View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Ghent University Academic Bibliography

Expansion Formulas for Terminating Balanced ${}_4F_3$-series from the Biedenharn-Elliot Identity for $\mathfrak{su}(1,1)$

S. Lievens

Department of Applied Mathematics and Computer Science,
University of Ghent, Krijgslaan 281-S9, B-9000 Gent, Belgium.
E-mail: Stijn.Lievens@rug.ac.be

Abstract

In a recent paper [7] George Gasper proved some expansion formulas for terminating balanced hypergeometric series of type ${}_4F_3$ with unit argument. In this article we show how one easily derives such expansion formulas from the Biedenharn-Elliot identity for the Lie algebra $\mathfrak{su}(1,1)$. Furthermore, we give a rather systematic method for determining when two apparently different expansion formulas are the same up to transformation formulas. This is a rather nice application of the so-called invariance groups of hypergeometric series. The method extends to other cases; we briefly indicate how it works in the case of expansion formulas for ${}_3F_2$-series. We conclude with some basic analogues and show their relation with the Askey-Wilson polynomials.

Keywords: Hypergeometric series; Biedenharn-Elliot identity; Expansion formula; Invariance group

1 Introduction

The Biedenharn-Elliot identity is well known in the quantum theory of angular momentum, i.e. in the context of the Lie algebra $\mathfrak{su}(2)$ [4–6]. This identity however is also valid and essentially the same for the Lie algebra $\mathfrak{su}(1,1)$; see [5, pp. 456–460] for a way to derive it. In fact, because of the different nature of the representation labels, it is, for mathematical applications (not necessarily related to the Biedenharn-Elliot identity), sometimes more convenient to work with $\mathfrak{su}(1,1)$ rather than with $\mathfrak{su}(2)$, as is for instance pointed out in [11, 16].

The Biedenharn-Elliot identity has some (mathematical) applications; it can be used to derive recurrences for $6j$-symbols (that have an interpretation in terms of Racah polynomials) and this leads to the three-term recurrence for Racah polynomials [14, 18]. In [11] it was used to derive a convolution identity for Wilson polynomials.

In the case of $\mathfrak{su}(1,1)$, the Biedenharn-Elliot identity is given by:

$$U^{k_{12},k_3,k_{13}}_{k_4,k_{14},k_{34}} U^{k_1,k_2,k_{12}}_{k_{34},k_{14},k_{24}} = \sum_{k_{23}} U^{k_1,k_2,k_{12}}_{k_3,k_{13},k_{23}} U^{k_1,k_{23},k_{13}}_{k_4,k_{14},k_{24}} U^{k_2,k_3,k_{23}}_{k_4,k_{24},k_{34}}, \tag{1}$$

where k_{23} is restricted to the range $k_2 + k_3$ up to $\min(k_{13} - k_1, k_{24} - k_4)$. The parameters k are the so-called representation labels of $\mathfrak{su}(1,1)$; they are positive real numbers, labelling the positive discrete series representations of $\mathfrak{su}(1,1)$ [14, Theorem 1.2]. Briefly, a Racah coefficient $U^{k_1,k_2,k_{12}}_{k_3,k,k_{23}}$ determines the basis transition between two so-called coupled bases. These coupled bases arise by decomposing the tensor product of three irreducible representations (k_1), (k_2) and (k_3) in two different ways. In case we work with positive discrete series representations all labels are positive real numbers and one has that the four differences

$$k_{12} - k_1 - k_2, \quad k_{23} - k_2 - k_3, \quad k - k_{12} - k_3, \quad \text{and} \quad k - k_1 - k_{23}$$

are nonnegative integers [14, 18]. This also determines the summation range in (1).

An explicit expression for the related recoupling or Racah coefficients of $\mathfrak{su}(1,1)$ is known [10,15]:

$$U^{k_1,k_2,k_{12}}_{k_3,k,k_{23}} = C\, {}_4F_3\left({k_1 + k_2 + k_{12} - 1, k_2 + k_3 + k_{23} - 1, k_1 + k_2 - k_{12}, k_2 + k_3 - k_{23} \atop 2k_2, k_1 + k_2 + k_3 + k - 1, k_1 + k_2 + k_3 - k} ; 1 \right), \tag{2}$$

where C is a numerical constant that is given by:

$$
\begin{aligned}
C = \big(&(2k_2)_{k_{12}-k_1-k_2}(2k_3)_{k-k_{12}-k_3}(2k_2)_{k_{23}-k_2-k_3}(2k_1)_{k-k_1-k_{23}} \\
&\times (k_1+k_{23}+k-1)_{k-k_1-k_{23}}\big)^{1/2}\big((2k_1)_{k_{12}-k_1-k_2}(k_1+k_2+k_{12}-1)_{k_{12}-k_1-k_2}(2k_{12})_{k-k_{12}-k_3} \\
&\times (k_{12}+k_3+k-1)_{k-k_{12}-k_3}(2k_3)_{k_{23}-k_2-k_3}(2k_{23})_{k-k_1-k_{23}}(k_2+k_3+k_{23}-1)_{k_{23}-k_2-k_3} \\
&\times (k_{12}-k_1-k_2)!(k-k_{12}-k_3)!(k-k_1-k_{23})!(k_{23}-k_2-k_3)!\big)^{-1/2} \\
&\times (k+k_1+k_2+k_3-1)_{k_{23}-k_2-k_3}(k-k_1-k_2-k_3)!.
\end{aligned}
\tag{3}
$$

