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AREA CONTRACTION IN THE PRESENCE OF FIRST INTEGRALS AND ALMOST GLOBAL CONVERGENCE

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Abstract. We investigate the evolution of the area of multi-dimensional surfaces along the flow of a dynamical system with known first integrals, and we formulate sufficient conditions for area contraction.

These results, together with known results about the Hausdorff dimension and the box-counting dimension of invariant sets, are applied to systems exhibiting almost global convergence/asymptotic stability. This leads to a generalization of a well-known result on almost global convergence of a system, based on the use of density functions. We conclude with an example.

1. Introduction.

Consider a region in the state space of a dynamical system. The changes in its volume, when moving along the flow of the dynamical system, are determined by the divergence of the vector field. If the divergence is positive (resp. negative) in the entire state space, then the volumes of all regions will be increasing (resp. decreasing) along the vector field and this implies that an invariant (measurable) region must have either zero or infinite volume. By introducing a density function, one can redefine volumes in this state space, and as a consequence the volume changes are determined by the divergence of the product of this density function with the vector field. This product can also be considered as a modified vector field with the same trajectories as the original vector field. The possibility of finding a density function for which the aforementioned divergence is positive (resp. negative) will thus allow to derive properties of the invariant sets of the vector field of the dynamical system.

In [6], A. Rantzer used this fact to investigate systems which exhibit almost global convergence of the origin, which means that the set of points in the state space that will not converge to the origin has zero volume. He showed that if one can find a density function such that the associated divergence is positive everywhere (except for the origin) and the volume of the entire state space (except for some neighborhood of the origin) has a finite volume, then almost all trajectories converge to the origin. The set of points that do not converge to the origin is invariant and if the origin is locally asymptotically stable this set is also bounded away from the

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origin. It then has a finite volume and it follows that this volume must be zero. In the case that the origin is not asymptotically stable, the considered set can be written as a (countable) union of invariant sets that are bounded away from the origin. It follows that each set has volume zero and so has their union. In case the origin is not stable, other techniques can be used to prove almost global convergence to the origin.

In an analogous way as for n-dimensional subsets of the (n-dimensional) state space, one can investigate the behavior of k-dimensional surfaces ($k \leq n$) when they move along with the flow of the system. The contraction or expansion of the area of k-dimensional surfaces everywhere in the state space implies that no invariant surfaces can exist with a finite area of a(n) (integer) dimension larger than or equal to k (as we will show in this paper). However, for the application to the aforementioned class of systems exhibiting almost global convergence, considering the area of regular surfaces is not sufficient. There is no way to make sure that any regular surface will have a finite area, even when one can freely choose a metric for the state space. Furthermore invariant sets need not be regular surfaces. The use of an extension of the notion of area to *Hausdorff measures* avoids these problems [1, 8, 5, 4]. A condition similar to the one for the contraction/expansion of the area of k-dimensional surfaces can be derived to guarantee that *Hausdorff d-measures* (d not necessarily integer) decrease/increase along the flow of the vector field, implying that the *Hausdorff dimension* of a bounded invariant set cannot be larger than d. Similar results can be obtained for the *box-counting dimension* [3, 5].

Physical systems often exhibit symmetries and conservation laws, allowing us to derive stronger results. In this paper we will generalize a result of [2] by showing that, if a system has p conservation laws, the contraction (resp. expansion) of k-dimensional surfaces will lead to contraction (resp. expansion) of $k-p$-dimensional surfaces in an arbitrary level set of the conservation laws. The previously mentioned results can then be applied to give an upper bound for the dimension of invariant sets in this level set.

Consider a system exhibiting almost global convergence to some invariant set. We will use the aforementioned results to give an upper bound for the Hausdorff dimension or the box-counting dimension of the set of points that do not converge to this attractive set. We discuss a problem that may arise and we show that for a certain class of systems with first integrals we are able to avoid this problem as will be illustrated with an example.

2. **Outline and preliminaries.**

We will consider a dynamical system in $\mathbb{R}^n$, given by the continuously differentiable vector field f. Its flow is denoted by ϕ_t. (We assume that the dynamical system has no finite escape time and thus ϕ_t is defined everywhere in $\mathbb{R}^n$ for all $t \in \mathbb{R}$.) There is a (positive definite) C^3 metric g, taking the form

$$g = \sum_{i,j} g_{ij} \mathrm{d}x^i \mathrm{d}x^j.$$

(We let g denote both the metric and the (symmetric) matrix consisting of the elements g_{ij}.) For a vector function $v(x)$ we will use the notation $\frac{\partial v}{\partial x}$ to denote the matrix with $\frac{\partial v_i}{\partial x^j}$ on the ith row and jth column.

In the following section we will derive an expression for the area of a k-dimensional parallelepiped with respect to a time-dependent metric, and we will give an upper bound for its time-derivative. In section 4 we will apply these results to give an expression for the area of a k-dimensional surface and an upper bound for its time-derivative when evolving under the flow of the dynamical system. This results in an upper bound for the dimension of regular bounded invariant sets. In order to extend this result to arbitrary bounded invariant sets, we will introduce the concept of Hausdorff measure in section 5, after which we will discuss its evolution under the flow of the dynamical system and the consequences for the Hausdorff dimension of invariant sets. Similar results will then be stated concerning the box-counting dimension. In section 7 we assume that the dynamical system has p first integrals and we show how the evolution of k-dimensional surfaces is related to the evolution of $k-p$-dimensional surfaces in the level set of the first integrals.

In section 8 we explain the result of [7] (which is a generalization of the result of [6]) and we relate it to the contraction/expansion of the area of multi-dimensional surfaces. Then we give an extension by using the results of the previous sections and we indicate a subtlety that can cause a practical problem and we show how, for some systems with first integrals, this problem can be avoided. We conclude with an example.

3. **Evolution of the volume of a parallelepiped.** In this section we will consider $\mathbb{R}^n$ as a vector space with a metric, represented by the symmetric, positive definite matrix G. We consider a parallelepiped P_k spanned by k ($k \leq n$) linearly independent vectors $w_1, \dots, w_k$. The length of a vector w_i equals

$$\sqrt{\langle w_i, w_i \rangle} = \sqrt{w_i^T G w_i}.$$

First we will assume a standard metric: $G = I_n$. Let B_k be an orthonormal basis in the k-dimensional subspace spanned by the vectors w_i. Define $W_n \in \mathbb{R}^{n \times k}$ and $W_k \in \mathbb{R}^{k \times k}$ by

$$W_n = \begin{bmatrix} w_1 & \cdots & w_k \end{bmatrix}, \qquad W_k = \begin{bmatrix} [w_1]_{B_k} & \cdots & [w_k]_{B_k} \end{bmatrix},$$

where $[w_i]_{B_k}$ is the column vector containing the coordinates of w_i with respect to the basis B_k. Then the k-dimensional area/volume $\sigma_{k,s}(P_k)$ (with respect to the standard metric) of the aforementioned parallelepiped can be written as

$$\sigma_{k,s}(P_k) = |\det W_k| = \sqrt{\det(W_k^T W_k)},$$

and since the element on row i, column j, equals $[w_i]_{B_k}^T [w_j]_{B_k} = \langle w_i, w_j \rangle = w_i^T w_j$,

$$\sigma_{k,s}(P_k) = \sqrt{\det(\langle w_i, w_j \rangle)} = \sqrt{\det(W_n^T I_n W_n)}.$$

From now on we let the metric be arbitrary. The expression $\sqrt{\det(\langle w_i, w_j \rangle)}$ also defines the k-dimensional area for a general metric G:

$$\sigma_k(P_k) = \sqrt{\det(\langle w_i, w_j \rangle)} = \sqrt{\det(W_n^T G W_n)}.$$

Assume that G is time-varying and consider the time-derivative of $(\sigma_k(P_k))^2$ for the case $k = 1$ ($W_n = w$):

$$\frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(w^T G w) = w^T \frac{\mathrm{d}G}{\mathrm{d}t} w,$$

which we rewrite as

$$\frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} = w^T G \mathcal{S} w,$$

with $\mathcal{S} = G^{-1}\frac{\mathrm{d}G}{\mathrm{d}t}$. We will show that we can bound this expression by the product of $\sigma_1(P_1)^2 = w^T G w$ and the largest eigenvalue of $\mathcal{S}$. First note that $G^{-\frac{1}{2}}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-\frac{1}{2}}$ is symmetric ($G^{\frac{1}{2}}$ is the positive definite matrix satisfying $(G^{\frac{1}{2}})^2 = G$), such that there exists a $Q_0 \in \mathbb{R}^{n\times n}$ with

$$G^{-\frac{1}{2}}\frac{\mathrm{d}G}{\mathrm{d}t}G^{-\frac{1}{2}}Q_0 = Q_0\Lambda, \qquad\qquad Q_0^T Q_0 = I_n,$$

where Λ is diagonal (and real) with $\Lambda_{11} \geq \cdots \geq \Lambda_{nn}$. Setting $Q_1 = G^{-\frac{1}{2}}Q_0$ we obtain

$$\mathcal{S}Q_1 = Q_1\Lambda, \qquad\qquad Q_1^T G Q_1 = I_n,$$

and the columns of Q_1 form a basis of orthonormal (with respect to G) eigenvectors of $\mathcal{S}$. By writing w as a linear combination of these eigenvectors we obtain

$$\begin{aligned}
\frac{\mathrm{d}(\sigma_1(P_1))^2}{\mathrm{d}t} &= w^T G\mathcal{S}w = w'^T Q_1^T G\mathcal{S}Q_1 w' && (\text{with } w' = Q_1^{-1}w)\\
&= w'^T Q_1^T G Q_1 \Lambda w' = w'^T\Lambda w' = \sum_i 2\lambda_i w_i'^2 && (\text{with } \lambda_i = \frac{1}{2}\Lambda_{ii})\\
&\leq \sum_i 2\lambda_1 w_i'^2 = 2\lambda_1 w'^T w' = 2\lambda_1 w'^T Q_1^T G Q_1 w'\\
&= 2\lambda_1 w^T G w = 2\lambda_1\sigma_1^2(P_1),
\end{aligned}$$

and thus

$$\frac{\mathrm{d}\sigma_1(P_1)}{\mathrm{d}t} \leq \lambda_1\sigma_1(P_1).$$

For general k-values we could write

$$\frac{\mathrm{d}(\sigma_k(P_k))^2}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\det(W_n^T G W_n) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_\tau \mathrm{sgn}(\tau)\prod_i (W_n^T G W_n)_{i\tau(i)}$$

