The Inferential and Representational Techniques in Galileo's Models of Uniformly Accelerated Motion

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Abstract: In this essay, I aim to scrutinize several of Galileo's *representational* and *inferential strategies* for dealing with naturally accelerated motion within the context of justification. Galileo's methodology succeeded in making the process of naturally accelerated motion intelligible *via* models. The focal point of the study of hand is the *Third Day* of the *Dialogues Concerning Two New Sciences*. My aim is, primarily, to provide a detailed historical case-study of the representational and inferential procedures in Galileo's mechanics. In this essay, I set out to bring these strategies and theoretical suppositions to the fore. I argue that the following inferential strategies were of key importance in Galileo's *Discorsi*: abstraction from and idealization of irrelevant factors, geo-infinitesimal representation, substitution of an inferentially recalcitrant type of motion for a less inferentially recalcitrant type of motion, physical interpretation by means of a previously established theorem, transference of geometrical relations to relations on motion, and proportionality as a proxy for otherwise unrelated motions. Furthermore, it will be shown that in Galileo's proto–mechanics a single unifying theoretical principle was absent. This limitation forced Galileo to use a heterogeneity of inferential strategies and theoretical assumptions. Additionally, Galileo's models illustrate how theoretical knowledge needs to be concretised by the introduction of specific models in order to obtain the desired inferential steps. In virtue of certain abstract properties pertaining to the models themselves, one is able to obtain novel results which are not derivable from the abstract theory alone. Another way of putting this, is that the information provided by a model helps to *constrain and concretize* the theoretical knowledge at hand.

1. Introduction

In this essay, I shall try to capture Galileo's scientific methodology as practiced in the *Third Day* of the *Dialogues Concerning Two New Sciences*[1] (first published: 1638) in which Galileo

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deals with naturally accelerated motion. Galileo's methodology succeeded in making the process of naturally accelerated motion intelligible *via* models[2]. Correspondingly, I shall study Galileo's techniques of *representing* natural phenomena and of *drawing inferences* from models (which represent these phenomena). Although, I agree that it is to some extent anachronistic to boldly claim that Galileo used "models" – in our contemporary sense – to represent naturally accelerated motion, I think that focussing on Galileo's models may help elucidating Galileo's reasoning[3] in the *Discorsi*. In addition to that, case-studies like these might also – on a more general level – help us to get more grip on the ways in which models represent in science. Few authors have actually engaged in a detailed analysis of how Galileo modelled the phenomenon of naturally accelerated motion. There are some notable exceptions from the history of science[4], but none of these connect such analysis with a systematic analysis of Galileo's representational and inferential techniques. In this essay, I shall focus on the ways in which models (or for Galileo "*una figura*" or "*un diagramma*") represented and helped Galileo to get a grasp on naturally accelerated motion. One *caveat* is in order here. I shall solely consider the context of justification: I shall not consider questions regarding the historical origin or emergence of Galileo's representational strategies for handling naturally accelerated motion. Correspondingly, I shall take Galileo's text at face value. The question at stake is: how did Galileo prove the things he proved (or attempted to prove)? In other words,

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[1] The original title is: *Discorsi e Dimostrazioni Matematiche, intorno à due nuoue scienze, Attenenti alla Mechanica & i Movimenti Locali* (for the standard Italian-Latin edition, see Galilei, 1968, VIII, pp. 49-313).

[2] Models are understood here as "the primary representational entities in science" (Giere, 1999, p. 5). Models are the entities scientists employ to represent a natural system. Models can be very broadly conceived: we can think of computer models, scale-models and mathematical models. Galileo's models clearly subsume under the class of mathematical models. For a general overview, see the introduction to Morgan & Morrison, 1999; for a recent volume that focuses on 3-D models, see de Chadarevian & Hopwood, 2004. To the reader who is left unsatisfied because of the absence of a more formal definition of a scientific model, I respond that it is advisable to study a considerable amount of case-studies first, before we engage in the activity of defining the notion "model".

[3] One remark with respect to "reasoning by means of a model" should be made from the outset. Some philosophers of science seem to suggest that, in contrast to verbal or discursive reasoning, there is a different type of reasoning occurring when scientists use models: "model-based reasoning" (e.g. Nersessian, 2002, pp. 135-143). I grant it that models help to facilitate reasoning, but continue to endorse the view that the *reasoning process itself remains discursive*.

[4] Winifred L. Wisan stressed Galileo's concerns of giving "a direct visual and exact visual demonstration" of scientific principles (Wisan, 1978, p. 40). The idea is that a principle is confirmed if we have a visual, mathematical model that proves the principle. According to Maurice Clavelin, explaining natural phenomena for Galileo was identical to establishing a model or "intelligible reproduction" of the relevant phenomena based on a number of principles and concepts (Clavelin, 1968, p. 456). For a detailed discussion on Galileo's early mathematization of nature, see Palmieri, 2003.

how do these proofs[5], or more precisely these models, function, represent and allow to draw the relevant inferences?