In (2) and (3) we have used the standard notation for (generalized) hypergeometric series and Pochhammer symbols [2, 13]. We draw attention to the fact that the ${}_4F_3$-series in (2) is terminating because the last two numerator parameters ($k_1+k_2-k_{12}$ and $k_2+k_3-k_{23}$) are negative integers. Although the last denominator parameter, $k_1+k_2+k_3-k$, is also a negative integer, the series is still well defined because this last parameter is smaller than the numerator parameter responsible for the termination. These facts follow from the representation theory of $\mathfrak{su}(1,1)$ [12]. Furthermore, the ${}_4F_3$-series is balanced or Saalschützian, meaning that the sum of the numerator parameters plus one equals the sum of the denominator parameters [1, Chapter 2].

For such balanced series an extensive transformation theory exists [1, 8]. In particular one has the following transformation (limit $q \uparrow 1$ in Sears' transformation [8, III.16]):

$$
\begin{aligned}
{}_4F_3\left({-n, a, b, c \atop d, e, f}; 1\right) = &\frac{(a)_n(e+f-a-b)_n(e+f-a-c)_n}{(e)_n(f)_n(e+f-a-b-c)_n} \\
&\times {}_4F_3\left({-n, e-a, f-a, e+f-a-b-c \atop e+f-a-b, e+f-a-c, 1-n-a}; 1\right),
\end{aligned}
\tag{4}
$$

provided $d+e+f = 1-n+a+b+c$.

In [7, Formula 2.9] Gasper gave the following expansion formula, which involves three of these Saalschützian series:

$$
\begin{aligned}
&{}_4F_3\left({\alpha, \beta, n+\nu, -n \atop \gamma, \delta, \epsilon}; 1\right) = \frac{(\gamma-\mu)_n(1+\nu-\gamma)_n}{(\gamma)_n(1+\nu+\mu-\gamma)_n} \\
&\quad\times \sum_{l=0}^{n} \frac{(\nu+\mu-\gamma)_l}{l!} \frac{(\mu)_l(\lambda+\delta-\alpha-\beta)_l(\lambda+\epsilon-\alpha-\beta)_l(n+\nu)_l(-n)_l(1+\nu+\mu-\gamma)_{2l}}{(1+\nu-\gamma)_l(\epsilon)_l(\delta)_l(1+\mu-\gamma-n)_l(n+1+\nu+\mu-\gamma)_l(\nu+\mu-\gamma)_{2l}} \\
&\quad\times {}_4F_3\left({\lambda-\alpha, \lambda-\beta, l+\nu+\mu-\gamma, -l \atop \mu, \lambda+\delta-\alpha-\beta, \lambda+\epsilon-\alpha-\beta}; 1\right) {}_4F_3\left({\alpha+\beta-\lambda, \lambda-\mu, n+l+\nu, l-n \atop \gamma-\mu, l+\delta, l+\epsilon}; 1\right),
\end{aligned}
\tag{5}
$$

when $\alpha+\beta+\nu+1 = \gamma+\delta+\epsilon$. More precisely, Gasper derived the q-analogue of (5) and some similar expansions. These expansions lead to q-analogues of Erdélyi's fractional integral representations of hypergeometric functions and extensions of them.

In Section 2 we show how to derive identities such as (5) in a straightforward way from the Biedenharn-Elliot identity. A problem then arises: how to recognize which identities are "essentially different" (this notion is made explicit in the beginning of Section 3). We give a possible way of handling this using a simple realization of the invariance group of (4) (and trivial permutations of numerator and denominator parameters); this is done in Section 3. In the last Section, we give a q-analogue of a (which we believe to be) new expansion formula and we also relate these expansion formulas to the Askey-Wilson polynomials rewriting them as connection coefficient or convolution identities.

2 Expansion formulas from the Biedenharn-Elliot identity

The expansion formula (5) has one ${}_4F_3$-series on the left side and two on the right side. The Biedenharn-Elliot identity on the other hand has two ${}_4F_3$-series on the left side and three on the right side. When we are able, by imposing a particular constraint on the parameters, to sum one ${}_4F_3$-series (hence reducing it to a simple numerical factor) on both the left and right side of (1), we will end up with a formula analogous to (5). Note that since we are only dealing with *terminating* series questions of convergence do not arise; thus when we say that one can "sum" a series, we mean that one can write it in closed form, i.e. without a summation sign.

The easiest way to ensure that a hypergeometric series is summable is to set one of its numerator parameters equal to zero. Consider the recoupling coefficients on the left side of (1):

$$U^{k_{12},k_3,k_{13}}_{k_4,k_{14},k_{34}} \to {}_4F_3\left(\begin{matrix} k_{12}+k_3+k_{13}-1, k_3+k_4+k_{34}-1, k_{12}+k_3-k_{13}, k_3+k_4-k_{34} \\ 2k_3, k_{12}+k_3+k_4+k_{14}-1, k_{12}+k_3+k_4-k_{14} \end{matrix}; 1\right) \quad (6)$$

$$U^{k_1,k_2,k_{12}}_{k_{34},k_{14},k_{24}} \to {}_4F_3\left(\begin{matrix} k_1+k_2+k_{12}-1, k_2+k_{34}+k_{24}-1, k_1+k_2-k_{12}, k_2+k_{34}-k_{24} \\ 2k_2, k_1+k_2+k_{34}+k_{14}-1, k_1+k_2+k_{34}-k_{14} \end{matrix}; 1\right). \quad (7)$$