(where the summation is over all permutations τ of $(1,\ldots,k)$ and the product is taken over all $i \in \{1,\ldots,k\}$)

$$\begin{aligned}
&= \sum_\tau \mathrm{sgn}(\tau)\sum_{j=1}^k \frac{\mathrm{d}}{\mathrm{d}t}(W_n^T G W_n)_{j\tau(j)}\prod_{i\neq j}(W_n^T G W_n)_{i\tau(i)}\\
&= \sum_\tau \mathrm{sgn}(\tau)\sum_{j=1}^k (W_n^T G\mathcal{S}W_n)_{j\tau(j)}\prod_{i\neq j}(W_n^T G W_n)_{i\tau(i)}
\end{aligned}$$

(setting $W_n' = Q_1^{-1}W_n$)

$$\begin{aligned}
&= \sum_\tau \mathrm{sgn}(\tau)\sum_{j=1}^k (W_n'^T Q_1^T G\mathcal{S}Q_1 W_n')_{j\tau(j)}\prod_{i\neq j}(W_n'^T Q_1^T G Q_1 W_n')_{i\tau(i)}\\
&= \sum_\tau \mathrm{sgn}(\tau)\sum_{j=1}^k (W_n'^T \Lambda W_n')_{j\tau(j)}\prod_{i\neq j}(W_n'^T W_n')_{i\tau(i)}.
\end{aligned}$$

Now we cannot just bound $(W_n'^T\Lambda W_n')_{j\tau(j)}$ by $2\lambda_1(W_n'^T W_n')_{j\tau(j)}$ since $\mathrm{sgn}(\tau)$ can be negative. But note that we have some freedom left in the choice of W_n or W_n'. We can perform column operations on W_n, since this corresponds to a right multiplication with some matrix $Q_2 \in \mathbb{R}^{k\times k}$, with $|\det Q_2| = 1$, which has no effect in the formula for $\sigma_k(P_k)$. We can use these column operations to make sure that the columns are orthogonal at some time t_0. (We don't want the column operations, or the matrix Q_2, to be time-dependent to avoid problems when taking the time-derivative.) Then, in the expressions for $(\sigma_k(P_k))^2$ and $\frac{\mathrm{d}(\sigma_k(P_k))^2}{\mathrm{d}t}$ at $t = t_0$, the only permutation that needs to be considered is the identity permutation, which has a positive sign. Even more, we can use these column operations to make sure that, at $t = t_0$, the jth column will only contain eigenvectors of $\frac{1}{2}\mathcal{S}$ corresponding to eigenvalues smaller than or equal to λ_j, leading eventually to a bound of the form

$$(W_n'^T \Lambda W_n')_{jj} \leq 2\lambda_j (W_n'^T W_n')_{jj}.$$

Redefine Λ and Q_1 as the time-invariant matrices associated with $G_0 = G|_{t=t_0}$ and $\mathcal{S}_0 = \mathcal{S}|_{t=t_0}$ ($\mathcal{S}_0 Q_1 = Q_1\Lambda$, $Q_1^T G_0 Q_1 = I_n$) and set $\lambda_i = \frac{1}{2}\Lambda_{ii}$, $\forall\, i \in \{1,\ldots,n\}$. Mathematically, the existence of the aforementioned column operations comes down to the following lemma, of which the proof is given in the appendix (section B).

Lemma 1. *With the notations introduced above we can find matrices $W_n'' \in \mathbb{R}^{n\times k}$ and $Q_2 \in \mathbb{R}^{k\times k}$ such that*

$$|\det Q_2| = 1, \qquad\qquad W_n Q_2 = Q_1 W_n'',$$

and W_n'' has the form

$$W_n'' = \begin{bmatrix} \times & 0 & \cdots & 0 \\ \times & \times & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \times & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ \times & \times & \cdots & \times \end{bmatrix},$$

and the property that if $i \neq j$, then

$$(W_n''^T W_n'')_{ij} = 0.$$

After applying the lemma, it follows that

$$(\sigma_k(P_k))^2 = \det((Q_2^{-1})^T W_n''^T Q_1^T G Q_1 W_n'' Q_2^{-1}) = \det(W_n''^T Q_1^T G Q_1 W_n''),$$

and we can now derive that

$$\left.\frac{\mathrm{d}(\sigma_k(P_k))^2}{\mathrm{d}t}\right|_{t=t_0} = \sum_\tau \operatorname{sgn}(\tau) \sum_{j=1}^k (W_n''^T \Lambda W_n'')_{j\tau(j)} \prod_{i\neq j} (W_n''^T W_n'')_{i\tau(i)}$$

(only when τ is the identity permutation we get something different from zero, because of the properties of W_n'')

$$= \sum_{j=1}^k (W_n''^T \Lambda W_n'')_{jj} \prod_{i\neq j} (W_n''^T W_n'')_{ii}$$

(using the special structure of W''_n)

$$= \sum_{j=1}^{k} \sum_{l=j}^{n} 2\lambda_l (W''_n)^2_{lj} \prod_{i \neq j} (W''^T_n W''_n)_{ii}$$

$$\leq \sum_{j=1}^{k} 2\lambda_j \sum_{l=j}^{n} (W''_n)^2_{lj} \prod_{i \neq j} (W''^T_n W''_n)_{ii}$$

$$= 2 \sum_{j=1}^{k} \lambda_j \prod_{i} (W''^T_n W''_n)_{ii}$$

(and since $\prod_i (W''^T_n W''_n)_{ii} = \det(W''^T_n W''_n) = (\sigma_k(P_k))^2|_{t=t_0}$)

$$= 2(\lambda_1 + \cdots + \lambda_k)\ (\sigma_k(P_k))^2\Big|_{t=t_0},$$

or

$$\left.\frac{\mathrm{d}\sigma_k(P_k)}{\mathrm{d}t}\right|_{t=t_0} \leq (\lambda_1 + \cdots + \lambda_k)\ \sigma_k(P_k)|_{t=t_0}.$$

Equality is reached for instance when w_i is the eigenvector of $\frac{1}{2}\mathcal{S}$ corresponding to λ_i $(1 \leq i \leq k)$.

This result can be formulated as follows.

Proposition 1. *Let G be a symmetric, positive definite and time-dependent matrix and let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $\frac{1}{2}\ G^{-1} \frac{\mathrm{d}G}{\mathrm{d}t}\big|_{t=t_0}$. Then*

$$\max_{\substack{W \in \mathbb{R}^{n\times k} \\ \det(W^T G(t_0) W) \neq 0}} \left.\frac{\frac{\mathrm{d}}{\mathrm{d}t} \det\left(W^T G W\right)}{\det\left(W^T G W\right)}\right|_{t=t_0} = 2(\lambda_1 + \cdots + \lambda_k).$$

4. **Evolution of the area of k-dimensional surfaces.** In the standard metric, the length $\sigma_{1,s}$ of a curve $\psi(V)$ in $\mathbb{R}^n$, represented by the function $\psi : V \to \mathbb{R}^n$, $V \subset \mathbb{R}$, is given by the well-known expression

$$\sigma_{1,s}(\psi(V)) = \int_{\psi(V)} \sqrt{\sum_i (\mathrm{d}x^i)^2} = \int_V \sqrt{\sum_i \left(\frac{\partial \psi^i}{\partial y}(y)\right)^2} \mathrm{d}y.$$

For a general metric the length σ_1 equals

$$\sigma_1(\psi(V)) = \int_{\psi(V)} \sqrt{\sum_{i,j} g_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j} = \int_V \sqrt{\sum_{i,j} g_{ij}(\psi(y)) \frac{\partial \psi^i}{\partial y}(y) \frac{\partial \psi^j}{\partial y}(y)\mathrm{d}y}.$$

This formula can be extended to an expression for the area of surfaces of larger dimensions in the following way. Let V be a region in $\mathbb{R}^k$ such that the function $\psi : V \to \mathbb{R}^n$ defines a (smooth) k-dimensional surface in $\mathbb{R}^n$. Then the k-dimensional area $\sigma_k(U)$ (with $U = \psi(V)$) can be found by replacing W_n by $\frac{\partial \psi}{\partial y}(y)\mathrm{d}y$ and G by $g(\psi(y))$ in the previous section and integrating over V:

$$\sigma_k(U) = \int_V \sqrt{\det\left(\frac{\partial \psi}{\partial y}^T (y) g(\psi(y)) \frac{\partial \psi}{\partial y}(y)\right)} \mathrm{d}y.$$

Now we let U evolve under the flow of the given dynamical system to obtain the time-variant surface $\phi_t(U) = \phi_t \circ \psi(V)$ and we consider its area:

$$\sigma_k(\phi_t(U)) = \int_V \sqrt{\det\left(\frac{\partial\psi}{\partial y}^T(y)\frac{\partial\phi_t}{\partial x}^T(\psi(y))g(\phi_t(\psi(y)))\frac{\partial\phi_t}{\partial x}(\psi(y))\frac{\partial\psi}{\partial y}(y)\right)}\mathrm{d}y.$$

To calculate the time-derivative $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k(\phi_t(U))$, we first consider the matrix

$$\begin{aligned}\mathcal{S}(f,g)(x) &= g^{-1}(x)\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\partial\phi_t}{\partial x}(x)\right)^T g(\phi_t(x))\frac{\partial\phi_t}{\partial x}(x)\right)\bigg|_{t=0}\\ &= g^{-1}(x)\frac{\partial f}{\partial x}^T(x)g(x) + g^{-1}(x)\sum_i f^i(x)\frac{\partial g}{\partial x^i}(x) + \frac{\partial f}{\partial x}(x),\end{aligned}$$

and denote the eigenvalues of $\frac{1}{2}\mathcal{S}(f,g)$ in $x \in \mathbb{R}^n$ by $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$. Then it follows from section 3 (with $W_n = \frac{\partial\psi}{\partial y}(y)\mathrm{d}y$ and $G(t) = (\frac{\partial\phi_t}{\partial x}(x))^T g(\phi_t(x))\frac{\partial\phi_t}{\partial x}(x))$ that

$$\begin{aligned}&\frac{\partial}{\partial t}\sqrt{\det\left(\frac{\partial\psi}{\partial y}^T(y)\frac{\partial\phi_t}{\partial x}^T(\psi(y))g(\phi_t(\psi(y)))\frac{\partial\phi_t}{\partial x}(\psi(y))\frac{\partial\psi}{\partial y}(y)\right)}\Bigg|_{t=0}\\ &\qquad\leq (\lambda_1(\psi(y)) + \cdots + \lambda_k(\psi(y)))\sqrt{\det\left(\frac{\partial\psi}{\partial y}^T(y)g(\psi(y))\frac{\partial\psi}{\partial y}(y)\right)}.\end{aligned}$$

Notice that for $k = n$ the inequality becomes an equality and we retrieve Liouville's theorem. Integrating the inequality over V leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_k(\phi_t(U))\bigg|_{t=0} \leq \sup_{x\in U}\left(\lambda_1(x) + \cdots + \lambda_k(x)\right)\sigma_k(U).$$

This means that the supremum of the sum $\lambda_1(x)+\cdots+\lambda_k(x)$ gives an upper bound for the rate at which k-dimensional surfaces can increase.