In a classic study, Maurice Clavelin has stressed the importance of the new conceptual outlook on motion Galileo introduced. By interpreting motion in terms of temporal and spatial relationships and looking for relations of proportionality between them, he had transformed it into the object of rational scientific enquiry which was ordered deductively and mathematically (Clavelin, 1968, 279-80). These new *interpretative notions* were primordial. Clavelin has done an excellent job in revealing these underlying notions. In dealing with uniformly acceleration motion Galileo conceived of velocity as an intensive magnitude, he stressed the concept of quantity of velocity and he postulated a continuum-idea, where motion is conceived as an infinitely indivisible process (ibid., chapter 6). By means of these concepts he was able to reduce all motions in free fall to one theoretical situation (where e.g. air resistance was absent). Explaining natural phenomena was essentially establishing a model or intelligible reproduction of the relevant phenomena based on a number of principles and concepts (ibid., 456). Physical necessity was then immediately reduced to mathematical necessity.[6] The explanation is used to establish the connections (*un rapport d'implication*) between the phenomena, which have become consequences of the model and the principles of rationality (ibid.). The physical truth is conceived as an imitation if the mathematical truth. However, a fundamental question concerning Galileo's methodology remains: in virtue of what does a model function as an "intelligible" reproduction of a phenomenon?

In my attempt in unravelling Galileo's methodology, I shall endorse a "models as mediators"-outlook on scientific models (Morgan & Morrison, 1999). This outlook will be my analytical tool. Let me first clarify what this outlook consists in. In the introduction to their volume on the use of models in science, Morrison and Morgan stress that in scientific praxis models function as autonomous agents (in the sense that they are partially independent of both theories and the empirical world) (Morrison & Morgan, 1999, 10). Because models are made up from a mixture of elements (elements that originate from outside of the original domain of investigation), they maintain this partially independent status (ibid., p. 14). The ways in which

[5] The word "proof" is used here two senses: (1) in the sense of being a test and (2) in the sense of being a deduction. Wisan has often pointed to the second sense: for Galileo true conclusions must be derived from true and evident principles (Wisan, 1978, p. 37). However, in Galileo's work the first sense also is present. In the *Discorsi*, for instance, he claims that a postulate will be established "when we find that the inferences from it correspond to and agree perfectly with experiment" (Galilei, 1954, p. 172).

[6] Clavelin is likely influenced by Heinrich Hertz' view on these matters.

models can function autonomously are various. The essential thing is that models are not wholly theory-driven nor data-driven. Models typically mediate between theory and data. Models replace physical system as the central objects of scientific inquiry and allow surrogate reasoning. Theories consist of general, abstract principles that govern the behaviour of a large set of phenomena; models are needed to apply these general principles to a number of different cases (ibid.). From Newton's second law (or in fact by all three) nothing much of interest follows (Giere, 1988, p. 66). We need additional information provided by a model (e.g. a two body-system, a simple pendulum) to actually represent a physical system. Laws of mechanics are like "general schemas that need to be filled in with a specific force function in order to carry information about the world" (ibid., p. 76). One remark should be added from the outset: Galileo, unlike Newton, did obviously not have a unified theory of motion. Therefore, it should come as no surprise that Galileo's theoretical resources are diverse and that what counts as theory fluctuates throughout the *Discorsi*.

Let me, finally, provide the structure of this essay. In section 2, I discuss and analyse several theorems from the third day of the *Discorsi*. In my analysis, I shall show that Galileo indeed used a variety of theoretical principles[7] and modelling procedures. Galileo's models help to crystallize abstract theoretical principles and replace the original physical system. In this process, they typically generate inferences which are not accessible from the given data or theoretical principles at hand. In great contrast to Newton's mechanics, Galileo's proto–mechanics was not founded on a set of generally accepted (theoretical) principles. Therefore, it should come as no surprise that Galileo's theoretical resources vary throughout the *Discorsi*. In part 3, I shall show how Galileo's methodology incorporated a broad myriad of very diverse inferential strategies. I shall discuss these strategies in view of the analyses carried out in part 2.

2. An Examination of Galileo's Models of Naturally Accelerated Motion

In this part, I shall scrutinize and interpret some of the models from the beginning of the *Third Day* in the *Discorsi*, where Galileo treats uniformly accelerated motion. Uniformly accelerated motion is motion that acquires, when starting from rest during, during equal time-intervals

[7] Throughout this paper, I use the term "theoretical principle" (or "theoretical knowledge") in the following sense: *a theoretical principle is a general statement that needs to be constrained by further model-specific information in order to yield useful and concrete information about the natural world*.

equal increments of speed, or more precisely, its momentum ("*celeritatis momenta*"[8]) receives equal increments in equal times (Galilei, 1954, p. 162, p. 169). This definition is not arbitrary: it fits the natural phenomenon of free fall (ibid., p. 160). This carefully selected set of models is to be understood as a representative sample of Galileo's modelling techniques. I shall use the resulting interpretation of these models as input for my account of Galileo's methodology in part 3.

