On the trivectors of a 6-dimensional symplectic vector space. II

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Abstract

Let V be a 6-dimensional vector space over a field $\mathbb{F}$, let f be a nondegenerate alternating bilinear form on V and let $Sp(V,f) \cong Sp_6(\mathbb{F})$ denote the symplectic group associated with (V,f). The group $GL(V)$ has a natural action on the third exterior power $\bigwedge^3 V$ of V and this action defines five families of nonzero trivectors of V. Four of these families are orbits for any choice of the field $\mathbb{F}$. The orbits of the fifth family are in one-to-one correspondence with the quadratic extensions of $\mathbb{F}$ that are contained in a fixed algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. In this paper, we divide the orbits corresponding to the separable quadratic extensions into suborbits for the action of $Sp(V,f) \subseteq GL(V)$ on $\bigwedge^3 V$.

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1 Introduction and main results

Let V be a 6-dimensional vector space over a field $\mathbb{F}$ which is equipped with a nondegenerate alternating bilinear form f. Let $Sp(V,f) \subseteq GL(V)$ denote the symplectic group associated with f. An ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of V is called a *hyperbolic basis* of (V,f) if $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i,j \in \{1,2,3\}$. Here, δ_{ij} denotes the Kronecker delta.

Let $\bigwedge^3 V$ denote the third exterior power of V. The elements of $\bigwedge^3 V$ are called the *trivectors* of V. For every $\theta \in GL(V)$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$ such that $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$ for all $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. Two trivectors α_1 and α_2 of V are called $GL(V)$*-equivalent* [resp., $Sp(V,f)$*-equivalent*] if there exists a $\theta \in GL(V)$ [resp., $\theta \in Sp(V,f)$] such that $\bigwedge^3(\theta)(\alpha_1) = \alpha_2$.

The subspaces $W := <\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3, \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3, \bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 - \bar{e}_3 \wedge \bar{f}_3), \bar{e}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 - \bar{e}_1 \wedge \bar{f}_1), \bar{f}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 - \bar{e}_1 \wedge \bar{f}_1), \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 - \bar{e}_2 \wedge \bar{f}_2) >$ and $\widetilde{W} := <\bar{e}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \bar{f}_1 \wedge (\bar{e}_2 \wedge \bar{f}_2 + \bar{e}_3 \wedge \bar{f}_3), \bar{e}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 + \bar{e}_1 \wedge \bar{f}_1), \bar{f}_2 \wedge (\bar{e}_3 \wedge \bar{f}_3 +$

$\bar{e}_1 \wedge \bar{f}_1), \bar{e}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2), \bar{f}_3 \wedge (\bar{e}_1 \wedge \bar{f}_1 + \bar{e}_2 \wedge \bar{f}_2) >$ of $\bigwedge^3 V$ are independent of the considered hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f). If $\mathbb{F}$ is a field of characteristic distinct from 2, then $\bigwedge^3 V = W \oplus \widetilde{W}$. If $char(\mathbb{F}) = 2$, then $\widetilde{W} \subseteq W$.

Let $\overline{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}$. Suppose $\mathbb{F}_1 \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $q(X) = X^2 - aX - b$ of $\mathbb{F}[X]$. Since $b = -q(0) \neq 0 \neq q(1) = 1 - a - b$, the values $\mu_1 := a + b - 1$ and $\mu_2 := \frac{1-a-b}{b}$ are nonzero. The field $\mathbb{F}_1$ is also the quadratic extension of $\mathbb{F}$ (contained in $\overline{\mathbb{F}}$) defined by the quadratic polynomial $\mu_2(X^2 - aX - b) = \mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1$. Observe that if $char(\mathbb{F}) \neq 2$, then the discriminant $a^2 + 4b$ of $q(X)$ is distinct from 0. If $char(\mathbb{F}) = 2$, then $a^2 + 4b = a^2 \neq 0$ if and only if the extension $\mathbb{F}_1$ of $\mathbb{F}$ is separable.

Proposition 1.1 (De Bruyn [3]) *Let $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ be a basis of V. Suppose $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1$ and $\mu_2' X^2 - (\mu_1'\mu_2' + \mu_1' + \mu_2')X + \mu_1'$ are two irreducible quadratic polynomials defining the respective quadratic extensions $\mathbb{F}_1 \subseteq \overline{\mathbb{F}}$ and $\mathbb{F}_1' \subseteq \overline{\mathbb{F}}$ of $\mathbb{F}$. Then the trivectors $\mu_1 \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \mu_2 \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$ and $\mu_1' \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \mu_2' \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$ are $GL(V)$-equivalent if and only if $\mathbb{F}_1 = \mathbb{F}_1'$.*

For every quadratic extension $\mathbb{F}_1$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, let $\chi^*_{\mathbb{F}_1}$ be a fixed trivector of the form $\mu_1 \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \mu_2 \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$, where $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ is some basis of V and $\mu_1, \mu_2 \in \mathbb{F}$ are chosen in such a way that $\mathbb{F}_1 \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$. The trivector $\chi^*_{\mathbb{F}_1}$ is not uniquely determined, but by Proposition 1.1, it is uniquely determined up to $GL(V)$-equivalence.

Classification results for $GL(V)$-equivalence classes of trivectors of V were obtained by a number of people.

- Reichel [7] obtained a complete classification of all trivectors of a 6-dimensional vector space, assuming the base field is the field of complex numbers.
- Cohen and Helminck [1] obtained a complete classification of all trivectors of a 6-dimensional vector space, assuming the base field is a perfect field of cohomological dimension at most 1.
- Revoy [8] obtained a complete classification of all trivectors of a 6-dimensional vector space, regardless of the structure of the underlying field.

The classification of the $GL(V)$-equivalence classes of trivectors of V can be found in the following proposition.

Proposition 1.2 ([1, 7, 8]) *Let $\{\bar{v}_1^*, \bar{v}_2^*, \ldots, \bar{v}_6^*\}$ be a fixed basis of V. Then every nonzero trivector of V is $GL(V)$-equivalent with precisely one of the following vectors:*
(A) $\bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_3^*$;
(B) $\bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_3^* + \bar{v}_1^* \wedge \bar{v}_4^* \wedge \bar{v}_5^*$;
(C) $\bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_3^* + \bar{v}_4^* \wedge \bar{v}_5^* \wedge \bar{v}_6^*$;

(*D*) $\bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_4^* + \bar{v}_1^* \wedge \bar{v}_3^* \wedge \bar{v}_5^* + \bar{v}_2^* \wedge \bar{v}_3^* \wedge \bar{v}_6^*$;
(*E*) $\chi_{\mathbb{F}_1}^*$ *for some quadratic extension* $\mathbb{F}_1$ *of* $\mathbb{F}$ *contained in* $\overline{\mathbb{F}}$.

Let $X \in \{A, B, C, D, E\}$. A nonzero trivector α of V is said to be of *Type (X)* if it is $GL(V)$-equivalent with the trivector described in (X) of Proposition 1.2. The description of the trivectors of Type (E) in terms of the parameters μ_1 and μ_2 is taken from De Bruyn [3].

Popov [6, Section 3] obtained a complete classification of all $Sp(V, f)$-equivalence classes of trivectors of V, assuming the underlying field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2. Popov's method heavily relies on the decomposition of $\bigwedge^3 V$ as a direct sum $W \oplus \widetilde{W}$ (which is only valid if $char(\mathbb{F}) \neq 2$) and invokes a result of Igusa [5] regarding the $Sp(V, f)$-equivalence classes of trivectors contained in the subspace $W \subset \bigwedge^3 V$. This result is only valid if the field is algebraically closed and of characteristic distinct from 2, see [5, p. 1026, Proposition 7]. In view of their applications to hyperplanes and projective embeddings of symplectic dual polar spaces, the authors are interested in the classification of all $Sp(V, f)$-equivalence classes of trivectors, regardless of the structure of the underlying field.

The $Sp(V, f)$-equivalence classes of the trivectors of Type (A), (B) and (C) were determined by De Bruyn and Kwiatkowski [4] for any field $\mathbb{F}$. The present paper is about trivectors of Type (E). By Propositions 1.1 and 1.2, if two trivectors of Type (E) are $Sp(V, f)$-equivalent, then they define the same quadratic extension $\mathbb{F}' \subseteq \overline{\mathbb{F}}$. So, it suffices to consider the following problem.

> Let $\mathbb{F}'$ be a fixed quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Let $\mathcal{E}_{\mathbb{F}'}$ denote the set of all trivectors of V which are $GL(V)$-equivalent with $\chi_{\mathbb{F}'}^*$. Then determine the $Sp(V, f)$-equivalence classes into which $\mathcal{E}_{\mathbb{F}'}$ splits.

In the present paper, we give a complete solution for the above problem in the case the quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ is separable. As before, let $a, b \in \mathbb{F}$ such that $\mathbb{F}'$ is the splitting field of the polynomial $X^2 - aX - b \in \mathbb{F}[X]$. Since we assume that the extension $\mathbb{F}'$ of $\mathbb{F}$ is separable, we have

$$a^2 + 4b \neq 0.$$

The extension $\mathbb{F}'$ of $\mathbb{F}$ is also a Galois extension and we denote by ψ the unique nontrivial automorphism of $\mathbb{F}'$ fixing each element of $\mathbb{F}$.

Before we can state our results, we need to define a number of trivectors. Consider a fixed hyperbolic basis $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ of (V, f).

- For all $h_1, h_2, h_3 \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$, let $\chi_1(h_1, h_2, h_3)$ be the trivector

$$2 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + a \cdot \Big(h_1 \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + h_2 \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + h_3 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^*\Big) + (a^2 + 2b) \cdot$$

$$\Big(h_1 h_2 \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + h_1 h_3 \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + h_2 h_3 \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*\Big) + h_1 h_2 h_3 a(a^2 + 3b) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*.$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_1(h_1,h_2,h_3)$ for some $h_1,h_2,h_3 \in \mathbb{F}^*$ is called a *trivector of Type (E1).*

• For all $k,h_1,h_2 \in \mathbb{F}$ with $k \neq 0$ and $h_1h_2(a^2+4b) \neq 1$, let $\chi_4(k,h_1,h_2)$ be the trivector

$$(1-h_1h_2(a^2+4b)) \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + (1+h_1h_2(a^2+4b)) \cdot \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$$
$$+k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b(1-h_1h_2(a^2+4b)) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big)$$
$$+h_1(1-h_1h_2(a^2+4b)) \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^* + (a^2+4b)h_2 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^*.$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_4(k,h_1,h_2)$ for some $k,h_1,h_2 \in \mathbb{F}^*$ satisfying $h_1h_2(a^2+4b) \neq 1$ is called a *trivector of Type (E4).* Notice that no trivector of type (E4) exists if the field $\mathbb{F}$ contains precisely two elements.

• For every $k \in \mathbb{F}^*$, let $\chi_2(k)$ be the trivector $\chi_4(k,0,0)$. So, $\chi_2(k)$ is equal to

$$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big).$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_2(k)$ for some $k \in \mathbb{F}^*$ is called a *trivector of Type (E2).*

• For all $k,h \in \mathbb{F}^*$, let $\chi_3(k,h)$ be the trivector $\chi_4(k,h,0)$. So, $\chi_3(k,h)$ is equal to

$$\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + k \cdot \Big(\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* - b \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big) + h \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*.$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_3(k,h)$ for some $k,h \in \mathbb{F}^*$ is called a *trivector of Type (E3).*

• For every $k \in \mathbb{F}^*$, let $\chi_5(k)$ be the trivector

$$\bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + 2 \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_2^* - a \cdot \bar{f}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + a \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^* + a \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + (a^2+b) \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^*$$
$$+k \cdot \Big(a \cdot \bar{e}_1^* \wedge \bar{f}_2^* \wedge \bar{e}_3^* - \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*\Big).$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_5(k)$ for some $k \in \mathbb{F}^*$ is called a *trivector of Type (E5).*

The following two theorems are the main results of this paper.