Assume that for some region $\Omega \subset \mathbb{R}^n$, it is true that

$$\lambda_1(x) + \cdots + \lambda_k(x) \leq 0, \qquad \forall\, x \in \Omega.$$

Then the area of any k-dimensional surface lying in Ω cannot increase under ϕ_t. This implies that if the k-dimensional surface under consideration is invariant under the flow of the dynamical system, then on this surface we must have that $\lambda_1(x)+\cdots+\lambda_k(x) = 0$. Under some extra conditions on the set $\{x \in \Omega : \lambda_1(x)+\cdots+\lambda_k(x) = 0\}$ (e.g. demanding that its dimension is smaller than k) we can conclude that there can be no invariant k-dimensional surfaces in Ω with a finite area.

If also

$$\sup_{x\in\Omega}\lambda_1(x) + \cdots + \lambda_k(x) < 0,$$

then there is uniform contraction of k-dimensional surfaces and the previous result can be extended to arbitrary (but still bounded) sets by using a result of Reitmann [8] (which is based on an article by Douady and Oesterlé [1]). To explain this result, we first need to recall the definition of the Hausdorff dimension and the box-counting dimension.

5. **The Hausdorff dimension and the evolution of Hausdorff measures.** Consider a totally bounded set S in $\mathbb{R}^n$. (A totally bounded set is a bounded set that can be covered with a finite number of balls of any predetermined radius $\epsilon > 0$.) Cover S with a countable number of balls of radius $r_i < \epsilon$, with $\epsilon > 0$. For a given $d \in [0, n]$ and $\epsilon > 0$, the *Hausdorff outer measure* $\mu_H(S, d, \epsilon)$ is defined as follows:

$$\mu_H(S, d, \epsilon) = \inf \sum_i r_i^d,$$

where the infimum is taken over all possible covers satisfying $r_i < \epsilon$, $\forall\, i$. Keeping d fixed, $\mu_H(S, d, \epsilon)$ as a function of ϵ is decreasing and non-negative. Therefore, the *Hausdorff d-measure*, equal to

$$\mu_H(S, d) = \lim_{\epsilon \to 0} \mu_H(S, d, \epsilon) \in \mathbb{R}^+ \cup \{+\infty\},$$

is well-defined. If S is a smooth k-dimensional surface, this measure has the property that $\mu_H(S, k)$ is proportional to the k-dimensional area of the surface and therefore it can be considered as an extension to the notion of length, (k-dimensional) area and volume. It also follows that for a general set S there exists a d^* such that

$$d < d^* \Rightarrow \mu_H(S, d) = +\infty,$$
$$d > d^* \Rightarrow \mu_H(S, d) = 0.$$

By definition, $d^* = \dim_H S$, the *Hausdorff dimension* of S. For instance, a two-dimensional surface in $\mathbb{R}^3$ will have $d^* = 2$ and the above inequalities can be interpreted by stating that it has an infinite length and zero (3-dimensional) volume.

For explaining the evolution of Hausdoff d-measures we will split d in an integer part k and a fractional part s and consider the linear interpolation between the sum of the k largest eigenvalues λ_i and the sum of the $k+1$ largest λ_i-values. From results in [8] and [5] one can then obtain the following:

Theorem 1. *Let Ω be a subset of $\mathbb{R}^n$ with*

$$\sup_{x \in \Omega} \lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0,$$

where $k \in \{1, \ldots, n-1\}$ and $s \in [0, 1]$, and let S be a totally bounded set, satisfying $\phi_t(S) \subset \Omega$, $\forall\, t \in \mathbb{R}$. Then, if we set $d = k + s$, for each $c > 0$, there exists a $T > 0$ and a $\epsilon_0 > 0$, such that for all $t > T$ and $\epsilon \in (0, \epsilon_0)$

$$\mu_H(\phi_t(S), d, \epsilon) \leq c\mu_H(S, d, \epsilon),$$

implying that

$$\mu_H(\phi_t(S), d) \leq c\mu_H(S, d).$$

Since we can choose c as small as we want, this means that, under similar conditions as for the contraction of k-dimensional surfaces, we also have that the d-dimensional Hausdorff outer measure will decrease under the flow of the dynamical system (for sufficiently large values of T). If S is invariant under ϕ_t, then we can choose $c < 1$ to obtain that for sufficiently small values of ϵ

$$\mu_H(S, d, \epsilon) = 0 \text{ and thus } \mu_H(S, d) = 0,$$

implying that

$$\dim_H S \leq d.$$

Therefore there can be no bounded invariant sets in Ω with a Hausdorff dimension higher than d.

Remark 1. Although the condition of S being bounded and the definition of Hausdorff measure will depend on the chosen metric, under some mild conditions the Hausdorff dimension will not. This allows for deriving better upper bounds for the Hausdorff dimension by choosing an appropriate metric. The same holds for the box-counting dimension, which is treated in the next section.

6. **The box-counting dimension.** We will define the box-counting dimension (or capacity dimension) more directly, although by introducing capacitive d-measures a similar definition can be obtained as for the Hausdorff dimension. For a totally bounded set S in $\mathbb{R}^n$ and a given $\epsilon > 0$ the covers considered now consist of a finite number of balls with radii *equal* to ϵ. Let $N(\epsilon)$ be the minimum number of balls of radius ϵ needed to cover S. The upper box dimension $\overline{\dim}_B S$ is defined as

$$\overline{\dim}_B S = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log(\epsilon)}.$$

An analogous definition holds for the lower box dimension $\underline{\dim}_B S$:

$$\underline{\dim}_B S = \liminf_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log(\epsilon)}.$$

If both are equal they are called the box(-counting) dimension. From the definitions it follows that $\dim_H S \leq \underline{\dim}_B S \leq \overline{\dim}_B S$.

From [3] now follows:

Theorem 2. *Let $S \subset \mathbb{R}^n$ be compact and invariant under ϕ_t with*

$$\lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0, \qquad \forall\, x \in S,$$

where $k \in \{1, \ldots, n-1\}$ and $s \in [0, 1]$. Then

$$\overline{\dim}_B S \leq k + s.$$

This implies that if

$$\lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0, \qquad \forall\, x \in \Omega,$$

for some compact set $\Omega \subset \mathbb{R}^n$ then there can be no invariant sets in Ω with a box dimension higher than $k + s$. (This follows from the fact that $S \subset \Omega$ implies that the closure $\bar{S}$ is compact (and of course $\bar{S}$ is invariant under f), and from the fact that $\overline{\dim}_B \bar{S} = \overline{\dim}_B S$, which can be derived from the definition of box-counting dimension.)

Note that the condition of theorem 2 is equivalent to

$$\sup_{x \in S} \lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0,$$

since S is compact.

7. **The presence of first integrals.** Assume there are p first integrals of the dynamical system, denoted by the column vector h, such that

$$\sum_i f^i \frac{\partial h}{\partial x^i} = 0,$$

and the matrix $\frac{\partial h}{\partial x}$ has full row rank everywhere in some region $\Omega \subset \mathbb{R}^n$. Then the level set

$$L_C = \{x : h(x) = C\},$$

with $C \in \mathbb{R}^p$, is invariant under ϕ_t and we can consider the restriction of the dynamical system to L_C. Let $\hat{g}$ be a metric in $L_C \cap \Omega$ which has to be determined yet.

For a neighborhood $U_x \subset L_C \cap \Omega$ of some $x \in L_C \cap \Omega$ we can define a $(n-p) \times (n-p)$-matrix $\mathcal{S}(f, \hat{g})$ for $\hat{g}$, such that $\frac{1}{2}\mathcal{S}(f, \hat{g})$ determines how the area of higher dimensional surfaces evolves under ϕ_t in U_x with respect to $\hat{g}$. Let $\hat{\lambda}_1(x) \geq \cdots \geq \hat{\lambda}_{n-p}(x)$ denote the eigenvalues of this matrix in x. Choose an integer k with $p \leq k < n$ and an $s \in (0, 1]$. Then we can prove the following.

Theorem 3. *Under the above conditions, one can choose $\hat{g}$ in such a way that*

$$\hat{\lambda}_1(x) + \cdots + \hat{\lambda}_{k-p}(x) + s\hat{\lambda}_{k-p+1}(x) \leq \lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x),$$

$\forall\, x \in L_C \cap \Omega$, $\forall\, C \in \mathbb{R}^p$.

This means that, in the presence of p first integrals, the contraction of k-dimensional surfaces (resp. Hausdorff d-measures) leads to contraction of $k-p$-dimensional surfaces (resp. Hausdorff $d-p$-measures) in any level set of the p first integrals (but with respect to another metric). The proof is given in the appendix (section C).

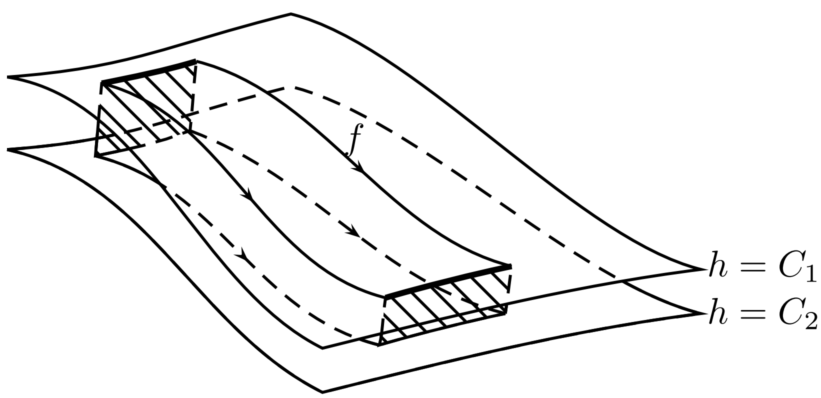

FIGURE 1. An example with one first integral h.

