Passivity-Preserving Parameterized Model Order Reduction using Singular Values and Matrix Interpolation

Elizabeth Rita Samuel, Francesco Ferranti, *Member, IEEE,*, Luc Knockaert, *Senior Member, IEEE,* and Tom Dhaene, *Senior Member, IEEE*

***Abstract*—We present a parameterized model order reduction method based on singular values and matrix interpolation. First a fast technique using grammians is utilized to estimate the reduced order and then common projection matrices are used to build parameterized reduced order models. The design space is divided into cells and a Krylov subspace is computed for each cell vertex model. The truncation of the singular values of the merged Krylov subspaces from the models located at the vertices of each cell yields a common projection matrix per design space cell. Finally, the reduced system matrices are interpolated using positive interpolation schemes to obtain a guaranteed passive parameterized reduced order model. Pertinent numerical results validate the proposed technique.**

***Index Terms*—Parameterized Model Order Reduction, Grammians, Singular Values, Projection Matrix, Interpolation, Passivity.**

I. Introduction

ELECTROMAGNETIC (EM) methods [1]–[3], have become indispensable analysis and design tools for a variety of complex high-speed systems. However, a major drawback of EM methods is that they usually generate very large systems of equations. The optimization and simulation of these large scale models is therefore computationally expensive, not to say unfeasible.

Therefore, model order reduction (MOR) techniques are crucial to reduce the complexity of large scale models and the computational cost of the simulations, while retaining the important physical features of the original system [4]–[9]. Over the past two decades active research has been focused on model reduction in the field of EM methods. Two main classes of MOR methods can be distinguished: 1) moment matching methods, and 2) balanced and Hankel norm methods. The moment matching methods for large-scale problems have led to the use of Krylov and rational Krylov subspace projection methods. The importance of producing passive (a.k.a. positive-real) reduced order models (ROMs) resulted in several algorithms that preserve passivity of *RLC* circuits [8], [9]. As Krylov techniques fail to generate models with provable error bounds [10], the balanced and Hankel norm approaches have gained attention in the MOR research area. Balanced and Hankel norm reduction methods, already well-developed in the control literature [11], have a very close connection to the singular value decomposition (*svd*) and have been receiving renewed attention in the electronic design automation (EDA) community. These methods preserve asymptotic stability and allow for global error bounds. As they rely upon dense matrix computations they do not scale well in terms of computational efficiency and numerical stability. A strong current trend aims at combining these two classes of methods and their corresponding advantages.

MOR techniques perform model order reduction only with respect to frequency. However, during the circuit design synthesis of large-scale applications, it is also essential to analyze the response of a circuit as a function of design parameters, such as geometrical and other characteristics. A typical design procedure includes optimization and design space exploration, and thus requires repeated simulations for different design parameter values. Parameterized model order reduction (PMOR) methods can reduce large systems of equations with respect to frequency and also other design parameters of the circuit and are therefore well suited to efficiently perform these design activities.

A number of PMOR methods have been developed in recent years. The multiparameter moment-matching methods presented in [12], [13] use a subspace projection approach. However, the resulting ROMs usually suffer from oversize when the number of moments to match is high, either because high accuracy is required or because the number of parameters is large. The parameterized interconnect macromodeling via a two-directional Arnoldi process (PIMTAP) algorithm presented in [14], preserves the passivity of parameterized *RLC* networks, but as all multiparameter moment-matching-based PMOR techniques, it is suitable only for a low-dimensional design space. The selection of the multidimensional expansion points and the number of multiparameter moments needs to be addressed in these methods. The technique presented in [15] combines traditional passivity-preserving MOR methods and interpolation schemes based on a class of positive interpolation operators. A PMOR method based on EM matrix parameterization and projection subspaces is proposed in [16]. Overall passivity of parameterized ROMs is guaranteed over the design space of interest in [15], [16]. A matrix interpolation-based technique [17], computes a set of reduced system matrices in a common subspace and interpolates them to generate a parameterized ROM. This technique avoids the oversize

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problem of multiparameter moment matching algorithms, but the reduced system matrices needed for interpolation must have the same reduced order and must be postprocessed for reprojection onto a common subspace. The passivity of parameterized ROM is not guaranteed with this approach.

This paper proposes a novel PMOR technique that remediates the shortcomings of the method in [17] by using a-priori reduced order estimation, common projection matrices (locally (cell by cell) or globally), design space decomposition and passivity-preserving parameterization schemes. A fast technique using grammians is first utilized to estimate the reduced order, after which projection matrices are used to build parameterized models. The design space is divided into cells and a Krylov subspace is computed for each cell and each cell vertex model. The truncation of the singular values of the merged Krylov subspaces computed from the models at the vertices of each cell generates a common projection matrix per cell for *local* approach. For *global* approach the whole design space is considered as a cell and a common projection matrix is computed globally. Next, the reduced system matrices are interpolated using positive interpolation schemes to obtain a passive parameterized ROM. The Krylov subspaces can be found using Krylov-based MOR methods. In this paper we use the Laguerre-SVD technique [9].

The paper is organized as follows. Section II describes the fast reduced order estimation algorithm. Section III describes the generation of the common projection matrices and proposes a flowchart with the logical steps of the novel technique. Next, multivariate interpolation schemes and passivity-preservation are described in Section IV. Finally some pertinent examples validate the proposed technique in Section V.

