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Routhian reduction for quasi-invariant Lagrangians

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In this paper, we describe Routhian reduction as a special case of standard symplectic reduction, also called Marsden–Weinstein reduction. We use this correspondence to present a generalization of Routhian reduction for quasi-invariant Lagrangians, i.e., Lagrangians that are invariant up to a total time derivative. We show how functional Routhian reduction can be seen as a particular instance of reduction in a quasi-invariant Lagrangian, and we exhibit a Routhian reduction procedure for the special case of Lagrangians with quasicyclic coordinates. As an application, we consider the dynamics of a charged particle in a magnetic field. © *2010 American Institute of Physics.* [doi:10.1063/1.3277181]

I. INTRODUCTION AND OUTLINE

In modern geometric approaches to Routhian reduction, it is often mentioned that this reduction technique is the Lagrangian analog of symplectic or Marsden–Weinstein (MW) reduction[14] (see, for instance, the introduction in Ref. 3). This assertion is usually justified by the fact that, roughly speaking, for Routhian reduction, one first restricts the system to a fixed level set of the momentum map and then reduces by taking the quotient with respect to the symmetry group. In this paper, we show, among other things, that the analogy between Routhian reduction and MW reduction holds at a more fundamental level: in fact, we will show that Routhian reduction is simply a special instance of general MW reduction. More specifically, by applying the MW reduction procedure to the tangent bundle of a manifold, equipped with the symplectic structure induced by the Poincaré–Cartan 2-form associated with a Lagrangian, we will show that the resulting reduced symplectic space is "tangent bundlelike," and that the reduced symplectic structure is again defined by a Poincaré–Cartan form, augmented with a gyroscopic 2-form. Of course, this symplectic description of the reduced system, obtained via the Routh's reduction method, is well known in literature. The difference with our approach, however, lies in the fact that we arrive at the reduced symplectic structure following the MW method. Until now, the symplectic nature of a Routh-reduced system was obtained either by reducing the variational principle (see Refs. 7 and 13 and references therein) or by directly reducing the second order vector field describing the given system (see Ref. 4).

The advantage of interpreting Routhian reduction in terms of MW reduction lies in the fact that we are able to extend the concept of Routhian reduction to quasi-invariant Lagrangian systems, i.e., Lagrangian systems that are invariant up to a total time derivative. Such a generalization lies at hand: it is well known that a quasi-invariant Lagrangian determines a strict invariant energy and a strict invariant symplectic structure on the tangent bundle. On the other hand, the actual

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reduction in quasi-invariant Lagrangians exploits the full power of MW reduction and is therefore, in our opinion, a very interesting application of this reduction procedure. The generalization to quasi-invariant Lagrangians is the main result of this paper.

A. Lagrangians with a quasicyclic coordinate

In the remainder of Sec. I, we illustrate some of the concepts used in this paper by means of a simple, but clarifying example: the case of a Lagrangian with a single quasicyclic coordinate. This is a generalization of the classical procedure of Routh dealing with Lagrangians with a cyclic coordinate and will serve as a conceptual introduction for the geometric techniques introduced later on, when we deal with the case of general quasi-invariant Lagrangians in Theorems 7 and 8.

We begin by recalling the classical form of Routh's result on the reduction in Lagrangians with cyclic coordinates (or, stated in a slightly different way, the reduction in Lagrangians that are invariant with respect to an Abelian group action). For simplicity, we confine ourselves to the case of one cyclic coordinate. Subsequently, we will illustrate how this theorem can be extended to cover the case of quasicyclic coordinates.

Given a Lagrangian $L:\mathbb{R}^{2n}\to\mathbb{R}$ for a system with n degrees of freedom $(q^1,\ldots,q^n)$ for which, say, q^1 is a *cyclic coordinate* (i.e., $\partial L/\partial q^1=0$). The momentum $p_1=\partial L/\partial\dot{q}^1$ is a first integral of the Euler–Lagrange equations of motion. If $\partial^2L/\partial\dot{q}^1\partial\dot{q}^1\neq 0$ holds, there exists a function ψ such that $p_1=\mu$ is equivalent to $\dot{q}^1=\psi(q^2,\ldots,q^n,\dot{q}^2,\ldots,\dot{q}^n)$.

Theorem 1: *[Routh reduction (Ref. 17)] Let $L:\mathbb{R}^{2n}\to\mathbb{R}$ be a regular Lagrangian for a system with n degrees of freedom $(q^1,\ldots,q^n)$. Assume that q^1 is a cyclic coordinate and that $\partial^2L/\partial\dot{q}^1\partial\dot{q}^1\neq 0$ so that $\dot{q}^1$ can be expressed as $\dot{q}^1=\psi(q^2,\ldots,q^n,\dot{q}^2,\ldots,\dot{q}^n)$. Consider the Routhian $R^\mu:\mathbb{R}^{2(n-1)}\to\mathbb{R}$ defined as the function $R^\mu=L-\dot{q}^1\mu$, where all instances of $\dot{q}^1$ are replaced by ψ. The Routhian is now interpreted as the Lagrangian for a system with $(n-1)$ degrees of freedom $(q^2,\ldots,q^n)$.*

Any solution $(q^1(t),\ldots,q^n(t))$ of the Euler–Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}^i}\right)-\frac{\partial L}{\partial q^i}=0,\quad i=1,\ldots,n$$

with momentum $p_1=\mu$, projects onto a solution $(q^2(t),\ldots,q^n(t))$ of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R^\mu}{\partial\dot{q}^k}\right)-\frac{\partial R^\mu}{\partial q^k}=0,\quad k=2,\ldots,n.$$

Conversely, any solution of the Euler–Lagrange equations for R^μ can be lifted to a solution of the Euler–Lagrange equations for L with momentum $p_1=\mu$.

The number of degrees of freedom of the system with Lagrangian R^μ is reduced by 1, and this technique is called Routh reduction. We now formulate a generalization of this theorem for a Lagrangian system with a *quasicyclic coordinate* q^1, i.e., there exists a function f depending on $(q^1,\ldots,q^n)$ such that

$$\frac{\partial L}{\partial q^1}=\dot{q}^i\frac{\partial f}{\partial q^i}.$$

If q^1 is quasicyclic, it is easy to show that there is an associated first integral of the Lagrangian system given by $F:=\partial L/\partial\dot{q}^1-f$. Note that if $\partial^2L/\partial\dot{q}^1\partial\dot{q}^1\neq 0$, we can again solve the equation $F=\mu$, where μ is a constant, to obtain an expression for $\dot{q}^1$ in terms of the remaining variables. In the next theorem, we now show how the classical procedure of Routh may be extended to cover the case of a Lagrangian with a quasicyclic coordinate. We defer the proof of this theorem to Sec. V A.

Theorem 2: *(Routh reduction for a quasicyclic coordinate) A regular Lagrangian $L:\mathbb{R}^{2n}\to\mathbb{R}$ for a system with n degrees of freedom $(q^1,\ldots,q^n)$ with a quasicyclic coordinate q^1 is Routh* *reducible if (i)* $\partial^2 L/\partial\dot{q}^1\partial\dot{q}^1 \neq 0$ *and if (ii) there exist* $(n-1)$ *functions* Γ_k *independent of* q^1 *such that*

$$\frac{\partial f}{\partial q^k} = \Gamma_k(q^2, \dots, q^n)\frac{\partial f}{\partial q^1}, \quad k = 2, \dots, n. \tag{1}$$

For μ *a constant, consider the Routhian* $R^\mu : \mathbb{R}^{2(n-1)} \to \mathbb{R}$ *defined as*

$$R^\mu = L - (\mu + f(q))(\dot{q}^1 + \Gamma_i \dot{q}^i),$$

where all instances of $\dot{q}^1$ *are replaced by the expression obtained from the equation* $\partial L/\partial\dot{q}^1 = \mu + f$. *The Routhian is independent of* q^1 *and can be seen as a Lagrangian for a system with* $(n-1)$ *degrees of freedom* $(q^2, \dots, q^n)$.

Then, any solution $(q^1(t), \dots, q^n(t))$ *of the Euler–Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n$$

such that $\partial L/\partial\dot{q}^1 - f = \mu$, *projects onto a solution* $(q^2(t), \dots, q^n(t))$ *of the Euler–Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R^\mu}{\partial \dot{q}^k}\right) - \frac{\partial R^\mu}{\partial q^k} = 0, \quad k = 2, \dots, n.$$

Conversely, any solution of the Euler–Lagrange equations for R^μ *can be lifted to a solution of the Euler–Lagrange equations for* L *for which* $\partial L/\partial\dot{q}^1 - f = \mu$.

Readers familiar with methods from differential geometry might recognize that the functions Γ_k determine a *connection* on the configuration space. Condition (ii) from the above theorem can be interpreted geometrically as the existence of a connection for which $\mathrm{d}f$ annihilates the horizontal distribution or, alternatively, such that f is covariantly constant: $Df = 0$ (with Df denoting the restriction of $\mathrm{d}f$ to the horizontal distribution). It turns out that this condition is essential to Routhian reduction in the context of quasi-invariant Lagrangians.

We note that the requirement that $\mathrm{d}f$ annihilates the horizontal distribution implies in this case that there exists an equivalent Lagrangian L' (i.e., a Lagrangian that differs from L by a total time derivative), which is strictly invariant so that Routhian reduction in the classical sense can be applied. However, we should warn against dismissing quasi-invariant Routh reduction too hastily since Routh reduction is possible also for quasi-invariant Lagrangians with nontrivial nonequivariance cocycle. We refer to Ref. 10 for a general discussion on quasi-invariant Lagrangian systems and, in particular, the property that the vanishing of this nonequivariance cocycle is a necessary condition for a quasi-invariant Lagrangian to be equivalent to a strict invariant Lagrangian.

To conclude this introduction, we note that the study of Routhian reduction for quasi-invariant Lagrangians was partially inspired by a technique called *functional Routhian reduction* described in Ref. 2, where it is used to obtain a control law for a three-dimensional bipedal robot. We will return to this example in Sec. V B.

B. Plan of the paper

In Secs. II and III we show that classical Routhian reduction is precisely MW reduction. We start with the well-known description of MW reduction in the cotangent bundle framework. Although a description of cotangent bundle reduction may be found in Ref. 12, we will elaborate on this and prove the results because this will be shown useful when considering quasi-invariant Lagrangians. Next, in Sec. IV we describe MW reduction for quasi-invariant Lagrangians. In Sec. V we conclude with a number of examples.

II. TANGENT AND COTANGENT BUNDLE REDUCTION

In this section, we recall some standard results on group actions and principal bundles and we formulate MW reduction theorem in its standard form. We then specialize to the reduction in a cotangent bundle with the canonical symplectic form or a tangent bundle with a symplectic form, which is obtained through pullback along the Legendre transformation. The material in this section is well known and more information can be found in Refs. 11 and 16.

