Symplectic polarities of buildings of type E_6

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Abstract

A symplectic polarity of a building Δ of type E_6 is a polarity whose fixed point structure is a building of type F_4 containing residues isomorphic to symplectic polar spaces. In this paper, we present two characterizations of such polarities among all dualities. Firstly, we prove that, if a duality θ of Δ never maps a point to a neighbouring symp, and maps some element to a non-opposite element, then θ is a symplectic duality. Secondly, we show that, if a duality θ never maps a chamber to an opposite chamber, then it is a symplectic polarity. The latter completes the programme for dualities of buildings of type E_6 of determining all domestic automorphisms of spherical buildings, and it also shows that symplectic polarities are the only polarities in buildings of type E_6 for which the Phan geometry is empty.

1 Introduction

A *domestic* automorphism of a spherical building is an automorphism which does not map any chamber to an opposite one. This notion arose from work of Abramenko & Brown [2], who proved that in non-spherical buildings, only the identity has bounded displacement on the set of chambers. In the spherical case, they proved that only the identity maps no flag to an opposite flag. This led Temmermans, Thas and the present author [10, 11, 12] to a more refined definition of J-domestic automorphism: this is an automorphism of a spherical building over the type set $I \supseteq J$ mapping no flag of type J to an opposite flag. For $J = I$, we obtain domesticity as defined above. It is clear that J-domesticity for any $J \subseteq I$ implies domesticity. Hence in order to understand all J-domestic automorphisms, one has to classify the domestic automorphisms. This was done for projective spaces in [10] and for generalized quadrangles in [11]. Partial—but important as we shall see in the proof of Lemma 5.3—results for polar spaces are contained in [12]. If the automorphism induces opposition on the type set, then it seems that domesticity is intimately related to the notion of geometric hyperplane in an apprpriate point-line geometry arising from the building. For instance, in projective 3-space, the domestic dualities fix a hyperplane of the line-Grassmanian (which is here the Klein quadric), namely a linear complex, also known as the set of lines fixed under a symplectic polarity. In a polar pace, an automorphism is line domestic if and only if it pointwise fixed a geometric hyperplane.

In the present paper, we investigate domestic dualties of spherical buidings Δ of type E_6 and we show that there is only one conjugacy class of such dualities, namely, the polarities that fix a large subbuilding of type F_4. In fact, this subbuilding is a geometric hyperplane of Δ in the ordinary point-line geometry related to Δ. This result is completely similar to the one in projective spaces, and therefore, we call these polarities also *symplectic polarities*. But of course, the proof is more involved as buildings of type E_6 are not so easily accessible as projective spaces. Also, maybe this result is not so surprising, but then this is compensated by the beauty of the geometric arguments, at the end giving a lot of insight into the structure of how this subbuilding of type F_4 sits in Δ. As a matter of fact, it turns out that this polarity admits another rather beautiful and less expected geometric characterization, which we can use in our proof. Indeed, it is well-known—see Section 3—that points and symps have three different possible mutual positions: (I) a point can be contained in a symp, (II) it can be outside the symp, but not opposite, and hence the point and the symp are incident with a unique common 5-space, or (III) they can be opposite. Then symplectic polarities are characterized by the fact that they are non-anisotropic dualities for which a point and its image are never in position (II) above. *Anisotropic* means that every element is mapped onto an opposite, hence the requirement of being non-anisotropic is rather weak and just says that some element must be mapped onto a non-opposite one.

Our motivation is not just curiosity and beauty. In Phan-Curtis-Tits theory, see [6], one is interested in involutions of spherical buildings (or more generally, twin buildings) and in the corresponding Phan geometry. The latter is exactly defined as the geometry induced on the set of chambers that are mapped onto an opposite chamber. Hence the first question in that theory is: when is the Phan geometry nonempty? Our Main Result 2.2 answers this for dualities in buildings of type E_6.

