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Finsler functions for two-dimensional sprays

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Abstract. I derive the general formula for a local Finsler function for any spray over a two-dimensional manifold specified by its geodesic curvature function relative to a given background Riemannian metric.

1. Introduction

It is well known that given any spray over a two-dimensional manifold there is a locally defined Finsler function of which it is the geodesic spray, up to projective equivalence (see for example [1, 3, 5]). The purpose of the present paper is to derive a general formula for such a Finsler function.

The historically-minded reader may object that this problem was solved long ago by Darboux. Indeed, if one takes advantage of the freedom of choice of parametrization implicit in the assumption of projective equivalence to choose one of the base coordinates as the curve parameter, then the problem is equivalent to the inverse problem of the calculus of variations in one dependent variable. The solution to the latter problem was given by Darboux in his *Leçons sur la Théorie Générale des Surfaces* of 1894 [7]. However, Darboux's approach is essentially analytical; here I mean to tackle the problem in a much more geometrical way.

The approach to be described below has two sources, both concerned with the particular case in which the base integral curves of the spray in question are Euclidean circles.

The first is a paper I wrote recently with Tom Mestdag, [4], in which we found all the Finsler functions of Randers type whose geodesics are Euclidean

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circles. We wrote the Finsler function in the form

$$F(x^1,x^2,y^1,y^2) = e^{\phi(x^1,x^2)}\sqrt{(y^1)^2+(y^2)^2} + a_1(x^1,x^2)y^1 + a_2(x^1,x^2)y^2,$$

so that (x^1,x^2) are isothermal coordinates for a Riemannian metric on the base manifold M; (y^1,y^2) are the corresponding coordinates on the fibre of the slit tangent bundle $T^\circ M$. The function

$$\kappa = e^{-2\phi}\left(\frac{\partial a_1}{\partial x^2} - \frac{\partial a_2}{\partial x^1}\right)$$

played an important role in the analysis: one of the conditions for the geodesics of F to be circles is that κ is constant. In fact (as we showed in [4]) for any F of this form and for any geodesic path of F, κ is its geodesic curvature with respect to the Riemannian part of F. Notice that $\kappa = 0$ just when the 1-form $a_1dx^1 + a_2dx^2$ is closed. In this case F differs from its Riemannian part by a total derivative (at least locally), and of course its geodesics are then the Riemannian geodesics.

The second source is a paper by Tabachnikov, [8], which contains (among other things) a general formula for the Finsler metrics in the plane whose geodesics are circles of a fixed radius. This is his result, slightly modified in both statement and notation to fit in with the present paper.

Theorem. *Every Lagrangian, homogeneous of degree 1 in the velocity, whose extremals are positively oriented circles of radius R can be represented, in polar coordinates, as follows:*

$$F(x^1,x^2,r,\theta) = r\left(\int_0^\theta \sin(\theta-\psi)\sigma(x^1 - R\sin\psi, x^2 + R\cos\psi)d\psi \right.$$
$$\left. + a_1(x^1,x^2)\cos\theta + a_2(x^1,x^2)\sin\theta\right)$$

where σ is a positive density function in the plane such that the center of mass of every circle of radius R is its center, and a_1, a_2 are two functions, satisfying

$$\frac{\partial a_1}{\partial x^2}(x^1,x^2) - \frac{\partial a_2}{\partial x^1}(x^1,x^2) = \frac{1}{R}\sigma(x^1,x^2+R).$$

By a 'Lagrangian homogeneous of degree 1 in the velocity' Tabachnikov means a Finsler function. The radius R is a fixed positive number. The polar coordinates referred to are coordinates in the fibre, so that $y^1 = r\cos\theta$, $y^2 = r\sin\theta$; x^1 and x^2 are Euclidean coordinates on the base M, which is the Euclidean plane. The spray whose base integral curves are positively oriented circles of radius R is given (up to projective equivalence) in these coordinates by

$$\Gamma = r\left(\cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2} + \frac{1}{R}\frac{\partial}{\partial\theta}\right).$$

The expression $\sigma(x^1 - R\sin\psi, x^2 + R\cos\psi)$ appearing in the integrand may be regarded as representing a function on $T^\circ M$ constant along the integral curves of Γ: to be precise, if one defines the function ρ on $T^\circ M$ by $\rho(x^1, x^2, r, \theta) = \sigma(x^1 - R\sin\theta, x^2 + R\cos\theta)$ then $\Gamma(\rho) = 0$. Notice that ρ is independent of r, or in other words is positively homogeneous of degree 0 with respect to the fibre coordinates. Conversely if ρ is a function on $T^\circ M$, positively homogeneous of degree 0, such that $\Gamma(\rho) = 0$ then σ is well-defined by this formula. The requirement that 'the center of mass of every circle of radius R is its center' turns out to be equivalent to the condition that

$$\int_0^{2\pi} \sin(\theta - \psi)\sigma(x^1 - R\sin\psi, x^2 + R\cos\psi)d\psi = 0$$

for all (x^1, x^2). The reader will no doubt have noticed that the term involving σ on the right-hand side of the final displayed equation in the theorem is obtained simply by setting $\theta = 0$.

We may in particular take ρ to be a constant, indeed to be 1; in this case Tabachnikov's formula produces a Randers metric whose Riemannian part is Euclidean, that is, for which in terms of the earlier remarks $\phi = 0$; the geodesic curvature function is then just $\kappa = 1/R$, and the condition on the partial derivatives of a_1 and a_2 is consistent with the formula for the geodesic curvature given previously.