We thus see that the constraints

$$k_{12} = k_1 + k_2 \qquad k_{34} = k_3 + k_4 \quad (8a)$$

$$k_{13} = k_{12} + k_3 \qquad k_{24} = k_{34} + k_2 \quad (8b)$$

immediately lead to a summable hypergeometric series on the left side. Two of these constraints (the ones on the first line) also lead to a numerator parameter zero in one of the hypergeometric series on the right side of (1). The other two constraints ensure that one of the numerator parameters equals a denominator parameter, reducing one ${}_4F_3$-series to a ${}_3F_2$-series with unit argument, that is summable because it is Saalschützian [13, Formula III.2].

Consider the constraint $k_{12} = k_1 + k_2$. We perform a shift on the summation variable and set $k_{23} = k_2 + k_3 + l$. Furthermore we set $k_{13} = n + k_3 + k_{12}$, with both l and n nonnegative integers. Finally, we assume that $k_{13} - k_1 \leq k_{24} - k_4$. The (remaining) variables are renamed in terms of α, β, $\ldots$, as follows:

$$
\begin{array}{lll}
k_1 = (\nu - \mu)/2 & k_2 = (1 + \mu - \gamma)/2 & k_3 = \gamma/2 \\
k_4 = (1 + \alpha + \beta - \gamma)/2 & k_{12} = (1 + \nu - \gamma)/2 & k_{13} = (1 + 2n + \nu)/2 \\
k_{14} = (\gamma + 2\delta - \alpha - \beta - \nu)/2 & k_{23} = (1 + 2l + \mu)/2 & k_{24} = (\gamma + 2\lambda - \alpha - \beta - \mu)/2 \\
k_{34} = (1 + \alpha - \beta)/2. & &
\end{array}
$$

The above substitution is the solution of a linear system of equations that ensures that the non-vanishing ${}_4F_3$ on the left side of (1) coincides with the left side of (5) and that introduces the variables μ and λ in a simple way on the right side.

A priori one has that certain linear combinations of α, β, $\ldots$ are nonnegative integers. E.g. in the process of the simplification of the numerical factors one encounters the following:

$$
\frac{(\gamma + \lambda - \alpha - \beta - \mu - 1)!}{(\gamma + \lambda - \alpha - \beta - \mu - 1 - l)!} = (-1)^l (1 + \alpha + \beta + \mu - \gamma - \lambda)_l, \tag{9}
$$

and

$$
\frac{(\delta + l)_{\gamma+\lambda-\alpha-\beta-\mu-1-l}}{(\delta)_{-\beta}(\delta - \beta)_{\gamma+\lambda-\alpha-\mu-1}} = \frac{(\delta + l)_{\gamma+\lambda-\alpha-\beta-\mu-1-l}}{(\delta)_{\gamma+\lambda-\alpha-\beta-\mu-1}} = \frac{1}{(\delta)_l}. \tag{10}
$$

Simplifying the numerical factors (3) in this way, the square roots also disappear and one arrives at the following expansion formula:

$$
\begin{aligned}
&{}_4F_3\left({\alpha, \beta, n + \nu, -n \atop \gamma, \delta, \epsilon}; 1\right) = \sum_{l=0}^{n} (-1)^l \frac{(-n)_l}{l!} \frac{(\lambda)_l (1 + \alpha + \beta + \mu - \gamma - \lambda)_l (n + \nu)_l (\mu)_l}{(\delta)_l (\epsilon)_l (\mu)_{2l}} \\
&\times {}_4F_3\left({\alpha, \beta, l + \mu, -l \atop \gamma, \lambda, 1 + \alpha + \beta + \mu - \gamma - \lambda}; 1\right) {}_4F_3\left({l + n + \nu, l + \lambda, 1 + l + \alpha + \beta + \mu - \gamma - \lambda, l - n \atop 1 + 2l + \mu, l + \delta, l + \epsilon}; 1\right),
\end{aligned} \tag{11}
$$

with $\alpha + \beta + \nu + 1 = \gamma + \delta + \epsilon$. In this formula n is a nonnegative integer and all other parameters are arbitrary real numbers because we are dealing with a rational identity in these parameters.

In an analogous way, considering the constraint $k_{13} = k_{12} + k_3$, setting $k_{23} = k_2 + k_3 + l$ and

$k_{12} = n + k_1 + k_2$, we arrive at the expansion formula:

$$
\begin{aligned}
{}_4F_3\left(\begin{matrix}\alpha,\beta,n+\nu,-n\\ \gamma,\delta,\epsilon\end{matrix};1\right) &= \frac{(\gamma+\delta+\lambda-\alpha-\beta)_n(1+\nu+\lambda-\delta)_n}{(\delta)_n(\epsilon)_n}\\
&\times \sum_{l=0}^{n}(-1)^l\frac{(-n)_l}{l!}\frac{(\gamma+\lambda-\alpha)_l(\gamma+\lambda-\beta)_l(1+\mu-\gamma)_l(\gamma-n-\nu)_l(\mu)_l}{(\gamma)_l(\gamma+\delta+\lambda-\alpha-\beta)_l(1+\nu+\lambda-\delta)_l(\mu)_{2l}}\\
&\times {}_4F_3\left(\begin{matrix}\lambda,\gamma+\lambda-\alpha-\beta,l+\mu,-l\\ 1+\mu-\gamma,\gamma+\lambda-\beta,\gamma+\lambda-\alpha\end{matrix};1\right)\\
&\times {}_4F_3\left(\begin{matrix}l+\gamma+\lambda-\alpha,l+\gamma+\lambda-\beta,1+l+n+\nu+\mu-\gamma,l-n\\ 1+2l+\mu,l+\gamma+\delta+\lambda-\alpha-\beta,1+l+\nu+\lambda-\delta\end{matrix};1\right),
\end{aligned}
\tag{12}
$$

with, again, $\alpha+\beta+\nu+1=\gamma+\delta+\epsilon$.

