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CHARACTERIZATION RESULTS ON WEIGHTED MINIHYPERS AND ON LINEAR CODES MEETING THE GRIESMER BOUND

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Abstract. We present characterization results on weighted minihypers. We prove the weighted version of the original results of Hamada, Helleseth, and Maekawa. Following from the equivalence between minihypers and linear codes meeting the Griesmer bound, these characterization results are equivalent to characterization results on linear codes meeting the Griesmer bound.

1. Linear codes meeting the Griesmer bound, minihypers, and blocking sets

A *linear* $[n, k, d]$*-code* C over the finite field $\mathbb{F}_q$ of order q is a k-dimensional subspace of the n-dimensional vector space $V(n, q)$ of vectors of length n over $\mathbb{F}_q$. The *minimum distance* d of the code C is the minimal number of positions in which two distinct codewords of C differ [17].

It is interesting to use linear codes having a minimal length n for given k, d, and q. The *Griesmer bound* is one of the many relations between the parameters n, k, d of a linear $[n, k, d]$-code C that exist, and states

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x [10, 18].

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Considering this lower bound on the length n for given values k, d, and q, the question arises whether there exists a linear $[n, k, d]$-code whose length n is equal to the lower bound $g_q(k, d)$. For some values of the parameters k, d, and q, linear codes of length equal to $g_q(k, d)$ are known to exist, for other values it is proved that no such codes exist.

Let $\mathrm{PG}(N, q)$ be the N-dimensional projective space over the finite field of order q. For $i \geq 0$, put $v_i = (q^i - 1)/(q - 1)$, which is the number of points in $\mathrm{PG}(i-1, q)$. A *weight function* w of $\mathrm{PG}(N, q)$ is a mapping from the point set of $\mathrm{PG}(N, q)$ to the set of non-negative integers. For a point P, the integer $w(P)$ is called the *weight* of the point P, and for a set M of points, its *weight* is the sum of the weights of its points. The sum of the weights of all points is the *total weight* of w. In principle, a minihyper is nothing else than such a weight function, but usually the definition is in the following way, which gives some information on w.

Definition 1.1. An $\{f, m; N, q\}$-*minihyper*, $f \geq 1$, $N \geq 2$, is a pair (F, w), where w is a weight function of $\mathrm{PG}(N, q)$ of total weight f, and F is the set of points of positive weight, and m is the minimum weight of the hyperplanes of $\mathrm{PG}(N, q)$.

Of course, the set F is determined by the weight function w. When the range of w is $\{0, 1\}$, the converse is true and then the minihyper is identified with F and called a *non-weighted minihyper*. Thus, a non-weighted $\{f, m; N, q\}$-minihyper of $\mathrm{PG}(N, q)$ is a set F of f points of $\mathrm{PG}(N, q)$ such that m is the minimum weight of the hyperplanes. This is the definition of a minihyper given by Hamada and Tamari in [15] and it was generalized to the definition of a weighted minihyper in [7].

Linear $[n, k, d]$-codes meeting the Griesmer bound can be linked with non-weighted minihypers in $\mathrm{PG}(k-1, q)$ when $d \leq q^{k-1}$ and with (weighted) minihypers in $\mathrm{PG}(k-1, q)$ when $d > q^{k-1}$. We explain first of all the link for $1 \leq d \leq q^{k-1}$. Then d can be written uniquely as $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$ such that:

(a) $0 \leq \lambda_1 \leq \cdots \leq \lambda_h < k - 1$,
(b) at most $q - 1$ of the values λ_i are equal to a given value.

Using this expression for d, the Griesmer bound for a linear $[n, k, d]$-code over $\mathbb{F}_q$ can be expressed as:

$$n \geq v_k - \sum_{i=1}^{h} v_{\lambda_i + 1}.$$

Hamada showed that in the case $d = q^{k-1} - \sum_{i=1}^{h} q^{\lambda_i}$, there is a one-to-one correspondence between the set of all non-equivalent $[n, k, d]$-codes meeting the Griesmer bound and the set of all projectively distinct $\{\sum_{i=1}^{h} v_{\lambda_i+1}, \sum_{i=1}^{h} v_{\lambda_i}; k - 1, q\}$-minihypers [11]. More precisely, the link is described in the following way.

Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n, k, d]$-code C, $d \leq q^{k-1}$, meeting the Griesmer bound. It can be shown that $g_i \neq \rho g_j$, $\rho \in \mathbb{F}_q^*$, for $i \neq j$. Then the set $\mathrm{PG}(k-1, q) \setminus \{g_1, \ldots, g_n\}$ is the minihyper linked to the code C meeting the Griesmer bound.

For $d > q^{k-1}$, the link between linear codes meeting the Griesmer bound and weighted minihypers is as follows.

Let $G = (g_1 \cdots g_n)$ be a generator matrix for a linear $[n,k,d]$-code over $\mathbb{F}_q$, $d > q^{k-1}$, meeting the Griesmer bound. We again look at a column of G as being the coordinates of a point in $PG(k-1,q)$. Let the point set of $PG(k-1,q)$ be $\{s_1,\ldots,s_{v_k}\}$. Let $m_i(G)$ denote the number of columns in G defining s_i. Let $m(G) = \max\{m_i(G) \| i = 1,2,\ldots,v_k\}$. Then $\theta = m(G)$ is uniquely determined by the code C and we call it the maximum multiplicity *of the code. Define the weight function $w : \mathrm{PG}(k-1,q) \to \mathbb{N}$ as $w(s_i) = \theta - m_i(G)$, $i = 1,2,\ldots,v_k$. Let $F = \{s_i \in \mathrm{PG}(k-1,q) \| w(s_i) > 0\}$, then (F,w) is a $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper with weight function w if $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \epsilon_i q^i$, with $0 \le \epsilon_i \le q-1, i = 0,\ldots,k-2$.*

Now the question arises how to construct linear codes meeting the Griesmer bound. The standard construction method is of Belov, Logachev, and Sandimirov [2]. This construction method is easily described by using the corresponding minihypers. We first of all describe the construction for non-weighted minihypers.