There is another, though more theoretic and abstract, motivation. Let us say that a domestic automorphism admits a *domesticity diagram* if all maximal flags mapped onto an opposite flag have the same type. In this case we can furnish the Dynkin or Coxeter diagram of the building with some extra circles as follows: we encircle the types of all minimal flags mapped onto an opposite. One can also consider the associated *Tits diagram* which encircles the types of the minimal flags which are fixed. Now, if we interpret the domesticity diagram as a Tits diagram, then with both type of diagrams we can associate a conjugacy class of automorphisms of the associated Coxeter system. Now it appears that, if a domestic automorphism admits a domesticity diagram and if the fixed element structure is a (weak) subbuilding, then the two mentioned conjugacy classes of automorphisms of the Coxeter system are related by opposition. This is, in fact, not so difficult to see. Now, such a situation—two Tits-diagrams related by involution—occurs in abundance in the Freudenthal-Tits magic square, more exactly in the South-East 3×3 corner, except for the $\mathsf{A}_2 \times \mathsf{A}_2$-entry. Our conjecture is that each of the five remaining pairs of entries (among which two identical pairs) can occur as pairs of domesticity diagram and Tits diagram of a domestic involution, which is unique under some domesticity condition (but this involution never corresponds with the Galois descent meant in the magic square). The present paper proves this conjecture for the pair corresponding to the entries of type E_6. The entries of type A_5 correspond precisely with the symplectic polarities, also characterized as being the only domestic dualities [10]. The entry of type D_6 (on the diagonal) corresponds with an involution of an 11-dimensional projective space $PG(11, \mathbb{K})$ fixing only odd-dimensional subspaces which form a $\mathsf{PG}(5, \mathbb{L})$, with $\mathbb{L}$ a quadratic Galois extension of $\mathbb{K}$, and inducing on a hyperbolic quadric $Q^+(11, \mathbb{K})$ a Hermitian variety $H(5, \mathbb{L})$ of maximal Witt index 3. This involution is characterized by point-domesticity, as can be deduced (with some minot additional work) from Theorem 4.2 of [12]. Hence our work should put some more magic into the Freudenthal-Tits magic square.

Let us finally remark that there exist domestic automorphisms admitting no domesticity diagram. Such automorphisms of generalized quadrangles are classified in [11]; there are exactly three conjugacy classes of these, they all have order 4 and exist in the small generalized quadrangles of orders (2,2), (2,4) and (3,5). It is an open question whether domestic automorphisms admitting no domesticity diagram exist in spherical buildings of rank at least 3.

2 Preliminaries and Main Results

Let Δ be a spherical building, i.e., a building with a finite Weyl group. For the definition of buildings, we refer to [1]. Let I be the set of types of Δ. As mentioned in the introduction, for a subset $J \subseteq I$, a J-*domestic* automorphism is an automorphism which does not map any flag of type J onto an opposite one. If $J = I$, then we simply talk about a *domestic* automorphism. If J is not stable under the opposition relation, then every automorphism is automatically J-domestic, hence we assume from now on that J is fixed under the opposition relation. It is clear that J-domesticity follows from J'-domesticity if $J' \subseteq J$. Hence the most general situation is that of domesticity. In the present paper, we concentrate on domestic dualities of buildings of type E_6.

So let Δ be a building of type E_6. If we label the types according to the Bourbaki conventions [4], then we call elements of type $1, 2, 3, 4, 5$ *points, 5-spaces, lines, planes, 4-spaces and symps*, respectively. This way, we in fact identify Δ with its shadow space corresponding with elements of type 1. The opposition relation on the types preserves the types 2 and 4 and switches type 1 with type 6 and type 3 with type 5.

Buildings of type E_6 are naturally associated to Chevalley groups of type E_6. Now, it is well known that every such group contains a maximal subgroup of type F_4, which is moreover fixed by an outer involution θ. This involution θ induces a nontrivial involution (in fact, the opposition relation) on the diagram and can hence be seen as a polarity of Δ. Geometrically, the subgroup of type F_4 stabilizes a subbuilding Δ' of type F_4, fully embedded in Δ, i.e., every point and every line in Δ of every plane of Δ' also belongs to Δ', and the same thing holds for every 5-space of Δ incident with a plane of Δ'. We can choose types in Δ' such that the points, lines planes and hyperlines of Δ' are points, lines, planes and 5-spaces, respectively, of Δ. The hyperlines of Δ' are symplectic polar spaces of rank 3. This motivates to call the polarity θ a symplectic polarity. It is unique up to conjugacy. Geometrically, this follows from the fullness of the embedding of Δ' in Δ.