The remaining two constraints lead (by a careful choice of the performed substitutions) to the same expansion formula as the constraint on the corresponding line of (8).

3 Comparison of expansion formulas using invariance groups

In the previous section, we derived two expansion formulas (11) and (12). Together with (5) we now have three of these formulas. But is this *really* the case? We mean the following: can one of these formulas be obtained from another by a renaming of the variables and (repeated) application of (4) together with permutations of the numerator and denominator parameters? Doing this by “trial and error” is not an easy task due to the many possibilities one has to consider. We describe here a (more or less) systematic method based on the knowledge of the invariance group of (4).

It is well known that transformations of hypergeometric series generate so-called invariance groups; for various transformations these invariance groups have been studied [17]. In [3] the invariance group of (4) is determined to be S_6, the permutation group on six elements. Explicitly: the function

$$
\begin{aligned}
&(x_1+x_2+x_3+x_4)_n(x_1+x_2+x_3+x_5)_n(x_1+x_2+x_3+x_6)_n\\
&\quad\times {}_4F_3\left(\begin{matrix}x_1+x_2,x_1+x_3,x_2+x_3,-n\\ x_1+x_2+x_3+x_4,x_1+x_2+x_3+x_5,x_1+x_2+x_3+x_6\end{matrix};1\right),
\end{aligned}
$$

with $x_1+x_2+x_3+x_4+x_5+x_6=1-n$ is symmetric in the variables $x_1,\ldots,x_6$.

We introduce twelve independent variables x_i and y_i with $i=1,\ldots,6$; the variables x_i (resp. y_i) correspond to the first (resp. second) hypergeometric series on the right side of the expansion formulas. Stated otherwise, we compare

$$
{}_4F_3\left(\begin{matrix} x_1+x_2, x_1+x_3, x_2+x_3, -l \\ x_1+x_2+x_3+x_4, x_1+x_2+x_3+x_5, x_1+x_2+x_3+x_6 \end{matrix}; 1\right)
\times {}_4F_3\left(\begin{matrix} y_1+y_2, y_1+y_3, y_2+y_3, l-n \\ y_1+y_2+y_3+y_4, y_1+y_2+y_3+y_5, y_1+y_2+y_3+y_6 \end{matrix}; 1\right)
$$

with the right side of (5) ((11) or (12)). This yields twelve independent linear equations between the nine free parameters on the right side of (5) (α, β, ν, γ, δ, μ, λ, n and the summation variable l) and the twelve variables x_i, y_i. This system is overdetermined and hence has no solutions for the parameters α, β, …, unless the twelve variables x_i, y_i satisfy three linear constraints; these are found be eliminating the nine free parameters from the twelve linear equations.

Doing this for the three formulas (5), (11) and (12) yields the following:

$$
\begin{cases} x_1-x_4+y_2-y_3=0 \\ x_4+x_6+y_3+y_5=1 \\ x_4+x_5+y_3+y_6=1 \end{cases}, \qquad
\begin{cases} x_1-x_4+y_4-y_3=0 \\ x_4+x_6+y_1+y_3=1 \\ x_4+x_5+y_2+y_3=1 \end{cases} \quad \text{and} \quad
\begin{cases} x_1-x_4+y_4-y_1=0 \\ x_4+x_6+y_1+y_3=1 \\ x_4+x_5+y_1+y_2=1 \end{cases} \tag{13}
$$

These constraints are of the same form, yet different and not equivalent. It is thus not possible to solve the parameters of one formula directly in terms of the parameters of another formula.

We remark that in the computation of the constraints (13), we have *fixed* the order of the numerator and denominator parameters as in the formulas (5) ((11) and (12)). We hold on to this fixed order for the rest of this section.

However, the permutation (in cycle notation) $g=(1,3,5)(2,6,4)$, when applied to the variables y_i, carries the constraints of (12) into those of (5). If we thus want the constraints of (12) to equal those of (5), we have to perform the transformation corresponding to the *inverse* permutation $g^{-1}=(1,5,3)(2,4,6)$ on the second ${}_4F_3$-series of (12). We have:

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} l+\gamma+\lambda-\alpha, l+\gamma+\lambda-\beta, 1+l+n+\nu+\mu-\gamma, l-n \\ 1+2l+\mu, l+\gamma+\delta+\lambda-\alpha-\beta, 1+l+\nu+\lambda-\delta \end{matrix}; 1\right) = \\
&\frac{(\gamma-2n-\nu-\mu)_{n-l}(l-n+\gamma+\delta-\beta-\nu)_{n-l}(l-n+\gamma+\delta-\alpha-\nu)_{n-l}}{(1+2l+\mu)_{n-l}(l+\gamma+\delta+\lambda-\alpha-\beta)_{n-l}(1+l+\nu+\lambda-\delta)_{n-l}} \\
&\times {}_4F_3\left(\begin{matrix} \delta-n-\nu-\lambda, 2\gamma+\delta+\lambda-1-n-\alpha-\beta-\nu-\mu, l-n+\gamma-\nu, l-n \\ \gamma-2n-\nu-\mu, l-n+\gamma+\delta-\beta-\nu, l-n+\gamma+\delta-\alpha-\nu \end{matrix}; 1\right).
\end{aligned} \tag{14}
$$