Theorem 1.3 *The trivectors of V that are $GL(V)$-equivalent with $\chi_{\mathbb{F}'}^*$ are precisely the trivectors of Type (E1), (E2), (E3), (E4) and (E5).*

Theorem 1.4 (1) *Let $i,j \in \{1,2,\dots,5\}$ with $i \neq j$. Then no trivector of Type (Ei) is $Sp(V,f)$-equivalent with a trivector of Type (Ej).*

(2) *Let $h_1,h_2,h_3,h_1',h_2',h_3' \in \mathbb{F}^*$. Then the trivectors $\chi_1(h_1,h_2,h_3)$ and $\chi_1(h_1',h_2',h_3')$ are $Sp(V,f)$-equivalent if and only if there exists a 3×3-matrix A over $\mathbb{F}'$ with*

determinant equal to 1 *such that* $A \cdot \mathrm{diag}(h_1, h_2, h_3) \cdot (A^\psi)^T$ *is equal to* $\mathrm{diag}(h'_1, h'_2, h'_3)$ *or* $\mathrm{diag}(-h'_1, -h'_2, -h'_3)$.

(3) *Let* $k, k' \in \mathbb{F}^*$. *Then the trivectors* $\chi_2(k)$ *and* $\chi_2(k')$ *are* $Sp(V, f)$*-equivalent if and only if* $k' \in \{k, -k\}$.

(4) *Let* $k, h, k', h' \in \mathbb{F}^*$. *Then the trivectors* $\chi_3(k, h)$ *and* $\chi_3(k', h')$ *are* $Sp(V, f)$*-equivalent if and only if there exists a* $\sigma \in \{1, -1\}$ *and an* $\eta \in \mathbb{F}' \setminus \{0\}$ *such that* $k' = \sigma k$ *and* $h' = \sigma \eta^{\psi+1} \cdot h$.

(5) *Let* $k, h_1, h_2, k', h'_1, h'_2 \in \mathbb{F}^*$ *such that* $h_1 h_2 (a^2 + 4b) \neq 1 \neq h'_1 h'_2 (a^2 + 4b)$. *Then the trivectors* $\chi_4(k, h_1, h_2)$ *and* $\chi_4(k', h'_1, h'_2)$ *are* $Sp(V, f)$*-equivalent if and only if* $h'_1 h'_2 = h_1 h_2$ *and there exist* $\eta_1, \eta_2 \in \mathbb{F}'$ *and a* $\sigma \in \{1, -1\}$ *such that* $k' = \sigma k$ *and* $\sigma h'_1 = \eta_1^{\psi+1} h_1 + \eta_2^{\psi+1} h_2$.

(6) *Let* $k, k' \in \mathbb{F}^*$. *Then the trivectors* $\chi_5(k)$ *and* $\chi_5(k')$ *are* $Sp(V, f)$*-equivalent if and only if* $k' \in \{k, -k\}$.

In Theorem 1.4(2), the matrix $\mathrm{diag}(h_1, h_2, h_3)$ denotes the diagonal matrix with diagonal entries equal to h_1, h_2 and h_3.

The conditions in Theorem 1.4(2),(4),(5) can be rephrased such that no reference is made any more to the extension field $\mathbb{F}'$. E.g., the condition in Theorem 1.4(4) that there exists a $\sigma \in \{1, -1\}$ and an $\eta \in \mathbb{F}' \setminus \{0\}$ such that $k' = \sigma k$ and $h' = \sigma h \cdot \eta^{\psi+1}$ can be rephrased as follows: there should exist a $\sigma \in \{1, -1\}$ and $\lambda, \mu \in \mathbb{F}$ such that $k' = \sigma k$ and $\lambda^2 + a\lambda\mu - b\mu^2 = \frac{h'}{\sigma h}$. Indeed, if $\delta \in \mathbb{F}'$ is a root of the polynomial $X^2 - aX - b \in \mathbb{F}[X]$, then $\delta + \delta^\psi = a$ and $\delta^{\psi+1} = -b$. Hence, $(\lambda + \delta\mu)^{\psi+1} = (\lambda + \delta\mu)(\lambda + \delta^\psi \mu) = \lambda^2 + a\lambda\mu - b\mu^2$.

Suppose $\mathbb{F}$ is the finite field $\mathbb{F}_q$ with q elements. Every quadratic extension of $\mathbb{F}$ is then separable. The number of $Sp(V, f)$-equivalence classes of trivectors of Type (E) can easily be deduced from Theorem 1.4. If $i \in \{1, 2, 3, 5\}$, then the total number of $Sp(V, f)$-equivalence classes of trivectors of Type (Ei) is equal to $q - 1$ if q is even and $\frac{q-1}{2}$ if q is odd. The total number of $Sp(V, f)$-equivalence classes of trivectors of Type (E4) is equal to $(q - 1)(q - 2)$ if q is even and $\frac{(q-1)(q-2)}{2}$ if q is odd.

We will prove Theorems 1.3 and 1.4 in Section 4. In Section 3, we do all the preparatory work. If we consider the 6-dimensional vector space V' over $\mathbb{F}'$ that naturally extends the 6-dimensional $\mathbb{F}$-vector space V, then the trivector $\chi^*_{\mathbb{F}'}$ is a trivector of Type (C) of V', see Lemma 3.8. In Section 2, we list all the $Sp(V', f')$-equivalence classes of trivectors of V', where f' is the alternating bilinear form on V' that naturally extends f. Although the trivectors of Type (E2), (E3) and (E4) can be uniformly described, we wish to look at them as if they belong to three distinct families. One of our motivations for doing so is that these trivectors belong to three distinct $Sp(V', f')$-equivalence classes of trivectors of V', see Corollary 4.2.

2 Classification results regarding trivectors of Type (C)

Let V' be a 6-dimensional vector space over a field $\mathbb{F}'$ which is equipped with a nondegenerate alternating bilinear form f'.

The following classification of the $Sp(V',f')$-equivalence classes of trivectors of Type (C) of V' was obtained in De Bruyn and Kwiatkowski [4].

Proposition 2.1 ([4, Theorem 1.5]) *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V',f'). Then every trivector of Type (C) of V' is $Sp(V',f')$-equivalent with at least one of the following trivectors:*

$(C1)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$*;*

$(C2)$ $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$*;*

$(C3)$ $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$*;*

$(C4)$ $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$*;*

$(C5)$ $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$*;*

$(C6)$ $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ *for some* $\lambda \in \mathbb{F}' \setminus \{0\}$ *and some* $\epsilon \in \mathbb{F}' \setminus \{0,-1\}$*.*

A trivector of V' is said to be of *Type (Ci)*, $i \in \{1,2,\dots,6\}$, if it is $Sp(V',f')$-equivalent with a trivector described in (Ci) of Proposition 2.1. Observe that there are no trivectors of Type (C6) if $|\mathbb{F}'| = 2$.

Proposition 2.2 ([4, Theorem 1.6]) *Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a given hyperbolic basis of (V',f').*

(1) *Let $i,j \in \{1,2,\dots,6\}$ with $i \neq j$. Then no trivector of Type (Ci) is $Sp(V',f')$-equivalent with a trivector of Type (Cj).*

(2) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + \lambda' \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ are $Sp(V',f')$-equivalent if and only if $\lambda' \in \{\lambda,-\lambda\}$.*

(3) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* - \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V',f')$-equivalent if and only if $\lambda = \lambda'$.*

(4) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ and $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \lambda' \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ are $Sp(V',f')$-equivalent if and only if $\lambda' \in \{\lambda,-\lambda\}$.*

(5) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V',f')$-equivalent if and only if $\lambda' \in \{\lambda,-\lambda\}$.*

(6) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ and $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + \lambda' \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ are $Sp(V',f')$-equivalent if and only if $\lambda' \in \{\lambda,-\lambda\}$.*

(7) *If $\lambda,\lambda' \in \mathbb{F}' \setminus \{0\}$ and $\epsilon,\epsilon' \in \mathbb{F}' \setminus \{0,-1\}$, then the trivectors $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon \cdot \bar{f}_3^*) + \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ and $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \epsilon' \cdot \bar{f}_3^*) + \lambda' \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ are $Sp(V',f')$-equivalent if and only if $\epsilon' = \epsilon$ and $\lambda' \in \{\lambda,-\lambda\}$.*

For a proof of the following lemma, see e.g. Lemma 5.7 of De Bruyn and Kwiatkowski [4].

Lemma 2.3 *Let* $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ *and* $\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_6\}$ *be two bases of* V'. *If* $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_4 \wedge \bar{w}_5 \wedge \bar{w}_6$, *then* $\{\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6\} = \{\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3, \bar{w}_4 \wedge \bar{w}_5 \wedge \bar{w}_6\}$.

3 Preliminary lemmas

Suppose $\mathbb{F}$ and $\mathbb{F}'$ are two fields such that $\mathbb{F}'$ is a separable quadratic extension of $\mathbb{F}$. Then $\mathbb{F}'/\mathbb{F}$ is also a Galois extension and we denote by ψ the unique nontrivial automorphism of $\mathbb{F}'$ fixing each element of $\mathbb{F}$.

Let V' be a 6-dimensional vector space over $\mathbb{F}'$ and let $B^* = \{\bar{v}_1^*, \bar{v}_2^*, \ldots, \bar{v}_6^*\}$ be a basis of V'. The set V of all $\mathbb{F}$-linear combinations of the elements of B^* can be given in a natural way the structure of a 6-dimensional vector space over $\mathbb{F}$. By definition, B^* is also a basis of this vector space V.

For every vector $\bar{x} = \sum_{i=1}^{6} \lambda_i \bar{v}_i^*$ of V', we define $\bar{x}^\psi := \sum_{i=1}^{6} \lambda_i^\psi \bar{v}_i^*$. The verification of the following lemma is straightforward.

Lemma 3.1 (1) *If* $\bar{x} \in V'$, *then* $\bar{x}^\psi = \bar{x}$ *if and only if* $\bar{x} \in V$.

(2) *If* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}'$ *and* $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \in V'$ *for some* $n \geq 1$, *then* $\Big(\sum_{i=1}^{n} \lambda_i \bar{v}_i\Big)^\psi = \sum_{i=1}^{n} \lambda_i^\psi \bar{v}_i^\psi$.

(3) *If* $B = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ *is a basis of* V *and* $\lambda_1, \lambda_2, \ldots, \lambda_6 \in \mathbb{F}'$, *then* $\Big(\sum_{i=1}^{6} \lambda_i \bar{v}_i\Big)^\psi = \sum_{i=1}^{6} \lambda_i^\psi \bar{v}_i$.

Every element of $GL(V)$ naturally extends to an element of $GL(V')$. In the sequel, we will identify each element of $GL(V)$ with its corresponding element of $GL(V')$.

Lemma 3.2 *If* $\bar{x} \in V'$ *and* $\theta \in GL(V)$, *then* $\theta(\bar{x})^\psi = \theta(\bar{x}^\psi)$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_6 \in \mathbb{F}'$ such that $\bar{x} = \sum_{i=1}^{6} \lambda_i \bar{v}_i^*$. Since $\theta \in GL(V)$, we have $\theta(\bar{v}_i^*) \in V$ and hence $\theta(\bar{v}_i^*)^\psi = \theta(\bar{v}_i^*)$. By Lemma 3.1, we have $\theta(\bar{x})^\psi = (\sum_{i=1}^{6} \lambda_i \theta(\bar{v}_i^*))^\psi = \sum_{i=1}^{6} \lambda_i^\psi \theta(\bar{v}_i^*) = \theta(\sum_{i=1}^{6} \lambda_i^\psi \bar{v}_i^*) = \theta(\bar{x}^\psi)$. ■

Now, let $\bigwedge^3 V$ denote the third exterior power of V and let $\bigwedge^3 V'$ denote the third exterior power of V'. We can and will regard $\bigwedge^3 V$ as a subset of $\bigwedge^3 V'$. The set $\mathcal{B}^* := \{\bar{v}_i^* \wedge \bar{v}_j^* \wedge \bar{v}_k^* \,|\, 1 \leq i < j < k \leq 6\}$ can be considered as a basis of the vector space $\bigwedge^3 V$ as well as a basis of $\bigwedge^3 V'$. For every vector $\alpha = \sum_{1 \leq i<j<k \leq 6} \lambda_{ijk} \cdot \bar{v}_i^* \wedge \bar{v}_j^* \wedge \bar{v}_k^*$ of $\bigwedge^3 V'$, we define $\alpha^\psi := \sum_{1 \leq i<j<k \leq 6} \lambda_{ijk}^\psi \cdot \bar{v}_i^* \wedge \bar{v}_j^* \wedge \bar{v}_k^*$. The following clearly holds.

