Complex Anal. Oper. Theory (2011) 5:83–95
DOI 10.1007/s11785-009-0038-9

Complex Analysis and Operator Theory

A Discrete Bochner–Martinelli Formula

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Received: 22 July 2009 / Accepted: 7 September 2009 / Published online: 17 September 2009

Abstract In the even dimensional case the discrete Dirac equation may be reduced to the so-called discrete isotonic Dirac system in which suitable Dirac operators appear from both sides in half the dimension. This is an appropriated framework for the development of a discrete Martinelli–Bochner formula for discrete holomorphic functions on the simplest of all graphs, the rectangular $\mathbb{Z}^m$ one. Two lower-dimensional cases are considered explicitly to illustrate the closed analogy with the theory of continuous variables and the developed discrete scheme.

Keywords Discrete Clifford analysis · Bochner–Martinelli transform

Mathematics Subject Classification (2000) Primary 30G35; Secondary 30G25

1 Continuous Clifford Analysis Setting

Clifford analysis in its original form (see the classical book [3]) deals with the study of the so-called monogenic functions (null solutions of the Dirac operator), which are an appropriate higher dimensional analogue of an holomorphic function on the complex plane. It is also a refinement of harmonic analysis, indeed the Dirac operator factors the m-dimensional Laplace operator.

Communicated by Michael Shapiro.

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This framework is introduced by endowing m-dimensional Euclidean space $\mathbb{R}^{0,m}$ with a non-degenerate quadratic form of signature $(0, m)$, and considering the corresponding orthonormal basis $(e_1 \cdots e_m)$. $\mathbb{R}_{0,m}$ denotes the real Clifford algebra constructed over $\mathbb{R}^{0,m}$.

Allowing complex constants produces the complex Clifford algebra, denoted by $\mathbb{C}_m$, which is also generated additively by elements of the form $e_A = e_{j_1} \cdots e_{j_k}$, where $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}$ is such that $j_1 < \cdots < j_k$, and so the complex dimension of $\mathbb{C}_m$ is 2^m. For $A = \emptyset$, $e_\emptyset = 1$, is the identity element. It is well known that on the Clifford algebra $\mathbb{C}_m$ one can consider the main involution $\tilde{e}_j = -e_j$, and for any $a, b \in \mathbb{C}_m$, $\widetilde{ab} = \tilde{a}\tilde{b}$.

If we identify the $\mathbb{R}^m$-vector $(x_1, \ldots, x_m)$ with the real Clifford vector $\underline{x} = \sum_{j=1}^m e_j x_j$, then $\mathbb{R}^m$ may be considered as a subspace of $\mathbb{C}_m$.

The multiplication of two Clifford vectors $\underline{x}$, $\underline{y}$ splits up into a scalar and a 2-vector part

$$\underline{x}\,\underline{y} = -\left\langle \underline{x}, \underline{y} \right\rangle + \underline{x} \wedge \underline{y},$$

where

$$\left\langle \underline{x}, \underline{y} \right\rangle = \sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_j e_k (x_j y_k - x_k y_j).$$

The Fourier dual of the vector $\underline{x}$, called Dirac operator, is the vector-valued first order differential operator given by $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$. If Ω is open in $\mathbb{R}^m$ and f is defined and differentiable in Ω, then f is said to be monogenic if $\partial_{\underline{x}} f = 0$ in Ω.

For our purpose we will assume that the dimension of the Euclidean space m is even whence we will put $m = 2n$ from now on.

Nowadays, several authors are much interested in finding discrete counterparts of various basic results of continuous Clifford analysis since finite difference operators are more suitable to an adequate numerical treatment of problems than continuous ones, see [4–6]. The greatest contribution was achieved in function theoretic results among which Cauchy's integral formula which is derived from a discrete version of Leibniz's rule.

The history of Bochner–Martinelli formula, for holomorphic functions of several complex variables in the continuous setting has been described in detail in [7]. It reduces to the traditional Cauchy's integral formula for the one-dimensional case. A nice presentation of the extensive bibliography can be found in the book [8].

In a series of recent papers the interest for proving different generalizations of the classical Bochner–Martinelli formula has emerged as a successful research field, see for instance [1,2,9–11,13–16].

Naturally a Bochner–Martinelli formula for discrete holomorphic functions of several variables taking values in an appropriate sub-structure of a complex Clifford algebra should play essential role in the further development of function theory, but has not yet been obtained.