We will try to provide some intuition. In figure 1 a system is shown that contracts the area of 2-dimensional surfaces (e.g. the shaded ones) with respect to the standard metric. In this standard metric though, 1-dimensional curves are not contracted (e.g. the thicker lines). However the system has a first integral h and we can define a new metric in the level surface $h = C_1$ by setting the length equal to (or proportional to) the area of the 2-dimensional surface that is formed by extending the curve (an infinitesimally small amount) in the direction of ∇h (again the shaded surfaces). Since this area decreases under the flow of the system, so will the (newly defined) length of 1-dimensional curves lying in the level sets of h.

8. **Application to almost global convergence.** The criterion from [6] mentioned in the introduction was generalized in [7] to include almost global convergence to an invariant set S, i.e. the set R of points not converging to S has measure zero. Consider again a C^1 vector field f with no finite escape time. Let d denote

the distance function associated with the standard metric and let S denote a closed set, invariant under f. Set $S_\epsilon = \{x \in \mathbb{R}^n : d(x,S) < \epsilon\}$. The theorem in [7] can then be formulated as follows:

Theorem 4. *Assume that* $\rho \in C^1(\mathbb{R}^n \setminus S, \mathbb{R}) \cap L^1(\mathbb{R}^n \setminus S_\epsilon)$ *for all* $\epsilon > 0$. *If* $\rho(x) > 0$ *and* $\nabla \cdot (\rho f)(x) > 0$ *for almost all* $x \in \mathbb{R}^n \setminus S$ *and* f *is bounded in* S_r *for some* $r > 0$, *then* $\lim_{t\to\infty} d(\phi_t(x), S) = 0$ *for almost all* $x \in \mathbb{R}^n$.

(Some property holds for almost all $x \in \mathbb{R}^n$ if the set of points where it does not hold has (Lebesgue) measure zero.) The density function ρ can be viewed as a new way to define volumes. For instance, by using the (not necessarily positive definite) metric $g = \rho^{\frac{2}{n}} I_n$ (with ρ satisfying the properties from theorem 4), the (n-dimensional) volume of a region $U \subset \mathbb{R}^n$ would become (with the notations from section 4, ψ the identity function, and $V = U$)

$$\sigma_n(U) = \int_U \sqrt{\det(\rho^{\frac{2}{n}}(y) I_n)}\mathrm{d}y = \int_U \rho(y)\mathrm{d}y.$$

For this metric the behavior of n-dimensional volumes is determined by the sum of all n eigenvalues of $\mathcal{S}(f,g)$, which equals its trace (assume $\rho(x) > 0$):

$$\begin{aligned}\lambda_1(x) + \cdots + \lambda_n(x) &= \operatorname{tr} S(f,g) \\ &= \operatorname{tr}\left(\frac{\partial f}{\partial x}^T(x) + \frac{\partial f}{\partial x}(x) + \frac{2}{n}\frac{1}{\rho(x)}\sum_i f^i \frac{\partial \rho}{\partial x}(x) I_n\right) \\ &= \frac{2}{\rho(x)} \nabla \cdot (\rho f)(x).\end{aligned}$$

The fact that f expands n-dimensional volumes, together with the fact that $\rho \in L^1(\mathbb{R}^n \setminus S_\epsilon)$ ($\int_{\mathbb{R}^n \setminus S_\epsilon} \rho(x)\mathrm{d}x$ is finite) for all $\epsilon > 0$ will guarantee that all invariant sets lying in $\mathbb{R}^n \setminus S_\epsilon$ for some $\epsilon > 0$ have n-dimensional volume (or Lebesgue measure) zero.

If the set S is (Lyapunov) stable, then it follows that the set R_ϵ, with

$$R_\epsilon = \{x \in \mathbb{R}^n : \limsup_{t\to\infty} d(\phi_t(x), S) > \epsilon\},$$

is contained in $\mathbb{R}^n \setminus S_\delta$, for some $\delta > 0$. Since the set R_ϵ is invariant under f it follows that it has Lebesgue measure zero and so has the set R, with

$$R = \bigcup_{\epsilon > 0} R_\epsilon = \{x \in \mathbb{R}^n : \lim_{t\to\infty} d(\phi_t(x), S) \neq 0\}.$$

We will not consider the case where S is not stable, since the arguments we will use for our extension do not apply in this case.

From now on assume that the set S is stable. Let g be a positive definite C^3 metric for which $\mathbb{R}^n \setminus S_\epsilon$ is bounded for all $\epsilon > 0$ and assume that (with $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ again the eigenvalues of $\frac{1}{2}\mathcal{S}(f,g)$ in $x \in \mathbb{R}^n$)

$$\inf_{x \in \mathbb{R}^n \setminus S} s\lambda_{n-k}(x) + \lambda_{n-k+1}(x) + \cdots + \lambda_n(x) > 0,$$

or equivalently, f expands Hausdorff $k+s$-measures, for some integer k and some $s \in [0,1]$. (This condition is the same as the condition for contraction of Hausdorff $k+s$-measures under $-f$.) Since $R_\epsilon \subset \mathbb{R}^n \setminus S_\delta$ for some $\delta > 0$ (by the stability of

S) and R_ϵ is invariant under f, it follows from section 5 that $\mu_H(R_\epsilon, k+s) = 0$. From the definition of the Hausdorff measure it follows that

$$\mu_H(R, k+s) = \mu_H\left(\bigcup_{i\in\mathbb{N}_0} R_{\frac{1}{i}}, k+s\right) \le \sum_{i\in\mathbb{N}_0} \mu_H\left(R_{\frac{1}{i}}, k+s\right) = 0,$$

and thus $\dim_H R \le k+s$.

If in addition $\mathbb{R}^n \setminus S_\epsilon$ is compact for all $\epsilon > 0$ and S is asymptotically stable (i.e. $\exists\, \epsilon > 0 : \lim_{t\to\infty} d(\phi_t(x), S) = 0, \forall\, x \in S_\epsilon$), then $R \subset \mathbb{R}^n \setminus S_\epsilon$ and from theorem 2 we obtain

$$\overline{\dim}_B R \le k+s.$$

As a result, if there is almost global convergence to S, then this approach allows to provide more information on the set R. An important difference with the approach in [7] however, is the condition needed for the expansion of Hausdorff $k+s$-measures. While the condition for the expansion of n-dimensional volumes comes down to

$$\lambda_1(x) + \cdots + \lambda_n(x) > 0, \qquad \text{for almost all } x \in \mathbb{R}^n \setminus S,$$

we now need

$$\inf_{x\in\mathbb{R}^n\setminus S} s\lambda_{n-k}(x) + \lambda_{n-k+1}(x) + \cdots + \lambda_n(x) > 0,$$

which can be hard to obtain, since often it may happen that $\lim_{|x|\to\infty} \lambda_i(x) = 0$, for all $i \in \{1, \dots, n\}$. (This is due to the fact that one might want to multiply the vector field of a system with a finite escape time with some function to make sure that for the new vector field f the ratio $\frac{|f(x)|}{|x|}$ is bounded, which would guarantee that the system determined by f has no finite escape time. This puts restrictions on the behavior of $f(x)$ and $\mathcal{S}(f,g)(x)$ as $|x| \to \infty$.) It is not clear to us whether the condition for contraction/expansion of Hausdorff $k+s$-measures can be relaxed or not for obtaining the same results concerning the Hausdorff dimension. For the box dimension however we were able to construct an example of a vector field f and an invariant set $\mathcal{C}$ where

$$\lambda_1(x) + 0.2\lambda_2(x) < 0, \qquad \forall\, x \in \mathcal{C},$$

suggesting that the box dimension of $\mathcal{C}$ would not exceed 1.2 (if this condition would have been sufficient), while it can be proven to be at least $\frac{4}{3}$. This example is described and investigated in the appendix (section A).

A class of systems where this problem does not arise is the set of systems with first integrals of which the level sets are compact. Then we can consider the restriction of the system to one of the level sets and derive results for the set of points lying in this level set and not converging to S. Since the level sets are compact there are no problems with finite escape times and there is no problem if $\lambda_i(x) \to 0$ as $|x| \to \infty$, for some i (since we need to consider the supremum over a level set). Because of the result of the previous section, we do not need to find coordinate systems for the level sets, but we can use contraction/expansion properties in the n-dimensional state space.

Assume that f is a C^1 vector field with flow ϕ_t and with p first integrals h_i, such that $\frac{\partial h}{\partial x}$ has full row rank everywhere in $\Omega \subset \mathbb{R}^n$. Choose a $C \in \mathbb{R}^p$ and assume that the level set $L_C = \{x : h(x) = C\}$ is compact and lies entirely in Ω. Let S denote a closed set, invariant under f and stable. Denote by R_C the set

$$R_C = \{x \in L_C : \limsup_{t\to\infty} d(\phi_t(x), S) \ne 0\}.$$

From the previous results immediately follows:

Theorem 5. *If there exists a* C^3 *metric* g *defined on* $\Omega \setminus S$ *such that*

$$\inf_{x \in L_C \setminus S} s\lambda_{n-k}(x) + \lambda_{n-k+1}(x) + \cdots + \lambda_n(x) > 0,$$

for some integer $k \in [p, n-1]$ *and some* $s \in (0, 1]$, *(*$\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ *are the eigenvalues of* $\frac{1}{2}\mathcal{S}(f, g)$*), then*

$$\mu_H(R_C, k+s-p) = 0,$$

implying that $\dim_H R_C \leq k+s-p$. *If in addition* S *is asymptotically stable, then*

$$\overline{\dim}_B R_C \leq k+s-p.$$

We will now illustrate this theorem with an example.