Lemma 3.3 (1) *If* $\alpha \in \bigwedge^3 V'$, *then* $\alpha^\psi = \alpha$ *if and only if* $\alpha \in \bigwedge^3 V$.

(2) *If* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}'$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \bigwedge^3 V'$ *(for some* $n \geq 1$*), then we have* $\left(\sum_{i=1}^n \lambda_i \alpha_i\right)^\psi = \sum_{i=1}^n \lambda_i^\psi \alpha_i^\psi$.

(3) *If* $\bar{v}_1$, $\bar{v}_2$ *and* $\bar{v}_3$ *are vectors of* V'*, then* $(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi = \bar{v}_1^\psi \wedge \bar{v}_2^\psi \wedge \bar{v}_3^\psi$.

(4) *If* $B = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ *is a basis of* V *and* $\lambda_{ijk} \in \mathbb{F}'$ *for all* $i, j, k \in \{1, 2, \ldots, 6\}$ *satisfying* $1 \leq i < j < k \leq 6$*, then* $\left(\sum_{1\leq i<j<k\leq 6} \lambda_{ijk} \cdot \bar{v}_i \wedge \bar{v}_j \wedge \bar{v}_k\right)^\psi = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk}^\psi \cdot \bar{v}_i \wedge \bar{v}_j \wedge \bar{v}_k$.

Every element of $GL(\bigwedge^3 V)$ naturally extends to an element of $GL(\bigwedge^3 V')$. In the sequel, we will identify each element of $GL(\bigwedge^3 V)$ with its corresponding element of $GL(\bigwedge^3 V')$. For every $\theta \in GL(V')$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V')$ such that $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$ for all $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$. If $\theta \in GL(V)$, then $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$.

Now, let f be a nondegenerate alternating bilinear form on V. Then f can be extended in a unique way to a nondegenerate alternating bilinear form f' on V'. Let $Sp(V, f) \cong Sp(6, \mathbb{F})$ and $Sp(V', f') \cong Sp(6, \mathbb{F}')$ denote the symplectic groups associated with these alternating bilinear forms. In view of $GL(V) \subset GL(V')$, we also have that $Sp(V, f)$ is a subgroup of $Sp(V', f')$.

The following lemma is known, see e.g. De Bruyn [2, Section 4].

Lemma 3.4 *For every hyperbolic basis* $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ *of* (V', f')*, let* π_B *denote the linear map from* $\bigwedge^3 V'$ *to* V' *defined by*

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_1, \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_1,$$

$$\pi_B(\bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_2, \pi_B(\bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_2,$$

$$\pi_B(\bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_3, \pi_B(\bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{f}_3.$$

Then π_B *is independent of the chosen hyperbolic basis* B *of* (V', f').

Put $\pi := \pi_B$, where B is an arbitrary hyperbolic basis of (V', f').

Lemma 3.5 *For every trivector* α *of* V*, we have* $\pi(\alpha^\psi) = \pi(\alpha)^\psi$.

Proof. Choose a hyperbolic basis $B = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_6)$ of (V, f). Then $\pi = \pi_B$. Let $\lambda_{ijk} \in \mathbb{F}'$ $(1 \leq i < j < k \leq 6)$ such that $\alpha = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk} \cdot \bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k$. Then $\pi_B(\alpha) = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk} \cdot \pi_B(\bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k)$ and hence $\pi_B(\alpha)^\psi = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk}^\psi \cdot \pi_B(\bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k)$ since $\pi_B(\bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k) \in V$. On the other hand, $\alpha^\psi = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk}^\psi \cdot \bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k$ and hence $\pi_B(\alpha^\psi) = \sum_{1\leq i<j<k\leq 6} \lambda_{ijk}^\psi \cdot \pi_B(\bar{g}_i \wedge \bar{g}_j \wedge \bar{g}_k)$. ■

Lemma 3.6 *For all $\bar{x}, \bar{y} \in V'$, we have $f'(\bar{x}^\psi, \bar{y}^\psi) = (f'(\bar{x}, \bar{y}))^\psi$.*

Proof. Put $\bar{x} = \sum_{i=1}^6 \lambda_i \bar{v}_i^*$ and $\bar{y} = \sum_{j=1}^6 \mu_j \bar{v}_j^*$. Then $f'(\bar{x}, \bar{y}) = \sum_{i=1}^6 \sum_{j=1}^6 \lambda_i \mu_j \cdot f'(\bar{v}_i^*, \bar{v}_j^*)$. Since $f'(\bar{v}_i^*, \bar{v}_j^*) \in \mathbb{F}$, we have $(f'(\bar{x}, \bar{y}))^\psi = \sum_{i=1}^6 \sum_{j=1}^6 \lambda_i^\psi \mu_j^\psi \cdot f'(\bar{v}_i^*, \bar{v}_j^*)$. On the other hand, since $\bar{x}^\psi = \sum_{i=1}^6 \lambda_i^\psi \bar{v}_i^*$ and $\bar{y}^\psi = \sum_{j=1}^6 \mu_j^\psi \bar{v}_j^*$, we have $f'(\bar{x}^\psi, \bar{y}^\psi) = \sum_{i=1}^6 \sum_{j=1}^6 \lambda_i^\psi \mu_j^\psi \cdot f'(\bar{v}_i^*, \bar{v}_j^*)$. Hence, $f'(\bar{x}^\psi, \bar{y}^\psi) = (f'(\bar{x}, \bar{y}))^\psi$. ∎

Let G be one of the groups $GL(V), GL(V'), Sp(V, f), Sp(V', f')$. Two trivectors α_1 and α_2 of V' are called *G-equivalent* if there exists a $\theta \in G$ for which $\bigwedge^3(\theta)(\alpha_1) = \alpha_2$. The following is obvious.

Lemma 3.7 (1) *Two $GL(V)$-equivalent trivectors of V' are also $GL(V')$-equivalent.*

(2) *Two $Sp(V, f)$-equivalent trivectors of V' are also $GL(V)$-equivalent, $Sp(V', f')$-equivalent and $GL(V')$-equivalent.*

(3) *Two $Sp(V', f')$-equivalent trivectors of V' are also $GL(V')$-equivalent.*

Let δ be a fixed element of $\mathbb{F}' \setminus \mathbb{F}$. Then δ is a root of an irreducible quadratic polynomial $X^2 - aX - b \in \mathbb{F}[X]$. If we put $\mu_1 = a + b - 1$ and $\mu_2 = \frac{1-a-b}{b}$, then δ is also a root of the polynomial $\mu_2 X^2 - (\mu_1\mu_2 + \mu_1 + \mu_2)X + \mu_1 \in \mathbb{F}[X]$. We define

$$\chi_{\mathbb{F}'}^* := \mu_1 \cdot \bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_3^* + \mu_2 \cdot \bar{v}_4^* \wedge \bar{v}_5^* \wedge \bar{v}_6^* + (\bar{v}_1^* + \bar{v}_4^*) \wedge (\bar{v}_2^* + \bar{v}_5^*) \wedge (\bar{v}_3^* + \bar{v}_6^*).$$

Observe that

$$\delta + \delta^\psi = a,\ \delta^{\psi+1} = -b,\ \delta^2 = a\delta + b,\ \delta^3 = (a^2 + b)\delta + ab.$$

Since $\mathbb{F}'$ is a separable extension of $\mathbb{F}$, the other root δ^ψ of $X^2 - aX - b$ is distinct from δ. We have

$$(\delta^\psi - \delta)^2 = (\delta^\psi + \delta)^2 - 4\delta^{\psi+1} = a^2 + 4b \neq 0.$$

Lemma 3.8 *There exist vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ such that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi\}$ is a basis of V' and $\chi_{\mathbb{F}'}^* = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1^\psi \wedge \bar{v}_2^\psi \wedge \bar{v}_3^\psi$.*

Proof. It is straightforward to verify that

$$\begin{aligned}
\chi_{\mathbb{F}'}^* &= \mu_1 \cdot \bar{v}_1^* \wedge \bar{v}_2^* \wedge \bar{v}_3^* + \mu_2 \cdot \bar{v}_4^* \wedge \bar{v}_5^* \wedge \bar{v}_6^* + (\bar{v}_1^* + \bar{v}_4^*) \wedge (\bar{v}_2^* + \bar{v}_5^*) \wedge (\bar{v}_3^* + \bar{v}_6^*) \\
&= \frac{\delta^\psi - 1}{\delta(\delta^\psi - \delta)} \cdot (\bar{v}_4^* + \delta\bar{v}_1^*) \wedge (\bar{v}_5^* + \delta\bar{v}_2^*) \wedge (\bar{v}_6^* + \delta\bar{v}_3^*) \\
&\qquad + \frac{1 - \delta}{\delta^\psi(\delta^\psi - \delta)} \cdot (\bar{v}_4^* + \delta^\psi\bar{v}_1^*) \wedge (\bar{v}_5^* + \delta^\psi\bar{v}_2^*) \wedge (\bar{v}_6^* + \delta^\psi\bar{v}_3^*),
\end{aligned}$$

where $\{\frac{\delta^\psi-1}{\delta(\delta^\psi-\delta)}(\bar{v}_4^* + \delta\bar{v}_1^*), \bar{v}_5^* + \delta\bar{v}_2^*, \bar{v}_6^* + \delta\bar{v}_3^*, \frac{1-\delta}{\delta^\psi(\delta^\psi-\delta)}(\bar{v}_4^* + \delta^\psi\bar{v}_1^*), \bar{v}_5^* + \delta^\psi\bar{v}_2^*, \bar{v}_6^* + \delta^\psi\bar{v}_3^*\}$ is a basis of V'. ∎

Lemma 3.8 implies that, although $\chi_{\mathbb{F}'}^*$ is a trivector of Type (E) of V, it is a trivector of Type (C) when regarded as an element of $\bigwedge^3 V'$.

Lemma 3.9 *Suppose $\chi \in \bigwedge^3 V$ can be written in the form $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1^\psi \wedge \bar{v}_2^\psi \wedge \bar{v}_3^\psi$, where $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi\}$ is a basis of V'. Then χ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$.*

Proof. Let $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ be vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V' and $\chi^*_{\mathbb{F}'} = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + \bar{u}_1^\psi \wedge \bar{u}_2^\psi \wedge \bar{u}_3^\psi$. Let θ be the element of $GL(V')$ defined by $\bar{v}_1 \mapsto \bar{u}_1$, $\bar{v}_2 \mapsto \bar{u}_2$, $\bar{v}_3 \mapsto \bar{u}_3$, $\bar{v}_1^\psi \mapsto \bar{u}_1^\psi$, $\bar{v}_2^\psi \mapsto \bar{u}_2^\psi$, $\bar{v}_3^\psi \mapsto \bar{u}_3^\psi$. Now, $\{\bar{v}_1 + \bar{v}_1^\psi, \delta\bar{v}_1 + \delta^\psi \bar{v}_1^\psi, \bar{v}_2 + \bar{v}_2^\psi, \delta\bar{v}_2 + \delta^\psi \bar{v}_2^\psi, \bar{v}_3 + \bar{v}_3^\psi, \delta\bar{v}_3 + \delta^\psi \bar{v}_3^\psi\}$ is a basis of V which is mapped by θ to the basis $\{\bar{u}_1 + \bar{u}_1^\psi, \delta\bar{u}_1 + \delta^\psi \bar{u}_1^\psi, \bar{u}_2 + \bar{u}_2^\psi, \delta\bar{u}_2 + \delta^\psi \bar{u}_2^\psi, \bar{u}_3 + \bar{u}_3^\psi, \delta\bar{u}_3 + \delta^\psi \bar{u}_3^\psi\}$ of V. So, $\theta \in GL(V)$. Since $\bigwedge^3(\theta)(\chi) = \chi^*_{\mathbb{F}'}$, the trivectors χ and $\chi^*_{\mathbb{F}'}$ are $GL(V)$-equivalent. ∎

Lemma 3.10 *Let χ be a trivector of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$. If $\chi = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6$ for some basis $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ of V', then we have $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi$.*

Proof. Let θ be an element of $GL(V)$ such that $\chi = \bigwedge^3(\theta)(\chi^*_{\mathbb{F}'})$. By Lemma 3.8, there exist vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V'$ such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V' and $\chi^*_{\mathbb{F}'} = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + \bar{u}_1^\psi \wedge \bar{u}_2^\psi \wedge \bar{u}_3^\psi$. By Lemma 3.2, we have $\chi = \bigwedge^3(\theta)(\chi^*_{\mathbb{F}'}) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3) + \theta(\bar{u}_1^\psi) \wedge \theta(\bar{u}_2^\psi) \wedge \theta(\bar{u}_3^\psi) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3) + \theta(\bar{u}_1)^\psi \wedge \theta(\bar{u}_2)^\psi \wedge \theta(\bar{u}_3)^\psi$. By Lemma 2.3, we have $\{\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6\} = \{\theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3), (\theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3))^\psi\}$. Hence, $\bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 = (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi$. ∎

For all $\bar{x}, \bar{y} \in V'$, we define $h(\bar{x}, \bar{y}) := f'(\bar{x}, \bar{y}^\psi)$. Then h is a nondegenerate skew-ψ-Hermitian form on V'. For all $\bar{x}, \bar{y} \in V'$, we have $h(\bar{x}^\psi, \bar{y}^\psi) = h(\bar{x}, \bar{y})^\psi$ and $h(\bar{y}, \bar{x}) = -h(\bar{x}, \bar{y})^\psi$.