Consider in $\mathrm{PG}(k-1,q)$ a union of pairwise disjoint ϵ_0 points $P_1, P_2, \ldots, P_{\epsilon_0}$, ϵ_1 lines $\ell_1, \ell_2, \ldots, \ell_{\epsilon_1}$, ϵ_2 planes, ϵ_3 solids, $\ldots$, ϵ_{k-2} $(k-2)$-dimensional subspaces $\pi_{k-2}^1, \ldots, \pi_{k-2}^{\epsilon_{k-2}}$, with $0 \le \epsilon_i \le q-1, i = 0,\ldots,k-2$. Then such a set defines a non-weighted $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper.

If one allows subspaces that are not mutually disjoint, then the union has to be replaced by the weight function, that is, the sum of the characteristic functions of the subspaces. These minihypers will correspond to linear codes with $d > q^{k-1}$.

Now that the standard examples of minihypers, or equivalently, of linear codes meeting the Griesmer bound, are known, the characterization problem on minihypers, or equivalently, on linear codes meeting the Griesmer bound, arises:

characterize (weighted) $\{f, m; k-1, q\}$-minihypers for given parameters $f = \sum_{i=0}^{k-2} \epsilon_i v_{i+1}$, $m = \sum_{i=0}^{k-2} \epsilon_i v_i$, k, and q.

Fundamental research on this characterization problem was performed by Hamada and Helleseth who studied minihypers thoroughly, and who developed many techniques that have proven to be very useful in the study of minihypers. Their main result can be formulated as follows.

Result 1.2. (Hamada, Helleseth, and Maekawa [13, 14])
A non-weighted $\{\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i; k-1, q\}$-minihyper, where $\sum_{i=0}^{k-2} \epsilon_i < \sqrt{q}+1$, is a union of ϵ_{k-2} hyperplanes, ϵ_{k-3} $(k-3)$-dimensional spaces, $\ldots$, ϵ_1 lines, and ϵ_0 points, which all are pairwise disjoint, so is of Belov-Logachev-Sandimirov type.

The main result of this paper generalizes this result of Hamada, Helleseth, and Maekawa to weighted minihypers. To state the result, we define a concept which generalizes the Belov, Logachev, and Sandimirov construction. Our definition is a generalization of a similar definition from [9]. Let $S_1, \ldots, S_u$ be subspaces of $\mathrm{PG}(k-1,q)$. Define as follows a weight function from the point set of $\mathrm{PG}(k-1,q)$ to the set of integers:

for each point P, its weight $w(P)$ is the number of indices i with $P \in S_i$.

In other words, w is the sum of the characteristic functions of the subspaces S_i. If F is the union of the subspaces S_i, then (F,w) is a minihyper, and we call it the *sum of the subspaces* $S_1, \dots, S_u$. We explicitly note that it is allowed that the list $S_1, \dots, S_u$ contains a subspace S_i several times.

With this definition, the original Belov, Logachev, and Sandimirov construction can be described as a sum of mutually skew subspaces. Our main theorem of this paper is the following.

Main Theorem. *A (weighted)* $\{\sum_{i=0}^{N-1} \epsilon_i v_{i+1}, \sum_{i=0}^{N-1} \epsilon_i v_i; N, q\}$*-minihyper* (F, w), $q \geq 4$, *where* $\epsilon_0, \dots, \epsilon_{N-1}$ *are non-negative integers satisfying* $\sum_{i=0}^{N-1} \epsilon_i < \sqrt{q} + 1$, *is a sum of* ϵ_{N-1} *hyperplanes,* ϵ_{N-2} $(N-2)$*-dimensional spaces,* $\dots, \epsilon_1$ *lines, and* ϵ_0 *points.*

2. Technical lemmas

Let q be a prime power. Every integer $f > 0$ can be uniquely written in the form $f = \sum_{i \geq k} \epsilon_i v_{i+1}$ for some integer $k \geq 0$ where $1 \leq \epsilon_k \leq q$ and $0 \leq \epsilon_i \leq q-1$ for $i > k$. This enables us to define (just for this paper) the *q-successor of f* as the integer $T_q(f) := \sum_{i \geq k} \epsilon_i v_i$. We also define $T_q(0) := 0$. Thus T_q is a map from the set of the non-negative integers to itself. When q is clear from the context, we simply write T. Applying T more than once to a number, results in numbers $T^i(f) = T(T^{i-1}(f))$. The $\{f, h; N, q\}$-minihypers related to Griesmer codes often have the property that $h = T(f)$. It is also important to note that many of these minihypers have the property that $T^j(f)$ is the minimum weight of the subspaces of codimension j. This motivates the following lemmas. We note that variants of the lemmas, either of smaller total weight or non-weighted versions or statements without proofs, can be found in the literature, e.g. in [9], [12], and [13].