II. Estimation of the Reduced Order Based on Grammians

An a-priori reduced order estimation makes the construction of parameterized ROMs much more efficient. The reduced order can be estimated by studying the so-called Hankel singular values which are based on the system grammians. The system grammians are positive-semidefinite matrices that express the controllability and observability properties of systems.

The reduced order is computed at the corner points of the design space.The design space is sampled as described in [16]. It contains all parameters except frequency. Two data grids are used in the modeling process: an estimation grid and a validation grid as shown in Fig. 1. Parametrized ROMs are estimated locally (cell by cell) or globally using the estimation grid and are validated over the validation grid. First, we estimate the reduced order as follows.

Let us consider a parameterized dynamical system with N design parameters $\mathbf{g} = (g^{(1)}, ..., g^{(N)})$ in descriptor state-space form :

$$\begin{aligned} \mathbf{C}(\mathbf{g})\frac{d\mathbf{x}(t,\mathbf{g})}{dt} &= -\mathbf{G}(\mathbf{g})\mathbf{x}(t,\mathbf{g}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t,\mathbf{g}) &= \mathbf{L}'\mathbf{x}(t,\mathbf{g}) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1}$$

The fast and efficient modified Smith technique [18], [19] enables to find the controllability grammian (W_c) and the observability grammian (W_o) of a large system. For the state-space model described in (1), the generalized grammians are defined as the unique solutions of the linear equations

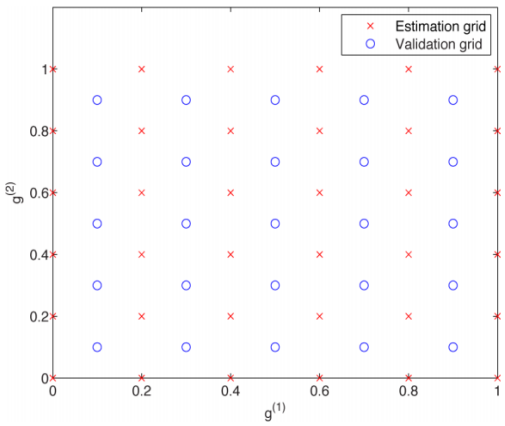

Fig. 1. Example of an uniformly sampled estimation and validation design space grids.

$$-\mathbf{C}(\mathbf{g})\mathbf{W_c}\mathbf{G}(\mathbf{g})' - \mathbf{G}(\mathbf{g})\mathbf{W_c}\mathbf{C}(\mathbf{g})' + \mathbf{B}\mathbf{B}' = 0 \tag{2}$$

$$-\mathbf{C}(\mathbf{g})'\mathbf{W_o}\mathbf{G}(\mathbf{g}) - \mathbf{G}(\mathbf{g})'\mathbf{W_o}\mathbf{C}(\mathbf{g}) + \mathbf{L}\mathbf{L}' = 0 \tag{3}$$

For every real scalar $p < 0$, the Stein equation [20] can be written for (2) as shown:

$$\mathbf{A}_p\mathbf{W_c}\mathbf{A}_p' - \mathbf{W_c} + \mathbf{B}_p\mathbf{B}_p' = 0 \tag{4}$$

where $\mathbf{A}_p = (p\mathbf{C}(g) - \mathbf{G}(g))^{-1}(p\mathbf{C}(g) + \mathbf{G}(g))$, and $\mathbf{B}_p = \sqrt{(-2p)}(p\mathbf{C}(g) - \mathbf{G}(g))^{-1}\mathbf{B}$. It follows that $\mathbf{W_c} = \sum_{j=0}^{\infty}\mathbf{A}_p^j\mathbf{B}_p\mathbf{B}_p'(\mathbf{A}_p')^j$ [20]–[22]. In practice the spectral radius of $\mathbf{A}_p$ should be minimized so that the power terms decay quickly and the infinite summation can be well approximated by finite terms.

$$\mathbf{W_c} \approx \sum_{j=0}^{k-1}\mathbf{A}_p^j\mathbf{B}_p\mathbf{B}_p'(\mathbf{A}_p')^j = \mathscr{K}_k(\mathbf{A}_p,\mathbf{B}_p)\mathscr{K}_k(\mathbf{A}_p,\mathbf{B}_p)^T \tag{5}$$

where $\mathscr{K}_k(\mathbf{A}_p,\mathbf{B}_p) = [\mathbf{B}_p\mathbf{A}_p\mathbf{B}_p...\mathbf{A}_p^{k-1}\mathbf{B}_p]$ is called the kth order Krylov matrix and serves as a Cholesky factor of $\mathbf{W_c}$. Similarly, taking $\tilde{\mathbf{A}}_p = (p\mathbf{C}(g) + \mathbf{G}(g))(p\mathbf{C}(g) - \mathbf{G}(g))^{-1}$, and $\mathbf{L}_p = \sqrt{(-2p)}\mathbf{L}(p\mathbf{C}(g) - \mathbf{G}(g))^{-1}$, the observability grammian $\mathbf{W_o}$ can be computed.

The value of k in (5) can be found from the convergence criterion:

$$\frac{\| \mathbf{W}_c^{k-1} - \mathbf{W}_c^{k}\|_2}{\| \mathbf{W}_c^{k-1}\|_2} \leq threshold \tag{6}$$

The Smith method is similar to the alternating direction implicit (ADI) method [11], [23], [24]. The Smith method is chosen because of its ease of exposition and also because it requires only one large-scale matrix inversion (in finding $\mathbf{A}_p$).