It is possible to start with a building Δ' of type F_4 having symplectic residues, and to define additional points using special substructures of Δ' so that one obtains Δ, and get the polarity θ for free. We will not carry out this rather cumbersome exercise here, but the properties we prove in the present paper should be very helpful for this purpose.

The automorphism group of Δ has three orbits on the point-symp-pairs: either the point is incident with the symp, or the point is opposite the symp, or the points and the symp are incident with a unique common 5-space. In the latter case, we say that the point and the symp are *neighbouring*. This notion is a standard one in the theory of Hjelmslev planes and is inspired by the fact that Δ can be described as a Hjelmslev-Moufang plane over split octonions, see [8].

We call a duality of Δ *anisotropic* if it maps every element to an opposite. Examples are provided by anisotropic forms of type E_6.

We can now state our main results.

Main Result 2.1 *If a duality of a building Δ of type E_6 maps no point to a neighbouring symp, then it is either anisotropic or a symplectic polarity.*

Main Result 2.2 *The only domestic dualities of a building Δ of type E_6 are the symplectic polarities.*

This shows the following result.

Corollary 2.3 *The only involutions of a building Δ of type E_6 with empty Phan geometry are the symplectic polarities.*

In the finite case, no duality can be anisotropic. Hence an immediate corollary of Main Result 2.1 is:

Corollary 2.4 *If a duality of a finite building Δ of type E_6 maps no point to a neighbouring symp, then it is a symplectic polarity.*

3 Some geometric properties of buildings of type E_6

In this section we gather some facts about buildings Δ of type E_6. Throughout, we number the diagram as in [4], and we choose to name the elements of type 1 *points*. We identify all other elements with the set of points incident with them. The elements of type 3 will be called *lines*, those of type 4 *planes*, those of type 5 will be called 4-spaces, those of type 6 *symps* and the elements of type 2 will be called 5-*spaces*. A $4'$-space is a hyperplane of a 5-space, but it does not conform to a type in Δ. Also, a 3-space is some 3-space in a 5-space, or, equivalently, in a 4-space.

Everything below is well-known, and we give precise references for most fact. Many things are contained in [13], but we also include references to [5], as the latter is easily accessible and provides an excellent source of information on buildings of type E_6. Let us also remark that some of the properties are stated, without proof, in [7], where they are seen as results of "reading" the diagram. We were unable to find Facts 3.5, 3.7 and 3.8 literally in the literature, but these (and also the others) can be verified by the reader himself by including two appropriate flags (mostly just two elements) in an apartment, and then the assertion becomes an assertion in a thin building of type E_6. Such a thin building $\mathcal{A}$ is provided by the following easy construction (see Paragraph 10.3.4 in [3]): the 27 points of $\mathcal{A}$ are the 27 points of the generalized quadrangle Q of order $(2,4)$ (arising from a nondegenerate bilinear form of Witt index 2 in a 6-dimensional projective space over the field of 2 elements). The lines of $\mathcal{A}$ are the non-collinear pairs of points of Q. The planes of $\mathcal{A}$ are the *triads* of Q (i.e., the triples of non-collinear points). The 4-spaces are the intersections $p^\perp \cap q^\perp$, where p and q are two non-collinear points and $x^\perp$ denotes the set of elements collinear with the point x in Q, but distinct from x. The 5-spaces through a point p are obtained by taking some point q not collinear with p in Q, and then the points in $q^\perp \setminus p^\perp$ together with p form a 5-space. A symp simply is $p^\perp$ for some point p of Q. Opposition is also easily defined in $\mathcal{A}$. Indeed, a point p is opposite the symp $p^\perp$; a line $\{p,q\}$ is opposite the 4-space $p^\perp \cap q^\perp$; the plane $\{x,y,z\}$ is opposite the plane $x^\perp \cap y^\perp \cap z^\perp$ and the 5-space $\{p\} \cup (q^\perp \setminus p^\perp)$ is opposite the 5-space $\{q\} \cup (p^\perp \setminus q^\perp)$. Also incidence can be stated in a simple way: an object is incident with another if and only if one is contained in the other, except if one object is a 5-space V, and the other is either a 4-space W or a symp Σ. Then V is incident with W if and only if $V \cap W$ is a 3-dimensional subspace of both V and W; V is incident with Σ is $V \cap \Sigma$ is a 4-dimensional (singular) subspace of both V and Σ, in which case it is a $4'$-space. The two families of maximal singular subspaces of Σ characterized by the property that subspaces from the same family meet each other in even-dimensional subspaces, and members of different families meet each other in odd-dimensional subspaces, are the family of 4-spaces and the family of $4'$-spaces contained in Σ.