Lemma 2.1. *Let w be a weight function from the point set of* $\mathrm{PG}(N, q)$ *to the set of non-negative integers of total weight f at most qv_{N+1}. Then some hyperplane has weight at most $T(f)$.*

Proof. This is trivial, if the total weight f is zero. Otherwise $f = \sum_{i=0}^{N} \epsilon_i v_{i+1}$, where for some k we have $\epsilon_i = 0$ for $i < k$, $1 \leq \epsilon_k \leq q$, and $0 \leq \epsilon_i \leq q-1$ for $i > k$. Then $T^j(f) = \sum_{i \geq j} \epsilon_i v_{i+1-j}$. Let h_j be the minimum weight of the subspaces of codimension j and put $h_j := T^j(f) + \delta_j$ for an integer δ_j. Then $h_0 = f$, so $\delta_0 = 0$. Also $h_{N+1} = T^{N+1}(f) = \delta_{N+1} = 0$. Considering for $j \geq 1$ all subspaces of codimension j on a fixed subspace of codimension $j+1$ and weight h_{j+1} gives $h_{j+1} + v_{j+1}(h_j - h_{j+1}) \leq f$. If one substitutes for f, h_j, and h_{j+1}, this reads as follows

$$\delta_{j+1} \geq \delta_j - \frac{-\delta_j + \sum_{i=0}^{j-1} \epsilon_i v_{i+1}}{v_{j+1} - 1}\ . \tag{1}$$

Here $\sum_{i=0}^{j-1} \epsilon_i v_{i+1} \leq q v_j = v_{j+1} - 1$. Assume that $\delta_1 > 0$. Then we find recursively that $\delta_{N+1} \geq \delta_N \geq \dots \geq \delta_1 > 0$. As $\delta_{N+1} = 0$, this is impossible. □

Lemma 2.2. *Let* (F, w) *be an* $\{f, T(f); N, q\}$*-minihyper with* $f = \sum_{i=0}^{N} \epsilon_i v_{i+1}$, $0 \leq \epsilon_i \leq q-1$ *for all* i.

(a) *For each j with $0 \leq j \leq N$, the minimum weight of the subspaces of codimension j is* $T^j(f) = \sum_{i=j}^{N} \epsilon_i v_{i+1-j}$.

(b) *Suppose that Δ is a subspace of codimension two of weight $T^2(f)$. Then for each of the $q+1$ hyperplanes $\pi_0, \dots, \pi_q$ on Δ, the restriction of w to π_j is a*

$\{\delta_j + T(f), T^2(f); N-1, q\}$*-minihyper inside* π_j*, where the* δ_j *are non-negative integers such that* $\sum_{j=0}^{q} \delta_j = \epsilon_0$*.*

Proof. (a) Using the same notations as in the previous lemma, we again find the inequalities (1). However, as $\epsilon_i \leq q-1$, we have this time that $\sum_{i=0}^{j-1} \epsilon_i v_{i+1} \leq v_{j+1} - 2$. As $\delta_0 = \delta_1 = 0$, we find recursively $\delta_{N+1} \geq \delta_N \geq \cdots \geq \delta_1 = 0$. As $\delta_{N+1} = 0$, it follows that $\delta_i = 0$ for all i. This proves (a).

(b) As $w(\pi_i) \geq T(f)$, then $w(\pi_i) = T(f) + \delta_i$ for non-negative integers δ_i. Using $\sum_i (w(\pi_i) - w(\Delta)) = f - w(\Delta)$, this gives $\sum_{i=0}^{q} \delta_i = \epsilon_0$. As every subspace of codimension 2 has weight at least $T^2(f)$ and as $w(\Delta) = T^2(f)$, the restriction of w to π_i gives a minihyper with the stated parameters. □

Proposition 2.3. *Let* (F, w) *be an* $\{f, T(f); N, q\}$*-minihyper,* $f = \sum_{i=0}^{N-1} \epsilon_i v_{i+1}$ *for non-negative integers* ϵ_i *satisfying* $\sum_{i=0}^{N-1} \epsilon_i \leq q-1$*. Let* U *be a subspace. If* $u := \dim(U) \leq N-2$*, then also suppose that* U *is not contained in* F*. Then the restriction of* w *to* U *is a* $\{\sum_{i=0}^{u-1} m_i v_{i+1}, \sum_{i=1}^{u-1} m_i v_i; u, q\}$*-minihyper for some non-negative integers* m_i *with* $\sum_{i=0}^{u-1} m_i \leq \sum_{i=0}^{N-1} \epsilon_i$*.*

Proof. We prove this by induction on the codimension of U. If $N = u$, the statement is trivial.

Now we study the case that U is a hyperplane. As $h := w(U) \leq f$, then $h = \sum_{i=k}^{N-1} m_i v_{i+1}$ with $k \geq 0$, $0 < m_k \leq q$, and $0 \leq m_i \leq q-1$ for $i > k$. Assume that $h > 0$. Consider a subspace S of U of dimension $N-2$. Then $h + q(T(f) - w(S)) \leq f$ and thus $h - qw(S) \leq \sum_{i=0}^{N-1} \epsilon_i$. As $h = qT(h) + \sum_{i=k}^{N-1} m_i$, it follows that

$$qT(h) + \sum_{i=k}^{N-1} m_i \leq qw(S) + \sum_{i=0}^{N-1} \epsilon_i. \tag{2}$$

As $\sum_{i=0}^{N-1} \epsilon_i < q$, this implies that $w(S) \geq T(h)$. Then Lemma 2.1 applied to the restriction of w onto U shows that $T(h)$ is the minimum weight of the $(N-2)$-subspaces of U. Considering such a subspace S in (2), we find $\sum_{i=k}^{N-1} m_i \leq \sum_{i=0}^{N-1} \epsilon_i$.