We define

$$M^* := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

For every $g \in \{f', h\}$ and every $k \geq 1$ vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ of V', let $M_g(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$ be the $(k \times k)$-matrix over $\mathbb{F}'$ whose (i,j)-th entry is equal to $g(\bar{x}_i, \bar{x}_j)$ $(i, j \in \{1, 2, \ldots, k\})$.

Lemma 3.11 *Let $\bar{u}_1, \bar{u}_2, \bar{u}_3$ be three vectors of V'. If $M = M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and $N = M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, then the matrix $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi)$ is equal to*

$$A = \begin{bmatrix} M & N \\ -N^T & M^\psi \end{bmatrix}.$$

As a consequence, $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V' if and only if A is nonsingular.

Proof. Since $h(\bar{x}, \bar{y}) = f'(\bar{x}, \bar{y}^\psi)$ and $f'(\bar{x}^\psi, \bar{y}^\psi) = f'(\bar{x}, \bar{y})^\psi$ for all $\bar{x}, \bar{y} \in V'$, the matrix $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi)$ is equal to A. ∎

Lemma 3.12 *Let $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{v}_1, \bar{v}_2, \bar{v}_3$ be vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ and $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi\}$ are two bases of V'. Put $M_1 := M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $N_1 := M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $M_2 := M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $N_2 := M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. Then the following holds.*

(1) *The element* θ *of* $GL(V')$ *defined by* $\bar{u}_1 \mapsto \bar{v}_1,\ \bar{u}_2 \mapsto \bar{v}_2,\ \bar{u}_3 \mapsto \bar{v}_3,\ \bar{u}_1^\psi \mapsto \bar{v}_1^\psi,\ \bar{u}_2^\psi \mapsto \bar{v}_2^\psi,\ \bar{u}_3^\psi \mapsto \bar{v}_3^\psi$ *is also an element of* $GL(V)$.

(2) *The element* θ *belongs to* $Sp(V, f)$ *if and only if* $M_1 = M_2$ *and* $N_1 = N_2$.

Proof. (1) Since θ maps the basis $\{\bar{u}_1 + \bar{u}_1^\psi, \delta\bar{u}_1 + \delta^\psi \bar{u}_1^\psi, \bar{u}_2 + \bar{u}_2^\psi, \delta\bar{u}_2 + \delta^\psi \bar{u}_2^\psi, \bar{u}_3 + \bar{u}_3^\psi, \delta\bar{u}_3 + \delta^\psi \bar{u}_3^\psi\}$ of V to the basis $\{\bar{v}_1 + \bar{v}_1^\psi, \delta\bar{v}_1 + \delta^\psi \bar{v}_1^\psi, \bar{v}_2 + \bar{v}_2^\psi, \delta\bar{v}_2 + \delta^\psi \bar{v}_2^\psi, \bar{v}_3 + \bar{v}_3^\psi, \delta\bar{v}_3 + \delta^\psi \bar{v}_3^\psi\}$ of V, θ must be an element of $GL(V)$.

(2) The element θ belongs to $Sp(V, f)$ if and only if it belongs to $Sp(V', f')$. The matrices representing f' with respect to the ordered bases $(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi)$ and $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi)$ are respectively equal to

$$A_1 = \begin{bmatrix} M_1 & N_1 \\ -N_1^T & M_1^\psi \end{bmatrix}, \qquad A_2 = \begin{bmatrix} M_2 & N_2 \\ -N_2^T & M_2^\psi \end{bmatrix}.$$

The element θ belongs to $Sp(V', f')$ if and only if $A_1 = A_2$, i.e. if and only if $M_1 = M_2$ and $N_1 = N_2$. ■

Lemma 3.13 *Let* $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{v}_1, \bar{v}_2, \bar{v}_3$ *be vectors of* V' *such that* $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ *and* $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi\}$ *are two bases of* V'. *Put* $M_1 := M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $N_1 := M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $M_2 := M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ *and* $N_2 := M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. *Then the two trivectors* $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$ *and* $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi$ *of* V *are* $Sp(V, f)$*-equivalent if and only if there exists a* (3×3)*-matrix* A *over* $\mathbb{F}'$ *with determinant 1 such that one of the following holds:*

(1) $A \cdot M_1 \cdot A^T = M_2$ *and* $A \cdot N_1 \cdot (A^\psi)^T = N_2$;

(2) $A \cdot M_1 \cdot A^T = (M_2)^\psi$ *and* $A \cdot N_1 \cdot (A^\psi)^T = (N_2)^\psi$.

Proof. By Lemma 2.3, the two trivectors $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$ and $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi$ are $Sp(V, f)$-equivalent if and only if there exist vectors $\bar{w}_1, \bar{w}_2, \bar{w}_3 \in V'$ and a $\theta \in Sp(V, f)$ such that at least one of the following holds:

(1) $\theta(\bar{u}_i) = \bar{w}_i$ for every $i \in \{1, 2, 3\}$ and $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$;

(2) $\theta(\bar{u}_i^\psi) = \bar{w}_i$ for every $i \in \{1, 2, 3\}$ and $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$.

There exist three vectors $\bar{w}_1, \bar{w}_2, \bar{w}_3$ of V' such that $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3$ if and only if there exists a (3×3)-matrix B with determinant 1 such that $[\bar{w}_1, \bar{w}_2, \bar{w}_3]^T = B \cdot [\bar{v}_1, \bar{v}_2, \bar{v}_3]^T$. If this is the case, then $M_{f'}(\bar{w}_1, \bar{w}_2, \bar{w}_3) = B \cdot M_2 \cdot B^T$ and $M_h(\bar{w}_1, \bar{w}_2, \bar{w}_3) = B \cdot N_2 \cdot (B^\psi)^T$. The lemma now easily follows from Lemma 3.12, taking into account that $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = M_1$, $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = N_1$, $M_{f'}(\bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi) = M_1^\psi$ and $M_h(\bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi) = N_1^\psi$. ■

Lemma 3.14 *Let* $A = (a_{ij})_{1 \le i,j \le 3}$ *be a matrix with determinant* 1. *Then* $A \cdot M^* \cdot A^T = M^*$ *if and only if* $a_{11} = 1$, $a_{12} = a_{13} = 0$ *and* $a_{22}a_{33} - a_{23}a_{32} = 1$.

Proof. If $a_{11} = 1$, $a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{23}a_{32} = 1$, then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a_{31}a_{23} - a_{21}a_{33} & a_{33} & -a_{23} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{32} & a_{22} \end{bmatrix}, \tag{1}$$

$A \cdot M^* = M^* \cdot (A^{-1})^T$ and hence $AM^*A^T = M^*$.

Conversely, suppose $AM^*A^T = M^*$. Let $\bar{e}_1, \bar{e}_2, \bar{f}_2$ be vectors of V' such that $f'(\bar{e}_1, \bar{e}_2) = f'(\bar{e}_1, \bar{f}_2) = 0$ and $f'(\bar{e}_2, \bar{f}_2) = 1$. Let $\bar{v}_1, \bar{v}_2, \bar{v}_3$ be the vectors of V' such that $[\bar{v}_1, \bar{v}_2, \bar{v}_3]^T = A \cdot [\bar{e}_1, \bar{e}_2, \bar{f}_2]^T$. Since $\det(A) = 1$, we have $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$. Since $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = A \cdot M_{f'}(\bar{e}_1, \bar{e}_2, \bar{f}_2) \cdot A^T = AM^*A^T = M^*$, we have that $f'(\bar{v}_1, \bar{v}_2) = f'(\bar{v}_1, \bar{v}_3) = 0$ and $f'(\bar{v}_2, \bar{v}_3) = 1$. Since $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$, we have $\bar{v}_1 = \pi(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \pi(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_1$. Since $\bar{v}_1 = a_{11}\bar{e}_1 + a_{12}\bar{e}_2 + a_{13}\bar{f}_2$, we have $a_{11} = 1$, $a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{23}a_{32} = 1$. ∎

Lemma 3.15 *Let $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{v}_1, \bar{v}_2, \bar{v}_3$ be vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ and $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1^\psi, \bar{v}_2^\psi, \bar{v}_3^\psi\}$ are two bases of V'. Put $M_1 := M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $N_1 := M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)$, $M_2 := M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $N_2 := M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. If the two trivectors $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$ and $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)^\psi$ of V are $Sp(V, f)$-equivalent, then $rank(M_1) = rank(M_2)$ and $rank(N_1) = rank(N_2)$. Moreover, if $M_1 = M_2 = M^*$, then $(N_2)_{11} \in \{(N_1)_{11}, ((N_1)_{11})^\psi\}$.*

Proof. The fact that $rank(M_1) = rank(M_2)$ and $rank(N_1) = rank(N_2)$ is a consequence of Lemma 3.13. Suppose now that $M_1 = M_2 = M^*$. Then there exists a (3×3)-matrix A with determinant 1 such that $A \cdot M^* \cdot A^T = M^*$. By Lemma 3.14, we have $a_{11} = 1$, $a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{23}a_{32} = 1$. If we put $B := ((A^{-1})^\psi)^T$ with A^{-1} as in (1), then $A \cdot N_1$ is equal to either $N_2 \cdot B$ or $N_2^\psi \cdot B$, implying that $(N_2)_{11} \in \{(N_1)_{11}, ((N_1)_{11})^\psi\}$. ∎

Lemma 3.16 *Let l be an arbitrary element of $\mathbb{F}^*$ and θ an arbitrary map from $\mathbb{F}^*$ to $\mathbb{F}$. If $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are three vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V', then precisely one of the following cases occurs.*

(1) *There exist $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ and $h_1, h_2, h_3 \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = diag(0, 0, 0)$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(h_1, h_2, h_3)$.*

(2) *There exist $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ and a $k \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, 0, 0)$.*

(3) *There exist $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ and $k, h \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, h, 0)$.*

(4) *There exist $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ and $k, h_1, h_2 \in \mathbb{F}^*$ with $h_1 h_2 (a^2 + 4b) \neq 1$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, h_1, h_2)$.*

(5) *There exist $\bar{v}_1, \bar{v}_2, \bar{v}_3 \in V'$ and a $k \in \mathbb{F}^*$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$,*
$$M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^* \text{ and } M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = \begin{bmatrix} 0 & 0 & k \\ 0 & \frac{l(\delta^\psi - \delta)}{k} & 0 \\ -k & 0 & (\delta^\psi - \delta) \cdot \theta(k) \end{bmatrix}.$$

Proof. Observe that by Lemma 3.15, at most one of the above five cases can occur. Let $\tilde{f}$ and $\tilde{h}$ denote the respective restrictions of f and h to the subspace $U := <\bar{u}_1, \bar{u}_2, \bar{u}_3>$.