In the next statements, we assume that Δ is a (thick) building of type E_6 over some field $\mathbb{K}$. Collinearity refers to points incident with (or, with our convention, contained in) a common line in Δ. *Opposite* elements are elements which are opposite in some apartment.

Note to start with that 4- and $4'$-spaces are really projective spaces of dimension 4 over $\mathbb{K}$, likewise 5-spaces are projective 5-spaces over $\mathbb{K}$, and symps are hyperbolic quadrics (quadrics of maximal Witt-index) defined in projective 9-space over $\mathbb{K}$.

A *flag* is a set of pairwise incident elements, and a chamber is a flag consisting of 6 elements (one of each type).

Fact 3.1 (Lemma 18.7.1 of [5], Statement 3.7 of [13]) *Any pair of non-collinear points is contained in a unique symp.*

Fact 3.2 (Proposition 18.7.2(vii) of [5], Statements 3.5.4 and 3.9 of [13]) *Given a point x and a symp Σ, then either $x \in \Sigma$, or x is opposite Σ, which is equivalent with "no point of Σ is collinear with x", or there is a unique 5-space V incident with both x and Σ. In the latter case, the intersection of V with Σ is precisely the set of points of Σ collinear with x.*

In the last case, namely when there is a unique 5-space incident with both a point x and a symp Σ, we say that Σ *neighbours* x (and x neighbours Σ).

The next fact is an immediate consequence of the fact that one can put a 4-space and a point in a common apartment, that in this apartment, there are exactly two symps incident with the 4-space, and that each point has exactly one opposite symp in every apartment.

Fact 3.3 *At least one symp through a given* 4*-space is not opposite a given point.*

Fact 3.4 (Proposition 18.7.2(v) of [5], Statement 3.5.3 of [13]) *Two* 5*-spaces are either disjoint, intersect in a point, or intersect in a plane. The latter case is equivalent with being incident with a common plane (namely, the intersection). In particular, every* 3*-space is contained in a unique* 5*-space.*

When two 5-spaces intersect in a plane, then we say that they are *adjacent*.

Fact 3.5 *Two disjoint* 5*-spaces are either opposite or both adjacent with a unique common* 5*-space.*

Fact 3.6 (Proposition 18.7.2(v) of [5], Statement 3.2 of [13]) *Every* 3*-space is contained in a unique* 4*-space.*

Fact 3.7 *Given a point x and a* 5*-space V, then either X and V are incident, or x is collinear with exactly one point of V, or x is collinear with all points of a unique* 3*-space of V. In the latter case, the space spanned by x and $x^\perp \cap V$ (i.e., the union of all lines through x meeting V) is a* 4*-space.*

Fact 3.8 *A point, line or plane is opposite a symp,* 4*-space, or plane, respectively, if and only if the collinearity relation between the two elements is empty. A* 5*-space is opposite another* 5*-space if and only if each point of the first is collinear to a unique point of the second* 5*-space.*

Now let F and F' be opposite flags in Δ, i.e., each element of F is opposite a unique element of F' and vice versa. For every chamber C containing F there is a unique chamber C' containing F' at minimal distance from C (where the distance of chambers is measured in the chamber graph, i.e., the graph with vertices the chambers, and two chambers are adjacent if they share 5 elements). We denote the map $C \mapsto C'$ by $\rho_{F,F'}$. The *residue of* F consists of all chambers containing F and carries the structure of a spherical building. It is well-known that $\rho_{F,F'}$ can be naturally extended to the elements incident with all elements of F, see Theorem 3.28 in [14].

Proposition 3.9 (Theorem 3.28 and Proposition 3.29 of [14]) *Let F and F' be opposite flags of Δ. Then $\rho_{F,F'}$ is an isomorphism from the residue of F into the residue of F' and the type of the image of an element of type i is the opposite in the residue of F of the opposite type of i in Δ. Also, some chambers $C \supseteq F$ and $C' \supseteq F'$ are opposite in Δ if and only if C' and $\rho_{F,F'}(C)$ are opposite in the residue of F'.*

There is a useful corollary.