The main purpose of this paper is to discuss an analogue of the Bochner–Martinelli formula for discrete holomorphic functions. It will be deduced directly from the isotonic approach.

2 A Discrete Function Theory

Throughout the paper we write $\mathbb{Z}^{2n}$ (a discrete analogous of $\mathbb{C}^n$) for the natural graph corresponding to the equidistant grid; thus a Clifford vector $\underline{x}$ will now only show integer co-ordinates.

2.1 Preliminaries Notations and Concepts

Concerning the contents of this section, we refer to [4], see also [5].

2.1.1 Discrete Dirac Operator

A pointwise discretization of the partial derivatives ∂_{x_j} is based on the traditional one-sided forward and backward differences, respectively, given by

$$\Delta_j^+[f](\underline{x}) = f(\underline{x}+e_j) - f(\underline{x}), \quad j = 1, \ldots, 2n,$$
$$\Delta_j^-[f](\underline{x}) = f(\underline{x}) - f(\underline{x}-e_j), \quad j = 1, \ldots, 2n.$$

With respect to the $\mathbb{Z}^{2n}$ neighbourhood of $\underline{x}$, the usual definition of the discrete Laplacian explicitly reads

$$\Delta^\dagger[f](\underline{x}) = \sum_{j=1}^{2n} \Delta_j^+ \Delta_j^- [f](\underline{x}) = \sum_{j=1}^{2n} \Delta_j^- \Delta_j^+ [f](\underline{x}).$$

The notation $\Delta^\dagger$ referring to this operator will be called the “cross Laplacian”.

When passing to the Dirac operator, we will combine each difference, forward or backward, with the corresponding forward or backward basis vector e_j^+ and e_j^-, $j = 1, \ldots, n$. To this end, we need to embed the Clifford algebra $\mathbb{R}_{0,2n}$ into a bigger one, denoted by $\mathbb{C}_{2n\pm}$, the complex Clifford algebra generated by the basic elements $\{e_j^\pm, e_{n+j}^\pm\}$, $j = 1, \ldots, n$. The $4n$ basis vectors are governed by the multiplication rules:

$$e_j^+ e_k^+ + e_k^+ e_j^+ = e_j^- e_k^- + e_k^- e_j^- = 0, \quad j \neq k,$$
$$e_j^+ e_k^- + e_k^- e_j^+ = 0, \quad j \neq k,$$

$$(e_j^+)^2 = (e_j^-)^2 = 0, \quad j = 1, \ldots, 2n,$$
$$e_j^+ e_j^- + e_j^- e_j^+ = -1, \quad j = 1, \ldots, 2n.$$

A feasible model for the forward and backward Clifford vectors can be given, in terms of so-called curvature vectors B_j, $j = 1, \ldots, 2n$, where B_j are subject to additional requirements, see [4] for more details.

We put

$$e_j^{\pm} = \frac{1}{2}(e_j \pm B_j), \quad j = 1, \ldots, 2n$$

ensuring that $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, 2n$.

The definition of a discrete Dirac operator is given by

Definition 2.1 The discrete Dirac operator $\partial_{\underline{x}}$ is the first order, Clifford vector valued difference operator given by $\partial_{\underline{x}} = \partial_{\underline{x}}^+ + \partial_{\underline{x}}^-$ where the forward and backward discrete operators $\partial_{\underline{x}}^{\pm}$ are given by $\partial_{\underline{x}}^{\pm} = \sum_{j=1}^{2n} e_j^{\pm} \Delta_j^{\pm}$.

Notice that $\partial_{\underline{x}}$ is a square root of the cross Laplacian $\partial_{\underline{x}}^2 = -\Delta^{\dagger}$. By abuse of notation, we continue to write $\partial_{\underline{x}}$ for Dirac operators independent of the case (continuous or discrete) that are considering.

2.1.2 Discrete Monogenic Functions

We consider a bounded set $\mathbf{B} \subset \mathbb{Z}^{2n}$ and its characteristic function $\chi_{\mathbf{B}}$ as well as the discrete operator $\check{\partial}_{\underline{x}} = \sum_{j=1}^{2n} (e_j^+ \Delta_j^- + e_j^- \Delta_j^+)$. The vector-valued function $\chi_{\mathbf{B}} \check{\partial}_{\underline{x}} = \sum_{j=1}^{2n} (e_j^+ \Delta_j^- [\chi_{\mathbf{B}}] + e_j^- \Delta_j^+ [\chi_{\mathbf{B}}])$ is said to be the oriented boundary measure of $\mathbf{B}$.