Next, the Hankel singular values σ_i, which quantifies the reachability and observability of a system, are defined as the square root of the eigenvalues of the product of the grammians as shown:

$$\sqrt{eig(\mathbf{W_c}\mathbf{W_o})} = \sigma_i \tag{7}$$

Here we define the reduced order q, based on the first q significant singular values, by setting a threshold for the ratio of the Hankel singular values and the largest singular value.

$$\frac{\sigma_i}{\sigma_{max}} \geq threshold_\sigma \ , i = 1, 2, ..., q \tag{8}$$

There are no a priori rules for setting the threshold, it can be adjusted to achieve the desired level of accuracy and compactness for the parameterized ROM.

Two strategies are proposed for the order estimation. First the reduced order is estimated at the corner points of the design space. Two strategies can be followed:

1) worst-case reduced order: the highest estimated reduced order at the corner points is extended to the entire design space. This approach can guarantee an accurate reduction over the design space.
2) best-case reduced order: the lowest estimated reduced order is extended to the design space. This approach can guarantee more compact models with respect to the worst-case, but the reduced order may be increased for some design space regions by a bottom-up approach to guarantee the desired accuracy.

III. Common projection matrix computation

For each point in the estimation design space grid as described in Section II, a Krylov-based MOR method is applied to the corresponding system and a set of projection matrices is obtained. In this paper, the Laguerre-SVD method [9] is used for this aim.

All the projection matrices will have the same dimension in the worst-case reduced order scenario, while it may have different dimensions for the best-case reduced order scenario. We propose two approaches for the construction of common projection matrices, namely *local* and *global*.

In the *local* approach, each design space cell has M vertices and for each cell an union of the vertex projection matrices is performed by column stacking

$$\mathbf{P}_{union} = [P_1, P_2, ...P_M] \tag{9}$$

In the *global* approach, the whole design space is considered as one cell and the projection matrices are computed for the estimation grid. All the projection matrices are merged by column stacking similarly as in (9). The dimension of $\mathbf{P}_{union}$ is $n \times w$ where n is the order of the system and $w = (q_1 + q_2 ... + q_M)$ with q_i the reduced order of the $i-th$ vertex of the cell. Then, the economy-size *svd* is computed for the union of the projection matrices

$$\mathbf{U}\Sigma\mathbf{V}' = svd(\mathbf{P}_{union}) \tag{10}$$

A common reduced order r for a cell is defined based on the first r significant singular values, by setting a threshold to the ratio of the singular values with respect to the largest singular value As in the case of the previous threshold value (8), there are no a priori rules for setting the threshold, it can be adjusted to achieve the desired level of accuracy and compactness for the parameterized ROM. Thus a common projection matrix $\mathbf{Q}_{comm}$ is obtained by the QR orthonormalization of $\mathbf{P}_{comm}$.

$$\mathbf{P}_{comm} = \mathbf{U}_r \Sigma_r \mathbf{V}_r{}'$$
$$[\mathbf{Q}_{comm}, \mathbf{R}] = qr(\mathbf{P}_{\mathbf{comm}}) \tag{11}$$

where $\mathbf{U}_r$, Σ_r and $\mathbf{V}_r$ have a truncated dimension of $n \times r$, $r \times r$ and $r \times r$ respectively. The congruence transformation using $\mathbf{Q}_{comm}$, the common projection matrix of dimension $n \times r$, on the original models of the design space gives the reduced system matrices for the specific cell. Using the *global* approach, means that one Q_{comm} is used over the entire design space.

Regarding the state-space equations of the system under study we assume that a topologically fixed discretization mesh is used and is independent of the specific design parameter values [16]. It preserves the size of the system matrices as well as the numbering of the mesh nodes and mesh edges. The mesh is only locally stretched or shrunk when shape parameters are modified. The matrices $\mathbf{B}$, $\mathbf{L}'$ are uniquely determined by the circuit topology and therefore remains constant, while the matrices $\mathbf{C}$ and $\mathbf{G}$ are defined as functions of the design parameters. Starting from a set of models in the estimation design space (generated with respect to a common space) using common projection matrices, it is straightforward to prove that all the reduced system matrices in the estimation grid are in the same subspace (locally or globally) and hence can be interpolated.

A flowchart that describes the different steps of the proposed technique is shown in Fig.2.

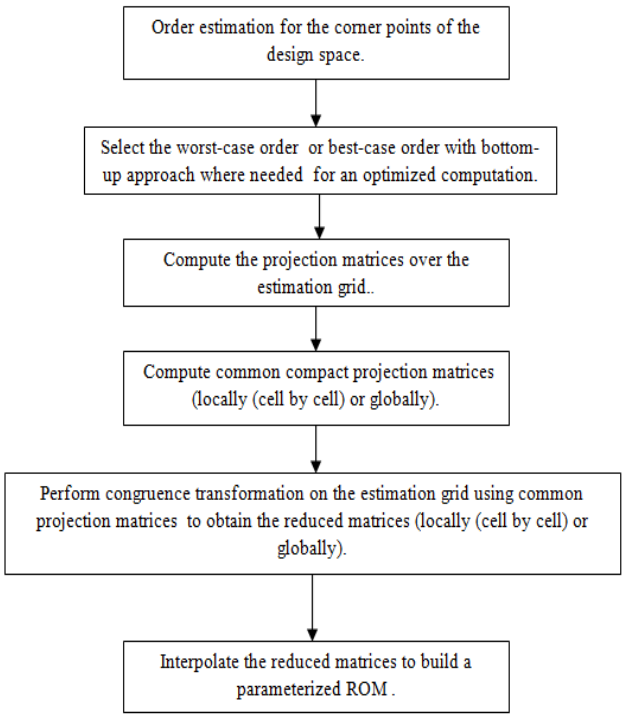

Fig. 2. Flowchart of the proposed technique.