Corollary 3.10 *Let θ be an automorphism of Δ. Let F and F^θ be opposite flags of Δ, and let $\sigma_{F,\theta}$ be the automorphism of the residue of F to itself mapping a chamber $C \supseteq F$ onto $\rho_{F',F}(C^\theta)$. If θ induces the natural opposition relation on the types of Δ, then so does $\sigma_{F,\theta}$ for the residue of F.*

4 Proof of Main Result 2.1

Let θ be a duality of a building Δ of spherical type E_6 and assume that

(N) no point is mapped to a neighbouring symp, and

(A) there is at least one absolute point.

Our aim is to show that, under these assumptions, θ is a symplectic polarity, i.e., the structure of fixed flags forms in a natural way the flag complex of a building of type F_4 with symplectic residues. After that, we will show how this implies Main Result 2.1.

Our first aim, however, is to show that θ is involutive, i.e., θ is a polarity. We start with a lemma producing many absolute points from just one.

Lemma 4.1 *If x is an absolute point, then so is each point of x^θ collinear with x.*

Proof Suppose x is an absolute point and let z be a point of x^θ collinear with x. Since x and z are collinear, the symps x^θ and z^θ intersect in a 4-space W. Since x^θ is a polar space, there are points of W collinear with z, hence z^θ is not opposite z. Assumption (N) implies that z belongs to z^θ. Hence z is absolute and the lemma is proved. $\square$

Lemma 4.2 *If x is an absolute point for θ, then $x^{\theta^2} = x$.*

Proof Suppose by way of contradiction that $y := x^{\theta^2} \neq x$, for an absolute point x. Then, in the symp x^θ, we can choose a point z collinear to x and not collinear to y. By Lemma 4.1, z belongs to z^θ. Since z belongs to x^θ, applying θ, we see that also y belongs to z^θ.

So we have shown that z^θ contains both y and z. This is also the case for x^θ. But since y and z are not collinear, they are, by Fact 3.1, contained in a unique symp, implying $x^\theta = z^\theta$. Hence $x = z$, a contradiction.

This completes the proof of the lemma. □

Next, we can show that every point is collinear with an absolute point.

Lemma 4.3 *Every non-absolute point is collinear with at least one absolute point.*

Proof Suppose by way of contradiction that the non-absolute point y is not collinear to any absolute point. Let x be any absolute point and let W be a 4-space in x^θ containing x. The dual of Lemma 4.1 says that every symp containing W is absolute. By Fact 3.3 we have thus found an absolute symp Σ neighbouring y. But then $y^\perp \cap (\Sigma^\theta)^\perp \cap \Sigma$ is nonempty and each of its elements is an absolute point. □

We can now prove that θ is a polarity.

Lemma 4.4 *The duality θ is a polarity.*

Proof It suffices to show that $y^{\theta^2} = y$ for every non-absolute point.

Let y be a non-absolute point. By Lemma 4.3, there is at least one absolute point x collinear with y. Applying θ, we see that x^θ and y^θ intersect in a 4-space which, by Lemma 4.1, contains a lot of absolute points. So we have found an absolute point z in y^θ. Applying θ^{-1}, we see that y is contained in $z^{\theta^{-1}} = z^\theta$. The point set $y^\perp \cap z^\perp \cap z^\theta$ is a non-degenerate polar space all of whose points are absolute by Lemma 4.1. Hence we have found two non-collinear absolute points z_1, z_2, both collinear with y. Applying θ, the intersection $W_i = y^\theta \cap z_i^\theta$, $i = 1, 2$, is a 4-space, and we find two 3-spaces $W_1 \cap z_1^\perp$ and $W_2 \cap z_2^\perp$ of y^θ consisting of absolute points. These 3-spaces have at most one point in common, as z_1^θ and z_2^θ meet in a unique point. Hence, since the maximal singular subspaces of y^θ are 4-dimensional, we find two non-collinear absolute points x_1, x_2 in y^θ.