The above concepts allow for defining the notion of discrete monogenic function.

Definition 2.2 Let $\mathbf{B}$ a bounded set in $\mathbb{Z}^{2n}$ and let the Clifford algebra-valued function f defined on $\mathbf{B} \cup \mathrm{supp}(\chi_{\mathbf{B}} \check{\partial}_{\underline{x}})$. Then f is called discrete monogenic in $\mathbf{B}$ if and only if $\partial_{\underline{x}}[f](\underline{x}) = 0$ for all $\underline{x} \in \mathbf{B}$.

2.1.3 Cauchy's Integral Formula

We have the discrete version of Cauchy's integral formula (for proof we refer again to [4]).

Theorem 2.1 *Let* $\mathbf{B}$ *a bounded set in* $\mathbb{Z}^{2n}$ *and let the function* f *be discrete monogenic on* $\mathbf{B}$, *then for all* $\underline{y} \in \mathbf{B}$ *we have*

$$-f(\underline{y}) = \int_{\partial \mathbf{B}} E(\underline{x} - \underline{y})(\chi_{\mathbf{B}} \check{\partial}_{\underline{x}}) f(\underline{x}) + \int_{\partial \mathbf{B}} GT(\underline{x}, \underline{y}) f(\underline{x}), \tag{1}$$

where $GT(\underline{x}, \underline{y})$ *is the so-called "grid tension" given by*

$$GT(\underline{x}, \underline{y}) = \sum_{j=1}^{2n} (\Delta_j^+[\chi_{\mathbf{B}}(\underline{x})]\Delta_j^+[E(\underline{x}-\underline{y}]e_j^- - \Delta_j^-[\chi_{\mathbf{B}}(\underline{x})]\Delta_j^-[E(\underline{x}-\underline{y}]e_j^+).$$

In the above formula the integration is a sum over the points belonging to

$$\partial\mathbf{B} = \text{supp}(\chi_{\mathbf{B}}\check{\partial}_{\underline{x}}) = \cup_{j=1}^{2n}\text{supp}(\Delta_j^{\pm}\chi_{\mathbf{B}}).$$

The Cauchy's integral formula for discrete monogenic functions is also given by

$$-f(\underline{y}) = \int_{\partial\mathbf{B}} ((E(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}})f(\underline{x}), \quad \underline{y} \in \mathbf{B} \setminus \partial\mathbf{B}. \tag{2}$$

The point is that between $(E(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}}$ and $E(\underline{x}-\underline{y})(\chi_{\mathbf{B}}\check{\partial}_{\underline{x}})(\underline{x})$ there is a difference in the discrete context, and this is precisely the grid-tension $GT(\underline{x}, \underline{y})$, so the Cauchy's integral formula (2) is an easier interpretation to understand the preceding formula (1). Notice that the formula (2) is only valid in the interior of **B** whereas the formula (1) holds everywhere on **B**. But for our purpose here it is for the moment sufficient to use the simplified formula (2).

2.2 Discrete Isotonic System

Let us introduce the following primitive idempotent for the discrete setting:

$$\mathcal{I}_d = \mathcal{I}\mathcal{J},$$

where

$$\mathcal{I} = \prod_{j=1}^{n} \mathcal{I}_j = \prod_{j=1}^{n} \frac{1}{2}(1 + i e_j e_{n+j}),$$

and

$$\mathcal{J} = \prod_{j=1}^{n} \mathcal{J}_j = \prod_{j=1}^{n} \frac{1}{2}(1 - i B_j B_{n+j}).$$

The following conversion relations hold:

$$e_{n+j}\,\mathcal{I} = i e_j\,\mathcal{I},$$
$$B_{n+j}\,\mathcal{J} = i B_j\,\mathcal{J}.$$

and consequently

$$e_{n+j}\,\mathcal{I}_d = i e_j\,\mathcal{I}_d,$$
$$B_{n+j}\,\mathcal{I}_d = i B_j\,\mathcal{I}_d.$$