For the induction step, we now consider the case that U has codimension at least two and a point of weight zero. Let H be a hyperplane on U. As we have proved the assertion for hyperplanes, we know that w induces in H a $\{\sum_{i=0}^{N-1} n_i v_{i+1}, \sum_{i=1}^{N-1} n_i v_i; N-1, q\}$-minihyper, with $\sum_{i=0}^{N-1} n_i \leq \sum_{i=0}^{N-1} \epsilon_i \leq q-1$. As U, and hence H, contains a point of weight zero, Lemma 2.2 applied to the minihyper in H shows that $n_{N-1} = 0$. Therefore we can apply the induction hypothesis to U considered as a subspace of H. This gives the desired minihyper in U where not only $\sum_{i=0}^{u-1} m_i \leq \sum_{i=0}^{N-1} \epsilon_i$ but even $\sum_{i=0}^{u-1} m_i \leq \sum_{i=0}^{N-2} n_i$. □

We need one more definition. If w is a weight function on $\mathrm{PG}(N, q)$, then for any subspace S of $\mathrm{PG}(N, q)$, we call the number $m := \min\{w(P) | P \in S\}$ the *multiplicity* of the subspace S; we also say that the subspace S occurs with multiplicity m in the minihyper. Of course, if S has dimension s and multiplicity m, then the weight $w(S)$ of S is at least $v_{s+1} m$.

Lemma 2.4. *Suppose that* w *is a weight function on the point set of* $\mathrm{PG}(N, q)$*, which is the sum of the characteristic functions of* $r \leq q-1$ *non-empty subspaces* $S_1, \ldots, S_r$*. Then for any non-empty subspace* T*, the following results hold.*

(a) *The multiplicity of* T *is equal to the number of subspaces* S_i *that contain* T*.*

(b) *If* $\dim(T) \geq 1$*, then there exists an* $(N-1)$*-subspace* Δ *that does not contain any of the subspaces* $T, S_1, \ldots, S_r$ *and such that* $T \cap \Delta$ *has the same multiplicity as* T*.*

Proof. (a) Let m be the number of subspaces S_i containing T. Then the remaining $r - m \leq q - 1$ subspaces S_i cannot cover T. Hence, T possesses a point that has weight m. Therefore, T has multiplicity m.

(b) At most $r \leq q-1$ non-empty and proper subspaces of T have the form $T \cap S_i$ for some i. Thus some hyperplane S of T will not contain any of these. Then every subspace S_i that contains S also contains T, so S and T have the same multiplicity. Also S is a proper subspace of all subspaces $\langle S, S_i \rangle$ and as $r \leq q-1$, this implies that some hyperplane Δ on S will not contain any of the subspaces $T, S_1, \ldots, S_r$. $\square$

3. $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; k-1, q\}$-minihypers

In this section, we prove the basic results on minihypers in $\mathrm{PG}(2, q)$. Since an $\{f, t; 2, q\}$-minihyper meets every line in at least t points, there is a strong connection to the theory of blocking sets in projective planes. Our first lemma generalizes a result on blocking sets of [1], in which the following lemma is proved for non-weighted minihypers without restriction on t.

Lemma 3.1. *A weighted* $\{f, t; 2, q\}$*-minihyper* (B, w)*, with* $1 \leq t < q-1$ *and* $q \geq 3$*, contains a line or satisfies* $f \geq tq + \sqrt{tq} + 1$*.*

Proof. We assume in the proof that $t \geq 2$. For, if $t = 1$, then B defines a blocking set w.r.t. the lines of $\mathrm{PG}(2, q)$. It follows from [4, 5] that every blocking set B, not containing a line of $\mathrm{PG}(2, q)$, contains at least $q + \sqrt{q} + 1$ points.

Defining $m := \sum_{P \in B}(w(P) - 1)$ and $s = f - tq - 1$, we have $|B| = f - m = qt + 1 + s - m$. Suppose that B contains no line. Then there exist points not in B. As the $q+1$ lines on such a point all have weight at least t, we find $f \geq (q+1)t$, that is, $s \geq t - 1$. Also, if l is a line, then considering a point of l that is not in B and the other q lines on this point, we find $w(l) \leq f - tq = s + 1$.