From the flowchart one can see that the technique can be a combination of best-case or worst-case reduced order strategy

TABLE I
COLUMN SIZE (w) OF THE PROJECTION MATRIX ($\mathbf{P}_{union}$)

Approach	Best-Case	Worst-Case
Local	$2^N(q_{min})+\sum_{i=1}^{2^N} q_{bu_i}$	$2^N(q_{max})$
Global	$M(q_{min})+\sum_{i=1}^{M} q_{bu_i}$	$M(q_{max})$

with *local* or *global* approach. Depending on the scenario selected, the computation complexity as well as the accuracy and compactness of the parameterized ROM change. For the *local* approach a parameterized ROM is built cell by cell in the design space. In this paper, a hypercube [15] is considered as elementary design space cell for the *local* approach and it has 2^N vertices. The vertices increase exponentially with the number of dimensions, but this number still remains smaller than the number of estimation points in the whole design space that are used in the *global* approach. From Table I we can obtain the dimension of the merged projection matrix (9), for the different approaches before computing the compact common projection matrix. The following notations are used in the table:
- q_{min} the minimum of the reduced order estimated at the corner points of the design space.
- q_{max} the maximum of the reduced order estimated at the corner points of the design space.
- N the number of design parameters.
- M the total number of estimation points and $M \geq 2^N$.
- q_{bu_i} the order by which the best-case order is increased at the $i-th$ design space point using a bottom-up approach.

Concerning the complexity of the proposed technique, it can be noted that the most expensive step is related to the Smith's technique for the order estimation at the corner points of the design space, where the inverse of $\mathbf{A}_p$ (4) is required and its complexity is $O(n^3)$ with n equal to the actual order of the system. Then the projection matrix can be computed at each estimation point using any model order reduction technique that influences the complexity of this step. After computing the merged projection matrix (9), the *svd* has to be performed to obtain the common projection matrix, which has a complexity of $O(4n^2w)$ where w is the column size of the merged projection matrix. Therefore, depending on the approach chosen, as stated in Table I, the complexity of *svd* varies. When the *local* approach is chosen, the model will be quite compact as only 2^N points are considered for each design space cell. It is important to note that each cell will have its own compact common projection matrix. While in the case of *global* approach, the projection matrix is computed using the projection matrices of all estimation points. Therefore, it will be less compact than the *local* approach. On the other hand, it is computed once for the entire design space and then only one *svd* computation must be performed. When the number of design parameters increases, it leads to increase the size of the merged projection matrix and the computational complexity of the related *svd* operation (10). In high dimensional design spaces, the *local* approach is more feasible since it works cell by cell. After obtaining the common projection matrix, congruence transformation has to be performed and its complexity is equivalent to matrix multiplication. Then, the complexity of the last step depends on the selected interpolation scheme.

IV. Multivariate Interpolation

Once the reduced matrices are computed, they are interpolated to build a parameterized ROM. Multivariate interpolation can be realized by means of tensor product [25] or tessellation methods [26]. Any interpolation scheme in the class of positive interpolation operators [15] can be used, e.g., multilinear and simplicial methods [27], to preserve overall passivity as described in the sequel.

For example considering multilinear interpolation, each interpolated matrix $\mathbf{T}(g^{(1)},...,g^{(N)})$ is

$$\begin{aligned}\mathbf{T}(g^{(1)},...,g^{(N)}) = \quad & \sum_{k_1=1}^{K_1}\cdots\sum_{k_N=1}^{K_N}\mathbf{T}_{(g_{k_1}^{(1)},...,g_{k_N}^{(N)})} \\ & l_{k_1}(g^{(1)})\cdots l_{k_N}(g^{(N)})\end{aligned} \tag{12}$$

where K_1 is the number of estimation points and the interpolation kernel $l_{k_i}(g^i)$ satisfies the following constraints

$$\begin{aligned} 0 \leq l_{k_i}(g^{(i)}) \leq 1, \\ l_{k_i}(g^{(i)}) = \delta_{k_i,i} \\ \textstyle\sum_{k=1}^{K_1} l_{k_i}(g^{(i)}) = 1 \end{aligned} \tag{13}$$

It should be noted that the interpolation kernel functions of these methods only depend on the design space grid points and their computation does not require the prior solution of a linear system to impose an interpolatory constraint. Positive interpolation schemes have already been used in [15], where a parameterized macromodel is built by interpolating a set of ROMs treated as input-output systems, while preserving overall stability and passivity. Therefore, interpolating systems, matrices or scalars does not make any difference for these *local* interpolation kernel functions.