Therefore

$$e^{\pm}_{n+j}\,\mathcal{I}_d = i e^{\pm}_j\,\mathcal{I}_d. \tag{3}$$

For $a \in \mathbb{C}_{n\pm}$, i.e., the complex Clifford algebra generated by the basic elements $\{e^{\pm}_j\}$, $j = 1, \dots, n$, we also have that

$$a\,\mathcal{I}_d = 0 \quad \Leftrightarrow \quad a = 0. \tag{4}$$

Introduce the following Clifford vectors and their corresponding Dirac operator

$$\underline{x}_1 = \underline{x}^+_1 + \underline{x}^-_1, \quad \underline{x}_2 = \underline{x}^+_2 + \underline{x}^-_2,$$

where

$$\underline{x}^{\pm}_1 = \sum_{j=1}^{n} e^{\pm}_j x^{\pm}_j; \quad \underline{x}^{\pm}_2 = \sum_{j=1}^{n} e^{\pm}_j x^{\pm}_{n+j},$$
$$\partial_{\underline{x}_1} = \partial^+_{\underline{x}_1} + \partial^-_{\underline{x}_1}; \quad \partial_{\underline{x}_2} = \partial^+_{\underline{x}_2} + \partial^-_{\underline{x}_2}$$

with

$$\partial^{\pm}_{\underline{x}_1} = \sum_{j=1}^{n} e^{\pm}_j \Delta^{\pm}_j; \quad \partial^{\pm}_{\underline{x}_2} = \sum_{j=1}^{n} e^{\pm}_j \Delta^{\pm}_{n+j}.$$

Consider two Clifford vectors $\underline{x}$, $\underline{y} \in \mathbb{C}_{2n\pm}$, which may be written as:

$$\underline{x} = \sum_{j=1}^{n}(e^+_j x^+_j + e^+_{n+j} x^+_{n+j}) + \left(\sum_{j=1}^{n} e^-_j x^-_j + e^-_{n+j} x^-_{n+j}\right),$$
$$\underline{y} = \sum_{j=1}^{n}(e^+_j y^+_j + e^+_{n+j} y^+_{n+j}) + \left(\sum_{j=1}^{n} e^-_j y^-_j + e^-_{n+j} y^-_{n+j}\right).$$

Applying now (3) yields

$$\underline{x} a \mathcal{I}_d = (\underline{x}_1 a + i\tilde{a}\underline{x}_2)\mathcal{I}_d. \tag{5}$$

By the above equality, we deduce that

$$\underline{x}\,\underline{y} a \mathcal{I}_d = (\underline{x}_1(\underline{y}_1 a + i\tilde{a}\underline{y}_2) + (a\underline{y}_2 - i\underline{y}_1\tilde{a})\underline{x}_2)\mathcal{I}_d. \tag{6}$$

Consider a discrete function $f : \mathbb{Z}^{2n} \to \mathbb{C}_{n\pm}$, then by (5) we obtain

$$\partial_{\underline{x}}(f\mathcal{I}_d) = (\partial_{\underline{x}_1} f + i\tilde{f}\partial_{\underline{x}_2})\mathcal{I}_d,$$

where

$$\partial_{\underline{x}} = \partial_{\underline{x}}^+ + \partial_{\underline{x}}^- = \sum_{j=1}^{n}(e_j^+\Delta_j^+ + e_{n+j}^+\Delta_{n+j}^+) + \sum_{j=1}^{n}(e_j^-\Delta_j^- + e_{n+j}^-\Delta_{n+j}^-).$$

We may conclude that $f\mathcal{I}_d$ is discrete monogenic if and only if

$$\partial_{\underline{x}_1} f + i\tilde{f}\partial_{\underline{x}_2} = 0 \hookleftarrow \textbf{Discrete isotonic system}.$$

3 Discrete Holomorphic Functions

In the continuous case, holomorphic functions of several complex variables can be interpreted as $\mathbb{C}$-valued solutions of the isotonic system

$$\partial_{\underline{x}_1} f + i\tilde{f}\partial_{\underline{x}_2} = \sum_{j=1}^{n}\sum_{A}(e_j e_A \partial_{x_j} f_A + i\tilde{e}_A e_j \partial_{x_{n+j}} f_A) = 0,$$

see [12] for more details.

Repeating the same idea in the discrete setting does not work; we now have to consider $2n$ generators e_j^+ and e_j^- and so, scalar valued solutions of the isotonic system will satisfy too many equations.

What we have to do is to combine e_j^+ with e_j^-; this can be done by considering the idempotents $E_j^+ = e_j^+ e_j^-$, $E_j^- = e_j^- e_j^+$ and instead of scalar valued solutions to consider solutions taking valued in the complex span of the primitive idempotents $E_1^\pm E_2^\pm \cdots E_n^\pm$, which itself form a 2^n-dimensional abelian subalgebra of the Clifford algebra.