How one specifies a Finsler function for a given spray depends of course on how one represents the spray. The idea pursued in this paper is to represent the spray by means of the geodesic curvature of its base integral curves with respect to some chosen Riemannian metric on the base. This approach should be well adapted to tackling problems like 'find a Finsler function for the horocycles in the Poincaré disk', where a Riemannian metric occurs naturally among the data; we did indeed give one solution to the horocycle problem in [4]. Given such a background Riemannian metric g_0, there is a unique member of the projective class of any spray whose base integral curves are parametrized with constant speed relative to g_0. There is then a well-defined function κ on $T^\circ M$, positively homogeneous of degree 0, such that $\kappa(x, y)$ is the geodesic curvature, with respect to g_0, at $x \in M$ of the base integral curve of Γ through x whose unit tangent vector there points in the direction of $y \in T_xM$. Note that in general κ is a function on $T^\circ M$; the cases discussed earlier, in which κ is at worst a function on M, are special. Such cases are discussed in Section 3.

As I shall show, there is a natural way of obtaining all of the local Finsler functions for the projective equivalence class of the spray with geodesic curvature function κ; the specification of any such function, when expressed in terms of isothermal coordinates for the background Riemannian metric, turns out to be a generalization of Tabachnikov's formula. I stress that this generalized Tabachnikov formula applies to arbitrary sprays, the original to one very limited class. In the general as in the particular case the freedom of choice in the Finsler function is parametrized by functions constant along the integral curves of the spray (or constants of motion, or first integrals, or functions on path space) — a fact known already to Darboux (see [7]); and also (as always) one is free to add a total derivative.

I have to admit to having been somewhat slipshod in my use of the term 'Finsler function' up to now. A Finsler function, in addition to being positively homogeneous, must be positive and strongly convex, and I have ignored the last two requirements. Elsewhere (for example in [5]) the term 'pseudo-Finsler function' has been used for a function that satisfies the conditions required of a Finsler function other than positivity and strong convexity, but whose Hessian with respect to fibre coordinates is quasi-regular, that is, defines a quadratic form which vanishes only along rays in the tangent space. The formula I shall derive gives, in the first instance, a pseudo-Finsler function whose Hessian with respect to the fibre coordinates is, in the terminology of [5], positive quasi-definite (and this is true of Tabachnikov's original formula as well). However, it is shown in the paper just referred to that given a pseudo-Finsler function whose Hessian is positive quasi-definite one may obtain locally a Finsler function, that is a function which is both positive and strongly convex, by the addition of a suitably chosen total derivative; this modification does not change the geodesics, of course. So the generalized Tabachnikov formula does effectively solve the local problem. That being the case, I shall sometimes, especially later in the paper, say 'Finsler function' where strictly speaking I should say 'pseudo-Finsler function whose Hessian is positive quasi-definite', with the understanding that finally a suitable total derivative may have to be added. Where no assumption about positivity and strong convexity, but only quasi-regularity of the Hessian, is made I shall use the term 'pseudo-Finsler function'. I shall also generally leave it to be understood that 'homogeneous' means 'positively homogeneous'.

2. The basic set-up

Let M be a two-dimensional Riemannian manifold with metric g_0; let F_0 be the corresponding Finsler function; and let Γ_0 be the canonical geodesic spray of F_0. (Symbols for geometric objects associated with the background metric g_0 will usually carry a subscript 0.) I denote by $T^\circ M$ the slit tangent bundle of M, by $T_1^\circ M$ the sub-bundle of $T^\circ M$ consisting of vectors of unit Riemannian length (the level set of F_0 of value 1), and by Δ the Liouville vector field on $T^\circ M$. I take a vertical vector field N_0 on $T^\circ M$ which is orthogonal to Δ and (like Δ) of length F_0 with respect to the fibre metric induced by g_0. Then $N_0(F_0) = 0$. At each point of $T^\circ M$ there are two possible choices of N_0; I can choose one consistently provided M is orientable, which I shall assume to be the case. Suppose we have chosen an orientation of M; then I shall choose N_0 such that N_0 is obtained from Δ by an anticlockwise rotation through a right angle, relative to the chosen orientation.

I shall usually write φ' for $N_0(\varphi)$, where φ is a function on $T^\circ M$.

I use coordinates x^i, $i = 1, 2$, on M (not assumed to be isothermal initially), with canonical fibre coordinates y^i; I shall always assume that the cordinates on M are positively oriented. I write g_{ij} for the components of g_0 (so $F_0 = \sqrt{g_{ij}y^iy^j}$). I denote by ε_{ij} the alternating symbol on M, so that $\varepsilon_{ji} = -\varepsilon_{ij}$ and $\varepsilon_{12} = \sqrt{\det g_0}$ (where $\det g_0$ is the determinant of the matrix representing g_0 with respect to the coordinates x^i): then the Riemannian volume form is $\nu_0 = \frac{1}{2}\varepsilon_{ij}dx^i \wedge dx^j$. Set $\varepsilon^{ij} = g^{ik}g^{jl}\varepsilon_{kl}$: then $\varepsilon^{ji} = -\varepsilon^{ij}$ and $\varepsilon^{12} = 1/\sqrt{\det g_0}$. Both ε_{ij} and ε^{ij} are (the components of) pseudo-tensors (they transform tensorially for orientation-preserving coordinate transformations). Each is covariantly constant. We have