Example 1. Consider the following vector field f in $\mathbb{R}^3$:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_2 x_3^2 - x_1 x_3^2, \\ f_2(x_1, x_2, x_3) &= -x_1 x_3^2 - x_2 x_3^2, \\ f_3(x_1, x_2, x_3) &= (x_1^2 + x_2^2) x_3. \end{aligned}$$

One can easily verify that the function h, with

$$h(x) = x_1^2 + x_2^2 + x_3^2,$$

is a first integral for the system $\dot{x} = f(x)$, and $\frac{\partial h}{\partial x}$ has full row rank everywhere in $\Omega = \mathbb{R}^3 \setminus \{0\}$. The level sets $L_C = \{x \in \mathbb{R}^3 : h(x) = C\}$, with $C > 0$, are compact and lie entirely in Ω. From the expression for f_1 and f_2 it follows that the set $S = \{x \in \mathbb{R}^3 : (x_1, x_2) = (0, 0)\}$ is stable. With the metric

$$g = \frac{1}{x_1^2 + x_2^2} I_3$$

one can derive that the eigenvalues of $\frac{1}{2}\mathcal{S}(f, g)$ satisfy

$$\lambda \left(\lambda^2 - (x_1^2 + x_2^2 + x_3^2)\lambda - (x_1^2 + x_2^2) x_3^2 \right) = 0,$$

and that, with $k = 2$ and $s > s_0 = 3 - 2\sqrt{2} \approx 0.17$,

$$\inf_{x \in L_C \setminus S} s\lambda_1(x) + \lambda_2(x) + \lambda_3(x) > 0.$$

We can conclude that R_C has a Hausdorff dimension smaller than or equal to $k + s_0 - p = 4 - 2\sqrt{2} \approx 1.17$.

Indeed, from the differential equations it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1^2 + x_2^2) = -(x_1^2 + x_2^2) x_3^2,$$

such that the only points in $\mathbb{R}^3$ that will not converge to S lie in the plane $\{x \in \mathbb{R}^3 : x_3 = 0\}$, and thus R_C is the circle in this plane around the origin with radius $\sqrt{C}$ and has a Hausdorff dimension of 1. Figure 2 shows 10 different trajectories belonging to the same level set ($C = 1$). They all start near the circle R_C in the (x_1, x_2)-plane and converge to the x_3-axis.

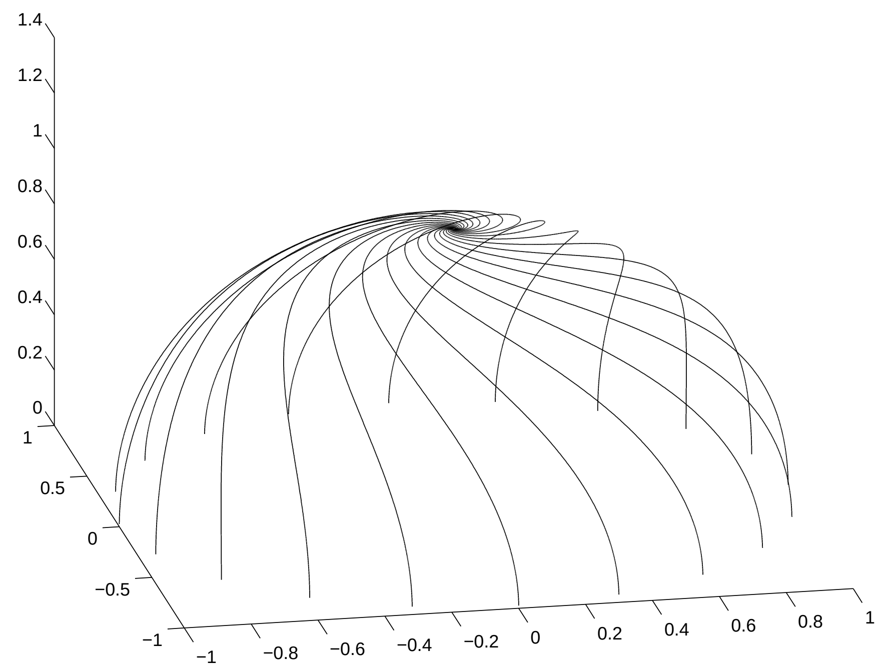

FIGURE 2. A plot of 10 different trajectories in the same level set.

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Appendix A. Counter-example for relaxing the conditions for an upper bound on the box dimension. Consider the vector field f in the x, y-plane, given by the following equations:

$$\begin{aligned}\dot{x} &= -y - 2x(x^2+y^2) + 2x^4 \\ \dot{y} &= x - 2y(x^2+y^2) - x^3(12y+1),\end{aligned}$$

or, in polar coordinates:

$$\begin{aligned}\dot{\theta} &= 1 - r^2\cos^4\theta(14r\sin\theta + 1) \\ \dot{r} &= -r^3(2+\cos^3\theta\sin\theta) + 2r^4\cos^3\theta(\cos^2\theta - 6\sin^2\theta).\end{aligned}$$

We will consider the part $\mathcal{C}$ of the unstable manifold of the saddle point $(1,0)$ that spirals towards the origin (see figure 3). We take $g = I_n$ (standard metric), and numerically calculate the eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ $(\lambda_1(x) \geq \lambda_2(x))$ of the matrix $\mathcal{S}(f, I_n)$ along the curve $\mathcal{C}$. From figure 4, where we have plot the ratio $-\frac{\lambda_1(x)}{\lambda_2(x)}$, together with the fact that

$$\lambda_1(x) + \lambda_2(x) = \nabla \cdot f(x) = -8x^2 - 8y^2 - 4x^3 < 0, \qquad \forall\,(x,y) \in \mathcal{C},$$

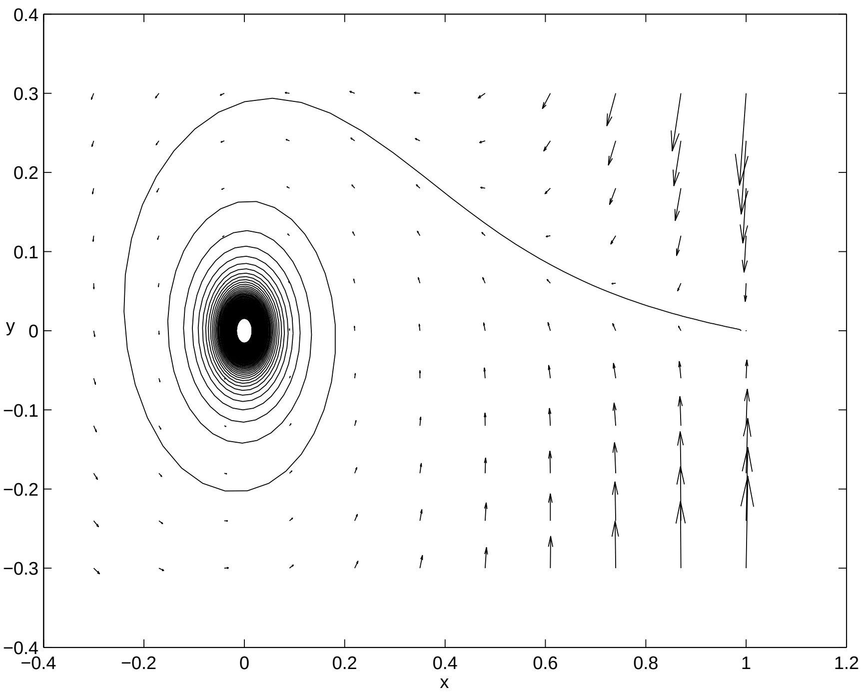

FIGURE 3. The part of the unstable manifold of (1,0) that spirals towards the origin.

we can conclude that $\lambda_1(x) + 0.2\lambda_2(x) < 0$ everywhere on $\mathcal{C}$. Note that $\lambda_1(0,0) = \lambda_2(0,0) = 0$, such that this condition does not hold anymore for the closure $\bar{\mathcal{C}}$, and $\sup_{x\in\mathcal{C}}(\lambda_1(x) + 0.2\lambda_2(x)) = 0$. This also implies that theorem 2 is not applicable. If the condition for this theorem could be relaxed to

$$\lambda_1(x) + \cdots + \lambda_k(x) + s\lambda_{k+1}(x) < 0, \qquad \forall\, x \in S,$$

for non-compact sets S, then we would be able to conclude that $\overline{\dim}_B\mathcal{C} \leq 1.2$, contradicting the fact that $\underline{\dim}_B\mathcal{C} \geq \frac{4}{3}$, which we will now show. Since this is a bit technical, we prefer to give a more intuitive approach before giving the rigorous proof. (Note that $\dim_H \mathcal{C} = 1 < 1.2$, leaving open whether or not a relaxation of the conditions for bounding the Hausdorff dimension is possible.)

It is clear that we only need to consider the part of $\mathcal{C}$ close to the origin, so we could say that $\dot{\theta} \approx 1$ (θ is chosen such that it increases along the spiral in the inward direction) and, after averaging out the goniometric term, we have that $\dot{r} \sim -r^3$. This leads to $r \sim 1/\sqrt{t} \sim 1/\sqrt{\theta}$. Now we consider the spiral as consisting of different arcs, each of them described by $\theta \in [n2\pi, (n+1)2\pi]$ for some $n \in \mathbb{N}$ and look at the part $r > R_1$ of the spiral in which the space between the arcs is large enough (of the order of ϵ, the radius of the covering discs) such that two discs, from coverings of different arcs, do not touch each other. For this to happen R_1 must be large enough, and can be approximated as the solution of: $|2\pi \mathrm{d}r/\mathrm{d}\theta| \sim \epsilon$. When substituting $\dot{\theta} \sim 1$, $\dot{r} \sim -r^3$ and $r \sim \theta^{-1/2}$ in this estimate, we obtain $\frac{1}{\theta^{3/2}} \sim r^3 \sim \epsilon$ or $R_1 \sim \epsilon^{1/3}$. The length of the part of $\mathcal{C}$ for which $r > R_1$ can be approximated