To illustrate how the scheme works we elaborate the case of one and two complex variable.

3.1 The Case of One Variable

By means of the defining relations for the Dirac operators $\partial_{\underline{x}_1}$, $\partial_{\underline{x}_2}$, we have

$$\partial_{\underline{x}_1} = e_1^+\Delta_1^+ + e_1^-\Delta_1^-,$$
$$\partial_{\underline{x}_2} = e_1^+\Delta_2^+ + e_1^-\Delta_2^-.$$

Hence, for a $\mathbb{C}_{1\pm}$-valued discrete function

$$f = f^+E_1^+ + f^-E_1^-, \tag{7}$$

where $E_1^{\pm} = e_1^{\pm} e_1^{\mp}$, the discrete isotonic system can be rewritten as

$$\partial_{\underline{x}_1} f + i f \partial_{\underline{x}_2} = [\Delta_1^+[f^-] + i\Delta_2^+[f^+]]e_1^+ + [\Delta_1^-[f^+] + i\Delta_2^-[f^-]]e_1^- = 0.$$

As we mentioned before, a discrete holomorphic function of one variable is defined by requiring f to be solution of the above system, which leads to

$$\begin{cases} \Delta_1^+[f^-] + i\Delta_2^+[f^+] = 0, \\ \Delta_1^-[f^+] + i\Delta_2^-[f^-] = 0. \end{cases} \tag{8}$$

3.2 The Case of Two Variables

In the sequel, we will consider in detail the special case of $n = 2$, then we have

$$\begin{aligned} \partial_{\underline{x}_1} &= e_1^+\Delta_1^+ + e_2^+\Delta_3^+ + e_1^-\Delta_1^- + e_2^-\Delta_3^-, \\ \partial_{\underline{x}_2} &= e_1^+\Delta_2^+ + e_2^+\Delta_4^+ + e_1^-\Delta_2^- + e_2^-\Delta_4^-. \end{aligned}$$

Let us introduce the following idempotents

$$E_1^{\pm} = e_1^{\pm} e_1^{\mp} \quad \text{and} \quad E_2^{\pm} = e_2^{\pm} e_2^{\mp},$$

and consider the set of functions $f : \mathbb{Z}^4 \to \mathbb{C}_{2\pm}$ given by

$$f = f^{+-}E_1^+E_2^- + f^{-+}E_1^-E_2^+ + f^{++}E_1^+E_2^+ + f^{--}E_1^-E_2^-. \tag{9}$$

The discrete isotonic system takes the form:

$$\begin{cases} \Delta_1^+[f^{-+}] + i\Delta_2^+[f^{++}] = 0, \\ \Delta_1^+[f^{--}] + i\Delta_2^+[f^{+-}] = 0, \\ \Delta_3^+[f^{+-}] + i\Delta_4^+[f^{++}] = 0, \\ \Delta_3^+[f^{--}] + i\Delta_4^+[f^{-+}] = 0, \\ \Delta_1^-[f^{+-}] + i\Delta_2^-[f^{--}] = 0, \\ \Delta_1^-[f^{++}] + i\Delta_2^-[f^{-+}] = 0, \\ \Delta_3^-[f^{-+}] + i\Delta_4^-[f^{--}] = 0, \\ \Delta_3^-[f^{++}] + i\Delta_4^-[f^{+-}] = 0, \end{cases} \tag{10}$$

because

$$\begin{aligned} \partial_{\underline{x}_1} f = -[&\Delta_1^+[f^{-+}](e_1^+e_2^+e_2^-) + \Delta_1^+[f^{--}](e_1^+e_2^-e_2^+) + \Delta_3^+[f^{+-}](e_1^+e_1^-e_2^+) \\ &+\Delta_3^+[f^{--}](e_1^-e_1^+e_2^+) + \Delta_1^-[f^{+-}](e_1^-e_2^-e_2^+) + \Delta_1^-[f^{++}](e_1^-e_2^+e_2^-) \\ &+\Delta_3^-[f^{-+}](e_1^-e_1^+e_2^-) + \Delta_3^-[f^{++}](e_1^+e_1^-e_2^-)] \end{aligned}$$

as well as

$$
\begin{aligned}
f\partial_{\underline{x}_2} = -[&\Delta_2^+[f^{+-}](e_1^+e_2^-e_2^+) + \Delta_2^+[f^{++}](e_1^+e_2^+e_2^-) + \Delta_2^-[f^{-+}](e_1^-e_2^+e_2^-)\\
&+\Delta_2^-[f^{--}](e_1^-e_2^-e_2^+) + \Delta_4^+[f^{-+}](e_1^-e_1^+e_2^+ + \Delta_4^+[f^{++}](e_1^+e_1^-e_2^+)\\
&+\Delta_4^-[f^{+-}](e_1^+e_1^-e_2^-) + \Delta_4^-[f^{--}](e_1^-e_1^+e_2^-)].
\end{aligned}
$$

