{s_1} \langle x_1 \wedge x_2, T_1 \wedge T_2\rangle^{s_2} \cdots \\ &\quad \langle x_1 \wedge \cdots \wedge x_M, T_1 \wedge \cdots \wedge T_M\rangle^{s_M} I_+, \\ F_- &= \langle x_1, T_1\rangle^{s_1} \langle x_1 \wedge x_2, T_1 \wedge T_2\rangle^{s_2} \cdots \\ &\quad \langle x_1 \wedge \cdots \wedge x_M, T_1 \wedge \cdots \wedge \bar{T}_M\rangle^{s_M} I_-, \end{aligned}$$

and the corresponding weights are $(s_1 + \cdots + s_M + \frac{1}{2}, \cdots, \pm(s_M + \frac{1}{2}))$. To obtain models for irreducible representations with integer weights one just leaves away the factors I_+, I_- in the above definition of F_+, F_-. For this choice of the highest weight vectors we now have that in case of the H-representation *both* the weight vectors

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1, \ldots, x_m; T_1, \ldots, T_M) \\ &= \langle x_1 T_1\rangle^{l_1 - l_2} \langle x_1 \wedge x_2 T_1 \wedge T_2\rangle^{l_2 - l_3} \ldots \langle x_1 \wedge \cdots \wedge x_M T_1 \wedge \cdots \wedge T_M\rangle^{l_M} \end{aligned}$$

and

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1, \ldots, x_m; T_1, \ldots, \bar{T}_M) \\ &= \langle x_1 T_1\rangle^{l_1 - l_2} \langle x_1 \wedge x_2 T_1 \wedge T_2\rangle^{l_2 - l_3} \ldots \langle x_1 \wedge \cdots \wedge x_M T_1 \wedge \cdots \wedge \bar{T}_M\rangle^{l_M} \end{aligned}$$

belong to one space $\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1, \ldots, x_M]$ of simplicial harmonic polynomials. Now the representation $s_1(1, 0, \ldots, 0) + s_2(1, 1, 0, \ldots, 0) + \cdots + s_M(1, \ldots, 1, \pm 1)$ is realized inside

$$E_{s_1,\ldots,s_{M-1},\pm s_M} = \mathrm{Sym}^{s_1}(1, 0, \ldots, 0) \otimes \mathrm{Sym}^{s_2}(1, 1, 0, \ldots, 0) \otimes \cdots \otimes$$

$$\mathrm{Sym}^{s_M}(1, \ldots, 1, \pm 1).$$

In case $s_M = 0$, the last symmetric tensor power in the above tensor product does not occur and we can immediately apply the argument with the Casimir operator as in the odd dimensional case, i.e.

$$\mathcal{H}^{N_M}_{l_1,\ldots,l_{M-1},0}[x_1, \ldots, x_M] \cong (l_1, \ldots, l_{M-1}, 0).$$

Consider now the case where the last number in the weight is positive. Because $(1, \ldots, 1, +1)$ generates only *half* of the M-linear alternating forms, the space $\mathrm{Sym}^{s_M}(1, \ldots, 1, +1)$ can *not* be identified with the space of simplicial polynomials $\mathcal{P}^{N_M}_{s_M,\ldots,s_M}[x_1, \ldots, x_M]$. This means that we cannot embed simplicial harmonics directly into $E_{s_1,\ldots,s_{M-1},s_M}$. The extra conditions are found as follows. By the Capelli identity the generalized Euler operator $\langle x_1 \wedge \cdots \wedge x_M, \partial_{x_1} \wedge \cdots \wedge \partial_{x_M} \rangle$ can be expressed as

$$\det \begin{pmatrix} \langle x_1, \partial_{x_1} \rangle + M - 1 & \langle x_1, \partial_{x_2} \rangle & \cdots & \langle x_1, \partial_{x_M} \rangle \\ \langle x_2, \partial_{x_1} \rangle & \langle x_2, \partial_{x_2} \rangle + M - 2 & \cdots & \langle x_2, \partial_{x_M} \rangle \\ \vdots & \vdots & & \vdots \\ \langle x_M, \partial_{x_1} \rangle & \langle x_M, \partial_{x_2} \rangle & \cdots & \langle x_M, \partial_{x_M} \rangle \end{pmatrix}$$

where the determinant of the $M \times M$-matrix of non commuting variables X_{ij} is given by

$$\det\,(X_{ij}) = \sum_{\sigma \in S_M} \mathrm{sign}\sigma X_{\sigma(1)1} \cdots X_{\sigma(M)M}.$$