[2.1] Galileo begins the *Third Day* with a proof of the following proposition[9]: the speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal (in a situation where there is no resistance, the planes are perfectly smooth, and the moving body is perfectly round) (ibid., pp. 169-70). Salviati, Galileo's spokesman, notes that he wishes "by experiment [which is repeated many times, as Galileo assures us] to increase the probability [of this theorem] to an extent which shall be little short of a rigid demonstration" (ibid., p. 170). The demonstration is based on an ideal pendulum and proceeds as follows. The purpose is to show that the momenta gained by fall through the arcs DB, GB and IB are equal (see figure 1). A nail, to which a lead bullet is suspended by a fine thread AB, is driven in a vertical wall. The bullet is set to swing from point C. It describes the arc CBD and approximately reaches point D which is equidistant from A as C is from A (if we abstract from the air resistance, it would do so exactly). Accepting this abstraction, we may infer that the impetus[10] on reaching B from C was sufficient to carry it to D at the same height. Then the experiment is performed with an extra nail inserted at a lesser height: at E or at F. In these cases, the bullet will also be carried to the line CD and the body will (nearly) describe path BG or BI respectively. (If the nail is placed so low that the remainder of the thread below it will not reach CD, the thread leaps over the nail and twists itself around it.) This shows that – given the abstraction from air resistance – the momenta, needed to carry a body of the same weight to equal heights along different arcs, are equal. So

[8] I have added some important terms in the original Latin or Italian from Galileo's *Opere* (Galilei, 1968, VIII, pp. 41-313.

[9] It is only in the 1656 edition (posthumously published by Vincenzo Viviani) that Galileo suggested a way of proving this principle from commonly accepted mechanical principles. See Galilei, 1968, VIII, pp. 214-219; Galilei, 1954, pp. 184-185. In this first proof of the principle, according to which the acquired speeds along different paths from equal heights are equal, Galileo had to resort to an analogy-argument to generalize this principle for inclined planes. In the presentation of **[2.5]**, we shall see that Galileo was able to directly apply a lemma (which we shall discuss in **[2.4]**) to inclined planes. This application entailed the desired result. See *infra*.

[10] Galileo uses the words impetus ("*impeto*"), ability ("*talento*"), energy ("*energia*"), speed ("*velocità*") weight ("*pondere*") and momentum ("*momento*") interchangeably (*ibid*., p. 18*n*).

conversely, it can be shown that the momenta acquired by fall through the arcs DB, GB and IB are equal.

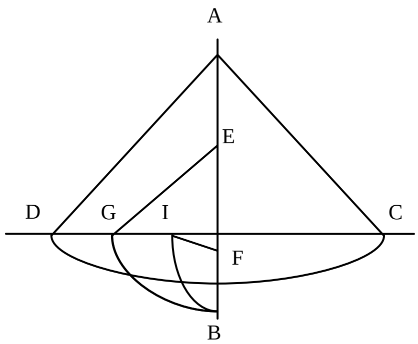

Figure 1.

The argument can be summarized as follows:

(1) $I_a(CB) = I_n(BD)$[11] [this follows from (a) the observation that a pendulum set to swing at C ascends (more or less) to D and (b) the underlying principle[12] according to which all bodies ascend to exactly the same height from which they initially acquired their momentum in descending (if we ignore air resistance)]

(2) $I_n(BD) = I_n(BG) = I_n(BI)$ [this generalisation follows (a) from the observation that the bobs – impeded by nails respectively inserted at E and F – set to swing from C approximately raise to G or to I, which are at the same height as D and (b) the previous principle]

(3) $I_a(DB) = I_a(GB) = I_a(IB)$ [this follows from (1) and (2)]

Galileo then extends this principle demonstrated for pendulums to inclined planes "by analogy". Sagredo admits that this experiment is well "adapted to establish the hypothesis" (ibid., p. 172). The reason that Galileo used this pendulum experiment is that it is impossible to produce these results on inclined planes, because at the lowest point the planes would form

[11] "I_a" stands for the impetus acquired along an arc; "I_n" for the impetus needed to be raised along an arc.

[12] This principle comes very close to Torricelli's principle which states that the centre of gravity cannot raise above itself. See Torricelli, 1919, II, p. 105, for Torricelli's own formulation.

angles and constitute an obstacle for the lead ball.[13] Galileo used the pendulum-model as an *idealization* of motion along inclined planes: the trajectory was deliberately distorted in order to obtain the inferential steps that he was interested in.

Bear in mind that Galileo claims to prove this proposition for motion along inclined planes without any resistance. For this purpose he uses an ideal pendulum (in which there is no resistance). The ideal pendulum allows the inferential steps that Galileo is interested in: proving that the momenta acquired along different arcs from equal heights are equal. The derivations above are exactly valid only in the ideal pendulum-model (it is only under these idealized conditions that the neat derivation in steps (1)-(3) obtains). The model is the ideal pendulum depicted by the drawing, where there is zero air resistance. *The theoretical principle, according to which all bodies ascend to exactly the same height from which they initially acquired their momentum in descending (if we ignore air resistance), remains silent on the specific trajectory a body describes*. This principle is applied to a specific case: the pendulum. This concretization is necessary to generate the relevant inferences. The inferences made from this model are then tested. Salviati takes "this as a postulate, the absolute truth of which will be established when we find that the inferences from it correspond to and agree perfectly with experiment" (ibid.).