When performing transient analysis, stability and passivity must be guaranteed. It is known that, while a passive or positive-real system is also stable, the reverse is not necessarily true [28], which is crucial when the macromodel is to be utilized in a time domain simulator. Passive systems cannot generate more energy than they absorb through their ports. When the system is terminated on any arbitrary passive load, none of them will cause the system to become unstable [29], [30].

A. Systems with a special state-space form

In the PRIMA and Laguerre-SVD methods, the original systems are assumed to be in the descriptor state-space form (1). If the following conditions are satisfied:

$$\begin{aligned} \mathbf{C} = \mathbf{C}' \geq 0 \\ \mathbf{G}+\mathbf{G}' \geq 0 \\ \mathbf{B} = \mathbf{L} \end{aligned} \tag{14}$$

the passivity of the system with transfer function $\mathbf{Y}(s) = \mathbf{L}'(s\mathbf{C}+\mathbf{G})^{-1}\mathbf{B}$ is guaranteed [31]. For this specific format, PRIMA and Laguerre-SVD methods guarantee the passivity of

the reduced model built by congruence transformation using the projection matrix $\mathbf{Q}_{comm}$

$$\begin{aligned}\mathbf{C}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{C}(\mathbf{g})\mathbf{Q}_{comm} \geq 0\\ \mathbf{G}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{G}(\mathbf{g})\mathbf{Q}_{comm} \geq 0\\ \mathbf{B}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{B}(\mathbf{g})\\ \mathbf{L}_r(\mathbf{g}) &= \mathbf{Q}_{comm}{}'\mathbf{L}(\mathbf{g})\end{aligned} \tag{15}$$

Since any nonnegative linear combination of positive semi-definite matrices is a positive semi-definite matrix, stability and passivity are preserved over the entire design space if positive interpolation operators are used to interpolate the reduced matrices.

B. System with a general state-space form

Consider the following state-space form

$$\begin{aligned}\frac{d\mathbf{x}(t,\mathbf{g})}{dt} &= \mathbf{A}(\mathbf{g})\mathbf{x}(t,\mathbf{g}) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t,\mathbf{g}) &= \mathbf{L}'\mathbf{x}(t,g) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{16}$$

To build passive parameterized reduced order models, some additional steps with respect to the previous case are required. A MOR technique that preserves passivity of systems in the form (16) by using the solution of linear matrix inequalities (LMI) to generate a descriptor state-space format has been proposed in [32]. The original systems after LMI matrix computations are in a descriptor form satisfying properties (14), and therefore the passivity-preserving interpolation previously described can be used to build a passive parameterized ROM. This method is less expensive than the passivity-preserving technique described in [33], since only a single LMI equation has to be solved.

V. Numerical Results

Some pertinent numerical examples are used to demonstrate the accuracy and efficiency of the proposed PMOR technique.

Let us define the weighted RMS error as

$$\begin{aligned}Err(\mathbf{Y}_1(s),\mathbf{Y}_2(s)) &= \sqrt{\frac{\sum_{k=1}^{K_s}\sum_{i=1}^{P_{in}}\sum_{j=1}^{P_{out}}\frac{\left|Y_{1,(ij)}(s_k)-Y_{2,(ij)}(s_k)\right|^2}{W_{(ij)}(s_k)}}{P_{in}P_{out}K_s}}\\ W_{(ij)}(s_k) &= |Y_{2,(ij)}(s_k)|^2\end{aligned} \tag{17}$$

In (17) K_s, P_{in} and P_{out} are the number of frequency samples, input and output ports of the system, respectively.

The worst case RMS error over the validation grid is chosen to assess the accuracy and the quality of parameterized ROMs

$$\begin{aligned}\mathbf{g}_{max} &= argmaxErr(g),\ \ \mathbf{g} \in validation\ grid\\ Err(g)_{max} &= Err(g_{max})\end{aligned} \tag{18}$$

and it is used in the numerical examples. The proposed technique was implemented in MATLAB R2009A [34] and all experiments were carried out on Windows platform equipped with Intel(R) Xeon(R) CPU E5504@2.0 GHz and 6GB RAM.

TABLE II
PARAMETERS OF THE EM MODEL

Parameter	Min	Max
Frequency ($freq$)	1 kHz	15 GHz
Horizontal spacing (S_x)	1 mm	2 mm
Vertical spacing (S_y)	2 mm	3 mm

A. EM: EM model

An EM model satisfying (14), of an interconnection composed of six conductors with length $L = 2\ cm$ and width $w = 1\ mm$ and thickness $t = 0.2\ mm$ has been modeled in this example. Fig.3 shows its cross section. S_x and S_y represents the horizontal and vertical spacings between the conductors and are the two parameters that vary in addition to frequency. Their corresponding ranges are shown in Table II. The order of the original models is 702. The design space is sampled

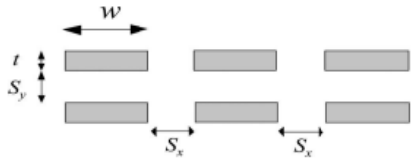

Fig. 3. EM: Cross section of multiconductor system.

uniformly over an estimation grid of 4×4 (S_x,S_y) samples and a validation grid of 3×3 (S_x,S_y) samples. The validation design space points are located in the center of each cell of the rectangular estimation grid as shown in Fig.1. The reduced order is estimated at the corners of the design space by the truncation of the Hankel singular values with a threshold.