Because simplicial polynomials $P(x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \cdots \wedge x_M)$ are annihilated by the vector fields $\langle x_i, \partial_{x_j} \rangle$, $j > i$, it follows that only the diagonal elements of this matrix contribute:

$$\begin{aligned} &\langle x_1 \wedge \cdots \wedge x_M, \partial_{x_1} \wedge \cdots \wedge \partial_{x_M} \rangle P_{l_1,\ldots,l_M}(x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \cdots \wedge x_M) \\ &\quad = \prod_{j=1}^{M} (l_j + M - j) P_{l_1,\ldots,l_M}(x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \cdots \wedge x_M), \end{aligned}$$

l_j being the degree of homogeneity in x_j. For $\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}$ acting on simplicial polynomials we have the Fischer decomposition

$$\begin{aligned} \mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1, \ldots, x_M] &= (\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1, \ldots, x_M] \cap \text{ Ker } (\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})) \\ &\quad \oplus_\perp x_1 \wedge \cdots \wedge x_M \mathcal{P}^{N_M}_{l_1-1,\ldots,l_M-1}[x_1, \ldots, x_M]. \end{aligned}$$

Let now P be a polynomial which belongs to the first space in the above decomposition. As the weight vector $w_{l_1,\ldots,l_M}$ reproduces the space of simplicial polynomials $\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1, \ldots, x_M]$, it follows that the Fischer inner product

$$\langle \partial_{x_1} \wedge \cdots \wedge \partial_{x_M} w_{l_1,\ldots,l_M}(x_1, \ldots, x_M; \cdot), P(\cdot) \rangle = 0.$$

Thus by the above Fischer decomposition

$$\partial_{x_1} \wedge \cdots \wedge \partial_{x_M} w_{l_1,\ldots,l_M}(x_1, \ldots, x_M; u_1, \ldots, u_M) = Q u_1 \wedge \cdots \wedge u_M$$

for some simplicial polynomial Q. This polynomial can be identified using the identity for the action of the generalized Euler operator on simplicial polynomials, i.e.

$$\begin{aligned} &\partial_{x_1} \wedge \cdots \wedge \partial_{x_M} w_{l_1,\ldots,l_M}(x_1, \ldots, x_M; u_1, \ldots, u_M) \\ &\quad = \prod_{j=1}^{M} (l_j + M - j) w_{l_1-1,\ldots,l_M-1}(x_1, \ldots, x_M; u_1, \ldots, u_M) \\ &\qquad u_1 \wedge \cdots \wedge u_M. \end{aligned}$$

Remark that in this way we can also inbed the integer weight representations of $Spin(m)$ in spaces of $\mathbb{C}_{m,M}$-valued simplicial monogenic polynomials. The identity above now clearly shows which conditions must be imposed on the simplicial harmonic polynomials to embed them in the tensor product $E_{s_1,\ldots,s_{M-1},+s_M}$. As a matter of fact, these polynomials must be null solutions of the scalar system of equations determined by the components of $P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$. Now we are in a situation to follow the same line of thinking as in the odd dimensional case and we have

$$\begin{aligned} \mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cap \text{Ker } P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}) &\cong (l_1,\ldots,l_{M-1},+l_M) \\ \mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cap \text{Ker } P_+(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}) &\cong (l_1,\ldots,l_{M-1},-l_M), \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cong (l_1,\ldots,l_{M-1},+l_M)\oplus \\ (l_1,\ldots,l_{M-1},-l_M), \quad (l_M > 0). \end{aligned}$$