$$N_0 = -\varepsilon^{ij}g_{jk}y^k\frac{\partial}{\partial y^i}.$$

Let H_i, $i = 1, 2$ be the horizontal lifts, relative to the Levi-Civita connection of g_0, of the coordinate vector fields on M. It will sometimes be convenient to denote the vertical field $\partial/\partial y^i$ by V_i. I shall introduce a vector field K_0 on $T^\circ M$ which is horizontal and satisfies $S(K_0) = N_0$ where S is the almost tangent structure tensor. Thus

$$K_0 = -\varepsilon^{ij}g_{jk}y^kH_i.$$

Then $\{\Gamma_0, K_0, \Delta, N_0\}$ is a global basis of vector fields on $T^\circ M$; these fields are pairwise orthogonal with respect to the Sasaki metric induced from g_0 and each is of length F_0. Furthermore, Γ_0 and K_0 are horizontal and homogeneous of degree 1, $\Delta = S(\Gamma_0)$ and $N_0 = S(K_0)$ are vertical and homogeneous of degree 0.

Lemma 1. *We have*

$$[N_0, \Gamma_0] = K_0, \quad [N_0, K_0] = -\Gamma_0, \quad [K_0, \Gamma_0] = -F_0^2 R_0 N_0$$

where R_0 is the Gaussian curvature of the metric.

Proof. The crucial point, so far as the first two formulae are concerned, is that $[H_i, N_0] = 0$. In fact for a vertical vector field V of the form $V = T^i_j y^j V_i$, $[H_i, V] = T^k_{j|i} y^j V_k$, so that indeed $[H_i, N_0] = 0$. Then

$$\begin{aligned} &[N_0, \Gamma_0] = [N_0, y^i H_i] = N_0(y^i) H_i = K_0 \\ &[N_0, K_0] = [N_0, N^i H_i] = N_0(N^i) H_i = -\Gamma_0 \quad \text{(where } N^i = -\varepsilon^{ij} g_{jk} y^k\text{)}; \end{aligned}$$

for the final step we have

$$N_0(N^i) = \varepsilon^{jk} g_{kl} y^l \frac{\partial}{\partial y^j} \left(\varepsilon^{ip} g_{pq} y^q\right) = \varepsilon^{jk} g_{kl} \varepsilon^{ip} g_{pj} y^l = \varepsilon_{pl} \varepsilon^{ip} y^l = -\delta^i_l y^l = -y^i$$

as required.

The expression for $[K_0, \Gamma_0]$ follows from the well-known formula

$$[H_i, H_j] = -R^l_{kij} y^k V_l. \qquad \square$$

Two sprays on $T^\circ M$ are projectively equivalent if and only if they differ by a multiple of Δ. Projectively equivalent sprays have the same base integral curves up to direction-preserving reparametrization. Any two sprays at all differ by a vertical vector field. It follows that any projective equivalence class of sprays over M has a representative of the form

$$\Gamma = \Gamma_0 + F_0 \kappa N_0$$

where κ is a function on $T^\circ M$ which is homogeneous of degree 0 (the factor F_0 is included to ensure that this is so). Such Γ is tangent to the level sets of F_0, and is the unique spray of its projective class with this property. The base integral curves of the restriction of Γ to $F_0 = 1$ are parametrized with arc-length relative to g_0.

Lemma 2. *For $(x, y) \in T^\circ M$ with $F_0(x, y) = 1$, $\kappa(x, y)$ is the geodesic curvature (with respect to g_0) at $x \in M$ of the base integral curve of Γ through x with unit tangent vector $y \in T_x M$.*

PROOF. Let us write ∇/ds for the operation of covariant differentiation along a curve parametrized by s with respect to the Levi-Civita connection of the Riemannian metric g_0. Then any base integral curve of the spray Γ, parametrized with Riemannian arc-length so that $F_0 = 1$ along it, satisfies

$$\frac{\nabla \dot{x}^i}{ds} = -\kappa(x,\dot{x})\varepsilon^{ij} g_{jk}\dot{x}^k.$$

It is clear that $\varepsilon^{ij} g_{jk}\dot{x}^k$ are the components of a unit normal field along the curve. Thus κ is the geodesic curvature of the base integral curve with respect to g_0. □

3. Magnetic flows and Randers spaces

A magnetic flow on a Riemannian manifold M, of any dimension, is a second-order differential equation field on TM determined by a 1-form α on M as follows. Let g be the metric and E its energy function (so that $E(x,y) = \frac{1}{2}g_x(y,y)$); let $\widehat{\alpha}$ be the function on TM, linear in the fibre coordinates, canonically defined by α. The magnetic flow of α is the Euler-Lagrange field of the Lagrangian $E + \widehat{\alpha}$. Its base integral curves are the solutions of the second-order differential equations

$$\frac{\nabla \dot{x}^i}{dt} = g^{ij}\left(\frac{\partial \alpha_j}{\partial x^k} - \frac{\partial \alpha_k}{\partial x^j}\right)\dot{x}^k.$$

It is clear that E is constant along any base integral curve. (It is also clear that since the right-hand side depends only on $d\alpha$ one could replace the 1-form α by a closed 2-form in the definition.)