Now, we prime the parameters of (5) and solve the parameters of the transformed formula (12) (using (14)) in terms of those primed parameters. We have the following system of linear equations:

$$\begin{cases} \lambda' - \alpha' = \lambda & \alpha' + \beta' - \lambda' = \delta - n - \nu - \lambda \\ \lambda' - \beta' = \gamma + \lambda - \alpha - \beta & \lambda' - \mu' = 2\gamma + \delta + \lambda - 1 - n - \alpha - \beta - \nu - \mu \\ l + \nu' + \mu' - \gamma' = l + \mu & n + l + \nu' = l - n + \gamma - \nu \\ \mu' = 1 + \mu - \gamma & \gamma' - \mu' = \gamma - 2n - \nu - \mu \\ \lambda' + \delta' - \alpha' - \beta' = \gamma + \lambda - \beta & l + \delta' = l - n + \gamma + \delta - \beta - \nu \\ \lambda' + \epsilon' - \alpha' - \beta' = \gamma + \lambda - \alpha & l + \epsilon' = l - n + \gamma + \delta - \alpha - \nu \end{cases} \tag{15}$$

Solving this system for the non-primed parameters, we find the unique solution:

$$\begin{cases} \alpha = \delta' - \alpha' & \delta = 1 - n + \beta' - \gamma' \\ \beta = 1 + \beta' + \nu' - \gamma' - \delta' & \epsilon = 1 - n - \alpha' \\ \nu = 1 - 2n - \gamma' & \mu = \nu' + \mu' - \gamma' \\ \gamma = 1 + \nu - \gamma' & \lambda = \lambda' - \alpha' \end{cases} \tag{16}$$

Substituting this in the transformed formula (12) gives on the right side the same hypergeometric series as in (5) and on the left side (omitting the primes):

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} \delta - \alpha, 1 + \beta + \nu - \gamma - \delta, 1 - n - \gamma, -n \\ 1 + \nu - \gamma, 1 - n + \beta - \gamma, 1 - n - \alpha \end{matrix} ; 1 \right) \\ &= \frac{(\gamma)_n (\delta)_n (\epsilon)_n}{(1 + \nu - \gamma)_n (\gamma - \beta)_n (\alpha)_n} {}_4F_3\left(\begin{matrix} \alpha, \beta, n + \nu, -n \\ \gamma, \delta, \epsilon \end{matrix} ; 1 \right). \end{aligned} \tag{17}$$

After substitution of (16) in and simplification of the numerical factors in (12), (14) and (17) we have deduced (5) from (12); these two are thus not really different. Note that, once the transformation (14) and the substitution (16) are known, the verification that (12) yields (5) is almost trivial. The method using the invariance group shows us these in a relatively easy way; indeed, finding the permutation g is much easier than finding (14) directly.

Next, we compare (11) and (5) in the same way. Performing the permutation $g = (1,5)(2,6,4)$ on the variables y_i takes the constraints for (11) into those of (5). Perform the transformation corresponding to the inverse permutation g^{-1} on the second ${}_4F_3$ of (11); prime the parameters of (5); solve the system of equations and you get the following unique solution:

$$\begin{cases} \alpha = \lambda' - \alpha' & \delta = 1 - n + \lambda' - \gamma' \\ \beta = \lambda' - \beta' & \epsilon = 1 - n + \lambda' - \alpha' - \beta' \\ \nu = 1 - 2n + \mu' - \gamma' & \mu = \nu' + \mu' - \gamma' \\ \gamma = \mu' & \lambda = \delta' + \lambda' - \alpha' - \beta' \end{cases} \tag{18}$$

At first sight it looks like we have a similar result as before. There is however one major difference: the solution for α, β, …, ϵ also contains references to μ' and λ' so that it is impossible to transform the new left side of (11) into that of (5). Of course, there are other permutations that carry the constraints of (11) into a set equivalent to those of (5), but not too many. All these permutations suffer from the same problem as the permutation g, so in this way we cannot rewrite (11) as (5).

In the process of trying to transform (11) into (5) one could also reverse the order of summation on the right side of (11), i.e. replace l by $n-l$. This changes the role of the variables x and y in the constraint. Here, again, there are some permutations that carry this constraint into that of (5), but again, none of them succeeds in rewriting (11) as (5), so therefore the expansion formulas (5) and (11) are essentially different.