[2.2] Theorem I, Proposition I is the mean-speed theorem or Mertonian rule which states that the "time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began" (ibid., p. 173). Let us look at the model Galileo used in his proof of this proposition (see figure 2). The aim is to show that, in equal times, a uniform motion with ½ overall momentum of an accelerated motion will traverse the same distance (neglecting at that point the question if such motions really exist). This proposition will be used as an inference-ticket or proxy in the following proposition (see **[3.3]**). AB represents the time in which the space CD is traversed (hence, the distance is the independent variable[14]) by a body that starts to fall at rest from C ("*Repraesentetur per existensionem AB tempus in quo a mobile*

[13] It is worth noting that Christiaan Huygens, in Proposition VI of the *Horologium Oscillatorium* (1673), simply abstracted from the resistance caused by the sudden inflexion and used such (fictive) inclined planes to demonstrate that "velocities acquired by bodies falling through variably inclined planes are equal if the elevations of the planes are equal" (Blackwell, 1986, pp. 43-44).

[14] E.J. Dijksterhuis remarks that Oresme had already used the traversed time as the independent variable (Dijksterhuis 1924, p. 257).

latione uniformiter accelerata ex quiete in C conficiatur spatium CD" (Galilei, 1968, VIII, p. 208)). The horizontal, parallel lines represent what we would today call the instantaneous velocity (or more precisely, "*crescentes velocitatis gradus post instans A"*[15]). The triangles and the rectangles represent the overall momentum acquired in a time-interval [t_0 *(=A)*, t_n *(=B)*] during uniformly accelerated motion (where the *gradus velocitatis* constantly increases) and during uniform motion (where the *gradus velocitatis* remains the same) respectively (Galilei, 1954, p. 173).

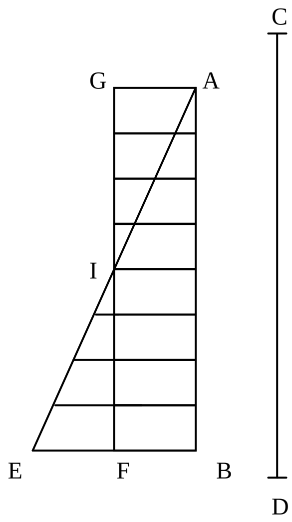

Figure 2.

The text proceeds as follows:

> Since each and every instant of time in the time-interval AB, from which points parallels drawn in and limited by the triangle AEB represent the increasing values of growing velocity, and since parallels contained within the rectangle represent the values of a speed which is not increasing, but constant, it appears, in like manner, that the momenta [*momenta*] assumed by the moving body may also be represented, in the case of the accelerated motion, by the increasing parallels of the triangle AEB, and, in the case of the uniform motion, by the parallels of the rectangle GB. For, what the momenta may lack in the first part of the accelerated

[15] This notion was never explicitly defined by Galileo. Michel Blay writes on Galileo's notion of degree of velocity: "While to a certain extent it prefigured the concept of instantaneous velocity, it nonetheless remained subject to the Galilean way of conceiving motion, which regarded velocity as an "intensive magnitude" increasing by successive additions of degrees." (Blay, 1998, p. 72).

> motion (the deficiency of the momenta being represented by the parallels of the triangle AGI) is made up by the momenta represented by the parallels of the triangle IEF. (ibid., pp. 173-74)

The parallels of "instantaneous" speed are contained ("*comprehensae*" or "*contentae*") in the triangle. The "aggregate" of all parallels contained in AEB equals the "aggregate" of the parallels contained in AGFB (Blay, 1998, p. 74). The degrees of speed that the uniformly accelerated motion lacks are made up during the second half (Dijksterhuis, 1924, p. 257). The relation between uniform motion and uniformly accelerated motion is established by the equality between the surfaces which represent them. An important premise, in Galileo's construction, is the mathematical assumption that an area is made up of an infinity of lines (ibid.). Galileo presupposed that the equality of the two infinite sets of moments of velocity establishes the equality of the corresponding overall speeds (Damerow, Freudenthal, McLaughlin and Renn 1992, p. 230). Galileo lacked the adequate tools to deal with this thoroughly (Clavelin, 1968, p. 316). He, however, did not find this assumption theoretically suspect and concluded:

> Hence it is clear that equal spaces will be traversed in equal times by two bodies, one of which, starting from rest, moves with a uniform acceleration, while the momentum of the other, moving with uniform speed, is on-half its maximum momentum under accelerated motion. (Galilei, 1954, p. 173)

Let me sum up how Galileo represents accelerated motion:

(1) AB, a line consisting of an infinite set of points, represents the time needed to traverse a distance CD; every point corresponds to an instant of time; A represents the starting point (t_0); B represents terminus (t_n)
(2) CD represents an arbitrary distance (hence, it is the independent variable)
(3) infinitesimal horizontal lines represents the (instantaneous) *crescentes gradus velocitatis*
(4) AEB represents the totality (*totidem velocitatis momenta*) of the increasing values of growing velocity (i.e., the aggregate of the constantly increasing speeds)

(5) AGFB represents the totality of the constant values of speed (i.e., the aggregate of the constant speeds)