Consider a point X. The sum of the weights of the lines on X is $f + q \cdot w(X)$. Hence, the sum of the numbers $w(l) - t$ for the lines l on X is $f + qw(X) - (q+1)t = s + 1 - t + qw(X)$. This we use to estimate the number

$$\Delta := \sum_{P \notin B} \sum_{Q \in B} w(Q) \cdot (w(PQ) - t),$$

where PQ denotes the line through the points P and Q. First of all, for $P \notin B$, we have

$$\begin{aligned} \sum_{Q \in B} w(Q) \cdot (w(PQ) - t) &= \sum_{P \in l} |l \cap B|(|l \cap B| - t) \\ &\leq \sum_{P \in l} (s+1)(|l \cap B| - t) = (s+1)(s+1-t), \end{aligned}$$

since this is a sum of weights of lines l on P, where each line l on P occurs $w(l) \leq s+1$ times. As $|B| = f - m$, we find

$$\Delta \leq (q^2 + q + 1 - f + m)(s+1)(s+1-t).$$

If $Q \in B$, then each line on Q has at least $q-s$ points not in B and thus we find similarly

$$\begin{aligned}\sum_{P \notin B} w(Q) \cdot (w(PQ) - t) &\geq (q-s) \sum_{Q \in l} w(Q)(|l \cap B| - t) \\ &= w(Q)(q-s)(s+1-t+qw(Q)),\end{aligned}$$

where the sum is over all lines l on Q. Using

$$\begin{aligned}\sum_{Q \in B} w(Q)(s+1-t+qw(Q)) &= f(s+1-t+q) + q \sum_{Q \in B} w(Q)(w(Q)-1) \\ &\geq f(s+1-t+q) + 2qm,\end{aligned}$$

we find a lower bound on Δ. Comparing both bounds yields

$$(q^2+q+1-f+m)(s+1)(s+1-t) \geq (q-s)(f(s+1-t+q)+2qm).$$

We may assume that $s \leq \sqrt{qt}$, since otherwise we are done. Using $2 \leq t < q-1$ and $s \leq \sqrt{tq} < (q+t)/2$, we see that the coefficient of m on the left hand side is smaller than the coefficient of m on the right hand side. Hence, the inequality remains true when deleting the m-terms. Using $f = qt+1+s$, the remaining inequality simplifies to

$$s^2 \geq tq + (t-1)(s-t).$$

If $s \geq t$, then we find $s \geq \sqrt{tq}$ as desired. Assume finally that $s < t$. Then $s = t-1$, $f = tq+t$, and every line on a point not in B must have exactly weight t. Since B contains no line, then every line has weight exactly t. But counting the total weight using the lines on a point X, we find $f = (q+1)t - qw(X)$, which implies $w(X) = 0$ for all points, that is, $f = 0$. As $t \geq 2$, this is a contradiction. $\square$

For weighted multiple blocking sets having at least one point of weight one, we can obtain better results using polynomial techniques. These improvements are described in [6], where they are used to prove characterization results on non-weighted minihypers. In [8], using polynomial techniques, the following corollary is proved. Together with Lemma 3.1, it will play an important role in characterizing planar minihypers.

Result 3.2. *Let (B,w) be a weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 2, q\}$-minihyper with $\epsilon_1+\epsilon_0 < q$. Then every point of (B,w) that lies on a line of weight ϵ_1 lies on at least $q+1-\epsilon_0-\epsilon_1$ different lines of weight ϵ_1.*

We can now prove a characterization result on certain weighted $\{f, m; N, q\}$-minihypers (F,w). The main goal will always be to prove that a minihyper is a sum of subspaces of the ambient projective space.

Lemma 3.3. *A weighted $\{\epsilon_1(q+1)+\epsilon_0, \epsilon_1; 2, q\}$-minihyper (F,w), with $\epsilon_1+\epsilon_0 < \sqrt{q}+1$ and $q \geq 4$, is a sum of ϵ_1 lines and ϵ_0 points.*

Proof. The case $\epsilon_0 = 0$ is discussed in [8, Theorem 2.5] and in [16, Theorem 20]. In [8, Theorem 2.5], it is proven that there does not exist a weighted minimal ϵ_1-fold blocking set (B,w) in $\mathrm{PG}(2,q)$ of size $|(B,w)| = \epsilon_1(q+1)+1$ for $\epsilon_1 < (q+1)/2$. So such a weighted ϵ_1-fold blocking set can be reduced to an ϵ_1-fold blocking set in $\mathrm{PG}(2,q)$ of size $\epsilon_1(q+1)$, which is a sum of ϵ_1 lines by the previous results.

Hence, from now on, we assume that $\epsilon_0 \geq 2$. For $q = 7$, this means that also an $\{q+1+2, 1; 2, q\}$-minihyper must be discussed. This is a 1-fold blocking set in

PG$(2,7)$. By results of Blokhuis [3], such a weighted blocking set is the sum of one line and two points. So suppose from now on that $q \geq 9$.

For $\epsilon_1 = 0$, this lemma is trivial. So suppose that $\epsilon_1 > 0$. By Lemma 3.1, F contains a line l. We shall show in the next paragraph that this line has weight at least $q + \epsilon_1$. Thus, removing l from (F, w) (that is, reducing the weight of every point of l by one) yields an $\{(\epsilon_1 - 1)(q+1) + \epsilon_0, \epsilon_1 - 1; 2, q\}$-minihyper. Then an induction argument on ϵ_1 completes the proof.

Assume that l has only weight $q + \epsilon_1 - \delta$ for some $\delta > 0$. As l is contained in F, then $\delta \leq \epsilon_1 - 1$. Now we reduce only the weight of $q + 1 - \delta$ different points from l by one; this results in a $\{(q+1)\epsilon_1' + \epsilon_0', \epsilon_1'; 2, q\}$-minihyper with $\epsilon_1' = \epsilon_1 - 1$ and $\epsilon_0' = \epsilon_0 + \delta$, in which l will have weight ϵ_1'. If we consider a point of l whose weight has not been reduced, then the other q lines on this point will have the same weight as before, that is, at least weight $\epsilon_1 = \epsilon_1' + 1$. This contradicts Result 3.2. Note that Result 3.2 requires $\epsilon_1' + \epsilon_0' < q$; as $\epsilon_1' + \epsilon_0' \leq 2(\epsilon_1 + \epsilon_0 - 1) < 2\sqrt{q}$, this follows from $q \geq 9$. □

We now generalize this to arbitrary dimensions.