A. Momentum maps and symplectic reduction

1. Notations

Throughout this paper we shall mainly adopt the notations from Refs. 3 and 15. Let M be a manifold on which a group G acts on the right. This action is denoted by $\Psi: M\times G \to M$ and is such that $\Psi_{gh}=\Psi_h\circ\Psi_g$ for all $g,h\in G$, with $\Psi_g :\equiv \Psi(\cdot,g)$. The action Ψ induces a mapping on the Lie-algebra level

$$\varphi: M\times\mathfrak{g}\to TM:(m,\xi)\mapsto\varphi_m(\xi)=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\Psi(m,\exp\,\epsilon\xi).$$

The mapping $\mathfrak{g}\to\mathfrak{X}(M)$ associated with a Lie-algebra element ξ the corresponding infinitesimal generator $\xi_M\in\mathfrak{X}(M): m\to\varphi_m(\xi)$ is a Lie-algebra morphism. The isotropy group $G_m < G$ of an element $m\in M$ is the subgroup of G determined by $\Psi(m,g)=m$. The Lie algebra of G_m is denoted by $\mathfrak{g}_m$. The orbit $\mathcal{O}_m$ of m is the subset of M consisting of the elements of the form $\Psi(m,g)$ with $g\in G$ arbitrary. Finally, we will sometimes consider the dual to φ_m, i.e., the map $\varphi_m^*: T_m^*M\to\mathfrak{g}^*$. With a slight abuse of notation, the symbol φ^* will also be used to map a 1-form to a $\mathfrak{g}^*$-valued function on M, pointwise defined by $\varphi^*(\alpha)(m)=\varphi_m^*(\alpha(m))$, with α a 1-form and $m\in M$ arbitrary.

We will often assume that the action on a manifold M is free and proper. This guarantees that the space of orbits M/G is a manifold and that the projection $\pi: M\to M/G$ is a principal fiber bundle.[8] We assume that the reader is familiar with the concept of associated bundles of a principal manifold and, in particular, the bundle $\tilde{\mathfrak{g}}$ associated with the Lie-algebra $\mathfrak{g}$ on which the group acts on the left by means of the adjoint action. The adjoint action of G on its Lie algebra $\mathfrak{g}$ is denoted by Ad_g and is defined as the differential at the identity of the conjugation mapping. The dual to the adjoint action is called the coadjoint action and is denoted by Ad_g^*, i.e., $Ad_g^*(\mu)\in\mathfrak{g}^*$ for $\mu\in\mathfrak{g}^*$. We denote elements in $\tilde{\mathfrak{g}}$ by $\tilde{\xi}$ and they represent orbits of points in $Q\times\mathfrak{g}$ under the action of G defined by $(q,\xi)\mapsto(qg,Ad_{g^{-1}}\xi)$ with $q\in Q$, $g\in G$, and $\xi\in\mathfrak{g}$ arbitrary. In this sense, we sometimes write $\tilde{\xi}=[q,\xi]_G$.

A principal connection on a manifold M on which G acts freely and properly is an equivariant $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on M such that, in addition, $\mathcal{A}(\xi_M)=\xi$ for all $\xi\in\mathfrak{g}$. The equivariance property is expressed by $\mathcal{A}_{\Psi_g(m)}(T\Psi_g(v_m))=Ad_{g^{-1}}(\mathcal{A}_m(v_m))$, for any $m\in M$, $v_m\in T_mM$, and $g\in G$. The kernel of $\mathcal{A}$ determines a G-invariant distribution on M which is called the *horizontal* distribution since it is complementary to the vertical distribution $V\pi=\ker T\pi$, with $\pi: M\to M/G$. In this paper, we will consider the dual of the linear map $\mathcal{A}_m: T_mM\to\mathfrak{g}$, which is understood to be a map $\mathcal{A}_m^*:\mathfrak{g}^*\to T_m^*M$. If $\mu\in\mathfrak{g}^*$, then the 1-form $\mathcal{A}^*(\mu): M\to T^*M$ is defined pointwise by $m\mapsto\mathcal{A}_m^*(\mu)$. Again, with a slight abuse of notation, we sometimes write $\mathcal{A}^*(\mu)=\mathcal{A}_\mu$.

Throughout the paper we encounter products of bundles over the same base manifold B, say $E_1\to B$ and $E_2\to B$. The fibered product $E_1\times_B E_2$ over the base manifold is often denoted simply by $E_1\times E_2$ and consists of pairs (e_1,e_2) with $e_1\in E_1$ and $e_2\in E_2$ such that e_1 and e_2 project onto the same point in B.

2. Symplectic reduction

Let (M,ω) be a symplectic manifold on which G acts freely on the right, $\Psi: M\times G\to M$. The action Ψ is *canonical* if $\Psi_g^*\omega=\omega$ for all $g\in G$. If the infinitesimal generators ξ_M are globally Hamiltonian vector fields, i.e., if there is a function J_ξ for any $\xi\in\mathfrak{g}$ such that $i_{\xi_M}\omega=-\mathrm{d}J_\xi$, then the map $J: M\to\mathfrak{g}^*$ is called a momentum map associated with the action.

Following Ref. 1, we define the *nonequivariance cocycle* associated with a momentum map of the canonical action,

$$\sigma: G \to \mathfrak{g}^*: g \mapsto J(mg^{-1}) - Ad^*_{g^{-1}}(J(m)),$$

where m is arbitrary in M. If M is connected, this definition is independent of the choice of the point m and determines a $\mathfrak{g}^*$-valued one-cocycle σ in G, i.e., for $g,h \in G$, it satisfies

$$\sigma(gh) = \sigma(g) + Ad^*_{g^{-1}}\sigma(h).$$

If M is not connected, we restrict the analysis to a connected component. Therefore, without further mentioning it, we will always assume that the manifolds we are considering are connected. Given another momentum map J' associated with the same action, its nonequivariance cocycle σ' determines the same element as σ in the first $\mathfrak{g}^*$-valued cohomology of G, i.e., $[\sigma]=[\sigma'] \in H^1(G,\mathfrak{g}^*)$. Note that for reasons of conformity, we haven chosen to define σ following Ref. 15 for left actions: recall that a right action composed with the group inversion is a left action.

If the moment map is not equivariant, one can show (see Ref. 16) that it becomes equivariant with respect to the *affine* action of G on $\mathfrak{g}^*$ determined using the cocycle σ and given by

$$(g,\mu) \mapsto Ad^*_g\mu + \sigma(g^{-1}).$$

Due to the fact that G acts freely on M—this is the only case we consider—any value of J is regular and, therefore, $J^{-1}(\mu)$ will be a submanifold of M for all $\mu \in J(M)$.[15]

Theorem 3: *(MW reduction) Let (M,ω) be a symplectic manifold with G acting freely, properly, and canonically on M. Let J be a momentum map for this action with nonequivariance cocycle σ. Assume that $\mu \in J(M)$, and denote by G_μ the isotropy of μ under the affine action of G on $\mathfrak{g}^*$. Then (M_μ,ω_μ), with $M_\mu = J^{-1}(\mu)/G_\mu$, is a symplectic manifold such that the 2-form ω_μ is uniquely determined by $i^*_\mu\omega = \pi^*_\mu\omega_\mu$, with $i_\mu: J^{-1}(\mu) \to M$ and $\pi_\mu: J^{-1}(\mu) \to M_\mu = J^{-1}(\mu)/G_\mu$.*

Let H denote a function on M, which is invariant under the action of G. Then, the Hamiltonian vector field X_H is tangent to $J^{-1}(\mu)$ and there exists a Hamiltonian h on M_μ with $\pi^*_\mu h = i^*_\mu H$ such that the restriction of X_H to $J^{-1}(\mu)$ is π_μ-related to X_h.

B. Cotangent bundle reduction

Consider now the case of a cotangent bundle T^*Q with its canonical symplectic structure $\omega_Q := d\theta_Q$, where θ_Q is the Cartan 1-form. [Let $\alpha \in T^*Q$, then $\theta_Q(\alpha)(X) = \langle \alpha, T\pi_Q(X)\rangle$ for arbitrary $X \in T_\alpha(T^*Q)$.] Let G be a Lie group acting freely and properly on Q from the right. Since a cotangent bundle is a special case of a symplectic manifold, the MW theorem obviously applies to T^*Q. However, because of the extra structure present on a cotangent bundle, much more can be said in this case than one would expect from the MW theorem (see Refs. 11 and 12).

The group G acts on Q by a right action Ψ and, hence, also on T^*Q by the cotangent lift of this action: $(g,\alpha) \mapsto T^*\Psi_{g^{-1}}(\alpha)$. The map $J := \varphi^*: T^*Q \to \mathfrak{g}^*$, defined by $\langle J(\alpha_q),\xi\rangle = \langle \alpha_q, \varphi_q(\xi)\rangle$, is a momentum map for this action. One can easily show that J is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, or in other words, $J \circ T^*\Psi_{g^{-1}} = Ad^*_g \circ J$.

Recall that we assume that the action of G is free and proper so that the quotient Q/G is a manifold. In this case, the quotient projection $\pi: Q \to Q/G$ defines a principal fiber bundle with structure group G. We denote the bundle of vertical vectors with respect to the projection π by $V\pi$. The subbundle $V^0\pi$ of T^*Q is defined as the annihilator of $V\pi$.

Fix a principal connection $\mathcal{A}$ on Q and let $\phi^\mu_{\mathcal{A}}$ be the map $J^{-1}(\mu) \to V^0\pi; \alpha_q \mapsto \phi^\mu_{\mathcal{A}}(\alpha_q) := \alpha_q - \mathcal{A}^*_q(\mu)$. This is an equivariant diffeomorphism with respect to the standard action of G_μ on $V^0\pi$, and its projection onto the quotient spaces is denoted by $[\phi^{\mathcal{A}}_\mu]: J^{-1}(\mu)/G_\mu \to V^0\pi/G_\mu$. The space $V^0\pi$ can be identified with $T^*(Q/G) \times Q$ and, consequently, the quotient space $V^0\pi/G_\mu$ can be identified with the product bundle $T^*(Q/G) \times Q/G_\mu$. We therefore conclude that the choice of a connection $\mathcal{A}$ allows us to identify $J^{-1}(\mu)/G_\mu$ with the bundle $T^*(Q/G) \times Q/G_\mu$ by means of the diffeomorphism $[\phi^{\mathcal{A}}_\mu]$.

$$\begin{tikzcd}
J^{-1}(\mu) \arrow[r, "\phi_\mu^{\mathcal{A}}"] \arrow[d, "\pi_\mu"'] & V^0\pi \arrow[d, "\pi_\mu^0"'] & \\
M_\mu \arrow[r, "{[\phi_\mu^{\mathcal{A}}]}"] & T^*(Q/G)\times Q/G_\mu \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Q/G_\mu \\
& T^*(Q/G) &
\end{tikzcd}$$

FIG. 1. Cotangent bundle reduction.

Next, the 1-form $\mathcal{A}_\mu$ [which is also denoted by $\mathcal{A}^*(\mu)$] determines a G_μ-invariant 1-form on Q. It is not hard to show that $d\mathcal{A}_\mu$ is a 2-form on Q, projectable to a 2-form $\mathcal{B}_\mu$ on Q/G_μ. This follows from the invariance under the action of G_μ and the annihilation of fundamental vector fields of the form ξ_Q with ξ in the Lie algebra $\mathfrak{g}_\mu$ of G_μ. In the following, we consider the 2-form on $T^*(Q/G)\times Q/G_\mu$ determined as the sum of

- the pullback to $T^*(Q/G)\times Q/G_\mu$ of $\omega_{Q/G}$ on $T^*(Q/G)$ and
- the pullback to $T^*(Q/G)\times Q/G_\mu$ of $\mathcal{B}_\mu$ on Q/G_μ.