When browsing through [7] one sees that there is one other formula in the article that *a priori* could be equivalent to (5) or (11), namely [7, Formula 3.4]. We repeat it here:

$$\begin{aligned}
&{}_4F_3\left(\begin{matrix}\alpha,\beta,n+\nu,-n\\ \gamma,\delta,\epsilon\end{matrix};1\right)=\frac{(\gamma-\lambda)_n(1+\nu-\gamma)_n}{(\gamma)_n(1+\nu+\lambda-\gamma)_n}\\
&\times\sum_{l=0}^{n}\frac{(\nu+\lambda-\gamma)_l(\lambda)_l(\epsilon-\alpha)_l(\delta-\mu)_l(n+\nu)_l(-n)_l(1+\nu+\lambda-\gamma)_{2l}}{l!(1+\nu-\gamma)_l(\delta)_l(\mu+\epsilon-\alpha)_l(1+\lambda-\gamma-n)_l(1+n+\nu+\lambda-\gamma)_l(\nu+\lambda-\gamma)_{2l}}\\
&\times{}_4F_3\left(\begin{matrix}\alpha-\mu,\beta,l+\nu+\lambda-\gamma,-l\\ \lambda,\delta-\mu,\epsilon\end{matrix};1\right){}_4F_3\left(\begin{matrix}\mu,\beta-\lambda,n+l+\nu,l-n\\ \gamma-\lambda,l+\delta,l+\mu+\epsilon-\alpha\end{matrix};1\right),
\end{aligned}\tag{19}$$

with, as usual, $\alpha+\beta+\nu+1=\gamma+\delta+\epsilon$. Calculating the three linear constraints for this expansion formula one gets

$$\begin{cases} x_2+x_4+y_1+y_3=0\\ x_4+x_6+y_3+y_5=1\\ x_1+x_5+y_2+y_6=1,\end{cases}\tag{20}$$

which is clearly of a different form than the constraints we have previously met. Therefore we conclude that (19) is essentially different from both (5) and (11).

Limit transitions from these expansion formulas for ${}_4F_3$-series give expansion formulas for ${}_3F_2$-series. E.g. letting $\nu\to\infty$ (and hence also $\epsilon\to\infty$) in (5) yields the following formula (which is [7, Formula 1.6]):

$$\begin{aligned}
{}_3F_2\left(\begin{matrix}\alpha,\beta,-n\\ \gamma,\delta\end{matrix};1\right)&=\sum_{l=0}^{n}\binom{n}{l}\frac{(\mu)_l(\lambda+\delta-\alpha-\beta)_l(\gamma-\mu)_{n-l}}{(\gamma)_n(\delta)_l}\\
\times{}_3F_2&\left(\begin{matrix}\lambda-\alpha,\lambda-\beta,-l\\ \mu,\lambda+\delta-\alpha-\beta\end{matrix};1\right){}_3F_2\left(\begin{matrix}\alpha+\beta-\lambda,\lambda-\mu,l-n\\ \gamma-\mu,l+\delta\end{matrix};1\right).
\end{aligned}\tag{21}$$

Different-looking formulas of this sort can be obtained by taking different limits. Also here, since there exists 72 transformations on terminating ${}_3F_2$-series with unit argument, it may be difficult to see which formulas are really different. The knowledge of transformation invariance groups is in this case also helpful. One knows [17] that the function

$$(x_1+x_2+x_3+x_4)_n(x_1+x_2+x_4+x_5)_n\,{}_3F_2\left(\begin{matrix}x_1+x_2,x_1+x_4,-n\\ x_1+x_2+x_3+x_4,x_1+x_2+x_4+x_5\end{matrix};1\right), \tag{22}$$

with $1-n=x_1+x_2+x_3+x_4+x_5+x_6$ is invariant (up to a sign factor) under the subgroup G of S_6 generated by $(2,4)$ and $(1,2,3,4,5,6)$.

Introducing again twelve variables x_i and y_i we will now end up, since we have ten linear equations and eight free parameters, with two linear constraints between these twelve variables. We can then proceed as in the ${}_4F_3$-case; we only have to pay attention to the fact that the permutation must belong to the group G.

Letting $\beta\to\infty$ (and hence also $\epsilon\to\infty$) in (11) yields, after rewriting ν as $\beta-n$, the following expansion formula for ${}_3F_2$-series:

$$\begin{aligned}{}_3F_2\left(\begin{matrix}\alpha,\beta,-n\\ \gamma,\delta\end{matrix};1\right)&=\sum_{l=0}^{n}\binom{n}{l}\frac{(\lambda)_l(\beta)_l(\mu)_l}{(\delta)_l(\mu)_{2l}}\\ &\times{}_3F_2\left(\begin{matrix}\alpha,l+\mu,-l\\ \gamma,\lambda\end{matrix};1\right){}_3F_2\left(\begin{matrix}l+\beta,l+\lambda,l-n\\ 1+2l+\mu,l+\delta\end{matrix};1\right).\end{aligned} \tag{23}$$

Using the method of invariance groups, it is easy to see that this formula is essentially different from (21) since the constraints for (21) and (23) are respectively given by

$$\begin{cases}x_3+x_6+y_4+y_5=1\\ x_1+x_6+y_2+y_5=1\end{cases}\quad\text{and}\quad\begin{cases}x_3+x_6+y_1+y_4=1\\ x_2+x_5+y_2+y_3=1\end{cases} \tag{24}$$