In this way solutions of the system (10) will be referred to as discrete holomorphic function of two variables. Note that a discrete holomorphic function is also discrete harmonic. Indeed, we have

$$
\begin{cases}
\Delta_1^+[f^{-+}] + i\Delta_2^+[f^{++}] = 0,\\
\Delta_1^-[f^{++}] + i\Delta_2^-[f^{-+}] = 0,
\end{cases}
$$

$\Longleftrightarrow$

$$
\begin{cases}
-i\Delta_2^-\Delta_1^+[f^{-+}] + \Delta_2^-\Delta_2^+[f^{++}] = 0,\\
\Delta_1^+\Delta_1^-[f^{++}] + i\Delta_1^+\Delta_2^-[f^{-+}] = 0,
\end{cases}
$$

so,

$$
(\Delta_1^+\Delta_1^- + \Delta_2^-\Delta_2^+)[f^{++}] = \Delta^\dagger[f^{++}] = 0.
$$

Analogously one can prove that

$$
\Delta^\dagger[f^{--}] = \Delta^\dagger[f^{+-}] = \Delta^\dagger[f^{-+}] = 0.
$$

4 Discrete Martinelli–Bochner Formula

Assume that f is a $\mathbb{C}_{n\pm}$-valued function defined in a bounded set $\mathbf{B} \subset \mathbb{Z}^{2n}$. By the discrete Cauchy's integral formula, we see that

$$
-f(\underline{y})\mathcal{I}_d = \int_{\partial\mathbf{B}} ((E(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}})f(\underline{x})\mathcal{I}_d, \quad \underline{y} \in \mathbf{B}\backslash\partial\mathbf{B}.
$$

There exists a scalar function $H(\underline{x})$ defined on $\mathbb{Z}^{2n}\backslash\{0\}$ such that $E(\underline{x}) = \check{\partial}_{\underline{x}}H(\underline{x})$, see [4].

Consequently, for $a \in \mathbb{C}_{n\pm}$ we have

$$
E(\underline{x})a\mathcal{I}_d = E_1(\underline{x})a\mathcal{I}_d + i\tilde{a}E_2(\underline{x})\mathcal{I}_d,
$$

where $E_j(\underline{x}) = \check{\partial}_{\underline{x}_j}H(\underline{x})$, $j = 1, 2$.

The equality (6) now implies

$$
\begin{aligned}
&((E(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}})f(\underline{x})\mathcal{I}_d\\
&\quad=[((E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}_1})f(\underline{x})+i(E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f(\underline{x})\dot{\check{\partial}}_{\underline{x}_2}\\
&\qquad+f(\underline{x})(\check{\partial}_{\underline{x}_2}(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}})-i\dot{\check{\partial}}_{\underline{x}_1}f(\underline{x})(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}})]\mathcal{I}_d,
\end{aligned}
$$

where the symbol "dot" over $\check{\partial}_{\underline{x}_j}$, $j=1,2$ means that the derivations do not act on f.

If we now apply (4), we get the following result

Theorem 4.1 *Let* $\mathbf{B}$ *be a bounded set in* $\mathbb{Z}^{2n}$ *and let* $f:\mathbb{Z}^{2n}\to\mathbb{C}_{n\pm}$ *be a discrete isotonic function on* $\mathbf{B}$*, then for all* $\underline{y}\in\mathbf{B}\backslash\partial\mathbf{B}$ *it hold that*

$$
\begin{aligned}
-f(\underline{y})=\int_{\partial\mathbf{B}}&[((E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}_1})f(\underline{x})+i(E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f(\underline{x})\dot{\check{\partial}}_{\underline{x}_2}\\
&+f(\underline{x})(\check{\partial}_{\underline{x}_2}(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})))-i\dot{\check{\partial}}_{\underline{x}_1}f(\underline{x})(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))].
\end{aligned}
$$