Let π_1, π_2, and p_μ be the projections $\pi_1 : T^*(Q/G)\times Q/G_\mu \to T^*(Q/G)$, $\pi_2 : T^*(Q/G)\times Q/G_\mu \to Q/G_\mu$, and $p_\mu : Q \to Q/G_\mu$, respectively. We further denote the natural injection $V^0\pi \to T^*Q$ by i_0. The above mentioned 2-form on $T^*(Q/G)\times Q/G_\mu$ equals

$$\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu.$$

Theorem 4: *(Cotangent bundle reduction) Given a free and proper action of G on Q and consider its canonical lift to T^*Q. Let μ be any value of the momentum map, with isotropy subgroup G_μ. By fixing a principal connection $\mathcal{A}$, the symplectic manifold (M_μ,ω_μ) is symplectomorphic to $(T^*(Q/G)\times Q/G_\mu, \pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}_\mu)$ with symplectomorphism $[\phi_\mu^{\mathcal{A}}]$.*

We can summarize this in the diagram presented in Fig. 1.

Although this result is not new and can be found, for instance, in Refs. 11 and 12, we include a proof because its method will turn out to be useful later on.

Proof: We know that $[\phi_\mu^{\mathcal{A}}]$ is a diffeomorphism, and therefore it only remains to show that the symplectic 2-form $\pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}_\mu$ is pulled back to ω_μ under this map. We use the fact that ω_μ is uniquely determined by $i_\mu^*\omega_Q = \pi_\mu^*\omega_\mu$, with $i_\mu : J^{-1}(\mu) \to T^*Q$ as the natural inclusion and $\pi_\mu : J^{-1}(\mu) \to M_\mu$ as the projection to the quotient space. Due to the uniqueness property, it is therefore sufficient to show that

$$\pi_\mu^*([\phi_\mu^{\mathcal{A}}]^*(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu)) = i_\mu^*\omega_Q. \tag{2}$$

We will slightly reformulate this condition by using the fact that $i_\mu^*\theta_Q = (\phi_\mu^{\mathcal{A}})^*(i_0^*(\theta_Q + \pi_Q^*\mathcal{A}_\mu))$ and $[\phi_\mu^{\mathcal{A}}]\circ\pi_\mu = \pi_\mu^0\circ\phi_\mu^{\mathcal{A}}$,

$$(\pi_\mu^0)^*(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu) = d i_0^*(\theta_Q + \pi_Q^*\mathcal{A}_\mu). \tag{3}$$

The latter equality follows easily from the properties of the maps involved: we have that (i) $(\pi_2\circ\pi_\mu^0)^*\mathcal{B}_\mu = (\pi_Q\circ i_0)^* d\mathcal{A}_\mu$ and (ii) $(\pi_1\circ\pi_\mu^0)^*\theta_{Q/G} = i_0^*\theta_Q$ hold. □

The above description of cotangent bundle reduction can be seen as a special case of the more general result, stating that if two symplectomorphic manifolds are both MW reducible for the same symmetry group and have compatible actions, then the reduced spaces are also symplectomorphic. More specifically, given two symplectic manifolds (P,Ω) and (P',Ω') and a symplectomorphism $f : P \to P'$, i.e., $f^*\Omega' = \Omega$. We assume in addition that both P and P' are equipped with a canonical free and proper action of G. Let $J : P \to \mathfrak{g}^*$ and $J' : P' \to \mathfrak{g}^*$ denote the corresponding momentum maps for these actions on P and P', respectively. We say that f is equivariant if

$f(pg)=f(p)g$ for arbitrary $p\in P$, $g\in G$. Note that the nonequivariance cocycles for J and J' are equal up to a coboundary. Without loss of generality, we assume $f^*J'=J$ and that the nonequivariance cocycles coincide. This, in turn, guarantees that the affine actions on $\mathfrak{g}^*$ coincide and that the isotropy group of an element $\mu\in\mathfrak{g}^*$ coincides for both affine actions. Finally, fix a value $\mu\in\mathfrak{g}^*$ of both J and J'.

Theorem 5: *If f is an equivariant symplectic diffeomorphism $P\to P'$ such that $J'=J\circ f$, then under MW reduction, the symplectic manifolds (P_μ,Ω_μ) and (P'_μ,Ω'_μ) are symplectically diffeomorphic under the map*

$$[f_\mu]:P_\mu\to P'_\mu;[p]_{G_\mu}\mapsto[f(p)]_{G_\mu}.$$

Proof: This is a straightforward result. Since f is a diffeomorphism for which $J'=J\circ f$, the restriction f_μ of f to $J^{-1}(\mu)$ determines a diffeomorphism from $J^{-1}(\mu)$ to $J'^{-1}(\mu)$. The equivariance implies that f_μ reduces to a diffeomorphism $[f_\mu]$ from $P_\mu=J^{-1}(\mu)/G_\mu$ to $P'_\mu=J'^{-1}(\mu)/G_\mu$. It is our purpose to show that $[f_\mu]^*\Omega'_\mu=\Omega_\mu$ or, since both π_μ and π'_μ are projections, that $\pi^*_\mu\Omega_\mu=f^*_\mu(\pi'^*_\mu\Omega'_\mu)$. The determining property for Ω_μ and Ω'_μ is $\pi^*_\mu\Omega_\mu=i^*_\mu\Omega$ (similar to Ω'_μ). From diagram chasing, we have that $i^*_\mu\Omega=f^*_\mu(i'^*_\mu\Omega')$. Then,

$$\pi^*_\mu\Omega_\mu=i^*_\mu\Omega=f^*_\mu(i'^*_\mu\Omega')=f^*_\mu(\pi'^*_\mu\Omega'_\mu)=\pi^*_\mu([f_\mu]^*\Omega'_\mu)$$

since by definition $\pi'_\mu\circ f_\mu=[f_\mu]\circ\pi_\mu$. This concludes the proof. □

C. Tangent bundle reduction

We start by recalling the symplectic formulation of Lagrangian systems on the tangent bundle TQ of a manifold Q and its relation to the canonical symplectic structure on T^*Q through the Legendre transform. Next, we shall consider Lagrangians invariant under the action of G and study a general MW reduction scheme for such systems.

Definition 1: A Lagrangian system is a pair (Q,L), where Q is called the configuration manifold and L is a smooth function on TQ. A Lagrangian system (Q,L) is said to be regular if the fiber derivative $\mathbb{F}L:TQ\to T^*Q;v_q\mapsto\mathbb{F}L(v_q)$ is a diffeomorphism. The map $\mathbb{F}L$ is called the Legendre transformation and is defined by

$$\langle\mathbb{F}L(v_q),w_q\rangle=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}L(v_q+\epsilon w_q)$$

for arbitrary $v_q,w_q\in T_qQ$.

Definition 2: Given a free and proper action Ψ of G on Q, then a Lagrangian system (Q,L) is said to be invariant if L is an invariant function for the lifted action $(v_q,g)\mapsto T\Psi_g(v_q)$.

Given a regular Lagrangian system (Q,L), one can define a symplectic structure on TQ by using the Legendre transform: we denote the 2-form on TQ obtained by pulling back ω_Q under $\mathbb{F}L$, by $\Omega^L_Q=(\mathbb{F}L)^*\omega_Q$. We will only consider regular Lagrangians throughout this paper. The following results are standard.

Theorem 6: *The lifted action $T\Psi$ of G on TQ is a canonical action for the symplectic manifold (TQ,Ω^L_Q). A momentum map is given by $J_L=J\circ\mathbb{F}L:TQ\to\mathfrak{g}^*$, and J_L is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$. Furthermore, the Legendre transformation is an equivariant symplectomorphism between the symplectic manifolds (TQ,Ω^L_Q) and (T^*Q,ω_Q).*

The above theorem guarantees that Theorem 5 is applicable. We are now ready to draw the diagram in Fig. 2 with $\mu\in\mathfrak{g}^*$.

Next, we will show that the manifold $J_L^{-1}(\mu)/G_\mu$ is diffeomorphic to the fiber product $T(Q/G)\times Q/G_\mu$ if L satisfies an additional regularity assumption. Lagrangians satisfying this condition are called *G -regular*. We shall compute the map $[\phi^{\mathcal{A}}_\mu]\circ[\mathbb{F}L_\mu]$ and show that it coincides with a Legendre transform for a function defined on $T(Q/G)\times Q/G_\mu$. This fact will eventually allow us to show that the reduced symplectic spaces are again originating from a Lagrangian

$$\begin{tikzcd}
(TQ,\Omega_Q^L) \arrow[r, "\mathbb{F}L"] \arrow[d, "\text{MW-red}"'] & (T^*Q,\omega_Q) \arrow[d, "\text{MW-red}"'] & \\
(J_L^{-1}(\mu)/G_\mu,\Omega_\mu) \arrow[r, "{[\mathbb{F}L_\mu]}"] & (J^{-1}(\mu)/G_\mu,\omega_\mu) \arrow[r, "{[\phi_\mu^{\mathcal{A}}]}"] & (T^*(Q/G)\times Q/G_\mu,\pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}_\mu)
\end{tikzcd}$$

FIG. 2. Diagram relating tangent and cotangent reduction.

system on Q/G_μ. The Lagrangian of this "reduced" Lagrangian system is precisely the Routhian known from classical Routhian reduction.

We use the fixed connection $\mathcal{A}$ on Q to identify TQ/G with the bundle $T(Q/G)\times\tilde{\mathfrak{g}}$ in the standard way. This identification is obtained as follows: let $[v_q]_G\in TQ/G$ be arbitrary and fix a representative $v_q\in TQ$. The image in $T(Q/G)\times\tilde{\mathfrak{g}}$ of $[v_q]_G$ is defined as the element $(T\pi(v_q),\tilde{\xi})$ with $\pi:Q\to Q/G$ and $\tilde{\xi}=[q,\mathcal{A}(v_q)]_G\in\tilde{\mathfrak{g}}$. This map is invertible and determines a diffeomorphism (see, for instance, Ref. 3). To define the inverse: let $(v_x,\tilde{\xi})$ be arbitrary in $T(Q/G)\times\tilde{\mathfrak{g}}$, and consider the tangent vector $v_q=(v_x)_q^h+\varphi_q(\xi)$ at $q\in\pi^{-1}(x)$, with $(v_x)_q^h$ the horizontal lift determined by $\mathcal{A}$ and ξ such that $\tilde{\xi}=[q,\xi]_G$. The inverse of $(v_x,\tilde{\xi})$ is the orbit $[v_q]_G\in TQ/G$ (the latter is well defined: one can show that it is independent of the point q, see also Ref. 3).

Completely analogous, one can show that TQ/G_μ is diffeomorphic to $T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$. Indeed, let $[v_q]_{G_\mu}\in TQ/G_\mu$ be arbitrary and fix a representative $v_q\in TQ$, then the image of $[v_q]_{G_\mu}$ is defined by $(v_x,p_\mu(q),\tilde{\xi})$, with $T\pi(v_q)=v_x$ and $\tilde{\xi}=[q,\mathcal{A}(v_q)]_G$ (recall that $p_\mu:Q\to Q/G_\mu$). The construction of the inverse map uses the previous diffeomorphism and is consist of three steps. Let $(v_x,y,\tilde{\xi})\in T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$ be arbitrary. First, we consider the element $[v_q]_G$ in TQ/G, which is the inverse of $(v_x,\tilde{\xi})\in T(Q/G)\times\tilde{\mathfrak{g}}$. Second, we take a representative v_q of $[v_q]_G$ at a point $q\in p_\mu^{-1}(y)$. Finally, we consider $[v_q]_{G_\mu}$. It is not hard to show that this inverse is well defined (i.e., independent of the chosen representative v_q).