4 q-analogues and interpretation in terms of Askey-Wilson polynomials

In [7] a q-analogue of (5) is given:

$$\begin{aligned}{}_4\varphi_3\left(\begin{matrix}\alpha,\beta,q^n\nu,q^{-n}\\ \gamma,\delta,\epsilon\end{matrix};q,q\right)&=\frac{(\gamma/\mu,q\nu/\gamma;q)_n}{(\gamma,\nu\mu q/\gamma;q)_n}\mu^n\\ &\times\sum_{l=0}^{n}\frac{(q^{-n},\nu\mu/\gamma,\mu,\lambda\delta/\alpha\beta,\lambda\epsilon/\alpha\beta,\nu q^n;q)_l(\nu\mu q/\gamma;q)_{2l}}{(q,\nu q/\gamma,\epsilon,\delta,\mu q^{1-n}/\gamma,\nu\mu q^{n+1}/\gamma;q)_l(\nu\mu/\gamma;q)_{2l}}\left(\frac{\epsilon\delta}{\lambda\nu}\right)^l\\ &\times{}_4\varphi_3\left(\begin{matrix}\lambda/\alpha,\lambda/\beta,q^l\nu\mu/\gamma,q^{-l}\\ \mu,\lambda\delta/\alpha\beta,\lambda\epsilon/\alpha\beta\end{matrix};q,q\right){}_4\varphi_3\left(\begin{matrix}\alpha\beta/\lambda,\lambda/\mu,q^{n+l}\nu,q^{l-n}\\ \gamma/\mu,q^l\delta,q^l\epsilon\end{matrix};q,q\right),\end{aligned} \tag{25}$$

with $\gamma\delta\epsilon=q\alpha\beta\nu$. In (25) we used the standard notation of Gasper and Rahman [8] for basic hypergeometric series and q-shifted factorials.

The Biedenharn-Elliot identity is also valid in the case $U_q(\mathfrak{su}(1,1))$; the Racah coefficients are in this case given by terminating balanced ${}_4\varphi_3$-series [10, Proposition 4.9]. The same method as before can be used to find a q-analogue of (11):

$$\begin{aligned}
{}_4\varphi_3\left(\begin{matrix}\alpha,\beta,q^n\nu,q^{-n}\\ \gamma,\delta,\epsilon\end{matrix};q,q\right) &= \sum_{l=0}^{n}(-1)^l q^{(l+1)l/2}\frac{(q^{-n};q)_l}{(q;q)_l}\frac{(\lambda,q\alpha\beta\mu/\gamma\lambda,q^n\nu,\mu;q)_l}{(\delta,\epsilon;q)_l(\mu;q)_{2l}}\\
\times\,{}_4\varphi_3\left(\begin{matrix}\alpha,\beta,q^l\mu,q^{-l}\\ \gamma,\lambda,q\alpha\beta\mu/\gamma\lambda\end{matrix};q,q\right) & {}_4\varphi_3\left(\begin{matrix}q^{l+n}\nu,q^l\lambda,q^{1+l}\alpha\beta\mu/\gamma\lambda,q^{l-n}\\ q^{1+2l}\mu,q^l\delta,q^l\epsilon\end{matrix};q,q\right),
\end{aligned}\tag{26}$$

with $\gamma\delta\epsilon = q\alpha\beta\nu$. Notice how (11) is indeed obtained from this by letting $q\uparrow 1$.

Both these formulas can be interpreted in terms of the Askey-Wilson polynomials [9]:

$$p_n(x;a,b,c,d|q) = (ab,ac,ad;q)_n a^{-n}{}_4\varphi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},a\exp(i\theta),a\exp(-i\theta)\\ ab,ac,ad\end{matrix};q,q\right),\tag{27}$$

with $x=\cos\theta$.

Indeed, if we rewrite the parameters of (25) as follows:

$$\begin{cases}\alpha = e\exp(i\theta) & \beta = e\exp(-i\theta) \quad \nu = efcd/q\\ \gamma = ef & \delta = ec \qquad\qquad \epsilon = ed\\ \mu = ab & \lambda = ae\end{cases}\tag{28}$$

then we get the following *connection coefficient* formula for Askey-Wilson polynomials:

$$\begin{aligned}
p_n(x;e,f,c,d|q) &= (ab/e)^n\frac{(ec,ed,cd,ef/ab;q)_n}{(abcd;q)_n}\\
&\times\sum_{l=0}^{n}(q/f)^l\frac{(q^{-n};q)_l}{(q;q)_l}\frac{(abcd/q,q^{n-1}efcd;q)_l(abcd;q)_{2l}}{(cd,ec,ed,q^nabcd,q^{1-n}ab/ef;q)_l(abcd/q;q)_{2l}}\\
&\times\,{}_4\varphi_3\left(\begin{matrix}e/a,e/b,q^{n+l-1}efcd,q^{l-n}\\ ef/ab,q^lec,q^led\end{matrix};q,q\right)p_l(x;a,b,c,d|q).
\end{aligned}\tag{29}$$

In this formula, two of the parameters (c and d) of the Askey-Wilson polynomials are identical, while the other two may vary. This is a special case of the connection coefficient formula [8, Formula 7.6.2], where only one parameter is identical.