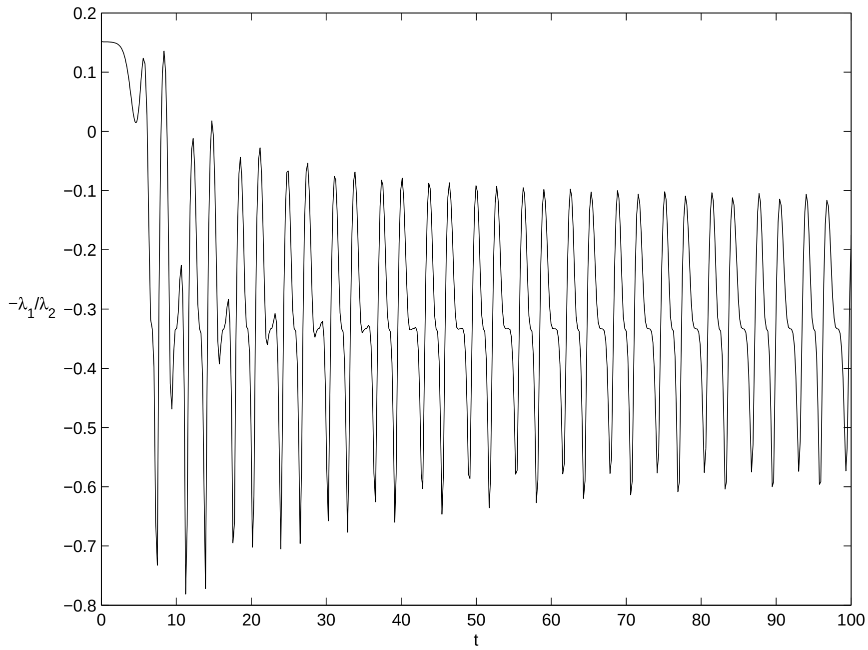

FIGURE 4. The ratio $-\lambda_1(x)/\lambda_2(x)$, calculated along $\mathcal{C}$.

by:

$$\int_1^{\Theta_1} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \sim \int_1^{\Theta_1} \frac{1}{\sqrt{\theta}}\mathrm{d}\theta \sim \sqrt{\Theta_1} \sim \frac{1}{R_1} \sim \epsilon^{-1/3}.$$

Thus, the part of $\mathcal{C}$ satisfying $r > R_1$ can be covered by a number of discs of the order $\epsilon^{-1/3}/\epsilon = \epsilon^{-4/3}$. We estimate the number of discs of radius ϵ, needed to cover the part of $\mathcal{C}$ in $r < R_1$, by the number needed to cover a disc of radius R_1:

$$\frac{\pi R_1^2}{\pi \epsilon^2} \sim \epsilon^{-4/3}.$$

Therefore, the box dimension can be estimated by

$$\dim_B \mathcal{C} \sim \lim_{\epsilon \to 0} \frac{\ln \epsilon^{-4/3}}{-\ln \epsilon} = \frac{4}{3} > 1.2,$$

what we wanted to show. Below we give a rigorous proof of the fact that $\underline{\dim}_B \mathcal{C} \geq \frac{4}{3}$.

Note that $|\cos^3\theta \sin\theta| \leq 1/2$. Choose C_1, C_2, C_3 and C_4 such that $0 < C_1 < 1 < C_2$, $0 < C_3 < 3/2$ and $5/2 < C_4$. Now there exists an $R_0 > 0$, such that for $r < R_0$:

$$C_1 < \dot{\theta} < C_2,$$
$$C_3 < -\frac{\dot{r}}{r^3} < C_4.$$

Denote the r-value of the ith intersection of $\mathcal{C}$ with the negative x-axis for which $r < R_0$ by r_i, such that $r_i > r_{i+1}$, $\forall\, i \in \mathbb{N}_0$. (This also defines the values θ_i and t_i, satisfying $t_i < t_{i+1}$.) From the above inequalities we obtain:

$$C_1(t_{i+1}-t_i) < 2\pi < C_2(t_{i+1}-t_i),$$
$$C_3(t_{i+1}-t_i) < \frac{1}{2r_{i+1}^2} - \frac{1}{2r_i^2} < C_4(t_{i+1}-t_i),$$

and thus

$$4\pi\frac{C_3}{C_2} < \frac{r_i^2 - r_{i+1}^2}{r_i^2 r_{i+1}^2} < 4\pi\frac{C_4}{C_1},$$

or

$$4\pi\frac{C_3}{C_2}\frac{r_i^2 r_{i+1}^2}{r_i + r_{i+1}} < r_i - r_{i+1} < 4\pi\frac{C_4}{C_1}\frac{r_i^2 r_{i+1}^2}{r_i + r_{i+1}},$$

and since $r_i > r_{i+1}$

$$2\pi\frac{C_3}{C_2}r_{i+1}^3 < r_i - r_{i+1} < 2\pi\frac{C_4}{C_1}r_i^2 r_{i+1}. \tag{1}$$

It follows that

$$r_i < r_{i+1} + 2\pi\frac{C_4}{C_1}r_i^2 r_{i+1},$$

or

$$\frac{1}{r_{i+1}} < \frac{1}{r_i} + 2\pi\frac{C_4}{C_1}r_i$$

$$\frac{1}{r_{i+1}^2} < \frac{1}{r_i^2} + 4\pi\frac{C_4}{C_1} + \left(2\pi\frac{C_4}{C_1}r_i\right)^2 \leq \frac{1}{r_i^2} + 4\pi\frac{C_4}{C_1} + \left(2\pi\frac{C_4}{C_1}r_1\right)^2.$$

If we set $\alpha = 4\pi\frac{C_4}{C_1} + \left(2\pi\frac{C_4}{C_1}r_1\right)^2$ and $\beta = 1/r_1^2 - \alpha$, by induction we obtain:

$$\frac{1}{r_i^2} \leq i\alpha + \beta \Longleftrightarrow r_i \geq \frac{1}{\sqrt{i\alpha+\beta}}.$$

For $\epsilon > 0$ sufficiently small, we have that

$$n := \left\lfloor \frac{1}{2\alpha}\left(\left(\frac{2\epsilon C_2}{\pi C_3}\right)^{-2/3} - \beta - \alpha\right)\right\rfloor > 0. \tag{2}$$

Then, from equation (1) it follows that for $i \in \{1,\ldots,2n\}$

$$\begin{aligned} r_i - r_{i+1} &> 2\pi\frac{C_3}{C_2}r_{i+1}^3 \geq 2\pi\frac{C_3}{C_2}r_{2n+1}^3 \\ &\geq 2\pi\frac{C_3}{C_2}\frac{1}{((2n+1)\alpha+\beta)^{3/2}} \geq 4\epsilon. \end{aligned}$$

If we denote by $\mathcal{C}_i$ the piece of $\mathcal{C}$ between the ith and the $i+1$th intersection with the negative x-axis (for which $r < R_0$), then $r_{i+1} \leq r(x,y) \leq r_i$, $\forall\,(x,y) \in \mathcal{C}_i$. Thus for $i \in \{1,\ldots,2n-1\}$ the covers with discs of radius ϵ of $\mathcal{C}_i$ and $\mathcal{C}_{i+2}$ will be disjunct, while, from simple geometric arguments, one can see that the number of discs needed to cover $\mathcal{C}_i$ is larger than $\pi r_{i+2}/\epsilon$.

To cover all $\mathcal{C}_i$ with $i \in \{1, 3, 5, \ldots, 2n-1\}$ we will need at least

$$\begin{aligned}\sum_{i=1}^{n} \frac{\pi r_{(2i-1)+2}}{\epsilon} &\geq \frac{\pi}{\epsilon} \sum_{i=1}^{n} \frac{1}{\sqrt{(2i+1)\alpha+\beta}} \\ &\geq \frac{\pi}{\epsilon} \int_{1}^{n+1} \frac{\mathrm{d}x}{\sqrt{(2x+1)\alpha+\beta}} \\ &= \frac{\pi}{\epsilon\alpha}\left(\sqrt{(2n+3)\alpha+\beta} - \sqrt{3\alpha+\beta}\right)\end{aligned}$$

discs. The total number of discs $N(\epsilon)$ needed to cover $\mathcal{C}$ satisfies

$$N(\epsilon) \geq \frac{\pi}{\epsilon\alpha}\left(\sqrt{(2n+3)\alpha+\beta} - \sqrt{3\alpha+\beta}\right),$$

and thus, considering equation (2), we can conclude that

$$\underline{\dim}_B \mathcal{C} = \liminf_{\epsilon \to 0} \frac{\log N(\epsilon)}{-\log \epsilon} \geq \frac{4}{3}.$$

Appendix B. **Proof of lemma 1.** Define the column vectors q_1^i $(i \in \{1, \ldots, n\})$ by setting $Q_1 = \begin{bmatrix} q_1^i & \cdots & q_1^n \end{bmatrix}$. Let W^1 be the k-dimensional subspace of $\mathbb{R}^n$ that consists of all linear combinations of the w_i's. The subspace $W^1 \cap q_1^\perp$ (orthogonality is considered with respect to G_0) is at least $k-1$-dimensional; let W^2 be a $k-1$-dimensional subspace of $W^1 \cap q_1^\perp$. (In the generic case $W^2 = W^1 \cap q_1^\perp$.) We apply a column transformation on W_n, represented by the matrix Q_2^1, leading to a matrix of which the first column vector is in $W^1 \cap (W^2)^\perp$ and the other column vectors are in W^2.

We now repeat this procedure, in the next step starting from W^2. In general, at step m, we will consider the intersection of the $k-m+1$-dimensional subspace W^m (where $W^m \perp \{q_1, \ldots, q_{m-1}\}$) with $q_m^\perp$, in which we choose the $k-m$-dimensional subspace W^{m+1}. (Again, in the generic case $W^{m+1} = W^m \cap q_m^\perp$.) We apply a column transformation, represented by the matrix Q_2^m, on the matrix $W_n Q_2^1 \cdots Q_2^{m-1}$, such that the first $m-1$ columns of the latter matrix remain unchanged, the mth column belongs to $W^m \cap (W^{m+1})^\perp$, and the other columns belong to W^{m+1}.

Setting $Q_2 = Q_2^1 \cdots Q_2^{k-1}$, we obtain the desired properties for the matrix $W'' = Q_1^{-1} W_n Q_2$.