(1) Suppose first that $\tilde{f} = 0$. We prove that $\tilde{h}$ is a nondegenerate skew-ψ-Hermitian form on U. Suppose $\bar{x}$ is a vector of U such that $\tilde{h}(\bar{x}, \bar{y}) = 0$, $\forall \bar{y} \in U$, i.e. $f'(\bar{x}, \bar{y}) = 0$, $\forall \bar{y} \in U^\psi = <\bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi>$. Since also $f'(\bar{x}, \bar{y}) = 0$, $\forall \bar{y} \in U$, we have $\bar{x} = \bar{o}$ since f' is nondegenerate. Hence, $\tilde{h}$ is nondegenerate.

Since $\tilde{h}$ is nondegenerate, there exists a vector $\bar{v}_1 \in U$ for which $\tilde{h}(\bar{v}_1, \bar{v}_1) \neq 0$. Since $\tilde{h}$ is nondegenerate, the restriction of $\tilde{h}$ to the subspace $\bar{v}_1^{\perp_{\tilde{h}}}$ is also nondegenerate. So, there exists a vector $\bar{v}_2 \in \bar{v}_1^{\perp_{\tilde{h}}}$ for which $\tilde{h}(\bar{v}_2, \bar{v}_2) \neq 0$. Now, let $\bar{v}_3$ be the unique vector of $\bar{v}_1^{\perp_{\tilde{h}}}$ for which $\tilde{h}(\bar{v}_2, \bar{v}_3) = 0$ and $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$. Since $\tilde{h}(\bar{v}_3, \bar{v}_1) = \tilde{h}(\bar{v}_3, \bar{v}_2) = 0$ and $\tilde{h}$ is nondegenerate, we have $\tilde{h}(\bar{v}_3, \bar{v}_3) \neq 0$. Now, put $h(\bar{v}_i, \bar{v}_i) = (\delta^\psi - \delta) \cdot h_i$ for every $i \in \{1, 2, 3\}$. The fact that h is skew-ψ-Hermitian implies that $h_1, h_2, h_3 \in \mathbb{F}^*$. Clearly, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = diag(0, 0, 0)$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(h_1, h_2, h_3)$.

In the sequel, we suppose that $\tilde{f} \neq 0$. Then $\bar{a} := \pi(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)$ is a nonzero vector in $Rad(\tilde{f})$.

(2) Suppose $h(\bar{a}, \bar{a}) \neq 0$ and $rank[M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)] = 1$. Then put $\bar{v}_1 := \bar{a}$ and let $\bar{v}_2$ and $\bar{v}_3$ be two vectors of $\bar{a}^{\perp_{\tilde{h}}}$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$. Since $rank[M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3)] = rank[M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)] = 1$, we have $h(\bar{v}_2, \bar{v}_2) = h(\bar{v}_3, \bar{v}_3) = h(\bar{v}_2, \bar{v}_3) = 0$. Since $\bar{v}_1 = \pi(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \pi(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)$, we have $f'(\bar{v}_2, \bar{v}_3) = 1$. Now, put $h(\bar{a}, \bar{a}) = k(\delta^\psi - \delta)$. Then $k \in \mathbb{F}^*$. Clearly, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, 0, 0)$.

(3) Suppose $h(\bar{a}, \bar{a}) \neq 0$ and $rank[M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)] = 2$. Then put $\bar{v}_1 := \bar{a}$, let $\bar{v}_3$ be an arbitrary nonzero vector of $Rad(\tilde{h})$ and let $\bar{v}_2$ be a vector of $\bar{v}_1^{\perp_{\tilde{h}}}$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$. Similarly as in (2), we have $f'(\bar{v}_2, \bar{v}_3) = 1$. Now, put $h(\bar{v}_1, \bar{v}_1) = (\delta^\psi - \delta)k$ and $h(\bar{v}_2, \bar{v}_2) = (\delta^\psi - \delta)h$. Then $k, h \in \mathbb{F}^*$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, h, 0)$.

(4) Suppose $h(\bar{a}, \bar{a}) \neq 0$ and $rank[M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)] = 3$. Then $\tilde{h}$ is nondegenerate. Put $\bar{v}_1 := \bar{a}$. Similarly as in (1), we can choose vectors $\bar{v}_2, \bar{v}_3 \in U$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $h(\bar{v}_2, \bar{v}_2) \neq 0 \neq h(\bar{v}_3, \bar{v}_3)$ and $h(\bar{v}_1, \bar{v}_2) = h(\bar{v}_1, \bar{v}_3) = h(\bar{v}_2, \bar{v}_3) = 0$. Since $\bar{v}_1 = \pi(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3)$, we have $f'(\bar{v}_2, \bar{v}_3) = 1$. If we put $h(\bar{v}_1, \bar{v}_1) = (\delta^\psi - \delta)k$ and $h(\bar{v}_i, \bar{v}_i) = (\delta^\psi - \delta) \cdot h_{i-1}$ for every $i \in \{2, 3\}$, then $k, h_1, h_2 \in \mathbb{F}^*$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $N := M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (\delta^\psi - \delta) \cdot diag(k, h_1, h_2)$. By Lemma 3.11, the matrix $\begin{bmatrix} M^* & N \\ -N^T & M^* \end{bmatrix}$ is nonsingular. This implies that $(a^2 + 4b)h_1 h_2 = (\delta^\psi - \delta)^2 h_1 h_2 \neq 1$.

(5) Suppose $h(\bar{a}, \bar{a}) = 0$. Put $W := \bar{a}^{\perp_{\tilde{h}}}$. Then $\bar{a} \in W$ and $W \neq U$ ($W = U$ would imply that $\bar{a} \in Rad(f') = \{\bar{o}\}$). We prove that $h(\bar{b}, \bar{b}) \neq 0$ for all $\bar{b} \in W \setminus <\bar{a}>$. If $h(\bar{b}, \bar{b})$ would be zero for a certain $\bar{b} \in W \setminus <\bar{a}>$, then $h(\bar{b}, \bar{b})$ would be zero for all $\bar{b} \in W \setminus <\bar{a}>$. This would imply that there exists a $\bar{b}_1 \in W \setminus <\bar{a}>$ such that $h(\bar{b}_1, \bar{c}) = 0$, $\forall \bar{c} \in U$. Then $\bar{b}_1^{\perp_{f'}} = <W, U^\psi>$. On the other hand, since $f'(\bar{a}^\psi, \bar{y}) = 0$ for all $\bar{y} \in <U^\psi, W>$, we have $\bar{b}_1^{\perp_{f'}} = (\bar{a}^\psi)^{\perp_{f'}}$ and hence $<\bar{b}_1> = <\bar{a}^\psi>$, an obvious contradiction.

Now, let $\bar{b}$ and $\bar{c}$ be two vectors such that $\bar{a} \wedge \bar{b} \wedge \bar{c} = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $\bar{b} \in W \setminus < \bar{a} >$ and $h(\bar{b}, \bar{c}) = 0$. As before, since $\bar{a} = \pi(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \pi(\bar{a} \wedge \bar{b} \wedge \bar{c})$, we have $f'(\bar{b}, \bar{c}) = 1$. Put $\bar{v}_1 := \bar{a}$, $\bar{v}_2 := \frac{\bar{b}}{\mu_1}$ and $\bar{v}_3 := \mu_1 \bar{c} + \mu_2 \bar{a}$, where $\mu_1 \in \mathbb{F}^*$ and $\mu_2 \in \mathbb{F}$ will be determined later. Since $f'(\bar{b}, \bar{c}) = 1$ and $f'(\bar{b}, \bar{a}) = 0$, we have $f'(\bar{v}_2, \bar{v}_3) = 1$.

Since $h(\bar{a}, \bar{c}) \neq 0 \neq h(\bar{b}, \bar{b})$, $h(\bar{v}_1, \bar{v}_3) = \mu_1^{\psi} \cdot h(\bar{a}, \bar{c})$ and $h(\bar{v}_2, \bar{v}_2) = \frac{h(\bar{b}, \bar{b})}{\mu_1^{\psi+1}}$, we can choose $\mu_1 \in \mathbb{F}^*$ such that $k := h(\bar{v}_1, \bar{v}_3) = l(\delta^{\psi} - \delta) \cdot h(\bar{v}_2, \bar{v}_2)^{-1}$. Clearly, $k \in \mathbb{F}^*$. We have $h(\bar{v}_1, \bar{v}_1) = h(\bar{v}_1, \bar{v}_2) = 0$, $h(\bar{v}_1, \bar{v}_3) = k$, $h(\bar{v}_2, \bar{v}_2) = (\delta^{\psi} - \delta)\frac{l}{k}$ and $h(\bar{v}_2, \bar{v}_3) = 0$. We now prove that $\mu_2 \in \mathbb{F}$ can be chosen in such a way that $h(\bar{v}_3, \bar{v}_3) = (\delta^{\psi} - \delta) \cdot \theta(k)$. We have $h(\bar{v}_3, \bar{v}_3) = h(\mu_1 \bar{c} + \mu_2 \bar{a}, \mu_1 \bar{c} + \mu_2 \bar{a}) = \mu_1^{\psi+1} h(\bar{c}, \bar{c}) + \mu_1 \mu_2^{\psi} h(\bar{c}, \bar{a}) + \mu_2 \mu_1^{\psi} h(\bar{a}, \bar{c}) = \mu_1^{\psi+1} h(\bar{c}, \bar{c}) + k\mu_2 - k\mu_2^{\psi}$. This is equal to $(\delta^{\psi} - \delta) \cdot \theta(k)$ if we take μ_2 equal to $\frac{\delta^{\psi}}{k(\delta^{\psi} - \delta)} \Big((\delta^{\psi} - \delta) \cdot \theta(k) - \mu_1^{\psi+1} \cdot h(\bar{c}, \bar{c}) \Big)$.

We conclude that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$ and $M_h(\bar{v}_1, \bar{v}_2, \bar{v}_3) = \begin{bmatrix} 0 & 0 & k \\ 0 & \frac{l(\delta^{\psi} - \delta)}{k} & 0 \\ -k & 0 & (\delta^{\psi} - \delta) \cdot \theta(k) \end{bmatrix}$. ■

4 Proofs of Theorem 1.3 and 1.4

4.1 Introduction

Suppose χ is a trivector of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$. Then by Lemmas 3.2 and 3.8, there exist vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V'$ such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^{\psi}, \bar{u}_2^{\psi}, \bar{u}_3^{\psi}\}$ is a basis of V' and $\chi = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^{\psi}$. So, precisely one of the five cases of Lemma 3.16 occurs.

In this section, we prove the following proposition.

Proposition 4.1 (1) *If case (1) of Lemma 3.16 occurs, then χ is $Sp(V, f)$-equivalent with the trivector $\chi_1(h_1, h_2, h_3)$ where $h_1, h_2, h_2 \in \mathbb{F}^*$ are as in Lemma 3.16(1). For all $h_1, h_2, h_3 \in \mathbb{F}^*$, the trivector $\chi_1(h_1, h_2, h_3)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and $Sp(V', f')$-equivalent with the trivector $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{e}_3^* + (\delta^{\psi} - \delta)^3 h_1 h_2 h_3 \cdot \bar{f}_1^* \wedge \bar{f}_2^* \wedge \bar{f}_3^*$ of V'.*

(2) *If case (2) of Lemma 3.16 occurs, then χ is $Sp(V, f)$-equivalent with the trivector $\chi_2(k)$ where $k \in \mathbb{F}^*$ is as in Lemma 3.16(2). For every $k \in \mathbb{F}^*$, the trivector $\chi_2(k)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and $Sp(V', f')$-equivalent with the trivector $\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + k(\delta^{\psi} - \delta) \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge \bar{f}_3^*$ of V'.*

(3) *If case (3) of Lemma 3.16 occurs, then χ is $Sp(V, f)$-equivalent with the trivector $\chi_3(k, h)$ where $k, h \in \mathbb{F}^*$ are as in Lemma 3.16(3). For all $k, h \in \mathbb{F}^*$, the trivector $\chi_3(k, h)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and $Sp(V', f')$-equivalent with the trivector $\bar{f}_1^* \wedge \bar{e}_3^* \wedge (\bar{e}_2^* + \bar{f}_3^*) - k(\delta^{\psi} - \delta) \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ of V'.*

(4) *If case (4) of Lemma 3.16 occurs, then χ is $Sp(V, f)$-equivalent with the trivector $\chi_4(k, h_1, h_2)$ where $k, h_1, h_2 \in \mathbb{F}^*$ are as in Lemma 3.16(4). If $k, h_1, h_2 \in \mathbb{F}^*$ such that $h_1 h_2 (a^2 + 4b) \neq 1$, then the trivector $\chi_4(k, h_1, h_2)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and*

$Sp(V',f')$-equivalent with the trivector $\bar{f}_1^* \wedge (\bar{e}_2^* + \bar{e}_3^*) \wedge (\bar{f}_2^* + \frac{1-(a^2+4b)h_1h_2}{(a^2+4b)h_1h_2}\bar{f}_3^*) - kh_1h_2(\delta^\psi - \delta)^3 \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^*$ of V'.