An invariant Lagrangian L determines a function on the quotient TQ/G, and under the identification determined above, a function l on $T(Q/G)\times\tilde{\mathfrak{g}}$. We define the fiber derivative $\mathbb{F}_{\tilde{\xi}}l:T(Q/G)\times\tilde{\mathfrak{g}}\to T(Q/G)\times\tilde{\mathfrak{g}}^*$ by

$$\langle\mathbb{F}_{\tilde{\xi}}l(v_x,\tilde{\xi}),(v_x,\tilde{\eta})\rangle \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} l(v_x,\tilde{\xi}+\epsilon\tilde{\eta}).$$

Definition 3: An invariant Lagrangian L is said to be G-regular if the map $\mathbb{F}_{\tilde{\xi}}l:T(Q/G)\times\tilde{\mathfrak{g}}\to T(Q/G)\times\tilde{\mathfrak{g}}^*$ *is a diffeomorphism.*

We remark here that according to the previous definition, G-regularity depends on the chosen connection $\mathcal{A}$. However, we mention here that G-regularity can alternatively be defined as a condition on L directly. We refer the reader to Ref. 9 for a detailed discussion on G-regularity.

A momentum value μ determines in the quotient spaces a mapping $\tilde{\mu}:Q/G_\mu\to\tilde{\mathfrak{g}}^*$ as follows: let $y\in Q/G_\mu$ be arbitrary

$$\langle\tilde{\mu}(y),\tilde{\xi}\rangle=\langle\mu,\xi\rangle$$

with ξ the unique representative of $\tilde{\xi}=[q,\xi]_G$ at a point $q\in p_\mu^{-1}(y)$. Recall that p_μ denotes the projection $p_\mu:Q\to Q/G_\mu$. Due to the identification $TQ/G_\mu\cong T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$ the manifold $J_L^{-1}(\mu)/G_\mu$ is a subset of $T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$. In the following lemma, we characterize this subset in terms of $\tilde{\mu}$ and $\mathbb{F}_{\tilde{\xi}}l$.

Lemma 1: There is a one-to-one correspondence between $J_L^{-1}(\mu)/G_\mu$ *and the subset of* $T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$ *determined as the set of points* $(v_x,y,\tilde{\xi})$ *that satisfy the condition* $\mathbb{F}_{\tilde{\xi}}l(v_x,\tilde{\xi})=(v_x,\tilde{\mu}(y))$.

Proof: Consider a point $[v_q]_{G_\mu}$ in $J_L^{-1}(\mu)/G_\mu$ and let v_q be a representative. Then, by definition, $J_L(v_q)=\mu$, and since L is invariant, we have $L(v_q)=l(v_x,\tilde{\xi})$, with $(v_x,\tilde{\xi})$ the element in

$$\begin{tikzcd}
(TQ,\Omega_Q^L) \arrow[r, "\mathbb{F}L"] \arrow[d, "\text{MW-red}"'] & (T^*Q,\omega_Q) \arrow[d, "\text{MW-red}"'] \\
(T(Q/G)\times Q/G_\mu,\tilde{\Omega}_\mu) \arrow[r, "{[\phi_\mu^{\mathcal{A}}]\circ[\mathbb{F}L_\mu]\circ\psi_\mu}"] & (T^*(Q/G)\times Q/G_\mu,\pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}_\mu)
\end{tikzcd}$$

FIG. 3. Diagram relating tangent and cotangent reduction for G-regular Lagrangians.

$TQ/G \cong T(Q/G)\times\tilde{\mathfrak{g}}$ corresponding to $[v_q]_G$. Using the definition of the momentum map J_L, we obtain

$$\langle J_L(v_q),\eta\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_0 L(v_q+\epsilon\varphi_q(\eta)) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_0 l(v_x,\tilde{\xi}+\epsilon\tilde{\eta}) = \langle \mathbb{F}_{\tilde{\xi}}l(v_x,\tilde{\xi}),(v_x,\tilde{\eta})\rangle$$

with $\tilde{\eta}=[q,\eta]_G$. □

Lemma 2: Let L be G -regular invariant Lagrangian. Then, there is a diffeomorphism between $J_L^{-1}(\mu)/G_\mu$ and $T(Q/G)\times Q/G_\mu$.

Proof: We define a map $J_L^{-1}(\mu)/G_\mu \to T(Q/G)\times Q/G_\mu$ and its inverse. Let $[v_q]_{G_\mu} \in J_L^{-1}(\mu)/G_\mu$, with $v_q \in J_L^{-1}(\mu)$ a representative at q. We again use the fixed connection $\mathcal{A}$ on Q, and we introduce the maps,

$$p_1([v_q]_{G_\mu}) \coloneqq T\pi(v_q), \quad p_2([v_q]_{G_\mu}) \coloneqq p_\mu(q), \quad p_3([v_q]_{G_\mu}) \coloneqq [q,\mathcal{A}(v_q)]_G$$

with $\pi: Q\to Q/G$ and $p_\mu: Q\to Q/G_\mu$. These maps $p_{1,2,3}$ are simply the restrictions to $J_L^{-1}(\mu)/G_\mu$ of the projections on the first, second, and third factors in the product $T(Q/G)\times Q/G_\mu\times\tilde{\mathfrak{g}}$. It is easily verified that $(p_1,p_2): J_L^{-1}(\mu)/G_\mu \to T(Q/G)\times Q/G_\mu$ is smooth.

We now define the inverse map ψ_μ of (p_1,p_2). Let $(v_x,y)\in T(Q/G)\times Q/G_\mu$ be arbitrary and define the element $\tilde{\xi}\in\tilde{\mathfrak{g}}_x$ such that $(v_x,\tilde{\xi})=(\mathbb{F}_{\tilde{\xi}}l)^{-1}(v_x,\tilde{\mu}(y))$ (here, we use the condition that L is G-regular). Now consider the tangent vector $v_q=(v_x)_q^h+\varphi_q(\xi)$, where ξ is such that $\tilde{\xi}=[q,\xi]_G$ and $q\in p_\mu^{-1}(y)$. By construction, we have, on the one hand, that $J_L(v_q)=\mu$ and, on the other hand, $(p_1,p_2)([v_q]_{G_\mu})=(v_x,y)$. □

Combined with Fig. 2, we can now draw the diagram in Fig. 3 below.

There is an interesting local criterion for a Lagrangian L to be G-regular. We say that L is *locally* G-regular if given a point $[v_q]_{G_\mu}$ in $J_L^{-1}(\mu)/G_\mu$, there is a neighborhood U of $[v_q]_{G_\mu}$ such that the restriction $(p_1,p_2)|_U$ is a diffeomorphism from U to its image $(p_1,p_2)(U)$.

Lemma 3: An invariant Lagrangian L is locally G-regular if one of the following two equivalent conditions hold:

(1) $T(J_L^{-1}(\mu))\oplus V_{J_L^{-1}(\mu)}\varphi = T_{J_L^{-1}(\mu)}(TQ)$, *with* $V\varphi\subset V\tau_Q\subset T(TQ)$ *defined as the set of tangent vectors of the form* $(\varphi_q(\xi))_{v_q}^v=\xi_Q^v(v_q)$, $\xi\in\mathfrak{g}$ *arbitrary, where* ξ_Q *is the fundamental vector field of the action on* Q *corresponding to* ξ *and* v *denotes the vertical lift* $TQ\times TQ\to T(TQ)$.

(2) *The "vertical" Hessian of* l, *defined as*

$$D^2 l(v_x,\tilde{\xi})(\tilde{\eta},\tilde{\eta}') \coloneqq \frac{\partial^2 l}{\partial\epsilon\,\partial\epsilon'}(v_x,\tilde{\xi}+\epsilon\tilde{\eta}+\epsilon'\tilde{\eta}')\big|_{\epsilon=\epsilon'=0}$$

for $v_x\in T(Q/G)$ *and* $\tilde{\xi},\tilde{\eta},\tilde{\eta}'\in\tilde{\mathfrak{g}}_x$ *is invertible.*

Proof: Note that for all $v_q\in J_L^{-1}(\mu)$,

$$\dim T_{v_q}(J_L^{-1}(\mu)) + \dim V\varphi(v_q) = (\dim TQ - \dim\mathfrak{g}) + \dim\mathfrak{g} = \dim T_{v_q}(TQ).$$

The direct-sum decomposition in (1) is therefore equivalent to the statement that $T(J_L^{-1}(\mu))\cap V_{J_L^{-1}(\mu)}\varphi=0$. We will now prove that this is equivalent to the vertical Hessian of l being invertible.

Assume that the intersection $T(J_L^{-1}(\mu))\cap V\varphi$ contains a nonzero element. Such an element is necessarily of the form $(\xi_Q)^v(v_q)$, where $\xi\in\mathfrak{g}$ and $\xi\neq 0$. Expressing the fact that this element is contained in $T(J_L^{-1}(\mu))$ implies that for every $\eta\in\mathfrak{g}$, $\langle TJ_L((\xi_Q)^v(v_q)),\eta\rangle=0$. This can be made more explicit as follows:

$$\langle TJ_L((\xi_Q)^v(v_q)),\eta\rangle=\frac{\mathrm{d}}{\mathrm{d}s}J_L(v_q+s\varphi_q(\xi))(\eta)|_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{F}_{\tilde{\xi}}l(v_x,\tilde{\zeta}+s\tilde{\xi})(\tilde{\eta})|_{s=0}=D^2l(v_x,\tilde{\zeta})(\tilde{\xi},\tilde{\eta}),$$

where we have decomposed v_q in its vertical and horizontal parts as $v_q=(v_x)_q^h+\varphi_q(\zeta)$. Since this holds for every $\eta\in\mathfrak{g}$, we conclude that $(\xi_Q)^v(v_q)$ is contained in the intersection $T(J_L^{-1}(\mu))\cap V_{J_L^{-1}(\mu)}\varphi$ if and only if the associated section $\tilde{\xi}$ is in the null space of $D^2l(v_x,\tilde{\zeta})$. Hence, the two statements in Lemma 3 are equivalent.

If $D^2l(v_x,\tilde{\zeta})$ is invertible, then via the implicit function theorem, the reduced Legendre transformation is locally invertible. The method of proof of the previous Lemma 2 can be used to show that, locally, (p_1,p_2) is invertible. □

Note in passing that if the given L is a mechanical Lagrangian, i.e., it is of type kinetic minus potential, then L is G-regular if the locked inertia tensor, defined by the restriction of the kinetic energy metric to the fundamental vector fields (see e.g., Ref. 13), is nondegenerate. In our language, the reduced locked inertia tensor coincides with $D^2l_x:\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}^*$ in the following sense:

$$D^2l_x(\tilde{\xi},\tilde{\eta})=J(q)(\xi,\eta)$$

with $\tilde{\xi}=[q,\xi]_G$ and $\tilde{\eta}=[q,\eta]_G$ arbitrary.

III. ROUTHIAN REDUCTION

In this section, we make a start with Routhian reduction. We consider a Lagrangian $L:TQ\to\mathbb{R}$, which is invariant under the action of a Lie group G and as before we consider a connection $\mathcal{A}$ in the bundle $\pi:Q\to Q/G$. Furthermore, let $\mu\in\mathfrak{g}^*$ be a fixed momentum value, and define the function R^μ as $R^\mu=L-\mathcal{A}_\mu$ (recall that $\mathcal{A}_\mu:TQ\to\mathbb{R}$ is the connection 1-form contracted with $\mu\in\mathfrak{g}^*$). By definition, R^μ is G_μ-invariant and, in particular, its restriction to $J_L^{-1}(\mu)$ is reducible to a function $[R^\mu]$ on the quotient $J_L^{-1}(\mu)/G_\mu$. In turn, we denote the function on $T(Q/G)\times Q/G_\mu$ corresponding to $[R^\mu]$ by $\mathcal{R}^\mu$, i.e., $\mathcal{R}^\mu=\psi_\mu^*[R^\mu]$. The function $\mathcal{R}^\mu$ is called the *Routhian*.