The natural phenomena being modelled here is uniformly accelerated motion.[16] The model is the “geo-infinitesimal” representation of the growth of speed of these motions depicted in figure 2. The mean-speed theorem is inferred from the equality between the surfaces of the triangle, which represents a uniformly accelerated motion, and the rectangle, which represents a uniform motion. The surfaces are considered as infinitesimally divisible: they both consist of an infinitude of horizontal lines. *The assumption that movements in time can be represented geo-infinitesimally is Galileo’s most central theoretical background assumption here*. This assumption functions as a *construction-rule* from which models can be derived and built. The inferential steps that Galileo is interested in follow from the mathematical properties of the model, viz. the equality between the surfaces. This model is not wholly data-driven: highly careful observation indeed shows that in free fall the acquired speed constantly increases, but there is no obvious reason why we should straightforwardly represent the increase of speed by a triangle. The model is not wholly construction-rule-driven either: treating speed and time geo-infinitesimally does not automatically lead to this particular model. We need extra assumptions here: for instance, that instants of time are depicted by points on a line; that the instantaneous degrees of velocity correspond to horizontal lines; that the triangular and rectangular surface stand for the growth or constancy of the *gradus velocitatis*, etc. These assumptions are precisely provided by the model.

[2.3] Theorem II, Proposition II is the squared-time law which states that the “spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time-intervals employed in traversing these distances” (ibid., pp. 175-176). The units of time (“*fluxus temporis*”) are represented on AB; the distances through which a body falls with a uniform acceleration starting from rest are represented by HI (see figure 3). Time AD corresponds to length HL, AE to HM, AF to HN and AG to HI. AC is constructed at an arbitrary angle on AB (“*quemcunque angulum”*). OD and PE represent the maximum speed at D and E.

[16] It is not until Proposition III that Galileo has demonstrated that uniformly accelerated motion corresponds to motion along an inclined plane or motion in free fall.

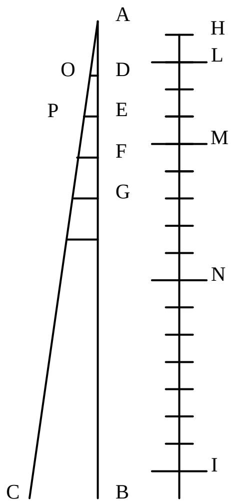

Figure 3.

The proof proceeds as follows (see also Wisan, 1974, pp. 286-288). From the mean-speed theorem it follows that the distances HM and HL are the same as those that would be traversed during AE and AD by a uniform motion with half the speeds of those by which OD and EP are represented. Since ratio AE is to AD as ½ PE is to ½ OD, or as PE to OD, the velocities are to each other as the time-intervals ($v \sim t$). Galileo replaces uniformly accelerated motions by uniform motions. From Theorem IV, Proposition IV (in the section on uniform motion) which states that "if two particles are carried with uniform motion, but each with a different speed, the distances covered by them during unequal intervals of time bear to each other the compound ratio of the speeds and time intervals", Galileo concludes: $x \sim (v \times t)$ (ibid., p. 157). Hence, the ratio of the spaces traversed is the same as the squared ratio of the time-intervals (hence: $x \sim t^2$). Galileo uses information about a simple situation (uniform motion) to a less simple situation (uniformly accelerated motion). Galileo then argues from his famous inclined plane experiments that the natural phenomena agree to this proposition. That the data correspond to the model is essential. Galileo seems, at least in the presentational or expositional part of his theory, not to spend much attention to the details of the experiments (for a recent study of Galileo's inclined plane experiment, see Romo, 2005). Let me sum up the elements of this model:

(1) AB, a line consisting of an infinite set of points, represents the time needed to traverse a distance HI; every point corresponds to an instant of time; A represents the starting point (t_0); B represents the terminal point (t_n); time-intervals AD, AE, AF and AG correspond to distances HL, HM, HN and HI

(2) OD and PE represent the *gradus velocitatis* at instants of time D and E

(3) HL, HM, HN, HI represent the distances traversed in time-intervals AD, AE, AF, AG

Data, model and the construction-rule are similar as in the previous theorem (except that uniform motion is now directly considered).[17]

[2.4] After the *scholium* to this proposition, a dialogue was inserted by Viviani at the suggestion of Galileo "for the better establishment on logical and experimental grounds, of the principle which we have above considered" (ibid., p. 180) a year after the publication of the *Discorsi* (Galileo was blind at that time). This insertion includes a lemma and a theorem (the theorem will be discussed in **[2.5]**). The lemma states that the ratio between the momentum of a body G along the vertical CF is to the momentum of the same body along the inclined plane FA as the inverse of that of the aforementioned lengths ($v_1/v_2 = x_2/x_1$) (ibid., p. 182). The impelling force acting on a body in descent ("*l'impeto del descendere*") is equal to the resistance or least force sufficient to hold it at rest (ibid.). To measure this force body G is connected to body H with a cord passing over F – see figure 4. We notice that, in order to hold G at rest, H must have a weight smaller in the same ratio as CF is smaller than FA (transcribed: W(G)/W(H) = FA/FC or $W_1/W_2 = x_1/x_2$). Galileo then writes:

> For if we consider the motion of the body G, from A to F, in the triangle AFC to be made up of a horizontal component AC and a vertical component CF, and remember that this body experiences no resistance to motion along the horizontal (because by such a motion the body neither gains nor loses distance from the common center of heavy things) it follows that resistance is met only in consequence of the body rising through the vertical distance CF. Since then the body G in moving from A to F offers resistance only in so far as it rises through the vertical distance CF, while the other body H must fall vertically through the entire distance FA, and since this ratio is maintained whether the motion be large or small, the two bodies being inextensibly connected, we are able to assert positively that, *in case of equilibrium (bodies at rest) the momenta, the velocities, or their tendency to motion, i.e. the spaces which would be traversed by them in*

[17] I omit a presentation of the corollaries which follow from this proposition, because they do not add anything substantial to the previous exposition.