Now $(y^\theta)^\theta$ is the intersection of x_1^θ with x_2^θ, hence of $x_1^{\theta^{-1}}$ with $x_2^{\theta^{-1}}$, which equals the inverse image of the unique symp containing x_1 and x_2, which is y^θ. Hence this inverse image is y and the lemma is proved. □

Our next aim is to show that θ fixes at least one chamber of Δ.

Lemma 4.5 *If θ fixes a plane π, then it fixes every 5-space incident with π. Also, every point and every line of π is absolute for θ.*

Proof Suppose, by way of contradiction, that θ fixes the plane π and maps a 5-space V incident with π to a 5-space W incident with π, with $W \neq V$. We know from Fact 3.4 that $V \cap W = \pi$. So we may choose a point $x \in V \setminus W$. Denote $U = x^{\perp} \cap W$ and notice that $\pi \subseteq U$ and U is a 3-space, by Fact 3.7.

Now, since x is incident with V, the symp x^{θ} is incident with W, i.e., x^{θ} contains a $4'$-space ξ' completely included in W. Note that ξ' does not contain π, or equivalently, x^{θ} is not incident with π, as otherwise, by taking (inverse) images under θ, the point x would be incident with π, contradicting our choice of x. Hence ξ' and π meet in a line L, and we see that x is collinear to all points of L, which is contained in $x^{\perp}$. Assumption (N) implies that $x \in x^{\theta}$. But then, since x^{θ} is a polar space, x is collinear in x^{θ} with a 3-space of points contained in ξ', and that 3-space must necessarily coincide with U, by uniqueness of U. But now $\pi \subseteq U$, a contradiction.

If x is a point of the fixed plane π, then x^{θ} is incident with π, hence x is absolute. Similarly, every line incident with π is absolute.

The proof of the lemma is complete. $\square$

Lemma 4.6 *If L is an absolute line for θ, then θ fixes every plane and every 5-space incident with both L and L^{θ}. Also, every point on L is absolute for θ.*

Proof In view of Lemma 4.5 it suffices to show that, if L is an absolute line for θ, then every plane incident with both L and L^{θ} is fixed by θ.

Let π be such a plane. Assume, by way of contradiction, that π is not fixed. Then π^{θ} is another plane through L in L^{θ}. Let x be a point in π not on L, and hence not in π^{θ}. Since x belongs to π, the symp x^{θ} contains π^{θ}, and since all points of π^{θ} are obviously collinear with x, Assumption (N) implies that x is absolute. Hence the 3-space in L^{θ} generated by x and π^{θ} is contained in x^{θ}. Fact 3.6 now implies that L^{θ} is contained in x^{θ}, hence $x \in L$, a contradiction.

The proof of the lemma is complete. $\square$

Lemma 4.7 *There is at least one absolute point, at least one absolute line, at least one fixed plane, and at least one fixed 5-space for θ.*

Proof Assumption (A) implies that there is at least one absolute point x. Lemma 4.1 implies that there is a line L through x inside x^θ consisting of absolute points. Let $y \in L$, $y \neq x$. Since $y \in x^\theta$, we have $x \in y^\theta$. Hence $L \subseteq x^\theta \cap y^\theta = L^\theta$. Hence L is absolute. Now Lemma 4.6 completes the proof of the lemma. $\square$

Definition 4.8 Now we define the following geometry Δ_θ. It consists of four types of elements: the *points* are the absolute points for θ, the *lines* are the absolute lines for θ, the *planes* are the fixed planes under θ, and the *hyperlines* are the fixed 5-spaces under θ. Incidence is given by the incidence relation in Δ. Note that Δ_θ is the restriction to the types {point, line, plane, 5-spaces} of the simplicial subcomplex of Δ consisting of all fixed simplices under θ. As such, it is immediate that Δ_θ is a spherical building, possibly degenerate.

In order to show that Δ_θ is a thick buidling of type F_4, we need to show that every rank 2 residue of Δ_θ is thick and of the appropriate type.

Lemma 4.9 *The residue of any hyperline of Δ_θ is a rank 3 symplectic polar space.*

Proof Let V be any fixed 5-space. Then the mapping $V \to V : x \mapsto x^\theta \cap V$ defines a polarity of V. Assumption (N) implies that every point x of V is absolute, hence θ induces a symplectic polarity in V and the lemma is proved. $\square$

Now Lemma 4.5 implies that the residue of any plane of Δ_θ is the direct product of a projective plane and a projective line, and Lemma 4.6 implies the same for any line of Δ_θ. Hence θ is a symplectic polarity. We can now finish the proof of Main Result 2.1.