Of course this characterization remains true if $l_M = 0$, but now the extra systems of scalar equations are satisfied in a trivial way and are actually redundant. In case $l_M > 0$, this splitting is very natural because $\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ is actually an irreducible $Pin(m)$-module corresponding to the weight $(l_1,\ldots,l_M)$. The important fact here is that in case of the Pin-representation either the M-frame $T_1 \wedge \cdots \wedge T_M$ or $T_1 \wedge \cdots \wedge \bar{T}_M$ generate the *whole* space of M-linear alternating forms, so the conditions arising from the components of $P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$ or $P_+(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$ do not occur. By regarding the irreducible $Pin(m)$-module $\mathcal{H}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ $(l_M > 0)$ as a $Spin(m)$-module, it splits as a sum of two irreducible $Spin$-modules: $(l_1,\ldots,l_{M-1},+l_M) \oplus (l_1,\ldots,l_{M-1},-l_M)$. The two extra $Spin(2M)$-invariant systems of scalar equations then simply identify the *sign* of the last number in the weight. Let us consider as an example the one variable case: $m = 2, M = 1$. Vectors $x \in \mathbb{R}^2$ are written as $x = e_1x_1 + e_2x_2$ while the Dirac operator is given by $\partial_x = e_1\partial_{x_1} + e_2\partial_{x_2}$. Irreducible representations of $Pin(2)$ are labelled by numbers $k \in \mathbb{N}$ and the corresponding models are given by harmonic polynomials (in $\mathbb{R}^2$) which are homogeneous of degree k. As $Spin(2)$-representations, the spaces of homogenous harmonic polynomials split into two pieces given by the kernel of the components of $P_-\partial_x$ or $P_+\partial_x$. Now

$$\begin{aligned} P_+\partial_x = (\frac{1-ie_{12}}{2})\partial_x &= \frac{e_1 - ie_2}{2}(\partial_{x_1} + i\partial_{x_2}) \\ P_-\partial_x = (\frac{1+ie_{12}}{2})\partial_x &= \frac{e_1 - ie_2}{2}(\partial_{x_1} - i\partial_{x_2}) \end{aligned}$$

and $P_+\partial_x P_-\partial_x = \frac{-1+ie_{12}}{2}(\partial^2_{x_1} + \partial^2_{x_2})$. Hence, as irreducible $Spin(2)$-modules, the harmonic polynomials should be annihilated by the appropiate Cauchy-

Riemann operators:

$$\begin{array}{lcl} (+k) & \cong & \text{anti holomorphic polynomials homogeneous of degree } k \\ (-k) & \cong & \text{holomorphic polynomials homogeneous of degree } k. \end{array}$$

In case of the L-representation we can follow the same procedure. Both the weight vectors

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1,\ldots,x_m;T_1,\ldots,T_M)I_1\ldots I_M \\ = \quad & \langle x_1T_1\rangle^{l_1-l_2}\langle x_1\wedge x_2T_1\wedge {T_2}^{l_2-l_3}\rangle\ldots \\ & \langle x_1\wedge\cdots\wedge x_MT_1\wedge\cdots\wedge T_M\rangle^{l_M}I_1\ldots I_M \end{aligned}$$

and

$$\begin{aligned} & w_{l_1,\ldots,l_M}(x_1,\ldots,x_m;T_1,\ldots,\bar{T}_M)I_1\ldots I'_M \\ = \quad & \langle x_1T_1\rangle^{l_1-l_2}\langle x_1\wedge x_2T_1\wedge T_2\rangle^{l_2-l_3}\ldots \\ & \langle x_1\wedge\cdots\wedge x_MT_1\wedge\cdots\wedge \bar{T}_M\rangle^{l_M}I_1\ldots I'_M \end{aligned}$$

belong to the same space $\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ of simplicial monogenic polynomials. Now the representation $s_1(1,0,\ldots,0)+s_2(1,1,0,\ldots,0)+\cdots+s_M(1,\ldots,1,\pm1)$ $+(\frac{1}{2},\ldots,\frac{1}{2},\pm\frac{1}{2}) = (l_1+\frac{1}{2},\ldots,l_{M-1}+\frac{1}{2},\pm(l_M+\frac{1}{2}))$ can be realized inside

$$E'_{s_1,\ldots,s_{M-1},\pm s_M} = E_{s_1,\ldots,s_{M-1},\pm s_M}\otimes(\frac{1}{2},\ldots,\frac{1}{2},\pm\frac{1}{2}),$$

or in the submodules of spinor valued simplicial harmonics

$$(l_1,\ldots,l_{M-1},+l_M)\otimes(\frac{1}{2},\ldots,\frac{1}{2},+\frac{1}{2}) \text{ and}$$

$$(l_1,\ldots,l_{M-1},-l_M)\otimes(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$$