(5) *If case (5) of Lemma 3.16 occurs, then χ is $Sp(V,f)$-equivalent with the trivector $\chi_5(k)$ where $k \in \mathbb{F}^*$ is as in Lemma 3.16(5) assuming that l and θ have been chosen in such a way that $l = 1$ and $\theta(x) := \frac{x}{a^2+4b}$, $\forall x \in \mathbb{F}^*$. For every $k \in \mathbb{F}^*$, the trivector $\chi_5(k)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and $Sp(V',f')$-equivalent with the trivector $\bar{e}_1^* \wedge \bar{e}_3^* \wedge (\bar{f}_3^* + \bar{f}_2^*) + k(\delta^\psi - \delta) \cdot \bar{e}_2^* \wedge \bar{f}_3^* \wedge (\bar{f}_1^* + \bar{e}_3^*)$ of V'.*

The following is an immediate consequence of Proposition 4.1.

Corollary 4.2 • *Every trivector of Type (E1) of V is a trivector of Type (C1) of V'.*
- *Every trivector of Type (E2) of V is a trivector of Type (C3) of V'.*
- *Every trivector of Type (E3) of V is a trivector of Type (C4) of V'.*
- *Every trivector of Type (E4) of V is a trivector of Type (C6) of V'.*
- *Every trivector of Type (E5) of V is a trivector of Type (C5) of V'.*

Theorem 1.3 is an immediate consequence of Proposition 4.1. In this section, we also prove Theorem 1.4. Observe that if $i, j \in \{1, 2, \ldots, 5\}$ with $i \neq j$, then no trivector of Type (Ei) is $Sp(V,f)$-equivalent with a trivector of Type (Ej) by Proposition 2.2(1), Lemma 3.7(2) and Corollary 4.2. This fact can also be derived in an alternative way by relying on Lemma 3.15.

4.2 Treatment of case (1) of Lemma 3.16

Let χ be a trivector of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and suppose there exist three vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V'$ such that the following hold:
- $\chi = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$;
- $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V';
- $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = diag(0,0,0)$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\delta^\psi - \delta) \cdot diag(h_1, h_2, h_3)$ for some $h_1, h_2, h_3 \in \mathbb{F}^*$.

Let $\bar{v}_1$, $\bar{w}_1$, $\bar{v}_2$, $\bar{w}_2$, $\bar{v}_3$ and $\bar{w}_3$ be the unique vectors of V such that

$$\bar{u}_1 = \bar{v}_1 + \delta\bar{w}_1, \qquad \bar{u}_2 = \bar{v}_2 + \delta\bar{w}_2, \qquad \bar{u}_3 = \bar{v}_3 + \delta\bar{w}_3.$$

Since $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V', the set $\{\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\}$ is a basis of V.

Since $f'(\bar{u}_1, \bar{u}_2) = f'(\bar{v}_1 + \delta\bar{w}_1, \bar{v}_2 + \delta\bar{w}_2) = 0$ and $h(\bar{u}_1, \bar{u}_2) = f'(\bar{v}_1 + \delta\bar{w}_1, \bar{v}_2 + \delta^\psi\bar{w}_2) = 0$, we have $f'(\bar{v}_1 + \delta\bar{w}_1, \bar{v}_2) = f'(\bar{v}_1 + \delta\bar{w}_1, \bar{w}_2) = 0$ and hence that $f'(\bar{v}_1, \bar{v}_2) = f'(\bar{w}_1, \bar{v}_2) = f'(\bar{v}_1, \bar{w}_2) = f'(\bar{w}_1, \bar{w}_2) = 0$. In a similar way, one proves that $f'(\bar{v}_1, \bar{v}_3) = f'(\bar{w}_1, \bar{v}_3) = f'(\bar{v}_1, \bar{w}_3) = f'(\bar{w}_1, \bar{w}_3) = f'(\bar{v}_2, \bar{v}_3) = f'(\bar{w}_2, \bar{v}_3) = f'(\bar{v}_2, \bar{w}_3) = f'(\bar{w}_2, \bar{w}_3) = 0$.

Since $(\delta^\psi - \delta)h_i = h(\bar{u}_i, \bar{u}_i) = h(\bar{v}_i + \delta\bar{w}_i, \bar{v}_i + \delta\bar{w}_i) = f'(\bar{v}_i + \delta\bar{w}_i, \bar{v}_i + \delta^\psi\bar{w}_i) = (\delta^\psi - \delta) \cdot f'(\bar{v}_i, \bar{w}_i)$, we have $f'(\bar{v}_i, \frac{\bar{w}_i}{h_i}) = 1$ for every $i \in \{1, 2, 3\}$.

So, $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3) := (\bar{v}_1, \frac{\bar{w}_1}{h_1}, \bar{v}_2, \frac{\bar{w}_2}{h_2}, \bar{v}_3, \frac{\bar{w}_3}{h_3})$ is a hyperbolic basis of (V, f). We have

$$\begin{aligned}
\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 &= (\bar{e}_1 + \delta h_1 \bar{f}_1) \wedge (\bar{e}_2 + \delta h_2 \bar{f}_2) \wedge (\bar{e}_3 + \delta h_3 \bar{f}_3) \\
&= \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + \delta \cdot (h_1 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + h_2 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_3 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) \\
&\quad +(a\delta + b) \cdot (h_1 h_2 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_1 h_3 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + h_2 h_3 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) \\
&\quad +h_1 h_2 h_3 \cdot ((a^2 + b)\delta + ab) \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3
\end{aligned}$$

and hence

$$\begin{aligned}
\chi &= \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi \\
&= 2 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + a \cdot (h_1 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + h_2 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_3 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) \\
&\quad +(a^2 + 2b) \cdot (h_1 h_2 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + h_1 h_3 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + h_2 h_3 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) \\
&\quad +h_1 h_2 h_3 (a^3 + 3ab) \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3.
\end{aligned}$$

So, χ is $Sp(V, f)$-equivalent with $\chi_1(h_1, h_2, h_3)$.

Reversing the above discussion, we see that the trivector $\chi_1(h_1, h_2, h_3)$ can be written in the form $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$, where $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are three vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V', $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = diag(0, 0, 0)$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\delta^\psi - \delta) \cdot diag(h_1, h_2, h_3)$. So, $\chi_1(h_1, h_2, h_3)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ by Lemma 3.9. We also have $\chi = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{e}'_3 + (\delta^\psi - \delta)^3 h_1 h_2 h_3 \cdot \bar{f}'_1 \wedge \bar{f}'_2 \wedge \bar{f}'_3$, where $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ is the hyperbolic basis $(\bar{u}_1, \frac{\bar{u}_1^\psi}{(\delta^\psi - \delta)h_1}, \bar{u}_2, \frac{\bar{u}_2^\psi}{(\delta^\psi - \delta)h_2}, \bar{u}_3, \frac{\bar{u}_3^\psi}{(\delta^\psi - \delta)h_3})$ of (V', f').

If $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{F}^*$, then by Lemma 3.13, the trivectors $\chi_1(h_1, h_2, h_3)$ and $\chi_1(h'_1, h'_2, h'_3)$ are $Sp(V, f)$-equivalent if and only if there exists a (3×3)-matrix A over $\mathbb{F}'$ with determinant 1 such that $A \cdot diag(h_1, h_2, h_3) \cdot (A^\psi)^T$ is equal to $diag(h'_1, h'_2, h'_3)$ or $diag(-h'_1, -h'_2, -h'_3)$. This proves Theorem 1.4(2).

4.3 Treatment of cases (2), (3) and (4) of Lemma 3.16

Let χ be a trivector of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and suppose there exist vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$ such that the following hold:

- $\chi = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$;
- $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V';
- $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = M^*$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\delta^\psi - \delta) \cdot diag(k, h_1, h_2)$ for some $k, h_1, h_2 \in \mathbb{F}$ satisfying $k \neq 0$ and $h_1 h_2 (a^2 + 4b) \neq 1$.

Let $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ be the basis of V such that

$$\bar{u}_1 = \bar{v}_1 + \delta \bar{v}_2, \quad \bar{u}_2 = \bar{v}_3 + \delta \bar{v}_4, \quad \bar{u}_3 = \bar{v}_5 + \delta \bar{v}_6.$$

Since $h(\bar{u}_1, \bar{u}_1) = (\delta^\psi - \delta)k$, $h(\bar{u}_2, \bar{u}_2) = (\delta^\psi - \delta)h_1$ and $h(\bar{u}_3, \bar{u}_3) = (\delta^\psi - \delta)h_2$, we have

$$f'(\bar{v}_1, \bar{v}_2) = k, \quad f'(\bar{v}_3, \bar{v}_4) = h_1, \quad f'(\bar{v}_5, \bar{v}_6) = h_2.$$

From $0 = f'(\bar{u}_1, \bar{u}_2) = f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4)$ and $0 = h(\bar{u}_1, \bar{u}_2) = f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta^\psi\bar{v}_4)$, it follows that $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3) = f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_4) = 0$ and hence that

$$f'(\bar{v}_1, \bar{v}_3) = f'(\bar{v}_2, \bar{v}_3) = f'(\bar{v}_1, \bar{v}_4) = f'(\bar{v}_2, \bar{v}_4) = 0.$$

Similarly, from $f'(\bar{u}_1, \bar{u}_3) = h(\bar{u}_1, \bar{u}_3) = 0$, we find that

$$f'(\bar{v}_1, \bar{v}_5) = f'(\bar{v}_2, \bar{v}_5) = f'(\bar{v}_1, \bar{v}_6) = f'(\bar{v}_2, \bar{v}_6) = 0.$$

We have

$$1 = f'(\bar{u}_2, \bar{u}_3) = f'(\bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = f'(\bar{v}_3, \bar{v}_5) + \delta(f'(\bar{v}_4, \bar{v}_5) + f'(\bar{v}_3, \bar{v}_6)) + \delta^2 \cdot f'(\bar{v}_4, \bar{v}_6) \tag{2}$$

and

$$0 = h(\bar{u}_2, \bar{u}_3) = f'(\bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta^\psi\bar{v}_6) = f'(\bar{v}_3, \bar{v}_5) + \delta \cdot f'(\bar{v}_4, \bar{v}_5) + \delta^\psi \cdot f'(\bar{v}_3, \bar{v}_6) + \delta^{\psi+1} \cdot f'(\bar{v}_4, \bar{v}_6). \tag{3}$$

From (3), it follows that

$$f'(\bar{v}_4, \bar{v}_5) = f'(\bar{v}_3, \bar{v}_6), \tag{4}$$
$$f'(\bar{v}_3, \bar{v}_5) + a \cdot f'(\bar{v}_4, \bar{v}_5) - b \cdot f'(\bar{v}_4, \bar{v}_6) = 0. \tag{5}$$

From (2), (4) and the fact that $\delta^2 = a\delta + b$, it follows that

$$f'(\bar{v}_3, \bar{v}_5) + b \cdot f'(\bar{v}_4, \bar{v}_6) = 1, \tag{6}$$
$$2 \cdot f'(\bar{v}_4, \bar{v}_5) + a \cdot f'(\bar{v}_4, \bar{v}_6) = 0. \tag{7}$$

The equations (5), (6) and (7) determine a linear system of three equations in the unknowns $f'(\bar{v}_3, \bar{v}_5)$, $f'(\bar{v}_4, \bar{v}_5)$ and $f'(\bar{v}_4, \bar{v}_6)$. Since the determinant of this linear system is equal to $-(a^2 + 4b) \neq 0$, there is a unique solution. We find:

$$f'(\bar{v}_4, \bar{v}_5) = f'(\bar{v}_3, \bar{v}_6) = -\frac{a}{a^2 + 4b}, \quad f'(\bar{v}_4, \bar{v}_6) = \frac{2}{a^2 + 4b}, \quad f'(\bar{v}_3, \bar{v}_5) = \frac{a^2 + 2b}{a^2 + 4b}.$$

Now, put $\bar{w}_1 := \bar{v}_1$, $\bar{w}_2 := \bar{v}_2$, $\bar{w}_3 := \bar{v}_3 + h_1(a\bar{v}_5 + (a^2 + 2b)\bar{v}_6)$, $\bar{w}_4 := \bar{v}_4$, $\bar{w}_5 := 2\bar{v}_5 + a\bar{v}_6 - (a^2 + 4b)h_2\bar{v}_4$ and $\bar{w}_6 := a\bar{v}_5 + (a^2 + 2b)\bar{v}_6$. Then $M_{f'}(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_6)$ is equal to

$$\begin{bmatrix} 0 & k & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - h_1h_2(a^2 + 4b) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & h_1h_2(a^2 + 4b) - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

So, there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that $\bar{w}_1 = \bar{e}_1$, $\bar{w}_2 = k\bar{f}_1$, $\bar{w}_3 = \bar{e}_2$, $\bar{w}_4 = \bar{e}_3$, $\bar{w}_5 = (1 - h_1h_2(a^2 + 4b))\bar{f}_2$ and $\bar{w}_6 = \bar{f}_3$.