Proposition 3.4. *An* $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; N, q\}$*-minihyper* (F, w)*,* $N \geq 2$*,* $q \geq 4$*, and* $\epsilon_1 + \epsilon_0 < \sqrt{q} + 1$*, is a sum of* ϵ_1 *lines and* ϵ_0 *points.*

Proof. We use induction on N; the case $N = 2$ being handled in the previous lemma. Now assume that $N \geq 3$ and that the statement holds for $N - 1$.

Consider a point $P \in \mathrm{PG}(N, q) \setminus F$. Projecting (F, w) from P onto a hyperplane π of PG(N, q), with $P \notin \pi$, yields in the following way a minihyper (F', w') in PG$(N-1, q) = \pi$. The set F' is defined as the projection of F from P, i.e. $P' \in F'$ if and only if the line PP' intersects the set F. For any point $P' \in F'$, we define $w'(P') = w(PP')$. It is clear that (F', w') is an $\{\epsilon_1(q+1) + \epsilon_0, \epsilon_1; N-1, q\}$-minihyper. By the induction hypothesis on N, (F', w') is a sum of ϵ_1 lines and ϵ_0 points.

Consider a line l' contained in the projection (F', w') that occurs m times in this sum. Then $w'(l') \geq (q+1)m$ and $w'(l') \leq qm + \epsilon_1 + \epsilon_0 < (m+1)q$. The plane τ generated by l' and P has weight $w(\tau) = w'(l')$. Proposition 2.3 shows that w induces in τ an $\{m(q+1) + n, m; 2, q\}$-minihyper (F'', w'') with $m + n < 1 + \sqrt{q}$. By Lemma 3.3, (F'', w'') is a sum of m lines and n points. Thus, the line l' is the projection of m lines contained in F.

Since this holds for all lines l' contained in (F', w'), we thus find ϵ_1 (not necessarily distinct) lines $l_1, \ldots, l_{\epsilon_1}$ contained in F that are projected to the lines of (F', w'). Our argument also shows the following. If a line l occurs x times in the list $l_1, \ldots, l_{\epsilon_1}$, then every point of l has weight at least x. We want to show that the sum of the lines $l_1, \ldots, l_{\epsilon_1}$ is contained in the minihyper (F, w). For this we have to show for each point X that $w(X)$ is equal or larger to the number of indices i with $X \in l_i$.

To see this, we select a line h on X that meets F only in X, and we project again as before but using this time for P a point of h that is different from X. As already noticed, the projection (F', w') of (F, w) contains a sum of ϵ_1 lines. As $\epsilon_0 + \epsilon_1 < \sqrt{q} + 1$, it is readily seen that these ϵ_1 lines are the images of the lines $l_1, \ldots, l_{\epsilon_1}$. As h meets F only in X, then the point X' onto which X is projected satisfies $w'(X') = w(X)$. Now, if X lies on x of the lines $l_1, \ldots, l_{\epsilon_1}$, then X' lies on x of the projected lines, and as (F', w') contains the sum of these lines, we have $w'(X') \geq x$. Hence, $w(X) \geq x$.

Now we have shown that (F, w) contains in fact the sum of the lines $l_1, \dots, l_{\epsilon_1}$, and is therefore the sum of these lines and of ϵ_0 points. □

4. Proof of the main theorem

We start with a lemma that will be used to find large subspaces in minihypers.

Lemma 4.1. *Let (F, w) be a $\{\sum_{i=0}^{N-1} \epsilon_i v_{i+1}, \sum_{i=1}^{N-1} \epsilon_i v_i; N, q\}$-minihyper with $q \geq 4$ and $\sum_{i=0}^{N-1} \epsilon_i < \sqrt{q} + 1$. Suppose that P is a point of F lying on two subspaces S_1 and S_2 of multiplicity m_1 and m_2 such that $m_1 + m_2 > w(P)$. Then the subspace $\langle S_1, S_2 \rangle$ is completely contained in F.*

Proof. As $m_1, m_2 \leq w(P)$, then m_1 and m_2 are positive and thus S_1 and S_2 are contained in F. Therefore the statement is trivial, if one of the subspaces S_1 and S_2 contains the other. Otherwise, $\langle S_1, S_2 \rangle$ is the union of planes $\langle l_1, l_2 \rangle$ with lines l_i on P and in S_i. It suffices thus to show that these planes are contained in F.