We begin by reconsidering some aspects from the reduction theory of tangent bundles, which we relate to the geometry of the Routhian. Recall from the diagram in Fig. 3 that we may write the symplectic 2-form $\tilde{\Omega}_\mu$ obtained from MW reduction as

$$([\phi_\mu^{\mathcal{A}}]\circ[\mathbb{F}L_\mu]\circ\psi_\mu)^*(\pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}_\mu).$$

Lemma 4: The map $[\phi_\mu^{\mathcal{A}}]\circ[\mathbb{F}L_\mu]\circ\psi_\mu$ is the fiber derivative of the Routhian $\mathcal{R}^\mu$, i.e., for $(v_x,y),(w_x,y)\in T(Q/G)\times Q/G_\mu$ arbitrary

$$\langle([\phi_\mu^{\mathcal{A}}]\circ[\mathbb{F}L_\mu]\circ\psi_\mu)(v_x,y),(w_x,y)\rangle=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\mathcal{R}^\mu(v_x+\epsilon w_x,y)=:\langle\mathbb{F}\mathcal{R}^\mu(v_x,y),(w_x,y)\rangle.$$

Proof: Fix elements $(v_x,y)\in T(Q/G)\times Q/G_\mu$ and fix a $v_q\in J_L^{-1}(\mu)$ that projects onto $\psi_\mu(v_x,y)$. By definition of the maps involved, we have

$$([\phi_\mu^{\mathcal{A}}]\circ[\mathbb{F}L_\mu]\circ\psi_\mu)(v_x,y)=(\pi_\mu^0\circ\phi_\mu^{\mathcal{A}})(\mathbb{F}L(v_q))=\pi_\mu^0(\mathbb{F}L(v_q)-\mathcal{A}_\mu(q)).$$

Fix a curve $\epsilon\mapsto\zeta(\epsilon)$ in $J_L^{-1}(\mu)$ that projects onto the curve $\epsilon\mapsto\psi_\mu(v_x+\epsilon w_x,y)$ in $J_L^{-1}(\mu)/G_\mu$ and such that $\zeta(0)=v_q$ and $\dot{\zeta}(0)$ is vertical to the projection $\tau_Q\circ i_\mu:J_L^{-1}(\mu)\to Q$. The existence of such a curve is best shown using Lemma 3 and some coordinate computations. For that purpose, fix a bundle adapted coordinate chart on $Q\to Q/G$, and let (x^i,g^a) denote the coordinate functions with $i=1,\dots,\dim Q/G$ and $a=1,\dots,\dim G$. From Lemma 3, where it was shown that $TJ_L^{-1}(\mu)$ is transversal to $V\varphi$, we deduce that (x^i, v^i, g^a) are (local) coordinate functions for $J_L^{-1}(\mu)$, with (x^i, v^i) a standard coordinate chart on $T(Q/G)$ associated to (x^i) on Q/G. In this coordinate chart we put $v_q=(x_0^i, v_0^i, g_0^a)$ and $w_x=(x_0^i, w_0^i)$, and we define the curve $\zeta(\epsilon)$ to be the curve $\epsilon \mapsto (x_0^i, v_0^i + \epsilon w_0^i, g_0^a)$. Then the tangent to ζ at $\epsilon=0$ is the vertical lift of some $w_q \in T_qQ$ with $T\pi(w_q)=w_x$.

Finally, from the definition of $\mathcal{R}^\mu$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{R}^\mu(v_x + \epsilon w_x, y) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} (L(\zeta(\epsilon)) - \mathcal{A}_\mu(\zeta(\epsilon))) = \langle \mathbb{F}L(v_q) - \mathcal{A}_\mu(q), w_q \rangle.$$

Since $\mathbb{F}L(v_q) - \mathcal{A}_\mu(q) \in V^0\pi$, the right-hand side of this equation can be rewritten as a contraction with (w_x, y),

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathcal{R}^\mu(v_x + \epsilon w_x, y) = \langle \pi_\mu^0(\mathbb{F}L(v_q) - \mathcal{A}_\mu(q)), (w_x, y) \rangle.$$

This concludes the proof. □

The above lemma allows us to compute the reduced symplectic 2-form on the manifold $T(Q/G) \times Q/G_\mu$,

$$(\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^*\omega_{Q/G} + \pi_2^*\mathcal{B}_\mu) = (\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^*\omega_{Q/G}) + \bar{\pi}_2^*\mathcal{B}_\mu$$

with $\bar{\pi}_2 : T(Q/G) \times Q/G_\mu \to Q/G_\mu$. In order to complete the symplectic reduction, we now study the energy function (this is the Hamiltonian function for the Euler–Lagrange equations). Recall that the energy E_L corresponding with the Lagrangian system (Q, L) is the function on TQ defined by $E_L(v_q) = \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q)$, for $v_q \in TQ$ arbitrary. The energy for the Routhian $\mathcal{R}^\mu$ is defined by

$$E_{\mathcal{R}^\mu}(v_x, y) = \langle \mathbb{F}\mathcal{R}^\mu(v_x, y), (v_x, y) \rangle - \mathcal{R}^\mu(v_x, y)$$

with $(v_x, y) \in T(Q/G) \times Q/G_\mu$ arbitrary.

Lemma 5: The energy $E_{\mathcal{R}^\mu}$ is the reduced Hamiltonian, i.e., it satisfies

$$((p_1, p_2) \circ \pi_\mu)^* E_{\mathcal{R}^\mu} = i_\mu^* E_L$$

with $\pi_\mu : J_L^{-1}(\mu) \to J_L^{-1}(\mu)/G_\mu$ and $i_\mu : J_L^{-1}(\mu) \to TQ$.

Proof: Let $v_q \in J_L^{-1}(\mu)$ such that $((p_1, p_2) \circ \pi_\mu)(v_q) = (v_x, y)$. Then,

$$\begin{aligned} i_\mu^* E_L(v_q) &= \langle \mathbb{F}L(v_q), v_q \rangle - L(v_q) \\ &= \langle \phi_\mu^{\mathcal{A}}(\mathbb{F}L_\mu(v_q)) + \mathcal{A}_q^*(\mu), v_q \rangle - L(v_q) \\ &= \langle ([\phi_\mu^{\mathcal{A}}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu)(v_x, y), (v_x, y) \rangle - \mathcal{R}^\mu(v_x, y). \end{aligned}$$

Using Lemma 4, this concludes the proof. □

We end this section with some additional definitions in order to interpret the MW reduced system as a Lagrangian system (we also refer to Ref. 9). For that purpose, consider a manifold M fibered over N with projection $\kappa : M \to N$. Roughly said, a Lagrangian L with configuration space M is said to be *intrinsically constrained* if it does not depend on the velocities of the fiber coordinates of $\kappa : M \to N$. This is made more precise in the following definition.

Definition 4: A Lagrangian system (M, L) on a fibered manifold $\kappa : M \to N$ is intrinsically constrained if L is the pullback of a function L' on $T_M N = TN \times_N M$ along the projection $TM \to T_M N$.

For notational simplicity, we will identify L with L'. If we fix a coordinate neighborhood (x^i, y^a) on M adapted to the fibration, we can write the Euler–Lagrange equations for this system. The fact that the Lagrangian is intrinsically constrained is locally expressed by the fact that $L(x, \dot{x}, y)$ is independent of $\dot{y}$, and the Euler–Lagrange equations then read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, n \quad \text{and} \quad \frac{\partial L}{\partial y^a} = 0, \quad a = 1, \ldots, k.$$

The latter k equations determine the constraints on the system. We now wish to write these equations as Hamiltonian equations with respect to a presymplectic 2-form on $T_M N$. For that purpose, we associate with the Lagrangian $L: T_M N \to \mathbb{R}$ a *Legendre transform* $\mathbb{F}L: T_M N \to T_M^* N$. The definition is given by, for $(v_n, m), (w_n, m) \in T_M N$ arbitrary

$$\langle \mathbb{F}L(v_n,m),(w_n,m)\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L(v_n + \epsilon w_n, m),$$

In the coordinates, $\mathbb{F}L(x^i, \dot{x}^i, y^a)$ simply reads as $(x^i, \partial L/\partial \dot{x}^i, y^a)$. Finally, if we write the projection $T_M^* N \to T^* N; (\alpha_n, m) \to \alpha_n$ by κ_1, then it is not hard to show that the pullback to $T_M N$ of the canonical symplectic form ω_N under the map $\kappa_1 \circ \mathbb{F}L: T_M N \to T^* N$ determines a presymplectic 2-form, locally equal to

$$\mathrm{d}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \wedge \mathrm{d}x^i.$$

We define the energy as the function

$$E_L: T_M N \to \mathbb{R}; (v_n, m) \to \langle \mathbb{F}L(v_n,m),(v_n,m)\rangle - L,$$

and the solutions $m(t)$ to the Euler–Lagrange equations solve the equation

$$(i_{\dot{\gamma}}(\kappa_1 \circ \mathbb{F}L)^* \omega_N = -\mathrm{d}E_L)|_\gamma$$

with $\gamma(t) = (\dot{n}(t), m(t))$ and $n(t) = \kappa(m(t))$ (see also Refs. 5 and 6).

If the original intrinsically constrained Lagrangian system (M, L) is nonconservative with a *gyroscopic force term*, i.e., a 2-form β on M is given and the force term is the function $TM \to T^* M; v_m \mapsto -i_{v_m}\beta_m$, then the Euler–Lagrange equations of motion are Hamiltonian with respect (pre)symplectic form $(\kappa_1 \circ \mathbb{F}L)^* \omega_N + \kappa_2^* \beta$ and with Hamiltonian E_L,

$$(i_{\dot{\gamma}}((\kappa_1 \circ \mathbb{F}L)^* \omega_N + \kappa_2^* \beta) = -\mathrm{d}E_L)|_\gamma.$$

Here, κ_2 denotes the projection to the second factor in $T_M N$, i.e., $\kappa_2: T_M N \to M$. In the case of Routhian reduction, the reduced space is of this type: the total space corresponds to Q/G_μ and the base space N to Q/G.

Theorem 7: *Given a G-invariant, G -regular Lagrangian L defined on the configuration space Q. Then, the MW reduction in the symplectic manifold (Q, Ω_L) for a momentum value $J_L = \mu$ is the symplectic manifold*

$$(T(Q/G) \times Q/G_\mu, (\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^* \omega_{Q/G}) + \bar{\pi}_2^* \mathcal{B}_\mu).$$

The reduced Hamiltonian of E_L is the energy $E_{\mathcal{R}^\mu}$. The equations of motion for this Hamiltonian vector field are precisely the Euler–Lagrange equations of motion for an intrinsically constrained Lagrangian system on $Q/G_\mu \to Q/G$ with Lagrangian $\mathcal{R}^\mu$ and gyroscopic force term determined by the 2-form $\mathcal{B}_\mu$ on Q/G_μ.

It is remarkable that the 2-form $\mathcal{B}_\mu$ is such that the presymplectic 2-form $(\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^* \omega_{Q/G}) + \bar{\pi}_2^* \mathcal{B}_\mu)$ is symplectic. A next step in Routhian reduction would be to identify $\mathcal{B}_\mu$ as a 2-from which is built up amongst others out of the curvature of $\mathcal{A}$ and a nondegenerate part on the fibers of $Q/G_\mu \to Q/G$. Since this is not the scope of this paper, we refer the reader to Refs. 9 and 13.