Next, we show that (26) can be rewritten as a sort of *convolution identity* for Askey-Wilson polynomials. First, we transform the second ${}_4\varphi_3$ on the right side to get rid of the factors q^l in the denominator parameters since we want the parameters of the Askey-Wilson polynomials to be independent of the summation variable l. Application of Sears' transformation [8, III.16] yields:

$$\begin{aligned}
&{}_4\varphi_3\left(\begin{matrix}q^{l+n}\nu,q^l\lambda,q^{1+l}\alpha\beta\mu/\gamma\lambda,q^{l-n}\\ q^{1+2l}\mu,q^l\delta,q^l\epsilon\end{matrix};q,q\right)\\
&= \frac{(q^l\lambda,q^{1-n}\alpha\beta/\gamma\lambda,\nu/\mu;q)_{n-l}}{(q^l\delta,q^{1+l}\alpha\beta\nu/\delta\gamma,q^{-l-n}/\mu;q)_{n-l}}{}_4\varphi_3\left(\begin{matrix}\delta/\lambda,\epsilon/\lambda,q^{-n-l}/\mu,q^{l-n}\\ q^{1-n}\alpha\beta/\gamma\lambda,\nu/\mu,q^{1-n}/\lambda\end{matrix};q,q\right).
\end{aligned}\tag{30}$$

Performing the following substitution:

$$\begin{cases} \alpha = a\exp(i\theta) & \beta = a\exp(-i\theta) & \nu = abc^2e^2/q \\ \gamma = ab & \delta = ace\exp(i\theta') & \epsilon = ace\exp(-i\theta') \\ \mu = abcd/q & \lambda = ac & \end{cases} \tag{31}$$

with $\cos\theta = x$ and $\cos\theta' = y$ yields the following convolution identity for Askey-Wilson polynomials:

$$\begin{aligned} &p_n(x; a, ce\exp(i\theta'), b, ce\exp(-i\theta')|q) \\ &\quad = \sum_{l=0}^{n} (-1)^n (ce)^{n-l} q^{l+\binom{n}{2}} \frac{(q^{-n}, q^{n-1}abc^2e^2; q)_l}{(q, abcdq^{l-1}; q)_l} \frac{(abq^l; q)_{n-l}}{(q^{1-n-l}/abcd; q)_{n-l}} \\ &\quad\quad \times p_l(x; a, b, c, d|q) p_{n-l}(y; e, q^{1-n}/ace, ce/d, q^{1-n}/bce|q). \end{aligned} \tag{32}$$

In the q-case one can also consider limit transitions and try to interpret these in terms of e.g. continuous dual q-Hahn polynomials or (on a lower level in the Askey-scheme) big q-Jacobi polynomials. We will not dwell on this.

Acknowledgements

The author would like to thank J. Van der Jeugt for a critical reading of the manuscript.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1999.

[2] W. Bailey, *Generalized Hypergeometric Series*, Hafner Publishing Company, New York, 1972.

[3] W. Beyer, J. Louck, and P. Stein, Group Theoretical Basis of Some Identities for the Generalized Hypergeometric Series, *J. Math. Phys.* **28** (1987) 497–508.

[4] L. C. Biedenharn, An Identity Satisfied by the Racah Coefficients, *J. Math. Phys.* **31** (1953) 287–293.

[5] L. C. Biedenharn and J. D. Louck, *The Racah-Wigner Algebra in Quantum Theory*, volume 9 of *Encyclopedia of Mathematics and its Applications*, Addison-Wesley Publishing Company, 1981.

[6] J. Elliot, Theoretical Studies in Nuclear Structure V: The Matrix Elements of Non-central Forces with an Application to the $2p$-shell, *Proc. Roy. Soc. Edinburgh Sect. A* **218** (1953) 370.

[7] G. Gasper, q-Extensions of Erdélyi's Fractional Integral Representations for Hypergeometric Functions and Some Summation Formulas for Double q-Kampé de Fériet Series, *Contemp. Math.* **254** (2000) 187–198.

[8] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, volume 35 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1990.

[9] R. Koekoek and R. F. Swarttouw, The Askey-scheme of Hypergeometric Orthogonal Polynomials and its q-analogue, Technical Report 98–17, Delft University of Technology, 1998.

[10] H. Koelink and J. Van der Jeugt, Convolutions for Orthogonal Polynomials from Lie and Quantum Algebra Representations, *SIAM J. Math. Anal.* **29** (1998) 794–822.

[11] S. Lievens and J. Van der Jeugt, $3nj$-coefficients of $\mathfrak{su}(1,1)$ as Connection Coefficients Between Orthogonal Polynomials in n Variables, *J. Math. Phys.* **43** (2002) 3824–3849.

[12] W. J. Miller, *Lie Theory and Special Functions*, volume 43 of *Mathematics in Science and Engineering*, Academic Press, New York and London, 1968.

[13] L. J. Slater, *Generalized Hypergeometric Functions*, Cambride University Press, 1966.

[14] J. Van der Jeugt, $3nj$-coefficients and Orthogonal Polynomials of Hypergeometric Type, in: H. T. Koelink and W. Van Assche (Eds.), *Lecture Notes of the Euro Summer School in Orthogonal Polynomials and Special Functions (Leuven, Belgium, 2002).*

[15] J. Van der Jeugt, Coupling Coefficients for Lie Algebra Representations and Addition Formulas for Special Functions, *J. Math. Phys.* **38** (1997) 2728–2740.

[16] J. Van der Jeugt, Hypergeometric Series Related to the 9-j Coefficient of $\mathfrak{su}(1,1)$, *J. Comput. Appl. Math.* **118** (2000) 337–351.

[17] J. Van der Jeugt and K. Srinivasa Rao, Invariance Groups of Transformations of Basic Hypergeometric Series, *J. Math. Phys.* **40** (1999) 6692–6700.

[18] N. J. Vilenkin and A. Klimyk. *Representation of Lie Groups and Special Functions*, volume 1, Kluwer Academic Publishers, 1991.