The equations above for a magnetic flow invite comparison with those for the geodesics of the Randers space defined by g and α, that is, the geodesics of the pseudo-Finsler function $F = \sqrt{E} + \widehat{\alpha}$. When expressed in terms of the Levi-Civita covariant derivative operator of g these are

$$\frac{\nabla \dot{x}^i}{dt} = g^{ij}\sqrt{E}\left(\frac{\partial \alpha_j}{\partial x^k} - \frac{\partial \alpha_k}{\partial x^j}\right)\dot{x}^k$$

(see Bao et al. [2] page 297). Again, E is constant along geodesics. The base integral curves of the magnetic flow for which $E = 1$ coincide with the geodesics of the Randers space for which $E = 1$; these curves are parametrized by Riemannian arc-length. (This equivalence is of course well known: it is discussed in [8], for example — though only in the case in which the Riemannian metric is actually Euclidean.)

In the light of subsequent developments it is worth noticing that the Hilbert 2-form of the Randers Finsler function $F = \sqrt{E} + \widehat{\alpha} = F_0 + \widehat{\alpha}$ is $\omega_0 + d\alpha$ where ω_0 is the Hilbert 1-form of F_0.

In the two-dimensional case the equations for Randers geodesics with unit Riemannian speed coincide with those for the base integral curves of Γ given in the proof of Lemma 2, with

$$\kappa = -\frac{1}{\sqrt{\det g_0}}\left(\frac{\partial \alpha_1}{\partial x^2} - \frac{\partial \alpha_2}{\partial x^1}\right).$$

In other words, $d\alpha = -\kappa\nu_0$. In this case κ is a function on M; we called sprays Γ with this property isotropic in [4]. Conversely, for any isotropic spray in two dimensions its base integral curves parametrized with arc-length relative to a background metric g_0 may be identified with those of a magnetic flow, and (at least locally) with the unit speed geodesics of a Randers space with Riemannian part g_0.

4. Hilbert forms

Let θ_0 be the Hilbert 1-form and $\omega_0 = d\theta_0$ the Hilbert 2-form of F_0. Then $i_{\Gamma_0}\omega_0 = i_\Delta\omega_0 = 0$. Let Ω_0 be the Hilbert 2-form of the energy of g_0: then

$$\Omega_0 = F_0\omega_0 + dF_0 \wedge \theta_0.$$

Let α^i, $i = 1, 2$ be 1-forms on $T^\circ M$ such that $\{dx^i, \alpha^i\}$ is the (local) basis of 1-forms dual to the basis $\{H_i, V_i\}$ of vector fields. Then

$$\Omega_0 = g_{ij}\alpha^i \wedge dx^j.$$

Let ν_0 be the pull-back to $T^\circ M$ of the volume form of g_0 on M with the chosen orientation (I have previously denoted the volume form itself by the same symbol: it seems unnecssary to make a notational distinction between the two).

Theorem 1. *Any Hilbert 2-form for* $\Gamma = \Gamma_0 + F_0\kappa N_0$ *may be written*

$$\omega = \rho(\omega_0 - \kappa\nu_0)$$

where ρ *is a non-vanishing function on* $T^\circ M$*, homogeneous of degree 0, such that*

$$\Gamma(\rho) + F_0\kappa'\rho = 0 = \Gamma_0(\rho) + F_0(\kappa\rho)'.$$

PROOF. I shall use the following criteria for a 2-form ω to be a Hilbert form for a spray Γ (Theorem 5 of [6]):

(1) the characteristic distribution of ω is the distribution spanned by the projective class of Γ, that is, $\langle\Gamma, \Delta\rangle$;

(2) $\mathcal{L}_\Gamma\omega = 0$;

(3) for any pair of vertical vector fields V_1, V_2, $\omega(V_1, V_2) = 0$;

(4) for any vector field H horizontal with respect to Γ and any pair of vertical vector fields V_1, V_2, $d\omega(H, V_1, V_2) = 0$.

It is evident that with ω as given in the statement of the theorem item 3 is satisfied. Moreover, $i_\Delta\omega = 0$ by inspection. Clearly $\mathcal{L}_\Delta\omega = 0$, so $i_\Delta d\omega = 0$. But in dimension 2 it is enough to take V_1 and V_2 in item 4 to be Δ and, say, N_0; so item 4 is satisfied. It remains to consider $i_\Gamma\omega$ and $\mathcal{L}_\Gamma\omega$.

We know that $i_{\Gamma_0}\omega_0 = 0$; we need to compute $i_{N_0}\omega_0$. From the formula $\Omega_0 = F_0\omega_0 + dF_0 \wedge \theta_0$, together with the facts that $N_0(F_0) = 0$ and $\theta_0(N_0) = 0$ (θ_0 is semi-basic) we see that $i_{N_0}\omega_0 = (1/F_0)i_{N_0}\Omega_0$. But

$$i_{N_0}\Omega_0 = \varepsilon_{ij}y^i dx^j = i_{\Gamma_0}\nu_0.$$

It follows that

$$i_\Gamma(\omega_0 - \kappa\nu_0) = 0.$$

Now in a 4-dimensional vector space $\mathcal{V}$ the set of 2-covectors which vanish on a given 2-dimensional subspace $\mathcal{W}$ of $\mathcal{V}$ is 1-dimensional: it consists of scalar multiples of the exterior product of any two linearly independent elements of $\mathcal{W}^\circ$, the annihilator of $\mathcal{W}$, a 2-dimensional subspace of $\mathcal{V}^*$. So a Hilbert 2-form for Γ must take the form $\omega = \rho\chi$, $\chi = \omega_0 - \kappa\nu_0$, for some non-vanishing function ρ on $T^\circ M$, which must be homogeneous of degree 0.