Suppose that θ is a duality in Δ which is not anisotropic. It suffices to show that θ contains a point x for which x^θ is not opposite x.

Since θ is not anisotropic, it maps some element A to a non-opposite one. If A is a point or a symp, then we are done. If A is a line or a plane, then A not opposite A^θ implies that some point $x \in A$ is collinear to some point y of A^θ (see Lemma 3.8). But A^θ belongs to x^θ, and so Lemma 3.8 implies that x is not opposite x^θ.

If A is a 5-space, then Lemma 3.8 and Lemma 3.7 imply that some point x of A is collinear with all points of a 3-space U of A^θ. Now $x \in A$ implies that x^θ and A^θ meet in a $4'$-space W. Now $U \cap W$ is nonempty and all points of it are collinear with x, implying that x is not opposite x^θ.

The proof of Main Result 2.1 is complete.

5 Proof of Main Result 2.2

In this section, we let θ be a domestic duality of a building Δ of spherical type E_6, i.e., we assume that

(D) no chamber is mapped to an opposite chamber.

We must show that θ is a symplectic polarity. Hence, by Main Result 2.1, it suffices to show that θ maps no point to a neighbouring symp (since θ is by Assumption (D) certainly not anisotropic). We first establish a sufficient condition for this to happen.

Lemma 5.1 *If θ is line-domestic, then it maps no point to a neighbouring symp.*

Proof Suppose that θ maps the point x to a neigbouring symp x^θ. By Fact 3.2, there is a unique 5-space S that contains x and which intersects x^θ in a $4'$-space V'. Then also S^θ intersects x^θ is a $4'$-space, say V''. In the polar space x^θ, there exists a 4-space V opposite both V' and V''. The inverse image $L := V^{\theta^{-1}}$ is a line that contains x, and that is not contained in S. We claim that L and V are opposite.

Indeed, suppose not. Then, by Fact 3.8 there is a point y on L that is collinear to some point z of V. Since $V \cap V' = \emptyset$, $y \neq x$. By Fact 3.2, y is collinear with all points of some $4'$-space of x^θ. Since two $4'$-spaces of the same symp always meet nontrivially, we obtain a point $z' \in V'$ collinear with y. By Fact 3.7, there is a 4-space W incident with both y and S. But then the image W^θ is a line contained in both V and V'', a contradiction to our choice of V.

The lemma is proved. □

So we need to show that θ is line-domestic. First we show that it is {point, line}-domestic.

Lemma 5.2 *The duality θ is {point,line}-domestic.*

Proof Suppose some flag $\{p, L\}$, with p a point and L a line, is mapped onto an opposite flag. The residue of $\{p, L\}$ is a 4-dimensional projective space, hence does not admit any domestic duality, see Theorem 3.1 in [10]. Consequently, by Proposition 3.9 and Corollary 3.10, there is a chamber C of that residue mapped onto an opposite chamber C^θ. Then the chamber $\{p, L\} \cup C$ of Δ is mapped onto an opposite chamber, contradicting Hypothesis (D).

The lemma is proved. □

We can now show that θ is line-domestic.

Lemma 5.3 *The duality θ is line-domestic.*

Proof Suppose that some line L is mapped onto an opposite 4-space L^θ. The same argument as in the proof of Lemma 5.2 implies that there is some symp Σ incident with L and mapped onto an opposite point Σ^θ. Now, since θ is {point, line}-domestic, by Proposition 3.9, $\sigma_{\Sigma,\theta}$ is also {point, line}-domestic. By Theorem 3.2 of [12], $\sigma_{\Sigma,\theta}$ is either point-domestic or line-domestic. Since we assumed that L is mapped onto an opposite 4-space, we deduce that $\sigma_{\Sigma,\theta}$ is point-domestic. Then Theorem 4.2 of [12] implies that $\sigma_{\Sigma,\theta}$ fixes a maximal singular subspace. But (the residue of) Σ is a polar space of type D_5, and hence the natural opposition relation induces a nontrivial involution on the types of the maximal singular subspaces, so that no maximal singular subspace can be fixed. This contradiction completes the proof of the lemma. $\square$

Using similar ideas, it is rather easy to show that θ is also plane-domestic and 5-space-domestic (and, of course, 4-space-domestic). But since we already attained our goal, we do not insist on this.