This threshold can be set based on the level of accuracy needed for the PMOR. For example we have set a threshold of 0.01 for (8), such that the weighted RMS error (17) at the corner points of the design space is not larger than 0.05. Depending on the accuracy and compactness required, one can increase or decrease the threshold. Then, the projection matrices are computed at the estimation points using Laguerre-SVD.

For *local* projection, the projection matrix is constructed cell by cell. For example, consider the cell with S_y varying from $2\ mm$ to $2.3\ mm$ and S_x varying from $1\ mm$ to $1.3\ mm$, the *local* projection matrix is found by truncating the singular values of the union of the projection matrices computed at the vertices of the cell. If the common projection matrix is generated by the mere union of the projection matrices computed at the vertices of the cell, then the reduced order is 178, but with the truncation of the merged projection matrices by a threshold of 0.01 as described in Section III, the reduced order is 66 as shown in Fig.4. Thus a more useful projection matrix is obtained locally for the specified cell using the novel PMOR technique. The parameterized model is obtained by multilinear interpolation of the reduced system matrices. Fig. 5 shows the crosstalk term $\mathbf{Y}_{16}(s,S_x,S_y)$ for $S_x = 1.2\ mm$. Fig.6 plots the magnitude of $\mathbf{Y}_{16}(s,S_x,S_y)$ at the validation points of this

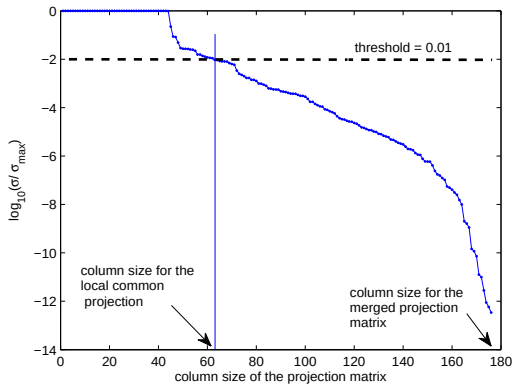

Fig. 4. EM: Singular values of the projection matrix (best-case *local* approach).

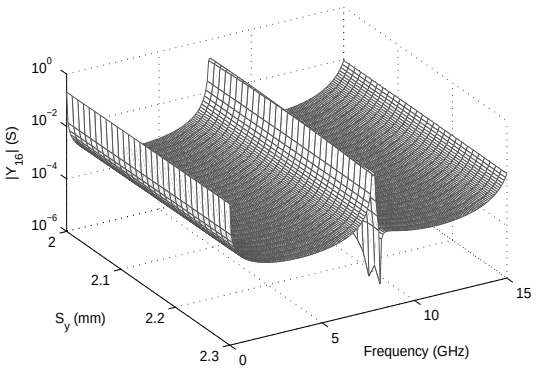

Fig. 5. EM: Magnitude of $\mathbf{Y}_{16}(s,S_x,S_y)$ for $S_x = 1.2\,mm$.

design space cell. The worst-case RMS error (18) for the *local* approach is 0.0416.

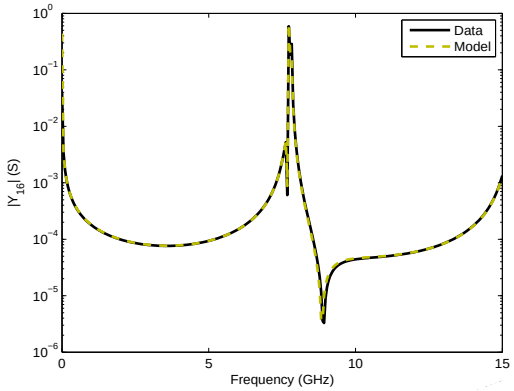

Fig. 6. EM: Magnitude of $\mathbf{Y}_{16}(s,S_x,S_y)$ for $S_x = 1.2\,mm$ and $S_y = 2.1\,mm$ using a best-case *local* projection.

The parameterized ROM can be built globally by computing a *global* common projection matrix for the entire design space. The whole design space is considered as one cell and then the projection matrices are found at the estimation points. The projection matrices are then merged and its singular values are truncated by a threshold of 0.01. It can be seen that the merged projection matrix, that is 708 has been reduced to 92 for the *global* common projection as shown in Fig.7. Thus we are able to obtain reduced order models at the estimation points globally.

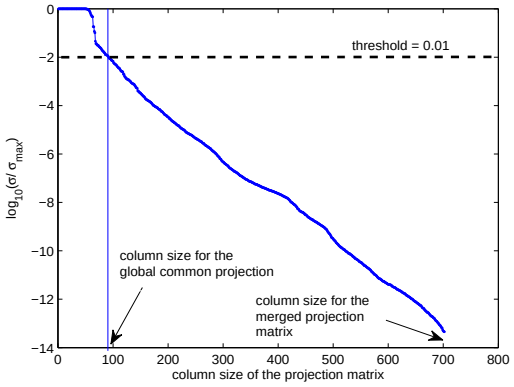

Fig. 7. EM: Singular values of the projection matrix (best-case *global* approach).