Two consequences of this theorem are the following

Theorem 4.2 *Let* $\mathbf{B}$ *be a bounded set in* $\mathbb{Z}^2$ *and let the function* $f:\mathbb{Z}^2\to\mathbb{C}_{1\pm}$ *of the form* (7) *be discrete holomorphic on* $\mathbf{B}$*, then for all* $\underline{y}\in\mathbf{B}\backslash\partial\mathbf{B}$ *it hold that*

$$
\begin{aligned}
f^+(\underline{y})=\int_{\partial\mathbf{B}}&\big[(\Delta_1^+ + i\Delta_2^+)(\Delta_1^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&-i(\Delta_1^- + i\Delta_2^-)(\Delta_2^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^+(\underline{x}),\\
f^-(\underline{y})=\int_{\partial\mathbf{B}}&\big[(\Delta_1^- + i\Delta_2^-)(\Delta_1^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&-i(\Delta_1^+ + i\Delta_2^+)(\Delta_2^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^-(\underline{x}).
\end{aligned}
$$

Proof The way out is to consider the discrete operators

$$
\begin{aligned}
\check{\partial}_{\underline{x}_1}&=e_1^+\Delta_1^- + e_1^-\Delta_1^+,\\
\check{\partial}_{\underline{x}_2}&=e_1^+\Delta_2^- + e_1^-\Delta_2^+
\end{aligned}
$$

for which the following relations hold:

$$
\begin{aligned}
&((E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}_1})f(\underline{x})\\
&\quad=\Delta_1^-(\Delta_1^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^-(\underline{x})E_1^- + \Delta_1^+(\Delta_1^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^+(\underline{x})E_1^+,\\
&(E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f(\underline{x})\dot{\check{\partial}}_{\underline{x}_2}\\
&\quad=\Delta_2^-(\Delta_1^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^-(\underline{x})E_1^- + \Delta_2^+(\Delta_1^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^+(\underline{x})E_1^+,
\end{aligned}
$$

$$
\begin{aligned}
&(f(\underline{x})(\check{\partial}_{\underline{x}_2}(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})))\\
&\quad=\Delta_2^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^+(\underline{x})E_1^+ + \Delta_2^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^-(\underline{x})E_1^-,\\
&\dot{\check{\partial}}_{\underline{x}_1}f(\underline{x})(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&\quad=\Delta_1^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^+(\underline{x})E_1^+ + \Delta_1^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f^-(\underline{x})E_1^-.
\end{aligned}
$$

The proof is completed by making use of Theorem 4.1. □

Theorem 4.3 *Let* $\mathbf{B}$ *be a bounded set in* $\mathbb{Z}^4$ *and let the function* $f:\mathbb{Z}^4\to\mathbb{C}_{2\pm}$ *of the form* (9) *be discrete holomorphic on* $\mathbf{B}$*, then for all* $\underline{y}\in\mathbf{B}\backslash\partial\mathbf{B}$ *it hold that*

$$
\begin{aligned}
f^{+-}(\underline{y}) = \int_{\partial\mathbf{B}} &\Big[\big[(\Delta_1^+ + i\Delta_2^+)(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^- + i\Delta_4^-)(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[(\Delta_1^- + i\Delta_2^-)(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + (\Delta_3^+ + i\Delta_4^+)\\
&\times(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\Big]f^{+-}(\underline{x}),\\
f^{++}(\underline{y}) = \int_{\partial\mathbf{B}} &\Big[\big[(\Delta_1^+ + i\Delta_2^+)(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^+ + i\Delta_4^+)(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[(\Delta_1^- + i\Delta_2^-)(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^- + i\Delta_4^-)(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\Big]f^{++}(\underline{x}),\\
f^{-+}(\underline{y}) = \int_{\partial\mathbf{B}} &\Big[\big[(\Delta_1^- + i\Delta_2^-)(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^+ + i\Delta_4^+)(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[(\Delta_1^+ + i\Delta_2^+)(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^- + i\Delta_4^-)(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\Big]f^{-+}(\underline{x}),\\
f^{--}(\underline{y}) = \int_{\partial\mathbf{B}} &\Big[\big[(\Delta_1^- + i\Delta_2^-)(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^- + i\Delta_4^-)(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[(\Delta_1^+ + i\Delta_2^+)(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&+(\Delta_3^+ + i\Delta_4^+)(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\Big]f^{--}(\underline{x}).
\end{aligned}
$$