Assume that such a plane $\pi := \langle l_1, l_2 \rangle$ is not contained in F. Then Proposition 2.3 gives $w(\pi) = a_1(q + 1) + a_0$, with integers $a_0, a_1 \geq 0$ and $a_1 + a_0 < \sqrt{q} + 1$, and such that every line of π has weight at least a_1. Since each point of l_i has weight at least m_i and since the other $q - 1$ lines of π on P each have weight at least a_1, we find

$$a_1(q + 1) + a_0 = w(\pi) \geq w(P) + qm_1 + qm_2 + (q - 1)(a_1 - w(P)).$$

Since $m_1 + m_2 \geq w(P) + 1$, it follows that $2a_1 + a_0 \geq q + 2w(P)$. As $w(P) \geq 1$, $q \geq 4$, and $2a_1 + a_0 < 2(\sqrt{q} + 1)$, this is impossible. □

Consider in $\mathrm{PG}(N, q)$ a $\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=0}^{t} \epsilon_i v_i; N, q\}$-minihyper (F, w), with $0 \leq t < N$ and $\sum_{i=0}^{t} \epsilon_i < \sqrt{q} + 1$. We want to show that it is a sum of subspaces, where the sum consists of ϵ_i subspaces of dimension i for $i = 0, \dots, t$. The proof is by induction on t. The case when $t = 0$ is trivial. The case $t = 1$ was proved in the last section. In the rest of this section, we prove the induction step. We thus suppose that $t \geq 2$ and that the assertion is true for smaller values of t. We also assume that $\epsilon_t > 0$, because otherwise we can immediately apply the induction hypothesis.

Lemma 4.2. *Consider a $(t - 1)$-subspace U with the property that there exists a hyperplane π_0 on U of weight $w(\pi_0) = \gamma_0 + \sum_{i=1}^{t} \epsilon_i v_i$, with $\gamma_0 \leq \epsilon_0$. Then $m(U) = \sum m(T)$ where the sum runs over all t-subspaces T containing U.*

Proof. We prove this by induction on $m(U)$. If $m(U) = 0$, this is trivial. Suppose now that $m(U) > 0$. By the induction hypothesis of this section, we see that the restriction of w to π_0 is a sum of $\gamma_0 + \epsilon_1 + \cdots + \epsilon_t < \sqrt{q} + 1$ subspaces. Lemma 2.4 shows that π_0 has a hyperplane Δ not containing any of the subspaces of this sum, that is, the weight of Δ is

$$\delta := \sum_{i=2}^{t} \epsilon_i v_{i-1},$$

and such that $S := U \cap \Delta$ has dimension $t - 2$ and multiplicity $m(U)$. This shows already that U is the only subspace of the sum $\pi_0 \cap (F, w)$ passing through S.

Consider the remaining hyperplanes $\pi_1, \ldots, \pi_q$ on Δ. By Lemma 2.2, the restriction of w to the hyperplanes π_i produces $\{\gamma_i + \sum_{j=1}^{t} \epsilon_j v_j, \sum_{j=2}^{t} \epsilon_j v_{j-1}; N-1, q\}$-minihypers in π_i, where $\gamma_i \geq 0$ and $\sum_{i=0}^{q} \gamma_i = \epsilon_0$. The global induction hypothesis of this section shows that these minihypers can be uniquely written as a sum of subspaces. The number of subspaces in this sum is $\gamma_i + \epsilon_1 + \cdots + \epsilon_t < \sqrt{q} + 1$. As $w(\Delta) = \delta$, we see that Δ does not contain any of the subspaces that occurs in the sum for π_i. In particular, S is not one of the subspaces occurring in the sum that makes up the minihyper in π_i.

Let U_i^j, $j = 1, \ldots, r_i$, be the different $(t-1)$-subspaces of π_i on S that have positive multiplicity $m_{i,j}$ (thus $r_0 = 1$ and $U_0^1 = U$ and $m_{0,1} = m(U) = m(S)$). Lemma 2.4 gives $\sum_j m_{i,j} = m(S)$ for each i.

First consider the case that $r_i = 1$ for all i, that is, U_i^1 is the only $(t-1)$-subspace of positive multiplicity of π_i on S, and $m(U_i^1) = m(S)$. By Lemma 4.1, the t-subspace $T := \langle U_1^1, U_2^1 \rangle$ is contained in F. Then $T \cap \pi_i$ is a $(t-1)$-subspace of π_i of positive multiplicity containing S. Hence, $T \cap \pi_i = U_i^1$ for all i and thus T is the union of the subspaces $U_0^1, \ldots, U_q^1$. As all these subspaces have multiplicity $m(S)$, we see that all points of T have weight at least $m(S)$, that is, T has multiplicity at least $m(S) = m(U)$. As the multiplicity of T cannot exceed the one of U, then $m(T) = m(U)$. By construction, no other t-subspace of positive multiplicity contains U (or S).

Now consider the case that some $r_i > 1$, say $r_1 > 1$. As $\sum_j m(U_1^j) = m(S)$, it follows that $m(U_1^j) < m(S) = m(U)$ for all j. Thus, the induction hypothesis applied to U_1^j gives $\sum m(T) = m(U_1^j)$ where the sum runs over all t-subspaces T containing U_1^j. As π_1 does not contain a t-subspace of (F, w), we see that different subspaces U_1^j produce different subspaces T, in fact that every t-subspace of positive multiplicity contains exactly one of the subspaces U_1^j. Therefore $\sum m(T) = \sum_j m(U_1^j) = m(S)$ where the first sum runs over all t-subspaces containing S. Every t-subspace of positive multiplicity occurring in this sum meets π_0 in a $(t-1)$-subspace of positive multiplicity and containing S. As U is the only such $(t-1)$-subspace of π_0, all these t-subspaces contain U and so we are done as $m(U) = m(S)$. $\square$

Lemma 4.3. *The weighted minihyper (F, w) contains a sum of ϵ_t subspaces of dimension t.*

Proof. Using Lemma 2.2, we find a subspace Δ of codimension two with the following property. If $\pi_0, \ldots, \pi_q$ are the hyperplanes on Δ, then the restriction of w to π_j is a $\{\gamma_j + \sum_{i=1}^{t} \epsilon_i v_i, \sum_{i=2}^{t} \epsilon_i v_{i-1}; N-1, q\}$-minihyper in π_j where $\gamma_j \geq 0$ and $\sum_{j=0}^{q} \gamma_j = \epsilon_0$. The global induction hypothesis of this section shows that the minihyper in π_j is a sum of subspaces, containing ϵ_i subspaces of dimension $i-1$ for $i = 2, \ldots, t$, and $\epsilon_1 + \gamma_j$ points.

