# Generalized Snell's law and its possible relation to coherent backscattering of ultrasonic waves

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The cause of backscattering in the Rayleigh angle has first been explained by means of finite beam models in which there are inherently present backward traveling components that explain the backscattering effect. Later, the nature of backscattered sound was proved to consist mostly of incoherent sound due to material anomalies. The present work shows that besides the well known real Snell's law: i.e., continuity of the frequency and continuity of the wave vector component along the interface, there is also the possibility of a complex solution of Snell's condition of continuity. The latter shows that it is possible that a part of the incident sound gets reflected into nonspecular directions including the backscatter direction. Furthermore, it is shown that this sound must have a different frequency than the incident frequency. 
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One of the most obscure phenomena in acoustics is perhaps that of seriously increased backscattering at the Rayleigh angle.[1,2] The most famous experiments of backscattering have been performed by de Billy *et al.*[3] Their results are shown in Fig. 1. The authors are aware of two theories that explain this phenomenon numerically. The first is the application of the Fourier theory to describe bounded beams. They almost always include some plane waves that travel in a backward direction.[4–6] These backward traveling waves are able to mathematically describe a backscattering phenomenon. The smaller the beam, the more backward waves are present when the backward scattering increases for smaller beams. The latter has been observed experimentally.[4] Second, it was assumed that backscattering occurs due to material defects such as dislocations and surface irregularities.[7,8] All of these "anomalies" are generating incoherent backscattering in all directions. At the Rayleigh angle, the backscattering power increases due to higher particle velocities near the surface. The increase of the measured backscattering amplitude for narrower beams follows directly from its incoherent nature. In Refs. 7 and 8, it is assumed that backscattered sound contains only incoherent waves and contains no coherent waves at all. This letter argues that this belief is doubtful.

We consider sound incident on a plane interface between a fluid and a solid. A schematic view and definition of Cartesian axes is shown in Fig. 2. The mathematical framework of what follows depends on ultrasonic complex inhomogeneous harmonic waves.[9–12] It will be shown that within this framework, it is possible that coherent backscattered sound exists having a frequency that differs from that of the incident sound.

Snell's law can be found in many versions that are all equivalent to each other.

For a plane wave

$$\mathbf{u}=\mathbf{P}\exp(i\mathbf{k}^{\text{inc}}\cdot\mathbf{r}-i\omega^{\text{inc}}t), \tag{1}$$

incident on $\mathbf{r}=x\mathbf{e}_x$, Snell's law states that for a reflected plane wave

$$k_x=k_x^{\text{inc}} \quad\text{and}\quad \omega=\omega^{\text{inc}}. \tag{2}$$

For complex inhomogeneous harmonic waves,[9–12] this has been generalized to the case where the wave vector and the frequency are both complex, whence for

$$\mathbf{k}=\mathbf{k}_1+i\mathbf{k}_2 \tag{3}$$

and

$$\omega=\omega_1+i\omega_2, \tag{4}$$

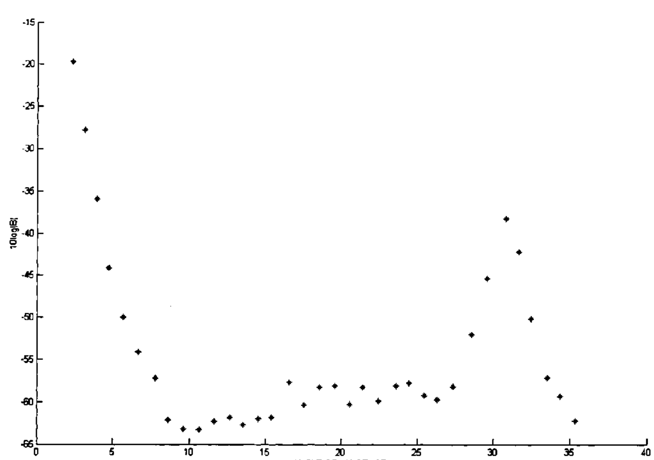


FIG. 1. Experimental results taken (from Ref. 3) for the amplitude of the backscattering coefficient for a water–stainless steel interface as a function of the angle of incidence.