Proof Direct calculation yields

$$
\begin{aligned}
&((E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\check{\partial}_{\underline{x}_1})f(\underline{x})\\
&\quad=-\big[\Delta_1^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) - \Delta_3^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{+-}(\underline{x})E_1^+E_2^-\\
&\qquad-\big[\Delta_1^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) - \Delta_3^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{++}(\underline{x})E_1^+E_2^+
\end{aligned}
$$

$$
\begin{aligned}
&-\big[\Delta_1^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{-+}(\underline{x})E_1^-E_2^+\\
&-\big[\Delta_1^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{--}(\underline{x})E_1^-E_2^+,
\end{aligned}
$$

$$
\begin{aligned}
&(E_1(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))f(\underline{x})\dot{\check{\partial}}_{\underline{x}_2}\\
&=-\big[\Delta_2^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{+-}(\underline{x})E_1^+E_2^-\\
&\quad-\big[\Delta_2^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{++}(\underline{x})E_1^+E_2^+\\
&\quad-\big[\Delta_2^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{-+}(\underline{x})E_1^-E_2^+\\
&\quad-\big[\Delta_2^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{--}(\underline{x})E_1^-E_2^+,
\end{aligned}
$$

$$
\begin{aligned}
&(f(\underline{x})(\check{\partial}_{\underline{x}_2}(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})))\\
&=-\big[\Delta_2^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{+-}(\underline{x})E_1^+E_2^-\\
&\quad-\big[\Delta_2^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{++}(\underline{x})E_1^+E_2^+\\
&\quad-\big[\Delta_2^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{-+}(\underline{x})E_1^-E_2^+\\
&\quad-\big[\Delta_2^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_4^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{--}(\underline{x})E_1^-E_2^+,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\check{\partial}}_{\underline{x}_1}f(\underline{x})(E_2(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\\
&=-\big[\Delta_1^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{+-}(\underline{x})E_1^+E_2^-\\
&\quad-\big[\Delta_1^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{++}(\underline{x})E_1^+E_2^+\\
&\quad-\big[\Delta_1^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{-+}(\underline{x})E_1^-E_2^+\\
&\quad-\big[\Delta_1^+(\Delta_2^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))-\Delta_3^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]f^{--}(\underline{x})E_1^-E_2^+.
\end{aligned}
$$

Combining these equalities we can assert that

$$
\begin{aligned}
f^{+-}(\underline{y})=\int_{\partial\mathbf{B}}&\big[\big[\Delta_1^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_3^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+i\big[\Delta_2^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_4^-(\Delta_3^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+\big[\Delta_2^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_4^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[\Delta_1^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_3^+(\Delta_4^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\big]f^{+-}(\underline{x}),\\
f^{++}(\underline{y})=\int_{\partial\mathbf{B}}&\big[\big[\Delta_1^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_3^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+i\big[\Delta_2^+(\Delta_1^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_4^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+\big[\Delta_2^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_4^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[\Delta_1^-(\Delta_2^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_3^-(\Delta_4^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\big]f^{++}(\underline{x}),\\
f^{-+}(\underline{y})=\int_{\partial\mathbf{B}}&\big[\big[\Delta_1^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_3^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+i\big[\Delta_2^-(\Delta_1^+H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))+\Delta_4^+(\Delta_3^-H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]
\end{aligned}
$$

$$
\begin{aligned}
&+\big[\Delta_2^+(\Delta_2^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_4^-(\Delta_4^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[\Delta_1^+(\Delta_2^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_3^-(\Delta_4^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\big] f^{-+}(\underline{x}),
\end{aligned}
$$

$$
\begin{aligned}
f^{--}(\underline{y}) = \int_{\partial\mathbf{B}} &\big[\big[\Delta_1^-(\Delta_1^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_3^-(\Delta_3^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+i\big[\Delta_2^-(\Delta_1^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_4^-(\Delta_3^+ H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&+\big[\Delta_2^+(\Delta_2^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_4^+(\Delta_4^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\\
&-i\big[\Delta_1^+(\Delta_2^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x})) + \Delta_3^+(\Delta_4^- H(\underline{x}-\underline{y})\chi_{\mathbf{B}}(\underline{x}))\big]\big] f^{--}(\underline{x}),
\end{aligned}
$$

and the theorem follows. □

Acknowledgments This paper was written while the first author was visiting the Department of Mathematical Analysis of Ghent University. He was supported by Research Foundation on Flanders (F.W.O.-project 31506208). Juan Bory Reyes wishes to thank the members of Clifford Research Group in Ghent for their kind hospitality.

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