I have to compute $\mathcal{L}_\Gamma\omega$; I use

$$\mathcal{L}_\Gamma(\rho\chi) = \Gamma(\rho)\chi + \rho\mathcal{L}_\Gamma\chi.$$

But

$$\mathcal{L}_\Gamma\chi = i_\Gamma d\chi + d(i_\Gamma\chi) = i_\Gamma(-d\kappa \wedge \nu_0) = -\Gamma(\kappa)\nu_0 + d\kappa \wedge i_\Gamma\nu_0$$

since $i_\Gamma\chi = 0$, $d\omega_0 = 0$ and $d\nu_0 = 0$. It follows directly that $i_\Gamma\mathcal{L}_\Gamma\chi = i_\Delta\mathcal{L}_\Gamma\chi = 0$, and so by the argument given earlier, $\mathcal{L}_\Gamma\chi$ is a scalar multiple of χ, say $\mathcal{L}_\Gamma\chi = \sigma\chi$. To evaluate σ I take the interior product with N_0:

$$\sigma i_{N_0}\chi = i_{N_0}(\mathcal{L}_\Gamma\chi) = N_0(\kappa)i_\Gamma\nu_0 = \kappa' i_\Gamma\nu_0.$$

But from the earlier calculations we know that $F_0 i_{N_0}\chi = i_{\Gamma_0}\nu_0 = i_\Gamma\nu_0$. It follows that $\sigma = F_0\kappa'$, and

$$\mathcal{L}_\Gamma(\rho\chi) = \Gamma(\rho)\chi + \rho\mathcal{L}_\Gamma\chi = (\Gamma(\rho) + F_0\kappa'\rho)\chi.$$

So in order that $\mathcal{L}_\Gamma(\rho\chi) = 0$ it is necessary and sufficient that

$$\Gamma(\rho) + F_0\kappa'\rho = 0. \qquad \square$$

Corollary 1. *If $\rho_1\chi$ and $\rho_2\chi$ are both Hilbert 2-forms for Γ then $\Gamma(\rho_1/\rho_2) = 0$; and conversely.*

Since ρ is of homogeneity degree zero, the defining condition $\Gamma(\rho)+F_0\kappa'\rho = 0$ holds for any Γ in the projective equivalence class if it holds for one.

5. Pseudo-Finsler functions

I next consider the relation between ρ and a pseudo-Finsler function for Γ, say F. As a first step, starting with F I shall determine that geodesic spray of F which is of the form $\Gamma_0 + F_0\kappa N_0$; that is to say, I shall find a formula for κ. I denote the Hilbert 1-form of F by θ and the Hilbert 2-form by ω. I shall evaluate $i_\Gamma\omega$, with $\Gamma = \Gamma_0 + F_0\kappa N_0$; κ will be determined by the condition $i_\Gamma\omega = 0$. Incidentally, it is clear that $i_\Delta\omega = 0$ and that the other conditions for a Hilbert 2-form are satisfied (it is a Hilbert 2-form after all).

The following results about K_0 will be useful; they are established by straightforward calculations.

(1) $K_0(F_0) = 0$ (indeed, this holds for any vector field which is horizontal with respect to the canonical geodesic spray of a Finsler function);
(2) $\theta_0(K_0) = \langle S(K_0), dF_0\rangle = N_0(F_0) = 0$;
(3) $\omega_0(N_0, K_0) = F_0$;
(4) $i_{K_0}\nu_0 = -F_0\theta_0$.

Let me write $F = pF_0$ where p is homogeneous of degree zero. Then

$$\theta = S^*(dF) = S^*(pdF_0 + F_0dp) = p\theta_0 + F_0S^*(dp).$$

The Hilbert 2-form is thus

$$\omega = p\omega_0 + dp\wedge\theta_0 + F_0d(S^*(dp)) + dF_0\wedge S^*(dp).$$

The one term which it is not immediately obvious how to deal with is the one involving $d(S^*(dp))$. For any vector fields X, Y,

$$d(S^*(dp))(X,Y) = X(S(Y)(p)) - Y(S(X)(p)) - S([X,Y])(p);$$

in particular,

$$\begin{aligned} d(S^*(dp))(\Gamma_0, Y) &= \Gamma_0(S(Y)(p)) - Y(\Delta(p)) - S([\Gamma_0, Y])(p) \\ &= [\Gamma_0, S(Y)](p)) + S(Y)(\Gamma_0(p)) - S([\Gamma_0, Y])(p) \\ &= (\mathcal{L}_{\Gamma_0} S)(Y)(p) + S(Y)(\Gamma_0(p)). \end{aligned}$$

Recall that $\mathcal{L}_{\Gamma_0} S$ acts as the identity on vertical vectors, as minus the identity on horizontal ones. Then

$$\begin{aligned} &d(S^*(dp))(\Gamma_0, K_0) = -K_0(p) + N_0(\Gamma_0(p)) = \Gamma_0(N_0(p)) = \Gamma_0(p') \\ &d(S^*(dp))(\Gamma_0, N_0) = N_0(p) = p'. \end{aligned}$$

We shall also require

$$\begin{aligned} d(S^*(dp))(N_0, K_0) &= N_0(S(K_0)(p)) - K_0(S(N_0)(p)) - S([N_0, K_0])(p) \\ &= p'' - 0 + \Delta(p) = p''. \end{aligned}$$

Finally, $d(S^*(dp))(X,Y) = 0$ when both arguments are vertical.