Appendix C. **Proof of theorem 3.** Let y^i, with $i \in \{1, \ldots, n-p\}$, denote coordinate functions for the level set L_C in some region $U \subset L_C \cap \Omega$. Let $\mathbf{x}$ denote the vector function that maps $y \in \mathbb{R}^{n-p}$ to the corresponding $x \in \mathbb{R}^n$ and let $\mathbf{y}$ denote the vector function that maps a $x \in U \subset \mathbb{R}^n$ to the corresponding $y \in \mathbb{R}^{n-p}$. Define Δ_h by

$$\Delta_h(x) = \det\left(\frac{\partial h}{\partial x}(x) g^{-1}(x) \frac{\partial h}{\partial x}^T(x)\right).$$

Since $\frac{\partial h}{\partial x}$ has full row rank in Ω, $\Delta_h(x)$ is positive in U. For a given $d \in (p, n]$, consider the metric, represented by the matrix $\hat{g}_d \in \mathbb{R}^{(n-p)\times(n-p)}$ with respect to the y^i's, given by

$$\hat{g}_d(y) = \Delta_h^{-\frac{1}{d-p}}(\mathbf{x}(y)) \frac{\partial \mathbf{x}}{\partial y}^T(y) g(\mathbf{x}(y)) \frac{\partial \mathbf{x}}{\partial y}(y).$$

We will prove that this metric satisfies the condition of the previous theorem. Set $\hat{\phi}_t = \mathbf{y} \circ \phi_t \circ \mathbf{x}$ and define $\hat{G}_d$ by

$$\hat{G}_d(y,t) = \frac{\partial \hat{\phi}_t}{\partial y}^T(y)\hat{g}_k(\hat{\phi}_t(y))\frac{\partial \hat{\phi}_t}{\partial y}(y),$$

which can be rewritten as

$$\begin{aligned}\hat{G}_d(y,t) &= \Delta_h(\mathbf{x}(\hat{\phi}_t(y)))^{-\frac{1}{k-p}}\frac{\partial \hat{\phi}_t}{\partial y}^T(y)\frac{\partial \mathbf{x}}{\partial y}^T(\hat{\phi}_t(y))g(\mathbf{x}(\hat{\phi}_t(y)))\frac{\partial \mathbf{x}}{\partial y}(\hat{\phi}_t(y))\frac{\partial \hat{\phi}_t}{\partial y}(y)\\ &= \Delta_h(\mathbf{x}\circ\hat{\phi}_t(y))^{-\frac{1}{k-p}}\frac{\partial(\mathbf{x}\circ\hat{\phi}_t)}{\partial y}^T(y)g(\mathbf{x}\circ\hat{\phi}_t(y))\frac{\partial(\mathbf{x}\circ\hat{\phi}_t)}{\partial y}(y)\end{aligned}$$

(with $\mathbf{x}\circ\hat{\phi}_t = \phi_t\circ\mathbf{x}$)

$$\begin{aligned}&= \Delta_h(\phi_t\circ\mathbf{x}(y))^{-\frac{1}{k-p}}\frac{\partial \mathbf{x}}{\partial y}^T(y)\frac{\partial \phi_t}{\partial x}^T(\mathbf{x}(y))g(\phi_t\circ\mathbf{x}(y))\frac{\partial \phi_t}{\partial x}(\mathbf{x}(y))\frac{\partial \mathbf{x}}{\partial y}(y)\\ &= \Delta_h(\phi_t(\mathbf{x}(y)))^{-\frac{1}{k-p}}\frac{\partial \mathbf{x}}{\partial y}^T(y)G(\mathbf{x}(y),t)\frac{\partial \mathbf{x}}{\partial y}(y).\end{aligned}$$

where we define G by

$$G(x,t) = \frac{\partial \phi_t}{\partial x}^T(x)g(\phi_t(x))\frac{\partial \phi_t}{\partial x}(x).$$

- First we will consider the case $d = k$, where k is an integer satisfying $p < k \le n$. From proposition 1 and the derivation in section 3, we obtain that

$$2\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p}(\mathbf{x}(y))\right) = \max_{\substack{W\in\mathbb{R}^{(n-p)\times(k-p)}\\ \det(W^T\hat{G}_k(y,0)W)\neq 0}} \left.\frac{\frac{\partial}{\partial t}\det\left(W^T\hat{G}_k(y,t)W\right)}{\det\left(W^T\hat{G}_k(y,t)W\right)}\right|_{t=0}.$$

Define H by

$$H(x,t) = \Delta_h(\phi_t(x))^{-\frac{1}{p}}\left(g(\phi_t(x))\frac{\partial \phi_t}{\partial x}(x)\right)^{-1}\frac{\partial h}{\partial x}^T(\phi_t(x)),$$

and consider the matrix

$$\begin{aligned}&\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y)W & H(\mathbf{x}(y),t)\end{bmatrix}^T G(\mathbf{x}(y),t)\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y)W & H(\mathbf{x}(y),t)\end{bmatrix}\\ &\quad= \begin{bmatrix}W^T\frac{\partial \mathbf{x}}{\partial y}^T(y)G(\mathbf{x}(y),t)\frac{\partial \mathbf{x}}{\partial y}(y)W & W^T\frac{\partial \mathbf{x}}{\partial y}^T(y)G(\mathbf{x}(y),t)H(\mathbf{x}(y),t)\\ H(\mathbf{x}(y),t)^TG(\mathbf{x}(y),t)\frac{\partial \mathbf{x}}{\partial y}(y)W & H(\mathbf{x}(y),t)^TG(\mathbf{x}(y),t)H(\mathbf{x}(y),t)\end{bmatrix}\end{aligned}$$

(noticing that $H(\mathbf{x}(y),t)^TG(\mathbf{x}(y),t)\frac{\partial \mathbf{x}}{\partial y}(y) = \frac{\partial h}{\partial x}(\phi_t(\mathbf{x}(y)))\frac{\partial \phi_t}{\partial x}(\mathbf{x}(y))\frac{\partial \mathbf{x}}{\partial y}(y) = \frac{\partial(h\circ\phi_t\circ\mathbf{x})}{\partial y}(y) = 0$)

$$= \begin{bmatrix}\Delta_h(x_t)^{\frac{1}{k-p}}W^T\hat{G}_k(y,t)W & 0\\ 0 & \Delta_h(x_t)^{-\frac{2}{p}}\frac{\partial h}{\partial x}(x_t)g^{-1}(x_t)\frac{\partial h}{\partial x}^T(x_t)\end{bmatrix}$$

(where $x_t = \phi_t(\mathbf{x}(y))$), of which the determinant equals

$$\det\left(\Delta_h(x_t)^{\frac{1}{k-p}} W^T \hat{G}_k(y,t) W\right) \det\left(\Delta_h(x_t)^{-\frac{2}{p}} \frac{\partial h}{\partial x}(x_t) g^{-1}(x_t) \frac{\partial h}{\partial x}^T(x_t)\right) = \det\left(W^T \hat{G}_k(y,t) W\right).$$

Thus we obtain

$$\frac{\partial}{\partial t} \det\left(W^T \hat{G}_k(y,t) W\right)\bigg|_{t=0} = \frac{\partial}{\partial t} \det\left(\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)\end{bmatrix}^T G(\mathbf{x}(y),t) \begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)\end{bmatrix}\right)\bigg|_{t=0}.$$

One can easily verify that in general, for time-dependent (and differentiable) matrices $A(t)$, $B(t)$ and $C(t)$ for which the product $A(t)B(t)C(t)$ is well-defined, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \det\left(A(t)B(t)C(t)\right)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \det\left(A(t)B(0)C(0)\right)\bigg|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t} \det\left(A(0)B(t)C(0)\right)\bigg|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t} \det\left(A(0)B(0)C(t)\right)\bigg|_{t=0}.$$

(This follows immediately from the formula $\det(A) = \sum_\sigma \operatorname{sgn}\sigma \prod_i A_{i\sigma(i)}$.) This allows us to write

$$\begin{aligned}
&\frac{\partial}{\partial t} \det\left(W^T \hat{G}_k(y,t) W\right)\bigg|_{t=0} \\
&= \frac{\partial}{\partial t} \det\left(\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),0)\end{bmatrix}^T G(\mathbf{x}(y),t) \begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),0)\end{bmatrix}\right)\bigg|_{t=0} \\
&+ \frac{\partial}{\partial t} \det\left(\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)\end{bmatrix}^T G(\mathbf{x}(y),0) \begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),0)\end{bmatrix}\right)\bigg|_{t=0} \\
&+ \frac{\partial}{\partial t} \det\left(\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),0)\end{bmatrix}^T G(\mathbf{x}(y),0) \begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)\end{bmatrix}\right)\bigg|_{t=0}.
\end{aligned}$$

We will prove that the second term is zero (and so is the third term, which is obviously equal to the second term).

Notice that again $H(\mathbf{x}(y),0)^T G(\mathbf{x}(y),0) \frac{\partial \mathbf{x}}{\partial y}(y) = \frac{\partial (h\circ \mathbf{x})}{\partial y}(y) = 0$.

$$\begin{aligned}
&\frac{\partial}{\partial t} \det\left(\begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)\end{bmatrix}^T G(\mathbf{x}(y),0) \begin{bmatrix}\frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),0)\end{bmatrix}\right)\bigg|_{t=0} \\
&= \frac{\partial}{\partial t} \det\begin{bmatrix} W^T \frac{\partial \mathbf{x}}{\partial y}^T(y) G(\mathbf{x}(y),0) \frac{\partial \mathbf{x}}{\partial y}(y) W & 0 \\ H(\mathbf{x}(y),t)^T G(\mathbf{x}(y),0) \frac{\partial \mathbf{x}}{\partial y}(y) W & H(\mathbf{x}(y),t)^T G(\mathbf{x}(y),0) H(\mathbf{x}(y),0)\end{bmatrix}\Bigg|_{t=0} \\
&= \det\!\left(W^T \frac{\partial \mathbf{x}}{\partial y}^T(y) G(\mathbf{x}(y),0) \frac{\partial}{\partial \mathbf{x}} y(y) W\right) \frac{\partial}{\partial t} \det\!\left(H(\mathbf{x}(y),t)^T G(\mathbf{x}(y),0) H(\mathbf{x}(y),0)\right)\bigg|_{t=0}.
\end{aligned}$$

From $\phi_{-t}(\phi_t(x)) = x$ it follows that

$$\frac{\partial \phi_{-t}}{\partial x}(\phi_t(x)) \frac{\partial \phi_t}{\partial x}(x) = I_n,$$

and we obtain that

$$\det\left(H(\mathbf{x}(y),t)^T G(\mathbf{x}(y),0) H(\mathbf{x}(y),0)\right)$$
$$= \det\left(\Delta_h(\mathbf{x}(y))^{-\frac{1}{p}} \Delta_h(x_t)^{-\frac{1}{p}} \frac{\partial h}{\partial x}(x_t) g^{-1}(x_t) \left(\frac{\partial \phi_t}{\partial x}^T(\mathbf{x}(y))\right)^{-1} \frac{\partial h}{\partial x}^T(\mathbf{x}(y))\right)$$
$$= \Delta_h(\mathbf{x}(y))^{-1} \Delta_h(x_t)^{-1} \det\left(\frac{\partial h}{\partial x}(x_t) g^{-1}(x_t) \left(\frac{\partial h}{\partial x}(\phi_{-t}(x_t)\frac{\partial \phi_{-t}}{\partial x}(x_t)\right)^T\right)$$