We have that $\bar{v}_1 = \bar{w}_1$, $\bar{v}_2 = \bar{w}_2$, $\bar{v}_3 = \bar{w}_3 - h_1\bar{w}_6$, $\bar{v}_4 = \bar{w}_4$ and

$$\begin{aligned}\bar{v}_5 &= \frac{1}{a^2+4b}((a^2+2b)\bar{w}_5 - a\bar{w}_6) + (a^2+2b)h_2\bar{w}_4,\\ \bar{v}_6 &= \frac{1}{a^2+4b}(-a\bar{w}_5 + 2\bar{w}_6) - ah_2\bar{w}_4.\end{aligned}$$

Hence,

$$\begin{aligned}\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 &= (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\\ &= (\bar{w}_1 + \delta\bar{w}_2) \wedge (\bar{w}_3 + \delta\bar{w}_4 - h_1\bar{w}_6) \wedge\\ &\quad \Big(\frac{1}{a^2+4b}\Big((a^2+2b-a\delta)\bar{w}_5 + (2\delta - a)\bar{w}_6\Big) + (a^2+2b-a\delta)h_2\bar{w}_4\Big).\end{aligned}$$

After some tedious calculations, we find

$$\begin{aligned}\chi &= \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi\\ &= \bar{w}_1 \wedge \bar{w}_3 \wedge \bar{w}_5 + (1 + (a^2+4b)h_1h_2) \cdot \bar{w}_1 \wedge \bar{w}_4 \wedge \bar{w}_6 + \bar{w}_2 \wedge \bar{w}_3 \wedge \bar{w}_6 + b \cdot \bar{w}_2 \wedge \bar{w}_4 \wedge \bar{w}_5\\ &\quad + a \cdot \bar{w}_2 \wedge \bar{w}_4 \wedge \bar{w}_6 - h_1 \cdot \bar{w}_1 \wedge \bar{w}_6 \wedge \bar{w}_5 + (a^2+4b)h_2 \cdot \bar{w}_1 \wedge \bar{w}_3 \wedge \bar{w}_4.\\ &= (1 - h_1h_2(a^2+4b)) \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 + (1 + h_1h_2(a^2+4b)) \cdot \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3\\ &\quad + k \cdot \Big(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 - b(1 - h_1h_2(a^2+4b)) \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + a \cdot \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3\Big)\\ &\quad + h_1(1 - h_1h_2(a^2+4b)) \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 + (a^2+4b)h_2 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3.\end{aligned}$$

So, χ is $Sp(V,f)$-equivalent with $\chi_4(k,h_1,h_2)$.

Reversing the above discussion, we see that the trivector $\chi_4(k,h_1,h_2)$ can be written in the form $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$, where $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are three vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V', $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = M^*$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\delta^\psi - \delta) \cdot diag(k, h_1, h_2)$. So, $\chi_4(k,h_1,h_2)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ by Lemma 3.9.

If $h_1 \neq 0 \neq h_2$, then $\chi_4(k,h_1,h_2) = \bar{f}'_1 \wedge (\bar{e}'_2 + \bar{e}'_3) \wedge (\bar{f}'_2 + \epsilon \bar{f}'_3) - kh_1h_2(\delta^\psi - \delta)^3 \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2$ where $\epsilon = \frac{1-(a^2+4b)h_1h_2}{(a^2+4b)h_1h_2}$ and $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ is the hyperbolic basis $(-\frac{\bar{u}_1^\psi}{(a^2+4b)h_1h_2k(\delta^\psi-\delta)}, (a^2+4b)h_1h_2 \cdot \bar{u}_1, -\frac{1}{(\delta^\psi-\delta)h_2}\bar{u}_3^\psi, (\delta^\psi - \delta)h_2 \cdot \bar{u}_2^\psi, \frac{\bar{u}_2}{(a^2+4b)h_1h_2} + \frac{\bar{u}_3^\psi}{(\delta^\psi-\delta)h_2}, \frac{1}{\epsilon}(\bar{u}_3 - (\delta^\psi - \delta)h_2 \cdot \bar{u}_2^\psi))$ of (V', f').

If $h_1 = h_2 = 0$, then $\chi_2(k) = \chi_4(k,0,0) = \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2 + k(\delta^\psi - \delta) \cdot \bar{f}'_1 \wedge \bar{e}'_3 \wedge \bar{f}'_3$, where $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ is the hyperbolic basis $(\bar{u}_1, \frac{\bar{u}_1^\psi}{k(\delta^\psi-\delta)}, \bar{u}_2, \bar{u}_3, \bar{u}_2^\psi, \bar{u}_3^\psi)$ of (V', f').

If $h_2 = 0$ and $h := h_1 \neq 0$, then $\chi_3(k,h) = \chi_4(k,h,0) = \bar{f}'_1 \wedge \bar{e}'_3 \wedge (\bar{e}'_2 + \bar{f}'_3) - k(\delta^\psi - \delta) \cdot \bar{e}'_1 \wedge \bar{e}'_2 \wedge \bar{f}'_2$, where $(\bar{e}'_1, \bar{f}'_1, \bar{e}'_2, \bar{f}'_2, \bar{e}'_3, \bar{f}'_3)$ is the hyperbolic basis $(-\frac{\bar{u}_1^\psi}{k(\delta^\psi-\delta)}, \bar{u}_1, (\delta^\psi - \delta)h\bar{u}_3^\psi, -\frac{\bar{u}_2^\psi}{(\delta^\psi-\delta)h}, \bar{u}_3, -\bar{u}_2 - (\delta^\psi - \delta)h\bar{u}_3^\psi)$ of (V', f').

Let $k, h_1, h_2, k', h'_1, h'_2$ be arbitrary elements of $\mathbb{F}$ such that $k \neq 0 \neq k'$ and $(a^2+4b)h_1h_2 \neq 1 \neq (a^2+4b)h'_1h'_2$. By Lemmas 3.13 and 3.14, the two trivectors $\chi_4(k, h_1, h_2)$ and $\chi_4(k', h'_1, h'_2)$ are $Sp(V, f)$-equivalent if and only if there exists a $\sigma \in \{1, -1\}$ and a (3×3)-matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

over $\mathbb{F}'$ with determinant 1 such that $B \cdot diag(k, h_1, h_2) = \sigma \cdot diag(k', h'_1, h'_2) \cdot ((B^\psi)^{-1})^T$. Since

$$((B^\psi)^{-1})^T = \begin{bmatrix} 1 & b_{31}^\psi b_{23}^\psi - b_{21}^\psi b_{33}^\psi & b_{21}^\psi b_{32}^\psi - b_{31}^\psi b_{22}^\psi \\ 0 & b_{33}^\psi & -b_{32}^\psi \\ 0 & -b_{23}^\psi & b_{22}^\psi \end{bmatrix},$$

this implies that $b_{21} = b_{31} = 0$, $k' = \sigma k$ and $A \cdot diag(h_1, h_2) \cdot (A^\psi)^T = \sigma \cdot diag(h'_1, h'_2)$, where A is the (2×2)-matrix $\begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}$ of determinant 1.

So, we conclude that the two trivectors $\chi_4(k, h_1, h_2)$ and $\chi_4(k', h'_1, h'_2)$ are $Sp(V, f)$-equivalent if and only if there exists a $\sigma \in \{1, -1\}$ and a (2×2)-matrix A over $\mathbb{F}'$ with determinant 1 such that $k' = \sigma k$, $A \cdot diag(h_1, h_2) \cdot (A^\psi)^T = \sigma \cdot diag(h'_1, h'_2)$. We can now already say the following.

- If $k, k' \in \mathbb{F}^*$, then the two trivectors $\chi_2(k) = \chi_4(k, 0, 0)$ and $\chi_2(k') = \chi_4(k', 0, 0)$ are $Sp(V, f)$-equivalent if and only if $k' \in \{k, -k\}$.

We will now focus on the trivectors of Type (E3) and (E4).

Lemma 4.3 *Let $\sigma \in \{1, -1\}$ and suppose that $h_1 \neq 0 \neq h'_1$. Then there exists a (2×2)-matrix A over $\mathbb{F}'$ with determinant 1 such that* $\mathrm{diag}(\sigma h'_1, \sigma h'_2) = A \cdot \mathrm{diag}(h_1, h_2) \cdot (A^\psi)^T$ *if and only if $h'_1h'_2 = h_1h_2$ and there exist $\eta_1, \eta_2 \in \mathbb{F}'$ such that $\sigma h'_1 = \eta_1^{\psi+1}h_1 + \eta_2^{\psi+1}h_2$.*

Proof. Suppose the matrix $A = \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix}$ has determinant 1 and satisfies $\mathrm{diag}(\sigma h'_1, \sigma h'_2)$ $= A \cdot \mathrm{diag}(h_1, h_2) \cdot (A^\psi)^T$. Calculating the determinants of the matrices at both sides of the equality, we find that $h'_1h'_2 = h_1h_2$. Comparing the elements in the first row and first column of the matrices at both sides of the equality, we find that $\sigma h'_1 = \eta_1^{\psi+1}h_1 + \eta_2^{\psi+1}h_2$.

Conversely, suppose that $h'_1h'_2 = h_1h_2$ and that there exist $\eta_1, \eta_2 \in \mathbb{F}'$ such that $\eta_1^{\psi+1}h_1 + \eta_2^{\psi+1}h_2 = \sigma h'_1$. Then the system of linear equations

$$\begin{aligned} \eta_1 \cdot \eta_4 - \eta_2 \cdot \eta_3 &= 1, \\ \eta_2^\psi h_2 \cdot \eta_4 + \eta_1^\psi h_1 \cdot \eta_3 &= 0, \end{aligned}$$

has a unique solution for $(\eta_3, \eta_4) \in \mathbb{F}' \times \mathbb{F}'$. The matrix $A := \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix}$ has determinant 1 and one can readily verify that $\mathrm{diag}(\sigma h'_1, \sigma h'_2) = A \cdot \mathrm{diag}(h_1, h_2) \cdot (A^\psi)^T$. ∎

Invoking Lemma 4.3, we now immediately see that the following hold.

- If $k, k', h, h' \in \mathbb{F}^*$, then the two trivectors $\chi_3(k,h) = \chi_4(k,h,0)$ and $\chi_3(k',h') = \chi_4(k',h',0)$ are $Sp(V,f)$-equivalent if and only if there exists a $\sigma \in \{1,-1\}$ and an $\eta \in \mathbb{F}' \setminus \{0\}$ such that $k' = \sigma k$ and $h' = \sigma \eta^{\psi+1} \cdot h$.