Theorem 2. *Let $F = pF_0$ be a pseudo-Finsler function on $T^\circ M$, Γ the geodesic spray of F such that $\Gamma(F_0) = 0$, so that $\Gamma = \Gamma_0 + F_0 \kappa N_0$. Then*

(1) *the function $\rho = p'' + p$ on $T^\circ M$ is non-vanishing;*

(2) *κ is given in terms of p by*

$$\kappa = \frac{K_0(p) - \Gamma_0(p')}{F_0 \rho};$$

(3) *the Hilbert 2-form ω of F is given by $\omega = \rho(\omega_0 - \kappa\nu_0)$;*

(4) $\Gamma(\rho) + F_0 \kappa' \rho = 0$.

Proof. Using the formulae derived above one finds that with $\Gamma = \Gamma_0 + F_0 \kappa N_0$, $\omega(\Gamma, \Gamma_0) = \omega(\Gamma, \Delta) = \omega(\Gamma, N_0) = 0$ identically, while

$$\omega(\Gamma, K_0) = F_0\big(\Gamma_0(p') - K_0(p) + F_0 \kappa (p'' + p)\big).$$

The condition $i_\Gamma\omega = 0$ reduces to the single requirement that $\omega(\Gamma, K_0) = 0$, and therefore

$$\Gamma_0(p') - K_0(p) + F_0\kappa(p'' + p) = 0.$$

But then ω must be of the form $\rho(\omega_0 - \kappa\nu_0)$. It must therefore be the case that

$$\omega(N_0, K_0) = F_0(p'' + p) = F_0\rho,$$

so that

$$\rho = p'' + p.$$

But ρ must never vanish. Thus

$$\kappa = \frac{K_0(p) - \Gamma_0(p')}{F_0\rho}.$$

Furthermore, it must be the case that $\mathcal{L}_\Gamma\omega = 0$, and therefore that

$$\Gamma(\rho) + F_0\kappa'\rho = 0. \qquad \square$$

Now suppose one is given a spray $\Gamma = \Gamma_0 + F_0\kappa N_0$, that is, a function κ, and one wishes to find the pseudo-Finsler functions of which it is a geodesic spray. The first step is to solve the equation $\Gamma(\rho) + F_0\kappa'\rho = 0$. Then any function pF_0 such that $p'' + p = \rho$ is a candidate to be a pseudo-Finsler function.

Lemma 3. *Let p be any function on $T^\circ M$, homogeneous of degree 0, such that $\Gamma(\rho) + F_0\kappa'\rho = 0$, where $\rho = p'' + p$ and ρ never vanishes. Set*

$$\tilde\kappa = \frac{K_0(p) - \Gamma_0(p')}{F_0\rho}.$$

Then $(\kappa - \tilde\kappa)\rho$ is the pullback to $T^\circ M$ of a function on M.

Proof. We have

$$\begin{aligned} N_0(F_0\tilde\kappa\rho) = F_0N_0(\tilde\kappa\rho) &= N_0(K_0(p)) - N_0(\Gamma_0(p')) \\ &= [N_0, K_0](p) + K_0(p') - [N_0, \Gamma_0](p') - \Gamma_0(p'') \\ &= -\Gamma_0(p) + K_0(p') - K_0(p') - \Gamma_0(p'') \\ &= -\Gamma_0(\rho) = F_0N_0(\kappa\rho). \end{aligned}$$

Since $\Delta(\tilde\kappa\rho) = 0 = \Delta(\kappa\rho)$, it follows that $V((\kappa - \tilde\kappa)\rho) = 0$ for all vertical V. $\square$

Theorem 3. *Suppose given a spray* $\Gamma = \Gamma_0 + F_0\kappa N_0$ *on* $T^\circ M$ *and a function* $\tilde{p}$ *such that* $\rho = \tilde{p}'' + \tilde{p}$ *is non-vanishing and satisfies* $\Gamma(\rho) + F_0\kappa'\rho = 0$*. Then over any contractible open subset of* M *there is a function* p *such that* $p'' + p = \rho$ *and* $F = pF_0$ *is a pseudo-Finsler function for* Γ*;* F *is determined by* ρ *up to the addition of a total derivative.*

Proof. From Theorem 2 we know that for any p the Hilbert 2-form of $F = pF_0$ differs from $(p'' + p)\omega_0$ by a semi-basic 2-form (a multiple of ν_0 in fact). So $\tilde{p}'' + \tilde{p} = p'' + p$ if and only if the Hilbert 2-forms of $\tilde{F} = \tilde{p}F_0$ and $F = pF_0$ differ by a semi-basic 2-form. It follows that $\tilde{F}$ and F differ by a term linear in the fibre coordinates, as I show next. In general the Hilbert 2-form of F can be written $h_{ij}dy^i \wedge dx^j + \ldots$ where the omitted terms are semi-basic; here h_{ij} are the components of the Hessian of F with respect to the fibre coordinates. Thus if the Hilbert 2-forms of $\tilde{F}$ and F differ by a semi-basic 2-form, as is the case here, the Hessians of $\tilde{F}$ and F must be the same, and so $\tilde{F}$ and F differ by a term linear in the fibre coordinates, say $\tilde{F} = F - \widehat{\phi}$ where ϕ is a 1-form on M and $\widehat{\phi}$ the corresponding fibre-linear function on $T^\circ M$. The Hilbert 2-forms $\tilde{\omega}$ and ω of $\tilde{F}$ and F are then related by $\tilde{\omega} = \omega - d\phi$ (I haven't distinguished notationally between $d\phi$ and its pullback to $T^\circ M$). Now the Hilbert 2-form of $\tilde{F}$ is $\rho(\omega_0 - \tilde{\kappa}\nu_0)$, and by the Lemma $\rho\tilde{\kappa} = \rho\kappa + f$ for some function f on M. Over a contractible open set in M we can find a 1-form ϕ such that $d\phi = f\nu_0$. Then

$$\omega = \tilde{\omega} + d\phi = \rho(\omega_0 - \tilde{\kappa}\nu_0) + f\nu_0 = \rho(\omega_0 - \kappa\nu_0).$$