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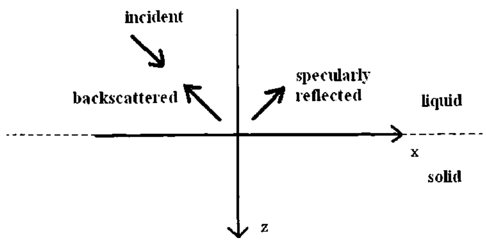


FIG. 2. A schematic view of the axes definition and the liquid–solid interface. The propagation direction of sound in the liquid is visualized.

the generalized law of Snell (also called the generalized law of Snell–Descartes) then becomes

$$k_{1,x}=k_{1,x}^{\rm inc}\qquad(5)$$
$$k_{2,x}=k_{2,x}^{\rm inc}$$

and

$$\omega_1=\omega_1^{\rm inc}\qquad(6)$$
$$\omega_2=\omega_2^{\rm inc}.$$

This generalized Snell's law does not predict backscattered sound.

An extension of Snell's law actually should not start from Eq. (2), but from Eq. (1). Hence, we must consider the foundation of Snell's law. This is done by demanding continuity of the argument of the exponential function in Eq. (1), whence

$$(k_x x-\omega t)=(k_x^{\rm inc}x-\omega^{\rm inc}t)\qquad(7)$$

from which

$$(k_{1,x}x-\omega_1 t)=(k_{1,x}^{\rm inc}x-\omega_1^{\rm inc}t)\qquad(8)$$
$$(k_{2,x}x-\omega_2 t)=(k_{2,x}^{\rm inc}x-\omega_2^{\rm inc}t).$$

Combining both equations in Eq. (8) leads to

$$(k_{1,x}-k_{1,x}^{\rm inc})(\omega_2^{\rm inc}-\omega_2)xt=(k_{2,x}-k_{2,x}^{\rm inc})(\omega_1^{\rm inc}-\omega_1)xt.\qquad(9)$$

Relation (9) is equivalent to Eq. (8) only if each value contained in the brackets differs from zero. Furthermore, by demanding that Eq. (9) must hold for each point in space-time (just as is done for the well known real solution of Snell's law), the generalized Snell's law becomes

$$(k_{1,x}-k_{1,x}^{\rm inc})(\omega_2^{\rm inc}-\omega_2)=(k_{2,x}-k_{2,x}^{\rm inc})(\omega_1^{\rm inc}-\omega_1).\qquad(10)$$

If this condition does not hold, then the real solution holds; i.e. Eqs. (5) and (6) are valid. If the condition does hold, then it is seen from Eq. (10) that there is a possibility for changing frequencies and changing wave vectors.

These generalized solutions have always been neglected, but they may become important in solving the backscattering phenomenon. First, it must be made clear that in nature, every possible solution is a solution. The total solution must be found as a linear combination of all possible solutions. For the reflected sound, there are six unknown variables; i.e., $k_{1,x}$, $k_{1,z}$, $k_{2,x}$, $k_{2,z}$, $\omega_1$, and $\omega_2$. There are three degrees of freedom, while one variable can be determined from the generalized Snell's law (10) or if the three freely chosen variables lead to one or more brackets being zero in Eq. (10), then the usual Snell's law [Eqs. (5) and (6)] needs to be used, while the remaining two unknown variables can be found from the dispersion relation for bulk waves, i.e.,

$$\mathbf{k}\cdot\mathbf{k}=\left(\frac{\omega}{\nu}-i\alpha_0\right)^2.\qquad(11)$$

Often one works with the complex slowness vector,[9–12] i.e.,

$$\mathbf{k}=\omega\mathbf{S}\rightarrow\begin{cases}\mathbf{k}_1=\omega_1\mathbf{S}_1-\omega_2\mathbf{S}_2\\ \mathbf{k}_2=\omega_2\mathbf{S}_1+\omega_1\mathbf{S}_2\end{cases}.\qquad(12)$$