- If $k, k', h_1, h_1', h_2, h_2' \in \mathbb{F}^*$ with $(a^2+4b)h_1h_2 \neq 1 \neq (a^2+4b)h_1'h_2'$, then the two trivectors $\chi_4(k,h_1,h_2)$ and $\chi_4(k',h_1',h_2')$ are $Sp(V,f)$-equivalent if and only if $h_1'h_2' = h_1h_2$ and there exist $\eta_1, \eta_2 \in \mathbb{F}'$ and a $\sigma \in \{1,-1\}$ such that $k' = \sigma k$ and $\sigma h_1' = \eta_1^{\psi+1} h_1 + \eta_2^{\psi+1} h_2$.

4.4 Treatment of case (5) of Lemma 3.16

Let χ be a trivector of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ and suppose there exist three vectors $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$ such that the following hold:
- $\chi = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$;
- $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V';
- $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = M^*$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \begin{bmatrix} 0 & 0 & k \\ 0 & \frac{\delta^\psi - \delta}{k} & 0 \\ -k & 0 & \frac{k(\delta^\psi - \delta)}{a^2+4b} \end{bmatrix}$.

The value of the matrix $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ corresponds to the choice $l = 1$ and $\theta : \mathbb{F}^* \to \mathbb{F}; x \mapsto \frac{x}{a^2+4b}$ in Lemma 3.16. The reason why we have made that choice is because this will lead to easier calculations later.

Let $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ be the basis of V such that

$$\bar{u}_1 = \bar{v}_1 + \delta \bar{v}_2, \quad \bar{u}_2 = \bar{v}_3 + \delta \bar{v}_4, \quad \bar{u}_3 = \bar{v}_5 + \delta \bar{v}_6.$$

From $h(\bar{u}_1, \bar{u}_1) = 0$, $h(\bar{u}_2, \bar{u}_2) = \frac{\delta^\psi - \delta}{k}$ and $h(\bar{u}_3, \bar{u}_3) = \frac{k(\delta^\psi - \delta)}{a^2+4b}$, we find

$$f'(\bar{v}_1, \bar{v}_2) = 0, \quad f'(\bar{v}_3, \bar{v}_4) = \frac{1}{k}, \quad f'(\bar{v}_5, \bar{v}_6) = \frac{k}{a^2+4b}.$$

From $f'(\bar{u}_1, \bar{u}_2) = h(\bar{u}_1, \bar{u}_2) = 0$, we obtain

$$f'(\bar{v}_1, \bar{v}_3) = f'(\bar{v}_1, \bar{v}_4) = f'(\bar{v}_2, \bar{v}_3) = f'(\bar{v}_2, \bar{v}_4) = 0.$$

From

$$0 = f'(\bar{u}_1, \bar{u}_3) = f'(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_5 + \delta \bar{v}_6) = f'(\bar{v}_1, \bar{v}_5) + \delta(f'(\bar{v}_2, \bar{v}_5) + f'(\bar{v}_1, \bar{v}_6)) + \delta^2 f'(\bar{v}_2, \bar{v}_6),$$

$$k = h(\bar{u}_1, \bar{u}_3) = f'(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_5 + \delta^\psi \bar{v}_6) = f'(\bar{v}_1, \bar{v}_5) + \delta f'(\bar{v}_2, \bar{v}_5) + \delta^\psi f'(\bar{v}_1, \bar{v}_6) + \delta^{\psi+1} \cdot f'(\bar{v}_2, \bar{v}_6),$$

and the facts that $\delta + \delta^\psi = a$ and $\delta^{\psi+1} = -b$, we deduce that

$$\begin{aligned} f'(\bar{v}_1, \bar{v}_6) - f'(\bar{v}_2, \bar{v}_5) &= 0, \\ 2f'(\bar{v}_1, \bar{v}_6) + af'(\bar{v}_2, \bar{v}_6) &= 0, \\ f'(\bar{v}_1, \bar{v}_5) + bf'(\bar{v}_2, \bar{v}_6) &= 0, \\ f'(\bar{v}_1, \bar{v}_5) + af'(\bar{v}_1, \bar{v}_6) - bf'(\bar{v}_2, \bar{v}_6) &= k. \end{aligned}$$

We find

$$f'(\bar{v}_1,\bar{v}_6)=f'(\bar{v}_2,\bar{v}_5)=\frac{ka}{a^2+4b},\quad f'(\bar{v}_2,\bar{v}_6)=\frac{-2k}{a^2+4b},\quad f'(\bar{v}_1,\bar{v}_5)=\frac{2bk}{a^2+4b}.$$

Similarly as in Section 4.3, the facts that $f'(\bar{u}_2,\bar{u}_3)=1$ and $h(\bar{u}_2,\bar{u}_3)=0$ imply that

$$f'(\bar{v}_4,\bar{v}_5)=f'(\bar{v}_3,\bar{v}_6)=\frac{-a}{a^2+4b},\quad f'(\bar{v}_4,\bar{v}_6)=\frac{2}{a^2+4b},\quad f'(\bar{v}_3,\bar{v}_5)=\frac{a^2+2b}{a^2+4b}.$$

Now, put $\bar{w}_1:=\bar{v}_1$, $\bar{w}_2:=\bar{v}_2$, $\bar{w}_3:=\bar{v}_3$, $\bar{w}_4:=\bar{v}_4$, $\bar{w}_5:=a\bar{v}_5-2b\bar{v}_6-k\bar{v}_3-ka\bar{v}_4$ and $\bar{w}_6=2\bar{v}_5+a\bar{v}_6-k\bar{v}_4$. Then $M_{f'}(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_6)$ is equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & 0 & -\frac{1}{k} & 0 & 0 & 0 \\ 0 & -k & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, there exists a hyperbolic basis $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ of (V,f) such that $\bar{w}_1=\bar{e}_2$, $\bar{w}_2=\bar{e}_3$, $\bar{w}_3=\bar{e}_1$, $\bar{w}_4=\frac{1}{k}\bar{f}_1$, $\bar{w}_5=k\bar{f}_3$ and $\bar{w}_6=k\bar{f}_2$.

We have $\bar{v}_1=\bar{w}_1$, $\bar{v}_2=\bar{w}_2$, $\bar{v}_3=\bar{w}_3$, $\bar{v}_4=\bar{w}_4$ and

$$\begin{aligned}\bar{v}_5 &= \frac{1}{a^2+4b}\Big(a\bar{w}_5+2b\bar{w}_6+ka\bar{w}_3+k(a^2+2b)\bar{w}_4\Big),\\ \bar{v}_6 &= \frac{1}{a^2+4b}\Big(-2\bar{w}_5+a\bar{w}_6-2k\bar{w}_3-ka\bar{w}_4\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\bar{u}_1\wedge\bar{u}_2\wedge\bar{u}_3 &= \frac{1}{a^2+4b}\cdot(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge\\ &\quad \Big(((a-2\delta)\bar{w}_5+(2b+a\delta)\bar{w}_6)+k\cdot((a-2\delta)\bar{w}_3+(a^2+2b-a\delta)\bar{w}_4)\Big)\\ &= \frac{1}{a^2+4b}\cdot(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge((a-2\delta)\bar{w}_5+(2b+a\delta)\bar{w}_6)\\ &\quad +k\cdot(\bar{w}_1+\delta\bar{w}_2)\wedge\bar{w}_3\wedge\bar{w}_4.\end{aligned}$$

After some tedious calculations, we find

$$\begin{aligned}\chi &= \bar{u}_1\wedge\bar{u}_2\wedge\bar{u}_3+(\bar{u}_1\wedge\bar{u}_2\wedge\bar{u}_3)^{\psi}\\ &= \bar{w}_1\wedge\bar{w}_3\wedge\bar{w}_6-\bar{w}_1\wedge\bar{w}_4\wedge\bar{w}_5+a\cdot\bar{w}_1\wedge\bar{w}_4\wedge\bar{w}_6-\bar{w}_2\wedge\bar{w}_3\wedge\bar{w}_5+a\cdot\bar{w}_2\wedge\bar{w}_3\wedge\bar{w}_6\\ &\quad -a\cdot\bar{w}_2\wedge\bar{w}_4\wedge\bar{w}_5+(a^2+b)\cdot\bar{w}_2\wedge\bar{w}_4\wedge\bar{w}_6+2k\cdot\bar{w}_1\wedge\bar{w}_3\wedge\bar{w}_4+ak\cdot\bar{w}_2\wedge\bar{w}_3\wedge\bar{w}_4.\\ &= \bar{f}_1\wedge\bar{e}_2\wedge\bar{f}_3+2\cdot\bar{e}_1\wedge\bar{f}_1\wedge\bar{e}_2-a\cdot\bar{f}_1\wedge\bar{e}_2\wedge\bar{f}_2+a\cdot\bar{f}_1\wedge\bar{e}_3\wedge\bar{f}_3+a\cdot\bar{e}_1\wedge\bar{f}_1\wedge\bar{e}_3\\ &\quad +(a^2+b)\cdot\bar{f}_1\wedge\bar{f}_2\wedge\bar{e}_3+k\cdot\Big(a\cdot\bar{e}_1\wedge\bar{f}_2\wedge\bar{e}_3-\bar{e}_1\wedge\bar{e}_2\wedge\bar{f}_2+\bar{e}_1\wedge\bar{e}_3\wedge\bar{f}_3\Big).\end{aligned}$$

So, χ is $Sp(V,f)$-equivalent with $\chi_5(k)$.

Reversing the above discussion, we see that the trivector $\chi_5(k)$ can be written in the form $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 + (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)^\psi$, where $\bar{u}_1, \bar{u}_2, \bar{u}_3$ are three vectors of V' such that $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_1^\psi, \bar{u}_2^\psi, \bar{u}_3^\psi\}$ is a basis of V', $M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) = M^*$ and $M_h(\bar{u}_1, \bar{u}_2, \bar{u}_3) = \begin{bmatrix} 0 & 0 & k \\ 0 & \frac{\delta^\psi - \delta}{k} & 0 \\ -k & 0 & \frac{k(\delta^\psi-\delta)}{a^2+4b} \end{bmatrix}$. So, $\chi_5(k)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ by Lemma 3.9. We also have $\chi_5(k) = \bar{e}_1' \wedge \bar{e}_3' \wedge (\bar{f}_3' + \bar{f}_2') + (\delta^\psi - \delta)k \cdot \bar{e}_2' \wedge \bar{f}_3' \wedge (\bar{f}_1' + \bar{e}_3')$, where $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is the hyperbolic basis $(\bar{u}_1, \frac{\bar{u}_3^\psi}{k} + \frac{\bar{u}_2}{\delta^\psi - \delta}, -\frac{1}{(\delta^\psi-\delta)k}\bar{u}_1^\psi, -(\delta^\psi - \delta)\bar{u}_3 + k\bar{u}_2^\psi, -\frac{\bar{u}_2}{\delta^\psi-\delta}, -k\bar{u}_2^\psi)$ of (V', f').

If $k, k' \in \mathbb{F}^*$, then by Lemma 3.13, the two trivectors $\chi_5(k)$ and $\chi_5(k')$ are $Sp(V,f)$-equivalent if and only if there exists a $\sigma \in \{1, -1\}$ and a (3×3)-matrix A over $\mathbb{F}'$ with determinant 1 such that $AM^*A^T = M^*$ and

$$A \cdot \begin{bmatrix} 0 & 0 & k \\ 0 & \frac{\delta^\psi - \delta}{k} & 0 \\ -k & 0 & \frac{k(\delta^\psi-\delta)}{a^2+4b} \end{bmatrix} \cdot (A^\psi)^T = \begin{bmatrix} 0 & 0 & k' \\ 0 & \sigma \cdot \frac{\delta^\psi - \delta}{k'} & 0 \\ -k' & 0 & \sigma \cdot \frac{k'(\delta^\psi-\delta)}{a^2+4b} \end{bmatrix}. \tag{8}$$

If we put $\sigma = -1$ and $A = diag(1, -1, -1)$, then we see that the trivectors $\chi_5(k)$ and $\chi_5(-k)$ are $Sp(V,f)$-equivalent.

Conversely, if the two trivectors $\chi_5(k)$ and $\chi_5(k')$ are $Sp(V,f)$-equivalent, then taking the determinants of the matrices at both sides of the equality (8), we see that $k' \in \{k, -k\}$. This finishes the proof of Theorem 1.4(6).

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