IV. QUASI-INVARIANT LAGRANGIANS

In this section we study a possible generalization of the Routhian reduction procedure to quasi-invariant Lagrangians. We refer the reader to Ref. 10 and references therein for further details on quasi-invariant Lagrangians. We assume throughout this section that Q is a connected manifold, which ensures that given a function f for which $\mathrm{d}f=0$ implies that f is constant.

A. Quasi-invariance and cocycles

We begin by defining what it means for a Lagrangian to be quasi-invariant under a group action. We then show that the transformation behavior of a quasi-invariant Lagrangian induces a certain cocycle on the space of 1-forms, and we study the properties of this cocycle.

Definition 5: A Lagrangian system (Q,L) *is quasi-invariant if the Lagrangian satisfies*

$$(T\Psi_g)^* L(v_q) = L(v_q) + \langle v_q, \mathrm{d}F_g(q)\rangle$$

with v_q *arbitrary and for some function* $F: G\times Q \to \mathbb{R}$. *We denote a quasi-invariant Lagrangian system as a triple* (Q,L,F).

Clearly, the function F is not arbitrary: from the fact that Ψ defines a right action, it follows that $(T\Psi_{gh})^* L = ((T\Psi_g)^* \circ (T\Psi_h)^*)L$ and one can see that $\mathrm{d}F: G \to \mathcal{X}^*(Q)$ should define a group 1-cocycle with values in the G-module of 1-forms on Q, i.e., for $g_1, g_2 \in G$ arbitrary

$$\Psi^*_{g_1}\mathrm{d}F_{g_2} - \mathrm{d}F_{g_1 g_2} + \mathrm{d}F_{g_1} = 0.$$

Consider the map $f: \mathfrak{g}\times Q \to \mathbb{R}$ defined by

$$f(\xi,q) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} F(\exp \epsilon\xi, q).$$

Clearly, f is linear in its first argument, and thus determines a map $Q \to \mathfrak{g}^*$ which is denoted by the same symbol. We now define a 1-cocycle with values in $\mathfrak{g}^*$.

Lemma 6: The map

$$\sigma_F : G \to \mathfrak{g}^* : g \mapsto Ad^*_{g^{-1}} f(q) - (\Psi^*_{g^{-1}} f)(q) + Ad^*_{g^{-1}}(\varphi^*_q(\mathrm{d}F_{g^{-1}}(q))).$$

does not depend on the chosen point q and determines a group 1-cocycle with values in $\mathfrak{g}^*$.

Proof: We first show that the differential of

$$q \mapsto f_{Ad_g\xi}(q) - \Psi^*_g f_\xi(q) + \langle (Ad_g\xi)_Q(q), \mathrm{d}F_g(q)\rangle$$

vanishes for arbitrary $\xi \in \mathfrak{g}$. This implies that the above definition of σ_F does not depend on the chosen point q.

We start from the cocycle property of the map $g \mapsto \mathrm{d}F_g$, i.e., we have $\Psi^*_{g_1}\mathrm{d}F_{g_2} - \mathrm{d}F_{g_1g_2} + \mathrm{d}F_{g_1} = 0$. Let $g_1 = g$ and $g_2 = \exp \epsilon\xi$, and take the derivative at $\epsilon = 0$, then

$$\Psi^*_g \mathrm{d}f_\xi - \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{d}F_{g \exp \epsilon\xi} = 0.$$

To compute the second term, we again use the cocycle property with $g_1 = g(\exp \epsilon\xi)g^{-1}$, $g_2 = g$, i.e., $\mathrm{d}F_{g\exp\epsilon\xi} = \mathrm{d}F_{(\exp \epsilon Ad_g\xi)g} = \Psi^*_{\exp \epsilon Ad_g\xi}\mathrm{d}F_g + \mathrm{d}F_{\exp \epsilon Ad_g\xi}$. The derivative with respect to ϵ at 0 equals

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{d}F_{g\exp\epsilon\xi}(q) = \mathrm{d}(\langle (Ad_g\xi)_Q, \mathrm{d}F_g\rangle)(q) + \mathrm{d}f_{Ad_g\xi}.$$

We conclude that the map $\langle \sigma_F(g), \xi\rangle = f_{Ad_{g^{-1}}\xi}(q) - \Psi^*_{g^{-1}} f_\xi(q) + \langle (Ad_{g^{-1}}\xi)_Q(q), \mathrm{d}F_{g^{-1}}(q)\rangle$ is independent of q and therefore is well defined. From straightforward computations, it follows that it is a group 1-cocycle with values in $\mathfrak{g}^*$: for g_1, g_2 arbitrary,

$$Ad^*_{g_1^{-1}}\sigma_F(g_2) - \sigma_F(g_1g_2) + \sigma_F(g_1) = 0.$$

This concludes the proof. □

This 1-cocycle induces a $\mathfrak{g}^*$-valued 1-cocycle on the Lie algebra, given by

$$\xi \mapsto -ad^*_\xi f + \xi_Q(f) - \varphi^*(\mathrm{d}f_\xi),$$

and hence also a real valued 2-cocycle $\Sigma_f(\xi,\eta) = \xi_Q(f_\eta) - \eta_Q(f_\xi) - f_{[\xi,\eta]}$. This is the cocycle used in the infinitesimal version of quasi-invariant Lagrangians discussed in, for instance, Ref. 10. If only an infinitesimal action is given, i.e., a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(Q); \xi \mapsto \xi_Q$; or by complete lifting, an infinitesimal action on TQ, then the above definition of 1-cocycle σ_F corresponds infinitesimally to Σ_f. It is often easier to compute Σ_f instead of σ_F in examples (see Sec. V).

B. The momentum map

As mentioned in Sec. I, Noether's theorem is applicable to quasi-invariant Lagrangians as well: for each Lagrangian that is quasi-invariant under a group action, there exists a momentum map which is conserved. In this section, we study the properties of this momentum map, with a view toward performing symplectic reduction later on.

We begin by investigating the equivariance of the Legendre transformation.

Lemma 7: Let (Q,L,F) denote a quasi-invariant system. Then, for $g \in G$ arbitrary, the Legendre map $\mathbb{F}L$ transforms as

$$\mathbb{F}L(T\Psi_g(v_q)) = T^*\Psi_{g^{-1}}(\mathbb{F}L(v_q) + \mathrm{d}F_g(q)) = T^*\Psi_{g^{-1}}(\mathbb{F}L(v_q)) - \mathrm{d}F_{g^{-1}}(qg).$$

Proof: To show this equality, fix an element $w_{qg} \in TQ$, and let $w_q = T\Psi_{g^{-1}}(w_{qg})$. Then, by definition of the fiber derivative,

$$\begin{aligned}\langle w_{qg}, \mathbb{F}L(T\Psi_g(v_q))\rangle &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L(T\Psi_g(v_q) + \epsilon w_{qg}) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} (L(v_q + \epsilon w_q) + \langle v_q + \epsilon w_q, \mathrm{d}F_g(q)\rangle) \\ &= \langle w_{qg}, T^*\Psi_{g^{-1}}(\mathbb{F}L(v_q) + \mathrm{d}F_g(q))\rangle.\end{aligned}$$

From $\Psi_g^*(\mathrm{d}F_{g^{-1}}) = -\mathrm{d}F_g$ (let $g_1 = g, g_2 = g^{-1}$ in the cocycle identity for $\mathrm{d}F$) we have the property that $T^*\Psi_{g^{-1}}(\mathrm{d}F_g(q)) = -\mathrm{d}F_{g^{-1}}(qg)$ for $q \in Q$ and $g \in G$ arbitrary. This concludes the proof. □

The above lemma justifies the next definition.

Definition 6: Let (Q,L,F) denote a quasi-invariant Lagrangian system. Then we define a right action Ψ_{aff} on T^*Q as follows. For $\alpha_q \in T^*Q$ arbitrary, we put

$$\Psi_{\mathrm{aff},g}(\alpha_q) = T^*\Psi_{g^{-1}}(\alpha_q + \mathrm{d}F_g(q)) = T^*\Psi_{g^{-1}}(\alpha_q) - \mathrm{d}F_{g^{-1}}(qg).$$

We say that Ψ_{aff} is the affine action on T^*Q associated with the 1-cocycle $\mathrm{d}F$.

We should check that the affine action is well defined. For that purpose, we need to verify that for g_1, g_2 arbitrary

$$T^*\Psi_{(g_1g_2)^{-1}}(\alpha_q + \mathrm{d}F_{g_1g_2}(q)) = T^*\Psi_{g_2^{-1}}(T^*\Psi_{g_1^{-1}}(\alpha_q + \mathrm{d}F_{g_1}(q)) + \mathrm{d}F_{g_2}(qg_1)).$$

This is a straightforward consequence from the fact that $\mathrm{d}F$ is a group 1-cocycle.

Lemma 8: Let (Q,L,F) denote a quasi-invariant Lagrangian system. Then,

(1) *the lifted action $T\Psi$ is a canonical action for the symplectic structure (TQ, Ω_Q^L);*
(2) *the map $J_L^f = \varphi^* \circ \mathbb{F}L - \tau_Q^* f : TQ \to \mathfrak{g}^*$ is a momentum map with nonequivariance cocycle σ_F and the energy E_L is an invariant function on TQ;*
(3) *the affine action Ψ_{aff} is a canonical action for the symplectic structure (T^*Q, ω_Q) ; the map $J^f = \varphi^* - \pi_Q^* f$ is a momentum map with nonequivariance cocycle σ_F; and*
(4) *$\mathbb{F}L$ is a symplectomorphism between (TQ, Ω_Q^L) and (T^*Q, ω_Q), and is equivariant with respect to the lifted action on TQ and the affine action on T^*Q associated to* $\mathrm{d}F$.

Proof: The affine action Ψ_{aff} on T^*Q acts by symplectic transformations, i.e., from local computations it follows that

$$(\Psi_{\mathrm{aff},g})^*\theta_Q = \theta_Q + \pi_Q^*\mathrm{d}F_g.$$

Together with Lemma 7, i.e., $\mathbb{F}L\circ T\Psi_g=\Psi_{\mathrm{aff},g}\circ\mathbb{F}L$, assertions (1) and (4) follow:

$$(T\Psi_g)^*\Omega_Q^L = \mathrm{d}(\mathbb{F}L\circ T\Psi_g)^*\theta_Q = \mathbb{F}L^*\mathrm{d}\Psi_{\mathrm{aff},g}^*\theta_Q = \Omega_L.$$

The latter equality holds since θ_Q is invariant under the affine action up to an exact form.