Consider π_0. Let $U_1, \ldots, U_s$ be the different $(t-1)$-subspaces occurring in the sum that make up the minihyper in π_0 and let U_i occur m_i times in this sum. Then $\sum_{i=1}^{s} m_i = \epsilon_t$. By Lemma 2.4, the subspace U_i has multiplicity m_i; also the U_i are all $(t-1)$-subspaces of π_0 of positive multiplicity. As $w(\pi_0) < v_{t+1}$, then π_0 contains no t-subspace of positive multiplicity. Lemma 4.2 applied to the U_i now shows the following. If $\mathcal{T}$ is the set of all t-subspaces of positive multiplicity, then $\sum_{T \in \mathcal{T}} m(T) = \epsilon_t$; moreover, we have $m(U) = \sum_{U \subset T \in \mathcal{T}} m(T)$ for every $(t-1)$-subspace U of π_0.

Now, if P is a point of π_0, then the sum of the $m(U)$ over all $(t-1)$-subspaces U of π_0 on P is less than or equal to $w(P)$ (because we know the structure of the minihyper in π_0). Therefore, the sum of the $m(T)$ over all t-subspaces $T \in \mathcal{T}$ that contain P is less than or equal to $w(P)$. Since the same property can be proved for the points of $\pi_1, \ldots, \pi_q$, it holds for all points. Thus using each $T \in \mathcal{T}$ exactly $m(T)$ times, then the sum of these ϵ_t t-subspaces is contained in (F, w). $\square$

Theorem 4.4. *A weighted $\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=0}^{t} \epsilon_i v_i; N, q\}$-minihyper (F, w), $q \geq 4$, where $t \leq N-1$, $0 \leq \epsilon_i \leq q-1$, $i = 0, \ldots, t$, $\sum_{i=0}^{t} \epsilon_i < \sqrt{q}+1$, is a sum of $\sum_{i=0}^{t} \epsilon_i$ subspaces, where for each i exactly ϵ_i of these subspaces have dimension i.*

Proof. By the preceding lemma, w contains a sum of ϵ_t non-necessarily distinct subspaces $T_1, \ldots, T_{\epsilon_t}$ of dimension t. This means for each point P that $w(P)$ is at least as large as the number of subspaces T_i that contain P. Let S_i be a (fixed) hyperplane of T_i. We define as follows a new weight function w'. For every point P, we define $w'(P)$ to be equal to $w(P)$ minus the number of affine spaces $T_i \setminus S_i$ that contain P. Note that $w'(P) \geq 0$ for all points P. Clearly the sum of the weights $w'(P)$ over all points is $\epsilon_t q^t$ less than the corresponding sum for the original function w, and thus it is

$$\sum_P w'(P) = \epsilon_t v_t + \sum_{i=0}^{t-1} \epsilon_i v_{i+1} =: f.$$

We analyze the w'-weight of the hyperplanes π. If π does not contain any of the S_i, then it also does not contain any of the t-subspaces T_i; in this case we have

$$w'(\pi) = w(\pi) - \epsilon_t q^{t-1} \geq \sum_{i=1}^{t} \epsilon_i v_i - \epsilon_t q^{t-1} = \epsilon_t v_{t-1} + \sum_{i=1}^{t-1} \epsilon_i v_i =: h.$$

If π contains a subspace S_i, then we have $w'(\pi) \geq v_t$, which is even better. Therefore Lemma 2.1 shows that w' defines an $\{f, h; N, q\}$-minihyper. The induction hypothesis in this section shows that this minihyper is a sum of subspaces, namely $\epsilon_t + \epsilon_{t-1}$ subspaces of dimension $t-1$ and, for $i < t-1$, another ϵ_i subspaces of dimension i. As $\sum_{i=0}^{t} \epsilon_i < \sqrt{q}+1$, the subspaces $S_1, \ldots, S_{\epsilon_t}$ must occur in this sum. It follows that (F, w) is the sum obtained from the previous sum when replacing S_i by T_i. $\square$

The following corollary now follows immediately, which is in fact the known result of Hamada, Helleseth, and Maekawa (Result 1.2).

Corollary 4.5. *A non-weighted $\{\sum_{i=0}^{t} \epsilon_i v_{i+1}, \sum_{i=1}^{t} \epsilon_i v_i; N, q\}$-minihyper F, $q \geq 4$, with $\sum_{i=0}^{t} \epsilon_i < \sqrt{q}+1$, is the union of ϵ_t t-dimensional subspaces, ϵ_{t-1} $(t-1)$-dimensional subspaces, …, ϵ_1 lines, and ϵ_0 points, which are all pairwise disjoint.*

Proof. In a non-weighted minihyper, we can define the weight function w by giving the points of F weight one, and the points not belonging to F weight zero. Then F can be described as a sum of the subspaces mentioned in the statement of this corollary, but since the points of F have weight one, these subspaces must be pairwise disjoint. $\square$

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