The Table III below summarizes, the dimension of the merged and common projection matrix along with the CPU time for computing the reduced order and the common projection matrix using the different approaches. In this example,

TABLE III
COLUMN SIZE OF PROJECTION MATRIX WITH COMPUTATION TIME FOR THE EM MODEL

Approach	Column size of $\mathbf{P}_{\mathbf{union}}$	Column size of $\mathbf{P}_{\mathbf{comm}}$	Computation time (sec)
Best-case *local*	178	66	41.58
Best-case *global*	708	92	82.71
Worst-case *local*	184	71	38.13
Worst-case *global*	736	128	57.64

Table III shows that for a compact model the best-case scenario can be selected and that for a faster performance the worst-case scenario can be selected. It should also be noted that the results of the *local* approach are related to the cell with S_y varying from $2\,mm$ to $2.3\,mm$ and S_x varying from $1\,mm$ to $1.3\,mm$. We recall that each design space cell has its own common projection matrix using the *local* approach. While for the *global* approach the common projection matrix can be used for the whole design space. The parameterized model is obtained by multilinear interpolation of the reduced system matrices. Fig. 8 compares the actual data and parameterized ROM obtained by interpolation for the spacing $S_y = \{2.2, 2.5, 2.9\}\,mm$ and $S_x = 1.3\,mm$. These specific spacing values have not been used for the estimation grid. The worst case RMS error (18) for the *global* approach is 0.0512. It is clear that, the parameterized ROM captures the behavior of the system very accurately with passivity guaranteed by construction.

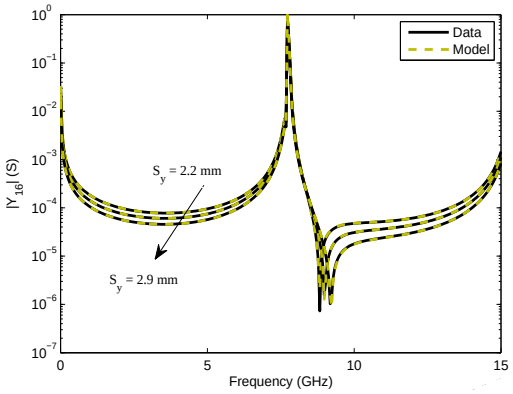

Fig. 8. EM: Magnitude of $\mathbf{Y}_{16}(s,S_x,S_y)$ for $S_x = 1.3$ *mm* and $S_y = \{2.2, 2.5, 2.9\}$ *mm* using a best-case *global* common projection matrix.

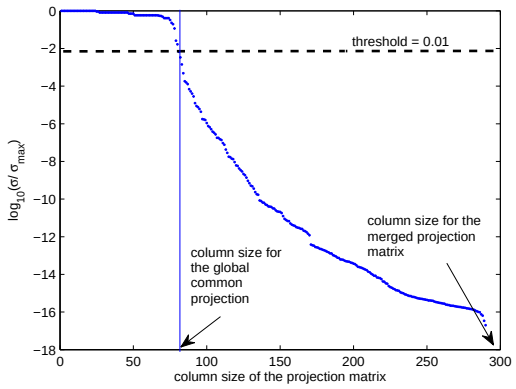

Fig. 10. 3MTL: Singular values of the projection matrix (best-case *global* approach).

B. 3MTL: Three coupled microstrip

TABLE IV
PARAMETERS OF THREE COUPLED MICROSTRIP

Parameter	Min	Max
Frequency ($freq$)	1 kHz	4 GHz
Spacing (S)	200 μm	400 μm
Length (L)	2 cm	6 cm

A three coupled microstrip structure can be modeled [9] starting from per-unit-length parameters. Fig.9 shows its cross

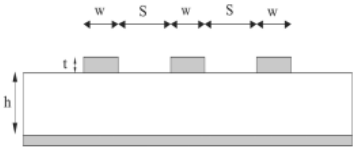

Fig. 9. 3MTL: Cross section of three coupled microstrip line.

section. The conductors have width $w = 100\,\mu m$ and thickness $t = 50\,\mu m$. The spacing S between the conductors and the length L are considered as variable parameters in addition to frequency. Their corresponding ranges are shown in Table IV. The $\mathbf{C},\mathbf{G},\mathbf{B},\mathbf{L}$ matrices are obtained for 5 values of S and 5 values of L. The original models are represented as in (1) and have an order of 10203. A 3×3 (S,L) estimation grid and a validation grid of 2×2 (S,L) samples is considered. The reduced order at the corner points of the design space is estimated by truncating the Hankel singular values, similar to the previous example a threshold of 0.01 is chosen.

For the *global* approach the projection matrices are computed at all the estimation points in the design space to obtain a common *global* projection matrix. Similar to the previous example, it can be seen that the size 298 for the merged projection matrix can be reduced to 81 by truncating the singular values as shown in Fig.10. The parameterized ROM is obtained using multilinear interpolation. Fig.11 plots the magnitude of $\mathbf{Y}_{11}(s,S,L)$ for $S = 200\,\mu m$. Fig.12 plots the magnitude of $\mathbf{Y}_{11}(s,S,L)$ for a $S = 200\,\mu m$ and $L = \{3,5\}$ *cm*. The worst case RMS error (18) for the *global* approach is 0.057.

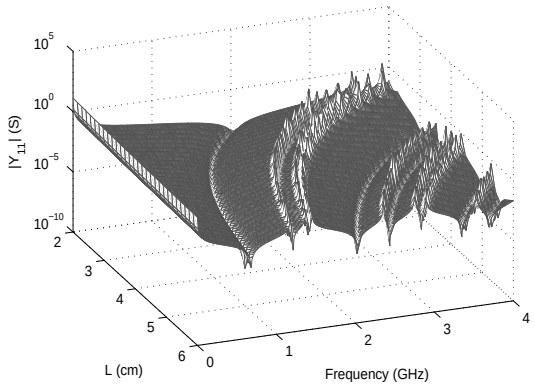

Fig. 11. 3MTL: Magnitude of $\mathbf{Y}_{11}(s,S,L)$ for $S = 200\,\mu m$.