To show that J_L^f is a momentum map, we use an argument involving coordinate expressions. Let $(q^i), i=1,\ldots,\dim Q$ denote coordinate functions on Q, and let $(q^i,\dot{q}^i)$ be the associated coordinate system on TQ. Then, it is not hard to show that

$$\xi_{TQ}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial f_\xi}{\partial q^i} - \frac{\partial \xi_Q^j}{\partial q^i}\frac{\partial L}{\partial \dot{q}^j}$$

holds, with $j=1,\ldots,\dim Q$ and ξ_Q^j the coordinate expression of $\xi_Q:\xi_Q=\xi_Q^j\partial_j$. From some tedious computations, it follows that

$$i_{\xi_{TQ}}\Omega_Q^L = -\,\mathrm{d}J_\xi^f$$

for $\xi\in\mathfrak{g}$ arbitrary. We now compute the nonequivariance cocycle of J_L^f. Fix any $\xi\in\mathfrak{g}$ and $v_q\in T_qQ$, then

$$\begin{aligned}
\langle J_L^f(T\Psi_g(v_q)),\xi\rangle &= \langle \mathbb{F}L(T\Psi_g(v_q)),\varphi_{qg}(\xi)\rangle - f_\xi(qg)\\
&= \langle T^*\Psi_{g^{-1}}(\mathbb{F}L(v_q)+\mathrm{d}F_g(q)),T\Psi_g(\varphi_q(Ad_g\xi))\rangle - f_\xi(qg)\\
&= \langle \mathbb{F}L(v_q),\varphi_q(Ad_g\xi)\rangle - f_{Ad_g\xi}(q) + (f_{Ad_g\xi}(q) - f_\xi(qg) + \langle \mathrm{d}F_g(q),\varphi_q(Ad_g\xi)\rangle)\\
&= \langle Ad_g^*J_L^f(v_q),\xi\rangle + \langle \sigma_F(g^{-1}),\xi\rangle.
\end{aligned}$$

Finally, the fact that the energy is invariant easily follows from Lemma 7, and from this, we conclude that (2) holds.

Since $\mathbb{F}L$ is a symplectic diffeomorphism and since $J^f\circ\mathbb{F}L=J_L^f$, we conclude that J^f is a momentum map with cocycle σ_F. This proves (3). □

The above lemma ensures that the equivariance conditions for Theorem 5 are satisfied. In that case we can study the MW reduction and the structure of the corresponding quotient spaces. If these quotient spaces are "tangent and cotangent bundlelike," we shall say that the MW reduction is a Routhian reduction procedure.

Following Theorem 5, we have that the reduced Legendre transformation $[\mathbb{F}L_\mu]$ is a symplectic diffeomorphism relating the symplectic structures on $(J_L^f)^{-1}(\mu)/G_\mu$ and $(J^f)^{-1}(\mu)/G_\mu$. The subgroup G_μ is the isotropy subgroup of the affine action of G on $\mathfrak{g}^*$ corresponding to the 1-cocycle σ_F. We now study the structure of the reduced manifolds $(J_L^f)^{-1}(\mu)/G_\mu$ and $(J^f)^{-1}(\mu)/G_\mu$, and their respective symplectic 2-forms.

Let $\mathcal{A}$ be a principal connection with horizontal projection operator $TQ\to TQ:v_q\mapsto v_q^h:=v_q-\varphi_q(\mathcal{A}_q(v_q))$. Similarly, we can restrict a covector α_q to horizontal tangent vectors: $T^*Q\mapsto T^*Q:\alpha_q\mapsto\alpha_q^h$, with $\langle v_q,\alpha_q^h\rangle=\langle v_q^h,\alpha_q\rangle$. Note that $\alpha_q^h=\alpha_q-(\mathcal{A}_q^*\circ\varphi_q^*)(\alpha_q)$. The covariant exterior derivative (see Ref. 8) of a function λ on Q is denoted by $D\lambda$ and is defined pointwise as $D\lambda_q=d\lambda_q^h$. We first study the symplectic structure of $(J^f)^{-1}(\mu)/G_\mu$. Similar to the invariant situation, we contract the connection 1-form on the Lie-algebra level with $\mu+f$ to obtain a 1-form $\mathcal{A}_\mu^f=q\mapsto\langle\mu+f(q),\mathcal{A}_q\rangle$ on Q.

Lemma 9: Consider a quasi-invariant Lagrangian system (Q,L,F), *for which there exists a principal connection* $\mathcal{A}$ *such that* $DF_g=0$, *for arbitrary* $g\in G$. *Then,*

(1) *the 2-form* $\mathrm{d}\mathcal{A}_\mu^f$ *is invariant under the action of* G_μ *on* Q *and is projectable to a 2-form on*

Q/G_μ denoted by $\mathcal{B}^f_\mu$ and

(2) *there exists a symplectic diffeomorphism*

$$[\phi^{\mathcal{A},f}_\mu]:((J^f)^{-1}(\mu)/G_\mu,\omega_\mu) \to (T^*(Q/G)\times Q/G_\mu,\pi_1^*\omega_{Q/G}+\pi_2^*\mathcal{B}^f_\mu).$$

Proof: For the proof of both statements, we rely on the following identities, for $g\in G_\mu$ and $q\in Q$:

$$\mathcal{A}^*_{qg} = T^*\Psi_{g^{-1}}\circ \mathcal{A}^*_q \circ Ad^*_{g^{-1}},$$

$$\mu = Ad^*_{g^{-1}}\mu + \sigma_F(g),$$

$$Ad^*_{g^{-1}}f(qg) = f(q) - (Ad^*_{g^{-1}}\circ \varphi^*_{qg})(\mathrm{d}F_{g^{-1}}(qg)) + \sigma_F(g),$$

$$Ad^*_{g^{-1}}\circ \varphi^*_{qg} = \varphi^*_q \circ T^*\Psi_g.$$

(1) The first statement is proven if we can show that $\mathcal{A}^f_\mu$ is invariant under G_μ up to an exact 1-form. Thus, consider any element $q\in Q$ and $g\in G_\mu$, then

$$(\Psi^*_g\mathcal{A}^f_\mu)(q) = \langle \mu + f(qg), Ad_{g^{-1}}\cdot \mathcal{A}_q\rangle = \langle(\mu-\sigma_F(g)) + (f(q) - (Ad^*_{g^{-1}}\circ\varphi^*_{qg})(\mathrm{d}F_{g^{-1}}(qg)) + \sigma_F(g)), \mathcal{A}_q\rangle = (\mathcal{A}^f_\mu)(q) + \mathrm{d}F_g(q).$$

The latter equality holds because $\mathrm{d}F^h_g(q)=DF_g(q)=0$. To show that the 2-form is projectable, we prove in addition that $i_{\xi_Q}\mathrm{d}\mathcal{A}^f_\mu=0$. This follows, on the one hand, from $\mathcal{L}_{\xi_Q}\mathcal{A}^f_\mu=\mathrm{d}f_\xi$ which is obtained using the previous equation with $g=\exp\epsilon\xi$ and, on the other hand, from $\mathcal{L}_{\xi_Q}=i_{\xi_Q}d+\mathrm{d}i_{\xi_Q}$,

$$i_{\xi_Q}\mathrm{d}\mathcal{A}^f_\mu = \mathcal{L}_{\xi_Q}\mathcal{A}^f_\mu - \mathrm{d}f_\xi = 0.$$

(2) Similar to the case of an invariant Lagrangian system we relate $(J^f)^{-1}(\mu)$ with $V^0\pi$ by means of the connection: $\phi^{\mathcal{A},f}_\mu:(J^f)^{-1}(\mu)\to V^0\pi;\alpha_q\mapsto\alpha_q-\mathcal{A}^*_q(\mu+f(q))$. The next step is to study the affine action of G_μ on $(J^f)^{-1}(\mu)$ through this diffeomorphism. Let $g\in G_\mu$, and $\alpha_q\in(J^f)^{-1}(\mu)$, then

$$\begin{aligned}\phi^{\mathcal{A},f}_\mu(T^*\Psi_{g^{-1}}(\alpha_q+\mathrm{d}F_g(q))) &= T^*\Psi_{g^{-1}}(\alpha_q+\mathrm{d}F_g(q)) - \mathcal{A}^*_{qg}(\mu+f(qg))\\ &= T^*\Psi_{g^{-1}}(\alpha_q - \mathcal{A}^*_q(\mu+f(q)) + \mathrm{d}F^h_g(qg)\ ,\end{aligned}$$

We conclude that $\phi^{\mathcal{A},f}_\mu$ is equivariant with respect to the affine action on $(J^f)^{-1}(\mu)$ and the standard lifted action on T^*Q restricted to $V^0\pi$ if the condition $DF_g=0$ holds. The reduced map is denoted by $[\phi^{\mathcal{A},f}_\mu]$ and maps $(J^f)^{-1}(\mu)/G_\mu$ to $T^*(Q/G)\times Q/G_\mu$. The fact that it is a symplectic map follows from analogous arguments as in the invariant case. □

C. The reduced phase space

We are now ready to take the final step toward a Routhian reduction procedure for quasi-invariant Lagrangians. It concerns the realization of $(J^f_L)^{-1}(\mu)/G_\mu$ as a tangent space $T(Q/G)\times Q/G_\mu$. We therefore reintroduce G-regular quasi-invariant Lagrangians. It should be clear that the definitions here are also valid in the strict invariant case. Let $R^\mu=L-\mathcal{A}^f_\mu$ denote the "Routhian" as a function on TQ. We first show that it is G_μ-invariant. For that purpose let $g\in G_\mu$ and $v_q\in T_qQ$, then

$$R^\mu(T\Psi_g(v_q)) = L(v_q) + \langle \mathrm{d}F_g(q), v_q\rangle - \langle (\Psi_g^* \mathcal{A}_\mu^f)(q), v_q\rangle = L(v_q) - \langle \mathcal{A}_\mu^f(q), v_q\rangle = R^\mu(v_q).$$

We know from the strict invariant case that TQ/G_μ can be identified with $T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}$. Let us denote $\mathfrak{R}^\mu$ denote the function on the latter space obtained from projecting R^μ. We now define the fiber derivative $\mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu$ of $\mathfrak{R}^\mu$ with respect to the $\tilde{\mathfrak{g}}$-fiber,

$$\langle \mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu(v_x, y, \tilde{\xi}), (v_x, y, \tilde{\eta})\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathfrak{R}^\mu(v_x, y, \tilde{\xi} + \epsilon\tilde{\eta}),$$

with $(v_x, y, \tilde{\xi}) \in T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}$ and $\tilde{\eta} \in \tilde{\mathfrak{g}}_x$ arbitrary.

Definition 7: Let (Q,L,F) denote a quasiregular Lagrangian system. We say that the system is G-regular if the function $\mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu : T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}} \to T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}^*$ is a diffeomorphism.

It is not so hard to show that there is a one-to-one identification with $(J_L^f)^{-1}(\mu)/G_\mu$ and the set of points $(v_x, y, \tilde{\xi})$ in $T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}$ for which $\mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu(v_x, y, \tilde{\xi}) = (v_x, y, 0)$. We consider the map $(p_1, p_2) : (J_L^f)^{-1}(\mu)/G_\mu \to T(Q/G)\times Q/G_\mu$ taking a point $[v_q]_G$ to the first two factors of the corresponding point $(T\pi(v_q), p_\mu(q), [q, \mathcal{A}(v_q)]_G)$ in the fibered product $T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}$.

Lemma 10: If (Q,L,F) is a G -regular quasi-invariant Lagrangian system, then the mapping $(p_1, p_2) : (J_L^f)^{-1}(\mu)/G_\mu \to T(Q/G)\times Q/G_\mu$ is a diffeomorphism with inverse ψ_μ.

The proof is completely analogous to the proof of Lemma 2: the inverse of (v_x, y) is defined as the point in $(J_L^f)^{-1}(\mu)/G_\mu$ that corresponds to $(\mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu)^{-1}(v_x, y, 0)$ in $T(Q/G)\times Q/G_\mu \times \tilde{\mathfrak{g}}$. Let $[R^\mu]$ denote the quotient of the restriction of R^μ to $(J_L^f)^{-1}(\mu)$. Similar to the previous case, we define $\mathcal{R}^\mu$ to be function on $T(Q/G)\times Q/G_\mu$ such that $(p_1, p_2)^*(\mathcal{R}^\mu) = [R^\mu]$. Note that $\mathcal{R}^\mu$ could also be obtained by $\mathcal{R}^\mu(v_x, y) = \mathfrak{R}^\mu(v_x, y, \tilde{\xi})$, with $(v_x, y, \tilde{\xi}) = (\mathbb{F}_{\tilde{\xi}}\mathfrak{R}^\mu)^{-1}(v_x, y, 0)$.