> *equal times, must be in the inverse ratio of their weights*. This is what has been demonstrated in every case of mechanical motion.[18] (Galilei, 1954, pp. 182-183; my emphasis)

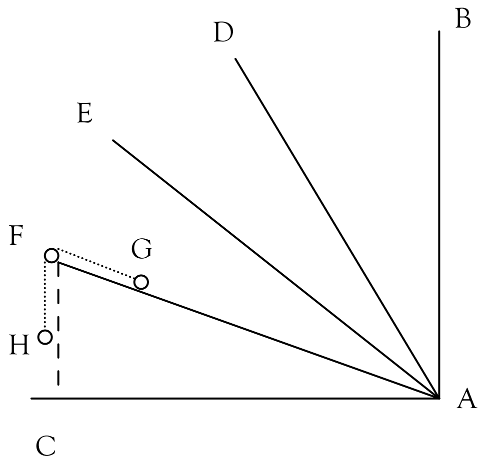

Figure 4.

Hence, in equilibrium, the velocities are to each other as the inverse ratio of the weights ($v_1/v_2 = W_2/W_1$). Notice that this involves the introduction of virtual velocities. This result combined with the previous ratio ($W_1/W_2 = x_1/x_2$) leads to the result: $v_1/v_2 = x_2/x_1$, which was to be demonstrated. The natural phenomenon being modelled here is the situation of "inclined equilibrium" depicted in figure 4. The theoretical principle involved is the proto-principle of virtual velocities. It clearly does not constitute a general theoretical framework: its scope is rather restricted and applies only to equilibriums. We need to introduce further information (in this case, the established empirical proportion $W_1/W_2 = x_1/x_2$) so that the theoretical principle generates the inferences we are interested in. This model mediates between the proto-principle of virtual velocities and the data: it is not wholly theory-driven, because it allows for empirical input; neither is it wholly data-driven, because it clearly adds theoretical, interpretative structure. The theoretical principle is concretized by the model: it is applied to a situation where unequal weights (one hanging vertically, the other along an inclined plane) balance each other. By applying the principle we can infer that the ratio of the momentum in free fall is to the momentum along an inclined plane as the inverse ratio of their distances. The inferential steps follow from applying the proto-principle of virtual work to this specific model (where a direct proportionality between the weights and the distances holds).

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[18] The translators point out that this principle is "a near approach" of the principle of virtual work formulated by Jean Bernoulli in 1717 (Galilei, 1954, p. 183*n*).

[2.5] The following theorem (which I shall refer to as the "equal-height-equal-momentum theorem") states that the (final) speeds at different angles along an inclined plane at equal heights are the same. From the construction it is given that: AD is the third proportional[19] to AB and AC (AB/AC = AC/AD) (ibid., p. 184). See figure 5. There are three important steps:

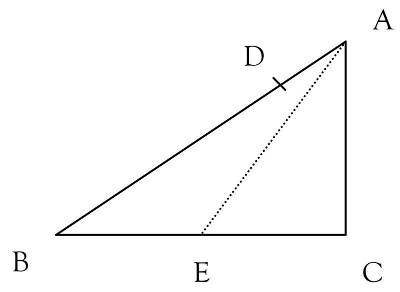

Figure 5.

(1) From the lemma (discussed in **[3.4]**), it follows that the impetus along AC is to that along AB as AB is to AC (I(AC)/I(AB) =[20] AB/AC). By combination of the given information that AD is the third proportional to AB and AC, it follows that the impetus along AC is to that along AD as AC is to AD (I(AC)/I(AD) = AC/AD).

(2) By the definition of naturally accelerated motion it follows that I(AB)/I(AD) = t(AB)/t(AD). From Corollary II to Theorem II, Proposition II, it follows that: t(AB)/t(AD) = AC/AD. Hence, I(AB)/I(AD) = AC/AD.

(3) From (1) and (2): I(AB)/I(AD) = I(AC)/I(AD), and thus finally, I(AB) = I(AC).

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[19] If A/B = B/C, then (1) B is the mean proportional to A and C and (2) C is the third proportional to A and B. In Book V of Euclid's *Elements*, proportionals are magnitudes which have the same ratio, i. e. "when the first of four magnitudes is said to have the same ratio to the second, that the third has to the fourth, when any equimultiples whatever of the first and the third being taken, and any equimultiples whatever of the second and the fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth, and if the multiple of the first is equal to that of the second, the multiple of the third is also equal to that of the fourth, and if the multiple of the first is greater than the first, the multiple of the third is also greater than that of the fourth" (Todhunter, 1967, p. 134).

[20] The relations denoted by the "="–sign are to be understood here as *purely proportional*.