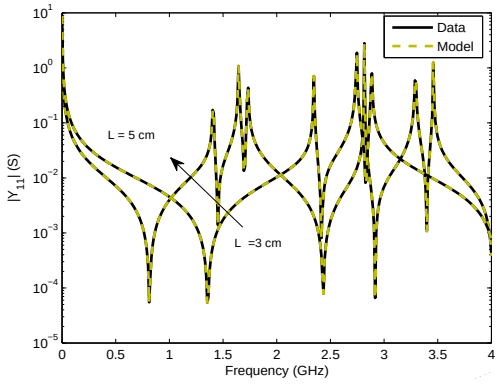

Fig. 12. 3MTL: Magnitude of $\mathbf{Y}_{11}(s,S,L)$ for $S = 200\,\mu m$ and $L = \{3,5\}$ *cm* using a best-case *global* common projection matrix.

As explained before, for *local* projection the projection matrix is found cell by cell. For example, the cell with S varying

from 200 μm to 297.44 μm and L varying from 2 cm to 4 cm, the *local* projection matrix is found by truncating the singular values of the union of the projection matrices computed at the vertices of the cell. When the common projection matrix is the union of the projection matrices computed at the vertices of the cell, then the reduced order will be 162, but with the truncation of the merged projection matrix by a threshold of 0.01 as described in Section III, the reduced order is 57 as shown in Fig.13. Thus a more useful projection matrix is obtained locally for each specified cell using the novel technique.

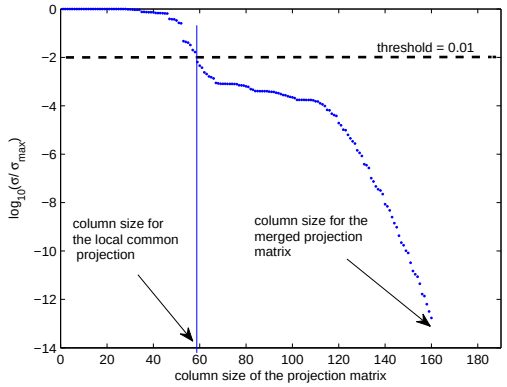

Fig. 13. 3MTL: Singular values of the projection matrix (best-case *local* approach).

Similarly to the previous example, Table V below summarizes the dimension of the merged and common projection matrix along with the CPU time for computing the reduced order and the common projection matrix using the different approaches. In this example, the results of the *local* approach

TABLE V
COLUMN SIZE OF PROJECTION MATRIX WITH COMPUTATION TIME FOR THREE COUPLED MICROSTRIP

Approach	Column size of $\mathbf{P}_{\mathbf{union}}$	Column size of $\mathbf{P}_{\mathbf{comm}}$	Computation time (sec)
Best-case *local*	162	57	1502.36
Best-case *global*	298	81	3104.11
Worst-case *local*	168	62	1456.42
Worst-case *global*	378	103	3002.75

shown in Table V are related to the cell with S varying from 200 μm to 297.44 μm and L varying from 2 cm to 4 cm.

Fig. 14 shows the magnitude of $\mathbf{Y}_{11}(s,S,L)$ for $S = 246.15\ \mu m$ and $L = 3\ cm$. The parameterized ROM with *local* approach has worst-case RMS error (18) equal to 0.0415.

As in the previous example, the parameterized ROM is able to accurately describe the parameterized behavior of the system with a common projection matrix locally and globally. The passivity of the system is guaranteed by construction.

VI. Conclusion

We have presented a novel PMOR method based on singular values and matrix interpolation. A fast technique using grammians is first utilized to estimate the reduced order and then

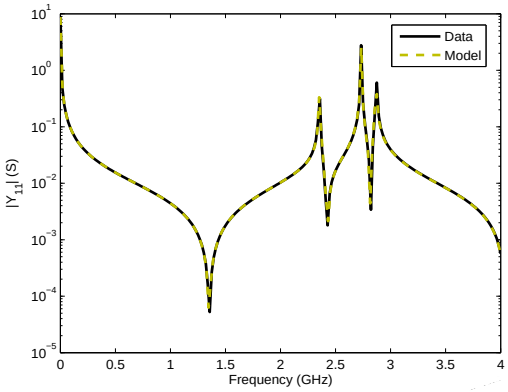

Fig. 14. 3MTL: Magnitude of $\mathbf{Y}_{11}(s,S,L)$ for $S = 246.15\ \mu m$ and $L = 3\ cm$ using a best-case *local* projection.

projection matrices are used to build parameterized reduced order models. The design space is divided into cells and a Krylov subspace is computed for each cell vertex model. The truncation of the singular values of the merged Krylov subspaces computed from the models at the vertices of each cell generates a common projection matrix per design space cell. The stability and passivity of the parameterized reduced order models are preserved using classical MOR methods and positive interpolation schemes. Pertinent numerical examples show that the proposed technique is able to build accurate and parameterized reduced order models of dynamic parametric systems.

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