Now Main Result 2.2 follows from Main Result 2.1.

6 Some consequences

Here we start by showing Corollary 2.4. So suppose that θ is an anisotropic duality of a finite buildings Δ of type E_6. Let π be any plane. Then, by anisotropy, π^θ is opposite π. By finiteness, the duality of π induced by $\sigma_{\pi,\theta}$ has at least one absolute point, contradicting the fact that θ is anisotropic.

More generally, this argument can be used to show that no automorphism of any finite thick irreducible building of rank at least 3 can be anisotropic. For rank 2, there are counterexamples in the case of finite generalized quadrangles (with non-classical parameters, see [11]).

Anyway, this proves Corollary 2.4.

Next we prove a result that is very useful for trying to prove geometrically that every building of type F_4 with symplectic residues is contained in a building of type E_6 as a geometric hyperplane.

Proposition 6.1 *Let Δ be a building of type E_6 and let θ be a symplectic polarity of Δ. Let Δ' be the building of type F_4 consisting of the absolute points, absolute lines, fixed* *planes and fixed* 5*-spaces for* θ*. Then a line* L *of* Δ *containing at least two points of* Δ' *is entirely contained in* Δ' *and either* L *is an absolute line, or* L *is a hyperbolic line in some fixed* 5*-space* V *(hyperbolic with respect to a symplectic polarity in* V *induced by* θ *by relating a point* $x \in V$ *to the intersection* $V \cap X^\theta$*).*

Proof Choose two absolute points p, q on L, and suppose that L is not contained in p^θ, nor in q^θ (otherwise L is absolute by Lemma 4.1 and the proof of Lemma 4.7). Now, p neighbours q^θ and q neighbours p^θ. hence there are unique 5-spaces V, V' incident with p and q^θ, and with q and p^θ, respectively. Clearly, V and V' are mapped onto each other. Since $V \cap V'$ contains L, Fact 3.4 implies that either $V = V'$ or $V \cap V'$ is a plane, which is fixed by θ. But in the latter case, Lemma 4.5 implies that both V and V' are fixed. We conclude that $V = V'$ in any case. Hence L is contained in a 5-space fixed by θ and the result now follows from Lemma 4.9. $\square$

Our last result implies a converse of Lemma 4.9.

Proposition 6.2 *Let* Δ *be a building of type* E_6 *and let* θ *be a symplectic polarity of* Δ*. Let* V *be a* 5*-space of* Δ*. Then every point of* V *is absolute for* θ *if and only if* V *is fixed under* θ*.*

Proof If V is fixed, then this follows from (the proof of) Lemma 4.9. Now suppose every point of V is absolute. If V is disjoint from V^θ, then by Fact 3.5, either V is opposite V^θ or there is a unique 5-space W adjacent with both V or V^θ. In the latter case W is fixed, and θ induces a symplectic polarity in W. But a symplectic polarity does not map any plane to an opposite one, contradicting $(V \cap W)^\theta = v^\theta \cap W$. The former case is impossible for symplectic polarities, as the polarity induced in V using Corollary 3.10 maps some line onto an opposite one, contradicting Lemma 5.3.

Now assume V and V^θ intersect in a plane. Then this plane is fixed under θ, and hence both V and V^θ are fixed by Lemma 4.5, a contradiction.

The last possibility is, in view of Fact 3.4, that V and V^θ intersect in a point x. Then V and V^θ are incident with a unique common symp x^θ, which contains x. The $4'$-spaces U and U' of x^θ, incident with V and V^θ, respectively, meet in x. It is easy to see that all points of $U \cup U'$ are absolute. We now claim that any point y of $V \setminus U$ is not absolute. Indeed, suppose by way of contradiction that y is absolute. Since y^θ is incident with V^θ, there is a $4'$-space A in both $V\theta$ and y^θ. Since y is not in x^θ, the $4'$-space is not incident with x. Now, y is collinear with all points of a 3-space of A, and also collinear with x. Hence y is collinear with all points of a hyperplane of V^θ, contradicting Fact 3.7.

This completes the proof of the proposition. $\square$

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