Lemma 11: Let (Q,L,F) denote a G -regular quasi-invariant Lagrangian system and let $\mathcal{A}$ be a principal connection such that $DF_g = 0$. Let μ denote a value of J_L^f. Then,

(1) *the map $[\phi_\mu^{\mathcal{A},f}] \circ [\mathbb{F}L_\mu] \circ \psi_\mu$ is the fiber derivative of $\mathcal{R}^\mu$ and*
(2) *the energy of $\mathcal{R}^\mu$ is the MW-reduced Hamiltonian of the energy E_L on the symplectic manifold (Q, Ω_Q^L).*

The proof is again completely similar to the proof of Lemmas 4 and 5. We conclude that the MW reduction in a G-regular quasi-invariant Lagrangian L is again a "Lagrangian" system on the manifold $T(Q/G)\times Q/G_\mu$, with Lagrangian $\mathcal{R}^\mu$: the symplectic structure is of the form $(\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^*\omega_{Q/G}) + \bar{\pi}_2^*\mathcal{B}_\mu^f$.

Theorem 8: *Let (Q,L,F) denote a G-regular quasi-invariant Lagrangian system and let $\mathcal{A}$ be a principal connection such that $DF_g = 0$. Let μ denote a value of J_L^f. Then, the MW reduction in the symplectic manifold (Q, Ω_L) for the regular momentum value μ is the symplectic manifold*

$$(T(Q/G)\times Q/G_\mu, (\mathbb{F}\mathcal{R}^\mu)^*(\pi_1^*\omega_{Q/G}) + \bar{\pi}_2^*\mathcal{B}_\mu^f).$$

The reduced Hamiltonian is the energy $E_{\mathcal{R}^\mu}$. The equations of motion for this Hamiltonian vector field are precisely the Euler–Lagrange equations of motion for an intrinsically constrained Lagrangian system on $Q/G_\mu \to Q/G$ with Lagrangian $\mathcal{R}^\mu$ and gyroscopic force term associated with the 2-form $\mathcal{B}_\mu^f$ on Q/G_μ.

V. EXAMPLES

A. Quasicyclic coordinates

We continue here the description started in the introduction of a Lagrangian L with a single quasicyclic coordinate. Recall that if $(q^1, \ldots, q^n)$ are coordinates on $Q = \mathbb{R}^n$ and $L(q^i, \dot{q}^i)$ is a Lagrangian, then we say that q^1 is quasicyclic if there exists a function $f(q^1, \ldots, q^n)$ such that

$$\frac{\partial L}{\partial q^1} = \dot{q}^i \frac{\partial f}{\partial q^i}.$$

The group $G=\mathbb{R}$ acts on $\mathbb{R}^n$ by translation in q^1. Since $\mathfrak{g}\equiv\mathbb{R}$, a principal connection $\mathcal{A}$ here becomes an ordinary G-invariant 1-form on $\mathbb{R}^n$. The infinitesimal version Σ_f of the definition of the cocycle σ_F is identically zero, and we can conclude that also σ_F vanishes. Since the group is Abelian, we have that $G_\mu=G$. The quotient space is $T(Q/G)$ and Q/G is labeled by the configuration space coordinates $(q^2,\ldots,q^n)$.

The condition that the system should be G-regular is locally expressed by $\partial^2 L/\partial\dot{q}^1\partial\dot{q}^1\neq 0$ and, second, the condition that there exists a (principal) connection $\mathcal{A}$ such that $Df=0$ (i.e., df restricted to the horizontal distribution should vanish) boils down to the condition that there should exist functions Γ_k, $k=2,\ldots,n$, independent of q^1, for which

$$\frac{\partial f}{\partial q^k} = \Gamma_k(q^2,\ldots,q^n)\frac{\partial f}{\partial q^1}, \quad k=2,\ldots,n.$$

This is precisely the condition (1) from Sec. I (cf. Theorem 2). The connection $\mathcal{A}$ then reads as $\mathcal{A}=\mathrm{d}q^1+\Gamma_k\mathrm{d}q^k$, with summation over $k=2,\ldots,n$. Note that $Df=0$ implies that the connection has vanishing curvature (the horizontal distribution is involutive because it is annihilated by an exact 1-form). Assume now that both of the above conditions hold and keep the value of the momentum $\mu=\partial L/\partial\dot{q}^1-f$ fixed. We solve this relation for $\dot{q}^1$ by writing $\dot{q}^1=\psi(q^k,\dot{q}^k)$, with $k=2,\ldots,n$. The Routhian then is the function

$$R^\mu(q^k,\dot{q}^k) = L-(\mu+f)(\dot{q}^1+\Gamma_k\dot{q}^k),$$

where all instances of $\dot{q}^1$ on the right-hand side have been replaced by the function ψ. It now remains to compute the 2-form $\mathcal{B}^f_\mu$ which is the projection of $\mathrm{d}[(\mu+f)(\mathrm{d}q^1+\Gamma_k\mathrm{d}q^k)]$. After some straightforward computations in which the condition $\mathrm{d}f^h=0$ is used, we obtain

$$\mathcal{B}^f_\mu = \frac{1}{2}(\mu+f)\left(\frac{\partial\Gamma_k}{\partial q^s}-\frac{\partial\Gamma_s}{\partial q^k}\right)\mathrm{d}q^k\wedge\mathrm{d}q^s.$$

The latter is identically zero since the connection has zero curvature due to $Df=0$. This also follows from the following:

$$\frac{\partial\Gamma_k}{\partial q^s} = \frac{1}{\partial f/\partial q^1}\frac{\partial^2 f}{\partial q^k\,\partial q^s} - \frac{1}{(\partial f/\partial q^1)^2}\frac{\partial f}{\partial q^k}\frac{\partial^2 f}{\partial q^s\,\partial q^1} = \frac{1}{\partial f/\partial q^1}\frac{\partial^2 f}{\partial q^k\,\partial q^s} - \frac{1}{\partial f/\partial q^1}\Gamma_k\Gamma_s\frac{\partial^2 f}{\partial q^1\,\partial q^1} = \frac{\partial\Gamma_s}{\partial q^k}.$$

We conclude that the Routhian reduction for Lagrangian systems with a single quasicyclic coordinate is the Lagrangian system on the reduced space with Lagrangian the Routhian $L-\mathcal{A}^f_\mu$. This concludes the proof of Theorem 2.

B. Functional Routhian reduction

Our motivation for studying Routh reduction for quasi-invariant Lagrangians was inspired from the reduction technique called *functional Routhian reduction* used in Ref. 2. We will argue here that functional Routhian reduction can be seen as Routhian reduction for a quasi-invariant Lagrangian. Consider a Lagrangian L of type kinetic minus potential energy define on a configuration space (locally) $(q^1,\ldots,q^{n-1},q^n)$. The coordinate q^n was denoted in Ref. 2 by ϕ and the coordinates q^k for $k=1,\ldots,n-1$ by θ^k. The Lagrangian L is of the form

$$L = \tfrac{1}{2}(M_{ij}(\theta)\dot{q}^i\dot{q}^j) - W(\theta,\dot{\theta},\phi) - V(\theta,\phi),$$

with $M_{ij}(\theta)$ mass-inertia functions depending only on θ^k and $W=(\lambda(\phi)/M_{nn}(\theta))M_{nk}(\theta)\dot{\theta}^k$ and $V=V_{\mathrm{fct}}(\theta)-\tfrac{1}{2}\lambda(\phi)^2/M_{nn}(\theta)$.

It should be immediately clear that ϕ is not a cyclic coordinate, nor a quasi-invariant cyclic coordinate. We will, however, define a "momentum map" J_L^λ associated to the would be cyclic coordinate ϕ,

$$J_L^\lambda(\theta,\dot{\theta},\phi) = \partial_{\dot{\phi}} L(\theta,\phi,\dot{\theta},\dot{\phi}) - \lambda(\phi) = M_{kn}(\theta)\dot{\theta}^k + M_{nn}(\theta)\dot{\phi} - \lambda(\phi).$$

Note that, since λ only depends on ϕ, we may use the standard connection $\mathcal{A}=d\phi$ when working in a local coordinate system. The Lagrangian L transforms as a quasi-invariant Lagrangian when restricted to the level set $J_L^\lambda=0$:

$$\left.\frac{\partial L}{\partial \phi}\right|_{J_L^\lambda=0} = \lambda'(\phi)\dot{\phi}.$$

Strictly speaking this example is not described in the theory outlined above. We hope however that it is clear to the reader that is an even more general type of Routh reduction for quasi-invariant Lagrangians that is valid only on a specific level set of the momentum map. The correspondence between both techniques is also seen from the fact that in Ref. 2 the authors defined the *functional Routhian* L_{fct} as the function

$$L_{\mathrm{fct}}(\theta,\dot{\theta}) = (L(q^i,\dot{q}^i) - \lambda(\phi)\dot{\phi})_{J_L^\lambda=0}.$$

This is precisely the function $\mathcal{R}^\mu$, with $\mu=0$ in our analysis of quasi-invariant Lagrangians. Note that all regularity conditions are satisfied and especially the horizontal condition $\mathrm{d}F_g^h=0$ is satisfied since λ is independent of θ.

C. Charged particle in a constant magnetic field

In Ref. 10 the example of a charged particle in a constant magnetic field B is studied. The Lagrangian for this system is $L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+eB(\dot{x}y-\dot{y}x)$. The coordinates x and y are quasicyclic, and from this, we may write that $f(x,y)=(-eBy,eBx)\in\mathbb{R}^2\cong\mathfrak{g}^*$. The 2-cocycle Σ_f is not vanishing and proportional to eB. The (infinitesimal) affine action on $\mathfrak{g}^*$ is completely determined by this 2-cocycle Σ_f and due to the Abelian nature of the group, the Lie algebra of isotropy subgroup $G_{(\mu_1,\mu_2)}$ is trivial since it is spanned by the kernel of Σ_f. In turn, $G_{(\mu_1,\mu_2)}=\{e\}$. The conserved momenta read as $m\dot{x}+2eBy=\mu_1$ and $m\dot{y}-2eBx=\mu_2$. Therefore the quotient space is $\mathbb{R}^2$. From the structure of the momenta equations it is immediately seen that the system is G-regular. Further the standard connection 1-form $\mathcal{A}=(\mathrm{d}x,\mathrm{d}y)^T$, with trivial horizontal distribution implies that $Df=\mathrm{d}f^h=0$. Therefore all conditions are met, and the Routhian is then a function on $\mathbb{R}^2$ depending only on x,y,

$$R^\mu = \frac{-1}{2m}((\mu_1-2eBy)^2+(\mu_2+2eBx)^2).$$

The symplectic 2-form $\mathcal{B}_\mu^f$ is precisely $2eB\mathrm{d}x\wedge\mathrm{d}y$. The Routhian reduced equations of motion the read: $i_{(\dot{x},\dot{y})}\mathcal{B}_\mu^f=\mathrm{d}R^\mu$, or simply the momentum equations $m\dot{x}+2eBy=\mu_1$ and $m\dot{y}-2eBx=\mu_2$.

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