As in the integer weight case, we now immediately have for $l_M=0$:

$$\mathcal{P}^{N_M}_{l_1,\ldots,l_{M-1},0}[x_1,\ldots,x_M]\,(\mathbb{C}^+_mI_+ - \text{valued}) \cong (l_1+\frac{1}{2},\ldots,l_{M-1}+\frac{1}{2},+\frac{1}{2})$$

$$\mathcal{P}^{N_M}_{l_1,\ldots,l_{M-1},0}[x_1,\ldots,x_M]\,(\mathbb{C}^+_mI_- - \text{valued}) \cong (l_1+\frac{1}{2},\ldots,l_{M-1}+\frac{1}{2},-\frac{1}{2}),$$

i.e., the $\pm\frac{1}{2}$ at the end of the weights are distinguished by considering the appropiate spinor values. Let us now consider the case where the last number in the weight is positive. To embed the spinor valued (corresponding to I) simplicial monogenics in $E'_{l_1,\ldots,l_{M-1},+l_M}$ we have to impose the condition that its harmonic components belong to the right space of simplicial harmonics. This means that these simplicial monogenics should be annihilated by the $Spin(2M)$-invariant scalar system of equations determined by the components of $P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$. Applying now the argument with the Casimir operator we thus get for spinor valued (corresponding to $\mathbb{C}_m^+ I_+$) polynomials:

$$\begin{aligned}&\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cap \text{ Ker (components of } P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}))\\ &\cong \quad (l_1 + \frac{1}{2}, \ldots, l_{M-1} + \frac{1}{2}, +(l_M + \frac{1}{2})).\end{aligned}$$

or in case of spinor values corresponding to $\mathbb{C}_m^+ I_-$:

$$\begin{aligned}&\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cap \text{ Ker (components of } P_+(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M}))\\ &\cong (l_1 + \frac{1}{2}, \ldots, l_{M-1} + \frac{1}{2}, -(l_M + \frac{1}{2})).\end{aligned}$$

We now conjecture that in both cases the system of scalar equations determined by the components of $P_-(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$ and $P_+(\partial_{x_1} \wedge \cdots \wedge \partial_{x_M})$ are actually redundant; they should be satisfied automatically by simplicial monogenics taking values in the appropiate spinor spaces. Now $\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M]$ ($\mathbb{C}_m I_+$ valued) is an irreducible $Pin(m)$-module corresponding to the weight $(l_1 + \frac{1}{2}, \ldots, l_{M-1} + \frac{1}{2}, l_M + \frac{1}{2})$. Regarding it as a $Spin(m)$-module, it splits as a direct sum of the two irreducible $Spin$-representations $(l_1 + \frac{1}{2}, \ldots, l_{M-1} + \frac{1}{2}, l_M + \frac{1}{2})$ and $(l_1 + \frac{1}{2}, \ldots, l_{M-1} + \frac{1}{2}, -(l_M + \frac{1}{2}))$, $l_M > 0$. On the level of Clifford algebra, this splitting comes from the decomposition of the *values* of the space of $\mathbb{C}_m I_+$-valued simplicial monogenics: $\mathbb{C}_m I_+ = \mathbb{C}_m^+ I_+ \oplus \mathbb{C}_m^- I_+ \cong \mathbb{C}_m^+ I_+ \oplus \mathbb{C}_m^+ I_- e_m$. Clearly the space of $\mathbb{C}_m^+ I_+$-valued simplicial monogenics contains the weight vector

$$w_{l_1,\ldots,l_M}(x_1,\ldots,x_m; T_1,\ldots,T_M) I_1 \ldots I_M$$

while the space of $\mathbb{C}_m^+ I_- e_m$-valued simplicial monogenics contains the inequivalent weight vector

$$w_{l_1,\ldots,l_M}(x_1,\ldots,x_m; T_1,\ldots,\bar{T}_M) I_1 \ldots I'_M e_m.$$

Hence for simplicial monogenics or half integer weight representations the consideration of the *appropiate* values makes the "extra" scalar system of equations needed in the integer weight case superfluous. We thus conclude:

$$\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cong (l_1 + \tfrac{1}{2}, \ldots, l_{M-1} + \tfrac{1}{2}, +(l_M + \tfrac{1}{2}))$$

where the polynomials are $\mathbb{C}_m^+ I_+$-valued, and

$$\mathcal{P}^{N_M}_{l_1,\ldots,l_M}[x_1,\ldots,x_M] \cong (l_1 + \tfrac{1}{2}, \ldots, l_{M-1} + \tfrac{1}{2}, -(l_M + \tfrac{1}{2}))$$

where the polynomials are $\mathbb{C}_m^+ I_-$-valued.

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