In the first step of this proof the lemma from **[3.4],** which serves as (part of our) theoretical knowledge here, is applied in order to infer the initial information that I(AB)/I(AC) = AC/AB. By the given third proportional, we are able to establish a relation between the impetus acquired along AC and the impetus acquired along AD: I(AC)/I(AD) = AC/AD. This proportion is further combined with inferences derived from other pieces of theoretical knowledge (the definition of naturally accelerated motion and Corollary II to Proposition II) in order to obtain the desired result: I(AB) = I(AC). The relation of (mean) proportionality is used to establish a relation between the impetuses acquired along AB, AC and AD. It is an abstract property of this mathematical model that facilitates the required inferential steps.[21]

[2.6] Theorem VI, Proposition VI – known as the theorem of the isochronism-of-the-chords – is also of considerable interest. This proposition states that if "from the highest or lowest point in a vertical circle there drawn any inclined planes meeting the circumference the times of descent along these chords are each equal to the other" (ibid., pp. 188-189). Galileo provides three different proofs of this theorem. I will only discuss the first one. AB and AC are the planes along which the motion is supposed to take place (see figure 6). AI is the mean proportional between AE and AD. We know by elementary geometry that: (1) the rectangles FA.AE and FA.AD are to each other as the squares of AC and AB, and that (2) FA.AE and FA.AD are to each other as AE to AD. From this we obtain that the squares of AC and AB are to each other as AE is to AD. Since AE is to AD as the squares of AI and AD, it follows that the squares of AC and AB are to each other as the squares of AI and AD (and hence, that AC is to AB as AI is to AD).

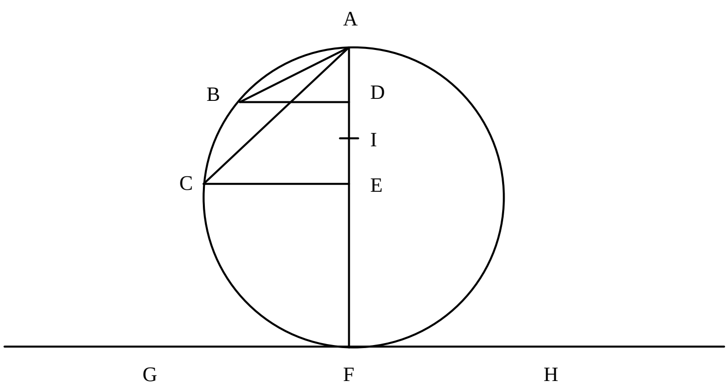

Figure 6.

[21] I want to thank Mauricio Suárez for pushing me towards that point.

Then Galileo continues as follows:

> But it has previously been demonstrated that the ratio of the time of descent along AC is to that along AB is equal to the product of the two ratios AC to AB and AD to AI; but this is the same as that of AB to AC. The ratio of these times is therefore unity [*igitur ratio eorumdem temporum ratio aequalitatis*]. (ibid., p. 189)

Hence, it follows that the times of descent along different chords on the same (vertical) circle will be equal. This shows something interesting about Galileo's reasoning: kinematical insight ($t_1/t_2 = (x_1.v_2)/(v_1.x_2)$) is used to interpret purely mathematically relations. Up until Galileo interprets the proportion of time needed to traverse AC to that of AB as the product of the two ratios AC/AB and AD/AI, the proof is purely mathematical. Notice that this mathematical model is only partially isomorphic to motion along inclined planes: only the two triangles inside the circle share their structure with an inclined plane. From the mathematical construction and the kinematical reading of it, it follows that this compound ratio equals 1. Hence, the times are equal. The inferences follow from combining purely mathematical relations with the kinematical knowledge that $t_1/t_2 = (x_1.v_2)/(v_1.x_2)$. Pure mathematics and kinematics allow the required inferential steps.

3. Galileo's Inferential Strategies

As can be seen from the previous section, Galileo used a broad myriad of models (each having different theoretical sources and relying on various modelling or inferential strategies). In this part, I will systematize Galileo's inferential strategies. The overview of models from Galileo's propositions concerning accelerated motion (provided in the previous section) suggests that the following strategies are important:

[3.1] ABSTRACTION FROM AND IDEALIZATION OF IRRELEVANT[22] FACTORS[23]

[22] Of course, both presuppose procedures for establishing that the *abstracted* or *idealized* features make no difference for our understanding of the real-world case.

[23] See McMullin, 1985 and Koertge, 1977 on Galilean abstraction and idealization.

In order to facilitate inferential steps one abstracts from aspects that do not sensibly disturb a body's motion (e.g. air resistance) and idealizes recalcitrant aspects of a body's motion (cf. the assumptions of a perfectly smooth surface, perfectly spherical balls, etc.).

This is especially clear from the example discussed in **[2.1]**. Galileo abstracts from air resistance which allows him to obtain the exact mathematical formulation he is interested in. In the same example, Galileo uses the ideal pendulum as an idealization of motion along inclined planes. The ideal pendulum perfectly allows the inferential steps that he is interested in. The inferential result is then transferred from pendulum motion to motion along inclined planes. The pendulum model turns out to be an idealization of the inclined plane situation: the trajectory along an inclined plane is deliberately distorted to allow the inferential steps that motion along an inclined plane by itself cannot provide. The neat result is only obtainable under the abstract and idealized conditions that hold in the model, but not in the natural world.