Then, Eq. (11) becomes

$$(\omega_1^2-\omega_2^2)(S_1^2-S_2^2)-4\omega_1\omega_2\mathbf{S}_1\cdot\mathbf{S}_2$$
$$=\frac{1}{\nu^2}(\omega_1^2-\omega_2^2)+2\frac{\omega_2}{\nu}\alpha_0-\alpha_0^2\qquad(13)$$

and

$$\omega_1\omega_2(S_1^2-S_2^2)+(\omega_1^2-\omega_2^2)\mathbf{S}_1\cdot\mathbf{S}_2=\frac{\omega_1\omega_2}{\nu^2}-\frac{\omega_1}{\nu}\alpha_0.\qquad(14)$$

Therefore, the reflected sound field is written as a linear combination of all possible solutions:

$$\mathbf{u}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(k_{1,x},k_{1,z},\omega_1)\{(1-\delta)\mathbf{P}$$
$$\times\exp i((k_{1,x}+ik_{2,x})x+(k_{1,z}+ik_{2,z})z-(\omega_1+i\omega_2)t)$$
$$+\delta\mathbf{P}\exp i((k_{1,x}^{\rm inc}+ik_{2,x}^{\rm nc})x+(k_{1,z}^{\rm inc}+ik_{2,z}^{\rm inc})z$$
$$-(\omega_1^{\rm inc}+i\omega_2^{\rm inc})t)\}dk_{1,x}dk_{1,z}d\omega_1,\qquad(15)$$

where $\delta$ is unity whenever the generalized Snell's law does not hold and is zero otherwise. When backscattering occurs, sound returns to the receiver, whence

$$(k_{1,x}=-k_{1,x}^{\rm inc})\ \text{ and }\ (k_{1,z}=-k_{1,z}^{\rm inc})\qquad(16)$$

and from Eq. (10) this means that

$$\omega_2=(\omega_1-\omega_1^{\rm inc})k_{2,x}^{\rm inc}/k_{1,x}^{\rm inc}+\omega_2^{\rm inc}.\qquad(17)$$

Hence, for the backscattered sound field only, Eq. (15) becomes

$$\mathbf{u}^{\rm BS}=\int_{-\infty}^{+\infty}A(-k_{1,x}^{\rm inc}-k_{1,z}^{\rm inc},\omega_1)\times\{(1-\delta)\mathbf{P}\exp i((-k_{1,x}^{\rm inc}$$
$$+ik_{2,x})x+(-k_{1,z}^{\rm inc}+ik_{2,z})z-(\omega_1+i\omega_2)t)\}d\omega_1.\qquad(18)$$

In other words, for all real frequencies $\omega_1$ that result in the generalized Snell's law, i.e., $\delta=0$, Eq. (18) will be different from zero. There will be real frequencies present in the backscattered sound that differ from the incident real frequency. This means that besides the non-coherent backscattering having the same frequency as the incident frequency, it is physically possible that other frequencies are also present due to coherent backscattering, as a consequence of the generalized Snell's law.

Nevertheless determining the coefficients $A(k_{1,x}$ $k_{1,z},\omega_1)$ numerically is something that can only occur if one considers cases like the usual Snell's law, since otherwise for example continuity of normal stress and normal displacement along the fluid-solid interface does not involve enough equations to find all (infinitely much) coefficients

$A(k_{1,x}, k_{1,z}, \omega_1)$. This is a major problem. However, the theoretical possibility of having coherent backscattered sound with different frequencies than the incident frequency is hereby proven. This means that whenever in reflection/transmission phenomena one observes different frequencies than the incident frequency, this phenomenon should not automatically be attributed to nonlinear effects, but perhaps also to the generalized Snell's law.

In this letter, it is shown that there is the possibility that sound is backscattered at frequencies different from the incident frequency. This effect is not due to any nonlinearity, but due to a generalized form of Snell's law. The total sound field is then an integration over all these frequencies and also contains the solutions that are always considered, i.e., the solutions of the usual Snell's law. It is therefore shown that even though noncoherent backscattering is probably the most important factor in the backscattering effect, that there is also the possibility of coherent backscattering due to this generalized Snell's law. Due to the infiniteness of the number of solutions of this generalized Snell's law, it is not possible to perform exact numerical simulations. However this does not harm the theoretical finding that frequency shifted coherent backscattering must be a topic of further research in the near future.

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