(with $\frac{\partial h}{\partial x}(\phi_{-t}(x_t))\frac{\partial \phi_{-t}}{\partial x}(x_t) = \frac{\partial (h\circ\phi_{-t})}{\partial x}(x_t) = \frac{\partial h}{\partial x}(x_t)$)

$$= \Delta_h(\mathbf{x}(y))^{-1} \Delta_h(x_t)^{-1} \det\left(\frac{\partial h}{\partial x}(x_t) g^{-1}(x_t) \frac{\partial h}{\partial x}^T(x_t)\right)$$
$$= \Delta_h(\mathbf{x}(y))^{-1},$$

which is independent of time. It follows that in the previously derived expression for $\frac{\partial}{\partial t}\det\left(W^T \hat{G}_k(y,t) W\right)\Big|_{t=0}$, only the first term remains, and, setting $W'(W,y) = \left[\frac{\partial \mathbf{x}}{\partial y}(y)W \quad H(\mathbf{x}(y),0)\right] \in \mathbb{R}^{n\times k}$, we eventually obtain

$$2\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p}(\mathbf{x}(y))\right)$$
$$= \max_{\substack{W\in\mathbb{R}^{(n-p)\times(k-p)} \\ \det(W^T\hat{G}_k(y,0)W)\neq 0}} \left.\frac{\frac{\partial}{\partial t}\det\left(W^T\hat{G}_k(y,t)W\right)}{\det\left(W^T\hat{G}_k(y,t)W\right)}\right|_{t=0}$$
$$= \max_{\substack{W\in\mathbb{R}^{(n-p)\times(k-p)} \\ \det(W^T\hat{G}_k(y,0)W)\neq 0}} \left.\frac{\frac{\partial}{\partial t}\det\left(W'^T(W,y)G(\mathbf{x}(y),t)W'(W,y)\right)}{\det\left(W'^T(W,y)G(\mathbf{x}(y),t)W'(W,y)\right)}\right|_{t=0}$$
$$\leq 2\Big(\lambda_1(\mathbf{x}(y)) + \cdots + \lambda_k(\mathbf{x}(y))\Big).$$

- Now we will consider the case where $d = k+s$, with $s \in [0,1]$ and $p < k \leq n$. From proposition 1 it follows that

$$2\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p}(\mathbf{x}(y)) + s\hat{\lambda}_{k-p+1}(\mathbf{x}(y))\right)$$
$$= 2(1-s)\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p}(\mathbf{x}(y))\right) + 2s\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p+1}(\mathbf{x}(y))\right)$$
$$= \max_{\substack{W_1\in\mathbb{R}^{(n-p)\times(k-p)} \\ \det(W_1^T\hat{G}_d(y,0)W_1)\neq 0 \\ W_2\in\mathbb{R}^{(n-p)\times(k-p+1)} \\ \det(W_2^T\hat{G}_d(y,0)W_2)\neq 0}} \left.\frac{\frac{\partial}{\partial t}\left(\left(\det W_1^T\hat{G}_d(y,t)W_1\right)^{1-s}\left(\det W_2^T\hat{G}_d(y,t)W_2\right)^s\right)}{\left(\det W_1^T\hat{G}_d(y,t)W_1\right)^{1-s}\left(\det W_2^T\hat{G}_d(y,t)W_2\right)^s}\right|_{t=0}.$$

With the previously derived expression for $\hat{G}_d(y,t)$ and the fact that

$$\Delta_h^{(1-s)\frac{k-p}{d-p}} \Delta_h^{s\frac{k-p+1}{d-p}} = \Delta_h = \Delta_h^{(1-s)\frac{k-p}{k-p}} \Delta_h^{s\frac{k-p+1}{k-p+1}}$$

we can rewrite this in terms of $\hat{G}_k$ and $\hat{G}_{k+1}$:

$$
\begin{aligned}
&2\left(\hat{\lambda}_1(\mathbf{x}(y)) + \cdots + \hat{\lambda}_{k-p}(\mathbf{x}(y)) + s\hat{\lambda}_{k-p+1}(\mathbf{x}(y))\right)\\
&= \max_{\substack{W_1\in\mathbb{R}^{(n-p)\times(k-p)}\\ \det(W_1^T\hat{G}_k(y,0)W_1)\neq 0\\ W_2\in\mathbb{R}^{(n-p)\times(k-p+1)}\\ \det(W_2^T\hat{G}_{k+1}(y,0)W_2)\neq 0}} \left.\frac{\dfrac{\partial}{\partial t}\left(\left(\det W_1^T\hat{G}_k(y,t)W_1\right)^{1-s}\left(\det W_2^T\hat{G}_{k+1}(y,t)W_2\right)^{s}\right)}{\left(\det W_1^T\hat{G}_k(y,t)W_1\right)^{1-s}\left(\det W_2^T\hat{G}_{k+1}(y,t)W_2\right)^{s}}\right|_{t=0}\\
&= \max_{\substack{W_1\in\mathbb{R}^{(n-p)\times(k-p)}\\ \det(W_1^T\hat{G}_k(y,0)W_1)\neq 0}} (1-s)\left.\frac{\frac{\partial}{\partial t}\det W_1^T\hat{G}_k(y,t)W_1}{\det W_1^T\hat{G}_k(y,t)W_1}\right|_{t=0}\\
&\quad + \max_{\substack{W_2\in\mathbb{R}^{(n-p)\times(k-p+1)}\\ \det(W_2^T\hat{G}_{k+1}(y,0)W_2)\neq 0}} s\left.\frac{\frac{\partial}{\partial t}\det W_2^T\hat{G}_{k+1}(y,t)W_2}{\det W_2^T\hat{G}_{k+1}(y,t)W_2}\right|_{t=0}\\
&= \max_{\substack{W_1\in\mathbb{R}^{(n-p)\times(k-p)}\\ \det(W_1^T\hat{G}_k(y,0)W_1)\neq 0}} (1-s)\left.\frac{\frac{\partial}{\partial t}\det\left(W'^T(W_1,y)G(\mathbf{x}(y),t)W'(W_1,y)\right)}{\det\left(W'^T(W_1,y)G(\mathbf{x}(y),t)W'(W_1,y)\right)}\right|_{t=0}\\
&\quad + \max_{\substack{W_2\in\mathbb{R}^{(n-p)\times(k-p+1)}\\ \det(W_2^T\hat{G}_{k+1}(y,0)W_2)\neq 0}} s\left.\frac{\frac{\partial}{\partial t}\det\left(W'^T(W_2,y)G(\mathbf{x}(y),t)W'(W_2,y)\right)}{\det\left(W'^T(W_2,y)G(\mathbf{x}(y),t)W'(W_2,y)\right)}\right|_{t=0}\\
&\leq 2(1-s)\Big(\lambda_1(\mathbf{x}(y)) + \cdots + \lambda_k(\mathbf{x}(y))\Big) + 2s\Big(\lambda_1(\mathbf{x}(y)) + \cdots + \lambda_{k-p+1}(\mathbf{x}(y))\Big)\\
&= 2\Big(\lambda_1(\mathbf{x}(y)) + \cdots + \lambda_k(\mathbf{x}(y)) + s\lambda_k(\mathbf{x}(y))\Big).
\end{aligned}
$$

- The case that remains to be investigated is $k = p$ and $s \in (0, 1]$. Intuitively, one could try to give a meaning to the previous derivations for the case $k = p$ by considering the limit $k \to p$ in the terms where this is possible. This would lead to the same reasoning as described below.

$$
2s\hat{\lambda}_1(\mathbf{x}(y)) = \max_{\substack{W_2\in\mathbb{R}^{(n-p)\times(k-p+1)}\\ \det(W_2^T\hat{G}_d(y,0)W_2)\neq 0}} \left.\frac{\dfrac{\partial}{\partial t}\left(\left(\det W_2^T\hat{G}_d(y,t)W_2\right)^{s}\right)}{\left(\det W_2^T\hat{G}_d(y,t)W_2\right)^{s}}\right|_{t=0}.
$$

We can rewrite this as

$$
2s\hat{\lambda}_1(\mathbf{x}(y)) = \max_{\substack{W_2\in\mathbb{R}^{(n-p)\times(k-p+1)}\\ \det(W_2^T\hat{G}_{k+1}(y,0)W_2)\neq 0}} \left.\frac{\dfrac{\partial}{\partial t}\left(\Delta_h^{-(1-s)}(x_t)\left(\det W_2^T\hat{G}_{k+1}(y,t)W_2\right)^{s}\right)}{\Delta_h^{-(1-s)}(x_t)\left(\det W_2^T\hat{G}_{k+1}(y,t)W_2\right)^{s}}\right|_{t=0}
$$

Since

$$
\Delta_h^{-1}(x_t) = \det\left(H^T(\mathbf{x}(y),t)G(\mathbf{x}(y),t)H(\mathbf{x}(y),t)\right),
$$

and again

$$
\frac{\partial}{\partial t}\det\left(H^T(\mathbf{x}(y),t)G(\mathbf{x}(y),0)H(\mathbf{x}(y),0)\right) = 0,
$$

we obtain that

$$\left.\frac{\frac{\partial}{\partial t}\left(\Delta_h^{-1}(x_t)\right)}{\Delta_h^{-1}(x_t)}\right|_{t=0} = \left.\frac{\frac{\partial}{\partial t}\left(H^T(\mathbf{x}(y),0)G(\mathbf{x}(y),t)H(\mathbf{x}(y),0)\right)}{(H^T(\mathbf{x}(y),0)G(\mathbf{x}(y),t)H(\mathbf{x}(y),0))}\right|_{t=0}$$
$$\leq 2\Big(\lambda_1(\mathbf{x}(y)) + \cdots + \lambda_p(\mathbf{x}(y))\Big).$$

The remainder of the proof is the same as before.

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