[3.2] GEO-INFINITESIMAL REPRESENTATION

Movements, which are processes that involve the decreasing, increasing of conserving of motion, are represented geo-infinitesimally, i.e. by geometrical entities consisting of an infinitude of points or lines.

This is especially clear in Galileo's demonstration of the mean-speed theorem (discussed in **[2.2]**). Galileo considered motion as an "intensive magnitude", i.e. as a magnitude that increases by instantaneous additions. Now, this assumption is rather abstract to infer useful knowledge from. We need a geo-infinitesimal model to further concretise this thesis. Geo-infinitesimal entities allow Galileo to represent instantaneous speeds and instants of time. Such representational strategies were of course crucial in the establishment of early-mechanics. In Theorem I, instantaneous velocity is represented by infinitesimal, horizontal lines and the aggregate of the increase or constancy of motion is represented by geometrical surfaces (by triangles and rectangles). Infinitesimal lines represent the instantaneous *gradus velocitatis*. The aggregate of such lines, an area, represents the totality of the *gradus velocitatis*. By introducing time into the representation (time is represented by a vertical line consisting of an infinitude of points which corresponds to the infinitude of instances of time), we can interpret the triangle and the rectangle as an accelerated motion or a uniform motion, respectively. This presupposes that time and velocity consist of an infinity of instants or degrees of velocity (or may at least be fruitfully considered as such). We notice that Galileo's geo-infinitesimal models helped to concretize his conception of motion as an "intensive magnitude".

[3.3] SUBSTITUTION OF AN INFERENTIALLY RECALCITRANT TYPE OF MOTION FOR A LESS INFERENTIALLY RECALCITRANT TYPE OF MOTION

In order to facilitate inferential steps concerning motions which are recalcitrant to generate the required inferential steps, one overcomes these limitations by substituting the inferentially recalcitrant motions for motions that are more easily treated.

In **[2.3]** we saw how Galileo derived the squared-time law by applying the mean-speed theorem. In Theorem II, accelerated motion is reinterpreted as uniform motion (by the mean-speed theorem). Now it becomes possible to derive the wanted proportionality by means of the relations that hold for uniform motion. By doing so Galileo succeeded in substituting uniformly accelerated motion for uniform motion and in paving the way for the inferential steps he was interested in. In **[2.1]** we have seen how Galileo similarly substituted movement along an inclined plane for movement along a pendulum.

[3.4] PHYSICAL INTERPRETATION BY MEANS OF A PREVIOUSLY ESTABLISHED THEOREM

A physical situation is interpreted by previously established theorems which now function as theoretical knowledge.

In **[2.5]** Galileo applied the lemma from **[2.4]** in order to establish that the impetus along AC is to that along AB inversely as their corresponding distances. The model is then used to further constrain these theoretical sources.

[3.5] TRANSFERENCE OF GEOMETRICAL RELATIONS TO RELATIONS ON MOTION

In order to establish a relation between motions, one uses information about the relations that hold between the geometrical entities that (by being isomorphic to the targeted physical system) represent these motions.

From the drawing presented in **[2.6]** Galileo infers from some elementary geometry that AC/AB = AI/AD. No kinematical knowledge enters the scene at all. But then Galileo suddenly adds information on the ratio between the times of descent along AC and AB. Here the mathematical figure becomes a kinematical model. The mathematically obtained proportion is then used in Galileo's kinematical interpretation.

[3.6] PROPORTIONALITY AS A PROXY FOR OTHERWISE UNRELATED MOTIONS[24]

(Third or mean) proportionals are used to establish relations between otherwise unconnected geometrical entities (and thus, indirectly, between otherwise unrelated motions).

A striking example of this can be found in **[2.5]**. Galileo only has information about the impetuses along AC and along AB at his disposal. By means of the third proportional AD, he is able to obtain a relation between the impetuses along AC and AD and to infer the further steps that he is interested in. Such proportionalities are properties that pertain to the models themselves.

These diverse modelling strategies, I would consider as being *prototypical* for Galileo, were crucial for him to get a grasp on the phenomenon of motion (and in particular free fall). Obviously, these strategies can be mutually combined, so that a very complex image of Galileo's models arises.

4. In Conclusion

One important lesson is that in Galileo's proto–mechanics there was no unifying theoretical principle. This "handicap" forced Galileo to use a heterogeneity of inferential strategies and

[24] As one can notice this is a specific case of the previous strategy. Because it is so prominently present in Galileo's reasoning, I have chosen to list it separately.

theoretical assumptions. Galileo’s models were derived from a broad myriad of *theoretical knowledge*. Galileo’s models amply show how theoretical knowledge needs to be concretised by models in order to obtain the desired inferential steps. In virtue of certain abstract properties pertaining to the models themselves (in Galileo’s very often proportionality), we are able to obtain novel results. Another way of putting it is that the information provided by a model helps to *constrain* the theoretical knowledge. In this paper, I have tried to bring the diversity of Galileo’s representational and inferential techniques to the fore.

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