Thus the Hilbert 2-form constructed from F is a Hilbert 2-form for Γ; that is to say, F is a pseudo-Finsler function for Γ. Moreover, ϕ is determined up to the addition of an exact 1-form; so F is determined up to the addition of a total derivative. □

6. Generalized Tabachnikov formula

I now specialize these results by taking isothermal coordinates for g_0 on M, that is, coordinates x^i with respect to which g_0 is conformally flat, so that

$$F_0(x^1, x^2, y^1, y^2) = e^{\phi(x^1,x^2)}\sqrt{(y^1)^2 + (y^2)^2}$$

for some function ϕ defined locally on M. I make a further coordinate transformation to the modified polar coordinates (r, θ) in the fibres such that

$$y^1 = e^{-\phi}r\cos\theta, \quad y^2 = e^{-\phi}r\sin\theta,$$

so that in fact $r = F_0$. It turns out that

$$N_0 = \frac{\partial}{\partial\theta}.$$

The function p in Theorem 3 can be expressed as a function of x^i and θ (it is independent of r by homogeneity). It must satisfy

$$\frac{\partial^2 p}{\partial\theta^2} + p = \rho.$$

The general solution of this equation (considered as an ordinary differential equation in θ, for fixed (x^1, x^2)), is

$$p(x^i, \theta) = \int_0^\theta \sin(\theta - \psi)\rho(x^i, \psi)d\psi + \alpha_1(x^i)\cos\theta + \alpha_2(x^i)\sin\theta.$$

It is required that p be periodic in θ: the necessary and sufficient condition for this is that

$$\int_0^{2\pi} \sin(\theta - \psi)\rho(x^i, \psi)d\psi = 0.$$

If this consition holds, and p is given by the expression above for any functions α_i, then $p'' + p = \rho$. However, the functions α_i are not free, but have to be chosen so that the formula

$$K_0(p) - \Gamma_0(p') = F_0\kappa\rho$$

from Theorem 2 is satisfied. In order to derive the consequences of this condition it is convenient to rewrite the expression for p above as

$$p(\theta) = \int_0^\theta \sin(\theta - \psi)\rho(\psi)d\psi + (a_1 e^{-\phi} + 1)\cos\theta + a_2 e^{-\phi}\sin\theta$$

(leaving the x-dependence to be understood). The reason for this odd-looking choice of expressions for the coefficients of $\cos\theta$ and $\sin\theta$ is easily explained: in the case in which $\rho = 1$ we have

$$\begin{aligned} p(\theta) &= \cos(\theta - \psi)|_{\psi=0}^{\theta} + (a_1 e^{-\phi} + 1)\cos\theta + a_2 e^{-\phi}\sin\theta \\ &= 1 + a_1 e^{-\phi}\cos\theta + a_2 e^{-\phi}\sin\theta, \end{aligned}$$

so that

$$F = pr = pF_0 = e^{\phi(x^i)}\sqrt{(y^1)^2 + (y^2)^2} + a_1 y^1 + a_2 y^2,$$

a Randers metric in the form given in the Introduction.

The Hessian of $F = rp$ with respect to the fibre coordinates y^i, expressed in terms of the modified polar coordinates, is

$$\frac{e^{2\phi}\rho}{r}\begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

where $\rho = p'' + p$. It follows that the Hessian is positive quasi-definite if and only if ρ is everywhere positive. (See [5] Section 7 for further discussion of this issue, and other matters concerning projective Finsler metrizability of two-dimensional sprays, such as the derivation of the integral condition on ρ for p to be periodic.)

Theorem 4. *Every Finsler function whose geodesic curvature function relative to the background metric g_0 is κ can be represented, in terms of isothermal coordinates in the base and the corresponding modified polar coordinates in the fibre, as follows:*

$$F(x^i, r, \theta) = r\Bigg(\int_0^\theta \sin(\theta-\psi)\rho(x^i,\psi)d\psi \\ + \big(a_1(x^i)e^{-\phi(x^i)} + 1\big)\cos\theta + a_2(x^i)e^{-\phi(x^i)}\sin\theta\Bigg)$$

where ρ is a positive solution of the equation

$$\Gamma(\rho) + F_0\kappa'\rho = 0 = \Gamma_0(\rho) + F_0(\kappa\rho)'$$

such that

$$\int_0^{2\pi} \sin(\theta-\psi)\rho(x^i,\psi)d\psi = 0,$$

and a_1, a_2 are two functions, satisfying

$$e^{-2\phi}\left(\frac{\partial a_1}{\partial x^2} - \frac{\partial a_2}{\partial x^1}\right) = (\kappa\rho)_{\theta=0}.$$

The proof is a calculation. It amounts in effect to carrying out explicitly in the given coordinates the adjustment required to pass from a general solution of $p'' + p = \rho$ to one which gives a pseudo-Finsler function, as specified in general terms in Theorem 3.

Recall the result of Corollary 4, which in this context states that if ρ_1 and ρ_2 are any two choices for the function ρ in the statement of the theorem then ρ_1/ρ_2 must be a first integral of Γ.

7. Circles

The base integral curves of a spray Γ are geodesic circles (with respect to g_0) if the geodesic curvature function κ satisfies $\Gamma(\kappa) = 0$. (For completeness we should allow geodesics of Γ_0, for which $\kappa = 0$ of course, to be regarded as geodesic circles.) When M with the metric g_0 has constant Gaussian curvature one can give an explicit solution of the equation $\Gamma(\rho)+F_0\kappa'\rho = 0$ in the case when the base integral curves of Γ are geodesic circles. With respect to isothermal coordinates these curves are Euclidean circles also, as we showed in [4].

Lemma 4. *Suppose that the Gaussian curvature R_0 of g_0 is constant, and that the geodesic curvature function κ of the spray $\Gamma = \Gamma_0 + F_0\kappa N_0$ satisfies $\Gamma(\kappa) = 0$. Then if*

$$\rho = \kappa^2 + R_0 - K_0(\kappa)/F_0,$$

ρ satisfies $\Gamma(\rho) + F_0\kappa'\rho = 0$ and is homogeneous of degree 0.

Proof. We have $\Gamma(\kappa) = 0$, $\Gamma(R_0) = 0$ and $\Gamma(F_0) = 0$; furthermore

$$\begin{aligned}
\Gamma(K_0(\kappa)) &= [\Gamma, K_0](\kappa) \\
&= [\Gamma_0, K_0](\kappa) + [F_0\kappa N_0, K_0](\kappa) \\
&= F_0^2 R_0 N_0(\kappa) - F_0\kappa\Gamma_0(\kappa) - F_0 K_0(\kappa)N_0(\kappa) \\
&= F_0^2 R_0\kappa' + F_0^2\kappa^2\kappa' - F_0 K_0(\kappa)\kappa' \\
&= F_0^2\kappa'\big(\kappa^2 + R_0 - K_0(\kappa)/F_0\big) = F_0^2\kappa'\rho.
\end{aligned}$$

It follows that $\Gamma(\rho) + F_0\kappa'\rho = 0$. □

In the isotropic case, when κ is a function on M, we have $N_0(\kappa) = 0$, and so $\Gamma_0(\kappa) = 0$ also. It is then clear that κ is a constant. The equation $\Gamma(\rho)+F_0\kappa'\rho = 0$ is satisfied by any constant, not just $\kappa^2 + R_0$ as provided by the formula in the Lemma; with $\rho = 1$ we obtain a Randers metric, as given in [4]. But it is now clear how to get the most general Finsler metric for each of the examples treated there: use the generalized Tabachnikov formula in Theorem 4, with ρ any first integral of Γ.

For another example I consider circles of arbitrary, not necessarily fixed, radius in some open subset M of the Euclidean plane. Let x^i be Euclidean coordinates, r and θ the corresponding polar coordinates in the fibres of $T_1^\circ M$ (we now have $\phi = 0$ of course). To facilitate comparison with Tabachnikov's original formula it seems natural to work in terms of radius of curvature rather than curvature (and now we do require the circles to be genuine circles of finite radius). So we have a positive function $R(x^i,\theta)$ on $T_1^\circ M$, with $R=\kappa^{-1}$ (R is not to be confused with R_0, which is zero here of course). Set

$$\Gamma = r\left(\cos\theta\frac{\partial}{\partial x^1}+\sin\theta\frac{\partial}{\partial x^2}+\frac{1}{R}\frac{\partial}{\partial\theta}\right),$$

and assume that $\Gamma(R)=0$, that is, that

$$R' = -R\left(\frac{\partial R}{\partial x^1}\cos\theta+\frac{\partial R}{\partial x^2}\sin\theta\right).$$

When this condition holds the integral curves of Γ with $r=1$ are

$$x^1=\xi^1+R\sin\theta,\quad x^2=\xi^2-R\cos\theta,\quad \theta=\frac{s}{R}+\theta_0,$$

with constants ξ^i. These are of course the parametric equations of a circle of radius R and centre (ξ^1,ξ^2) in the plane, described in the anticlockwise sense: note that θ is the direction of the unit tangent vector, not the radius vector. Now

$$K_0 = r\left(-\sin\theta\frac{\partial}{\partial x^1}+\cos\theta\frac{\partial}{\partial x^2}\right);$$

so Lemma 4 gives

$$\rho=\frac{1}{R^2}\left(1-\frac{\partial R}{\partial x^1}\sin\theta+\frac{\partial R}{\partial x^2}\cos\theta\right).$$

Since R is itself a first integral of Γ we may ignore the overall factor $1/R^2$ in applying Theorem 4. We have the following generalization of Tabachnikov's formula for circles in the plane.

Theorem. *The most general Finsler function for Γ is given by*

$$F(x^i,r,\theta)=r\left(\int_0^\theta \sin(\theta-\psi)\rho(x^i,\psi)d\psi+a_1(x^i)\cos\theta+a_2(x^i)\sin\theta\right)$$

where ρ is a positive function on $T_1^\circ M$ of the form

$$\rho=\left(1-\frac{\partial R}{\partial x^1}\sin\theta+\frac{\partial R}{\partial x^2}\cos\theta\right)\sigma(x^1-R\sin\theta,x^2+R\cos\theta)$$

for some function σ of two variables, such that $\int_0^{2\pi}\sin(\theta-\phi)\rho(\psi)d\psi=0$, and a_1 and a_2 are functions on M which satisfy

$$\frac{\partial a_1}{\partial x^2}-\frac{\partial a_2}{\partial x^1}=(\rho/R)_{\theta=0}.$$

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